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Common Fixed Points for Non-Self Mappings in Menger PM-Spaces Under a Weak Segment Inequality

Moosa Gabeleh [1,2,*] and Maggie Aphane [2]

[1] Department of Mathematics, Faculty of Basic Sciences, Ayatollah Boroujerdi University, Boroujerd 69199-69737, Iran
[2] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa 0204, South Africa
* Correspondence: gabeleh@abru.ac.ir

Abstract

The objective of this study is to determine whether a nontrivial common fixed point theorem for pairs of non-self mappings in Menger probabilistic metric spaces can be obtained without the strict convexity requirements imposed in earlier work. We establish a common fixed point theorem for pairs of non-self mappings in complete Menger probabilistic metric spaces under continuous convex paths, a non-strict segment inequality, and strictly increasing nontrivial distance distributions. The result does not assume the endpoint-scaling identities traditionally used to define strict convexity. An explicit reparametrized convex structure satisfies these assumptions but fails the strict convexity identities. A second example on a non-strictly convex normed plane shows that even the strict segment inequality is unnecessary. As motivation, we re-examine the geometric hypotheses of a theorem in *Mathematica Slovaca* **2020**, *70*, 1367–1380: its printed definition restricts the space to a singleton and its additional strict inequality is inconsistent on the diagonal. The published metric-induced example also fails the strict inequality for distinct endpoints. We provide a nontrivial probabilistic model, a complete proof for the nonlinear generalized contraction and a single-map corollary.

Keywords: Menger PM-space; non-self mappings; common fixed point; convex structure; weak segment inequality; nonlinear contraction; strict convexity

MSC: 47H10; 54H25

1. Introduction

Fixed point theorems for non-self mappings go back to Assad and Kirk [1]. Their extensions to pairs of mappings and to nonlinear contractions have developed in both deterministic and probabilistic metric spaces; see, for example, [2–5]. Convex structures in Menger probabilistic metric (PM) spaces were introduced by Hadžić [6]. A strict convexity framework was developed for intuitionistic fuzzy metric spaces by Ješić [7] and subsequently in Menger PM-spaces in [8]. Related non-self mapping results appear in [9–11].

The contributions of these references address distinct aspects of the subject. Imdad and Kumar [4] established common fixed point theorems for pairs of non-self mappings in metrically convex metric spaces by using suitable weak commutativity conditions; their setting is deterministic rather than probabilistic. Schweizer and Sklar [5] developed the foundations of probabilistic metric spaces, including the distribution-function approach to distance and its associated topology. Gabeleh [2] revisited a probabilistic best proximity

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result based on a control function and gave a proof through a fixed point principle for probabilistic contractions. More recently, Gabeleh, Ekici and Aphane [3] investigated the best proximity points of cyclic Kannan-type mappings in metrically convex Menger PM-spaces, using probabilistic proximal geometry. Thus, these works concern commuting non-self mappings, foundational PM-space theory, probabilistic best proximity arguments, and cyclic proximal geometry, respectively; none addresses the weak segment condition used in our common fixed point theorem.

For further perspectives, the classical contraction principle of Sehgal and Bharucha-Reid [12] treats fixed points of self-mappings in complete Menger PM-spaces, while Hicks [13] studied metric representations of probabilistic fixed point conditions. In a different generalized-distance setting, Younis [14] developed fixed point results for convex contractions in suprametric spaces, with applications to fractional discrete neural networks. That work uses second-iterate contractions and a suprametric distance; the present work instead treats two non-self maps under a probabilistic min-type nonlinear contractive inequality and weak convex segment geometry. The results are therefore methodologically distinct, and no implication between the two sets of hypotheses is claimed.

Our principal result is a common fixed point theorem for the generalized nonlinear contraction of [11], but under geometric conditions that do not require the endpoint-scaling identities defining strict convexity. The essential assumptions are continuity of the convex paths, a *weak* (non-strict) segment inequality for their points and a strict increase in nontrivial probabilistic distance distributions. The weak inequality is sufficient because the estimates in the proof need lower bounds, not strict separation. This observation permits a stronger and more transparent result than a formal correction of a printed definition.

A motivating issue is that Definition 10 and condition (3.3) in [11], as printed, are incompatible with nontrivial applications. The two endpoint formulas printed in the definition force the underlying space to be a singleton. Independently, the strict condition cannot hold on the diagonal of a convex structure. A metric-induced example used in [11] violates the strict condition even away from the diagonal. The first two obstructions are logically independent.

This motivates our re-examination of the geometric hypotheses of the theorem published in [11]: the purpose is not merely to amend an endpoint formula, but to identify consistent assumptions under which the proposed non-self common fixed point mechanism has nontrivial models. In particular, the proof below replaces the universal strict segment inequality by a non-strict estimate and works with explicitly stated path continuity and distribution monotonicity conditions. Compared with the published argument, the geometric setting is changed and the required recursive estimates are verified directly.

The main difference from the non-self mapping theorem of [11] lies in the geometric assumptions, not in the nonlinear contractive condition or the boundary and range requirements. We show why the published strict conditions cannot yield a nontrivial result and replace them with a weak segment inequality, continuity of convex paths and strict increase in nontrivial distance distributions. In contrast to the strict-convexity setting of [9,11], the resulting theorem does not require the endpoint-scaling identities. The reparametrized convex structure in Proposition 4 and the normed-plane example in Example 3 show that these distinctions are substantive rather than changes of terminology.

The correctly paired endpoint identities already appear in Definition 3.6 of [9]. The strict segment inequality and its application to a step-type distribution, however, require independent verification. The related result of [15] provides additional context. We consider the min-type contraction and the precise range/boundary conditions of [11].

We first examine the geometric assumptions of the published theorem and construct a smooth probabilistic model. We then prove the common fixed point theorem and exhibit

a reparametrized convex structure that satisfies its hypotheses while failing the endpoint-scaling identities. A single-map corollary and two examples of non-self mappings follow, including an application on a non-strictly convex normed plane.

More specifically, Section 2 records the probabilistic-metric and convex-structure definitions and recalls the published theorem. Section 3 identifies the geometric obstructions, and Section 4 constructs and examines a nontrivial corrected model. Section 5 proves the new theorem, gives its consequences and examples, and discusses the role of the segment condition. Section 6 summarizes the contributions and indicates possible further directions.

2. Preliminaries

We recall only the notions needed in the sequel. For further background on probabilistic metric spaces and their topology, we refer to [5].

Throughout, a *nontrivial distance distribution* means $F_{u,v}$ with $u \neq v$. A *weak segment inequality* refers to the non-strict comparison in (14); it is imposed on the points generated by a convex structure, not on numerical segments of the distribution-function parameter. Continuity of a convex path always means continuity in the (ε,λ)-topology recalled below.

Let D^+ denote the family of all distribution functions $G : \mathbb{R} \to [0,1]$ that are non-decreasing and left-continuous, satisfy $G(0) = 0$, and have supremum one. Its maximal element is

$$\varepsilon_0(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

A continuous t-norm is a continuous mapping $T : [0,1]^2 \to [0,1]$ that is associative, commutative, nondecreasing in each variable, and satisfies $T(a,1) = a$ for every $a \in [0,1]$.

A triple (X,F,T) is called a *Menger PM-space* if X is nonempty, T is a continuous t-norm, and $F : X \times X \to D^+$ satisfies

$$F_{x,y} = \varepsilon_0 \text{ if and only if } x = y, \qquad F_{x,y} = F_{y,x},$$

and

$$F_{x,z}(s+t) \geq T\big(F_{x,y}(s), F_{y,z}(t)\big), \tag{1}$$

for all $x,y,z \in X$ and $s,t \geq 0$, where $F_{x,y}$ denotes $F(x,y)$.

The usual (ε,λ)-topology on X is generated by

$$N_x(\varepsilon,\lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1-\lambda\}, \qquad \varepsilon > 0, \quad \lambda \in (0,1).$$

A sequence (x_n) is Cauchy if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists n_0 such that

$$F_{x_n,x_m}(\varepsilon) > 1-\lambda, \qquad n,m \geq n_0,$$

and (X,F,T) is complete if every Cauchy sequence converges in this topology.

A subset $K \subseteq X$ will be called *probabilistically bounded* if, for every $\lambda \in (0,1)$, there exists $t > 0$ such that

$$F_{x,y}(t) > 1-\lambda, \qquad x,y \in K.$$

This is the form of probabilistic boundedness used below.

A mapping $S : X \times X \times [0,1] \to X$ is called a *convex structure* if

$$S(x,y,0) = y, \qquad S(x,y,1) = x,$$

and

$$F_{S(x,y,\theta),z}(2t) \geq T\left(F_{x,z}\left(\frac{t}{\theta}\right), F_{y,z}\left(\frac{t}{1-\theta}\right)\right), \tag{2}$$

for all $x, y, z \in X$, $\theta \in (0,1)$, and $t > 0$. A subset $K \subseteq X$ is *convex* if

$$S(x, y, \theta) \in K, \qquad x, y \in K, \quad \theta \in [0,1].$$

The strict convexity identity as *printed* in [8,11] requires that, for arbitrary $x, y \in X$ and $\theta \in (0,1)$, the point $p = S(x, y, \theta)$ be the unique element satisfying

$$F_{x,y}\left(\frac{t}{\theta}\right) = F_{p,y}(t), \qquad F_{x,y}\left(\frac{t}{1-\theta}\right) = F_{p,y}(t), \qquad t > 0. \tag{3}$$

For later use, note that every t-norm satisfies $T(a,a) \leq a$. Hence, the condition

$$T(a,a) \geq a, \quad \text{for each } a \in [0,1],$$

implies

$$T(a,a) = a, \quad \text{for each } a \in [0,1].$$

If $a \leq b$, monotonicity gives $T(a,a) \leq T(a,b) \leq T(a,1)$, so $T(a,b) = a$. Consequently,

$$T(a,b) = \min\{a,b\}, \quad \text{for each } a, b \in [0,1].$$

Under this condition, (1) takes the form

$$F_{x,y}(s+t) \geq \min\{F_{x,z}(s), F_{z,y}(t)\}. \tag{4}$$

Let $K \subseteq X$, and let $f, g : K \to X$. Following [11], the pair (f,g) is said to be *coincidentally commuting* if, whenever

$$f(x) = g(x) \in K,$$

one has

$$f(g(x)) = g(f(x)).$$

We also use the elementary fact that if $\varphi : (0,+\infty) \to (0,+\infty)$ is continuous and satisfies $\varphi(t) < t$ for every $t > 0$, then

$$\varphi^n(t) \longrightarrow 0, \qquad t > 0. \tag{5}$$

Indeed, $\varphi^n(t)$ is decreasing, and a positive limit would contradict continuity and the strict inequality $\varphi(s) < s$.

The main theorem considered here is the following result from [11].

Theorem 1 (Theorem 4.1 of [11]). *Let (X, F, T) be a strictly convex, complete Menger PM-space with a convex structure S satisfying*

$$F_{S(x,y,\theta),z}(t) > \min\{F_{z,x}(t), F_{z,y}(t)\}, \tag{6}$$

for all $x, y, z \in X$, $\theta \in (0,1)$, and $t > 0$, and suppose that $T(a,a) \geq a$ for every $a \in [0,1]$. Let $K \subseteq X$ be nonempty, closed and probabilistically bounded, and let $f, g : K \to X$ satisfy

$$\begin{aligned} F_{f(x),f(y)}(\varphi(t)) \geq \min\{&F_{g(x),g(y)}(t), F_{g(x),f(x)}(t), F_{g(y),f(y)}(t), \\ &F_{g(x),f(y)}(2t), F_{g(y),f(x)}(2t)\}, \end{aligned} \tag{7}$$

for all $x, y \in K$ and $t > 0$, where $\varphi : (0,+\infty) \to (0,+\infty)$ is continuous, nondecreasing, and satisfies

$$\varphi(t) < t, \qquad t > 0. \tag{8}$$

Assume further that

$$\begin{aligned} &\partial K \subseteq g(K), \qquad f(K) \cap K \subseteq g(K), \\ &g(x) \in \partial K \text{ implies } f(x) \in K, \qquad x \in K, \\ &g(K) \text{ is closed in } X. \end{aligned}$$

Then, f and g have a coincidence point in K. If, in addition, f and g are coincidentally commuting, then they have a unique common fixed point.

3. The Geometric Assumptions

We first examine the definition of strict convexity appearing in (3).

Proposition 1. *If a Menger PM-space with a convex structure satisfies the definition of strict convexity in* (3)*, then X consists of a single point.*

Proof. Let $x, y \in X$. Choose $\theta = 1/3$ and put $p = S(x, y, 1/3)$. From the two equalities in (3),

$$F_{x,y}(3t) = F_{p,y}(t) = F_{x,y}\left(\frac{3t}{2}\right), \qquad t > 0.$$

Setting $s = 3t/2$, we obtain

$$F_{x,y}(2s) = F_{x,y}(s), \qquad s > 0.$$

Hence, for every $n \geq 1$,

$$F_{x,y}(s) = F_{x,y}(2^n s), \qquad s > 0.$$

Since $F_{x,y}$ is nondecreasing and has supremum one, $F_{x,y}(2^n s) \to 1$ as $n \to +\infty$. Therefore

$$F_{x,y}(s) = 1, \qquad s > 0,$$

so $F_{x,y} = \varepsilon_0$. Thus $x = y$. Since x and y were arbitrary, X is a singleton. □

The additional condition (6) is incompatible with a convex structure even without using strict convexity.

Proposition 2. *No nonempty Menger PM-space endowed with a convex structure can satisfy* (6) *under the assumptions of Theorem 1.*

Proof. Every convex structure satisfies

$$S(x, x, \theta) = x, \qquad x \in X, \quad 0 < \theta < 1. \tag{9}$$

Indeed, by (2),

$$F_{S(x,x,\theta),x}(2t) \geq T\left(F_{x,x}\left(\frac{t}{\theta}\right), F_{x,x}\left(\frac{t}{1-\theta}\right)\right) = 1, \qquad t > 0.$$

Thus $F_{S(x,x,\theta),x} = \varepsilon_0$, and (9) follows. Putting $y = x$ in (6) now gives

$$F_{x,z}(t) > F_{x,z}(t), \qquad t > 0,$$

which is impossible. □

Propositions 1 and 2 show that there is no nonempty Menger PM-space satisfying all the geometric assumptions of Theorem 1. The theorem as printed is thus vacuous; this does *not* amount to a counterexample to its logical implication. The metric-generated example used in [11] does not resolve these difficulties.

Example 1. *Consider the usual metric on* $\mathbb{R}$ *and the induced probabilistic metric*

$$F_{x,y}(t) = \varepsilon_0(t - |x - y|), \qquad T = T_{\min},$$

with

$$S(x, y, \theta) = \theta x + (1 - \theta)y.$$

For $x = 0$, $y = 2$, $\theta = 1/4$, and $t = 1$, we have $S(x, y, \theta) = 3/2$, while

$$F_{0,2}\left(\frac{4}{3}\right) = 0, \qquad F_{3/2,2}(1) = 1.$$

Thus the second identity in (3) fails.

Even if (6) is restricted to $x \neq y$, it still fails in this example. Indeed, with

$$x = 0, \qquad y = 2, \qquad \theta = \frac{1}{2}, \qquad z = 4, \qquad t = 1,$$

one has $S(x, y, \theta) = 1$ and

$$F_{1,4}(1) = F_{4,0}(1) = F_{4,2}(1) = 0.$$

Hence the asserted strict inequality reduces to $0 > 0$.

The natural correction of (3) is

$$F_{x,y}\left(\frac{t}{\theta}\right) = F_{S(x,y,\theta),y}(t), \qquad F_{x,y}\left(\frac{t}{1-\theta}\right) = F_{S(x,y,\theta),x}(t), \qquad t > 0, \tag{10}$$

and a formally consistent strict version of (6) must at least exclude $x = y$. The fixed point theorem in Section 5 uses only the corresponding *non-strict* inequality, including on the diagonal.

Remark 1. *The same definition of strict convexity and the same strict structural condition also occur in [15]. Consequently, Propositions 1 and 2 apply to that setting as well. The present paper is concerned only with the non-self mapping result in [11].*

Remark 2 (Historical context and the metric-induced example)**.** *The correctly paired Formula (10) already appear in Definition 3.6 of [9]. In Example 3.8 of that paper, however, a strict metric inequality is used to infer a strict inequality for the step distribution ε_0. This implication does not hold in general: when both arguments of the step function are nonpositive, both values are zero. Example 1 provides explicit numerical values. Independently, Proposition 2 shows that the universal strict segment inequality is impossible.*

4. A Nontrivial Corrected Geometric Setting

We next show that the corrected geometric conditions are satisfied in a nontrivial Menger PM-space.

Lemma 1. *Define F on $\mathbb{R}$ by*

$$F_{x,y}(t) = \begin{cases} 0, & t \leq 0, \\ \dfrac{t}{t + |x - y|}, & t > 0, \end{cases} \tag{11}$$

and let $T = T_{\min}$. Then $(\mathbb{R}, F, T)$ is a Menger PM-space. Moreover,

$$S(x, y, \theta) = \theta x + (1 - \theta)y, \qquad x, y \in \mathbb{R}, \tag{12}$$

is a convex structure on $\mathbb{R}$.

Proof. For each $x, y \in \mathbb{R}$, the function $F_{x,y}$ belongs to D^+: it is nondecreasing and left-continuous, satisfies $F_{x,y}(0) = 0$, and tends to one as $t \to +\infty$. The symmetry and separation axioms are immediate. Let $s, t > 0$, and set

$$a = |x - y|, \qquad b = |y - z|.$$

Since $|x - z| \le a + b$,

$$F_{x,z}(s+t) \ge \frac{s+t}{s+t+a+b}.$$

The elementary inequality

$$\frac{s+t}{s+t+a+b} \ge \min\left\{\frac{s}{s+a}, \frac{t}{t+b}\right\},$$

follows by considering whether $sb \le ta$ or $sb > ta$. Hence, the Menger triangle inequality holds. The cases $s = 0$ or $t = 0$ are immediate.

To verify the convexity inequality, let

$$a = |x - z|, \qquad b = |y - z|, \qquad M = \max\{\theta a, (1-\theta)b\}.$$

Then,

$$|S(x,y,\theta) - z| \le \theta a + (1-\theta)b \le 2M.$$

Consequently,

$$\begin{aligned} F_{S(x,y,\theta),z}(2t) &\ge \frac{2t}{2t+2M} \\ &= \frac{t}{t+M} \\ &= \min\left\{\frac{t}{t+\theta a}, \frac{t}{t+(1-\theta)b}\right\} \\ &= \min\left\{F_{x,z}\left(\frac{t}{\theta}\right), F_{y,z}\left(\frac{t}{1-\theta}\right)\right\}. \end{aligned}$$

The endpoint conditions for S are immediate, so (12) is a convex structure. Moreover,

$$F_{x,y}(\varepsilon) > 1 - \lambda \quad \text{if and only if} \quad |x - y| < \frac{\lambda\varepsilon}{1-\lambda}, \qquad \varepsilon > 0, \quad \lambda \in (0,1).$$

Hence, the (ε, λ)-topology generated by (11) is the usual topology on $\mathbb{R}$. □

Proposition 3. *The space in Lemma 1, equipped with the affine structure (12), satisfies the corrected strict convexity identities (10), and $S(x,y,\theta)$ is the unique point satisfying these identities. In addition, for $x \ne y$,*

$$F_{S(x,y,\theta),z}(t) > \min\{F_{z,x}(t), F_{z,y}(t)\}, \tag{13}$$

for all $z \in \mathbb{R}$, $\theta \in (0,1)$, and $t > 0$.

Proof. Let $p = S(x,y,\theta)$. Since

$$|p - y| = \theta|x - y|, \qquad |p - x| = (1-\theta)|x - y|,$$

we obtain directly from (11) that

$$F_{x,y}\left(\frac{t}{\theta}\right) = F_{p,y}(t), \qquad F_{x,y}\left(\frac{t}{1-\theta}\right) = F_{p,x}(t), \qquad t > 0.$$

If a point $q \in \mathbb{R}$ satisfies the same two identities, then

$$|q - y| = \theta|x - y|, \qquad |q - x| = (1-\theta)|x - y|.$$

The sum of these two distances is $|x - y|$, so q lies between x and y, and has the prescribed distances from the endpoints. Hence $q = p$.

Now assume $x \ne y$. Since p lies strictly between x and y, for every $z \in \mathbb{R}$,

$$|p - z| < \max\{|x - z|, |y - z|\}.$$

Because $r \mapsto t/(t+r)$ is strictly decreasing on $[0,+\infty)$,

$$F_{p,z}(t) > \frac{t}{t+\max\{|x-z|,|y-z|\}} = \min\{F_{z,x}(t), F_{z,y}(t)\}.$$

This proves (13). □

Remark 3. *The equivalence at the end of Lemma 1 also shows that a sequence is Cauchy for (11) exactly when it is Cauchy in the usual metric. Thus $(\mathbb{R}, F, T_{\min})$ is complete. Moreover, whenever $x \neq y$, the function $t \mapsto F_{x,y}(t)$ is strictly increasing on $(0,+\infty)$, and the affine paths $\theta \mapsto S(x,y,\theta)$ are continuous.*

5. A Common Fixed Point Theorem Without Strict Convexity

We now establish the main result. Neither the endpoint-scaling identities (10) nor the strict structural inequality of [11] is assumed. The non-strict segment inequality below suffices for the recursive probabilistic diameter estimates. Continuity of convex paths permits the boundary construction, while strict increase in the nontrivial distance distributions supplies the comparisons needed in the contractive argument. These assumptions describe a different geometric setting: the segment condition is weaker, whereas path continuity and strict monotonicity are additional requirements.

The proof follows a boundary-adjusted iteration. Continuity of the convex paths provides the required points on the boundary, while the weak segment inequality and the nonlinear contraction give lower bounds for the probabilistic distances along the iteration. Completeness then supplies a limit, and strict increase in nontrivial distance distributions is used to establish coincidence and uniqueness.

Theorem 2. *Let (X,F,T) be a complete Menger PM-space with a convex structure S, and suppose that $T(a,a) \geq a$ for every $a \in [0,1]$. Assume that*

1. *for every $u,v \in X$, the path $\theta \mapsto S(u,v,\theta)$ is continuous on $[0,1]$ in the (ε,λ)-topology;*
2. *for all $u,v,w \in X$, all $\theta \in (0,1)$, and all $t>0$,*

$$F_{S(u,v,\theta),w}(t) \geq \min\{F_{w,u}(t), F_{w,v}(t)\}; \tag{14}$$

3. *for every $u \neq v$, the function $F_{u,v}$ is strictly increasing on $(0,+\infty)$.*

Let $K \subseteq X$ be nonempty, closed and probabilistically bounded, and let $f,g : K \to X$ satisfy (7), where $\varphi : (0,+\infty) \to (0,+\infty)$ is continuous, nondecreasing, and satisfies (8). Suppose further that

$$\begin{aligned} &\partial K \subseteq g(K), \qquad f(K) \cap K \subseteq g(K), \\ &g(x) \in \partial K \text{ implies } f(x) \in K, \qquad x \in K, \\ &g(K) \text{ is closed in } X. \end{aligned} \tag{15}$$

Then, f and g have a coincidence point in K. If, in addition, f and g are coincidentally commuting, then they have a unique common fixed point.

Proof. As observed after (8), the assumption on T gives $T = T_{\min}$, so (4) is available throughout the proof.

We construct recursively sequences

$$(x_n) \subseteq K, \qquad (z_n) \subseteq K,$$

and a sequence $(y_n) \subseteq X$ satisfying

$$y_n = f(x_{n-1}), \qquad n \geq 1,$$

and

$$z_n = g(x_n), \qquad n \geq 0.$$

First suppose that $K = X$. Choose $x_0 \in K$ arbitrarily and set

$$z_0 = g(x_0).$$

Define

$$y_1 = f(x_0).$$

Since $K = X$, we have $y_1 \in K$. Hence,

$$y_1 \in f(K) \cap K \subseteq g(K),$$

so there exists $x_1 \in K$ such that

$$z_1 = g(x_1) = y_1.$$

Proceeding inductively, once $x_{n-1} \in K$ has been chosen, define

$$y_n = f(x_{n-1}).$$

Since $K = X$, we have $y_n \in K$, and therefore

$$y_n \in f(K) \cap K \subseteq g(K).$$

Thus, there exists $x_n \in K$ such that

$$z_n = g(x_n) = y_n.$$

Consequently, when $K = X$,

$$z_n = y_n, \qquad n \geq 1.$$

Now suppose that $K \neq X$. Since the mappings

$$[0,1] \ni \theta \longmapsto S(u, v, \theta),$$

are continuous for every $u, v \in X$, the space X is path connected and hence connected. In view of the fact that K is a nonempty proper closed subset of X, we obtain $\partial K \neq \emptyset$. Choose $z_0 \in \partial K$. Since $\partial K \subseteq g(K)$, there exists $x_0 \in K$ such that $z_0 = g(x_0)$. By the boundary condition,

$$y_1 = f(x_0) \in K.$$

Hence,

$$y_1 \in f(K) \cap K \subseteq g(K),$$

and therefore there exists $x_1 \in K$ such that

$$z_1 = g(x_1) = y_1.$$

Assume inductively that $x_{n-1} \in K$ and $z_{n-1} = g(x_{n-1}) \in K$ have been constructed for some $n \geq 2$, and define

$$y_n = f(x_{n-1}).$$

If $y_n \in K$, then

$$y_n \in f(K) \cap K \subseteq g(K),$$

so there exists $x_n \in K$ such that

$$z_n = g(x_n) = y_n.$$

Suppose instead that $y_n \notin K$. Then, $z_{n-1} \in \operatorname{int} K$. Indeed, if $z_{n-1} \in \partial K$, then

$$g(x_{n-1}) = z_{n-1} \in \partial K,$$

would imply

$$y_n = f(x_{n-1}) \in K,$$

which is a contradiction. Since $z_{n-1} \in K$ and K is closed, we therefore have

$$z_{n-1} \in K \setminus \partial K = \operatorname{int} K.$$

Now consider the continuous path

$$\gamma_n(\theta) = S(z_{n-1}, y_n, \theta), \qquad 0 \le \theta \le 1.$$

Since

$$\gamma_n(0) = y_n \notin K, \qquad \gamma_n(1) = z_{n-1} \in \operatorname{int} K,$$

the path meets ∂K. Hence, there exists $\theta_n \in (0,1)$ such that

$$z_n = S(z_{n-1}, y_n, \theta_n) \in \partial K.$$

Since $\partial K \subseteq g(K)$, there exists $x_n \in K$ such that $z_n = g(x_n)$. Moreover,

$$g(x_n) = z_n \in \partial K,$$

and therefore

$$y_{n+1} = f(x_n) \in K.$$

Thus, in both cases, the construction yields

$$y_n = f(x_{n-1}), \qquad z_n = g(x_n) \in K, \qquad n \ge 1,$$

with

$$y_n \in K \quad \text{implies} \quad z_n = y_n, \tag{16}$$

and

$$y_n \notin K \quad \text{implies} \quad z_n \in \partial K \quad \text{and} \quad y_{n+1} \in K, \qquad n \ge 1. \tag{17}$$

In particular, no two consecutive terms of (y_n) lie outside K.

The estimates below do not depend on excluding an early coincidence point; hence, the argument proceeds uniformly even when one occurs.

Suppose that $\ell \ge 2$ and $y_\ell \notin K$. Then, by the construction, $z_\ell \in \partial K$. Moreover, $y_{\ell-1} \in K$. Indeed, if $y_{\ell-1} \notin K$, then (17) would imply $y_\ell \in K$, a contradiction. Hence, by (16),

$$z_{\ell-1} = y_{\ell-1}.$$

Applying (7) to $x_{\ell-2}$ and $x_{\ell-1}$ and using $z_{\ell-1} = y_{\ell-1}$, we obtain

$$\begin{aligned} F_{y_{\ell-1},y_\ell}(\varphi(t)) \ge \min\{&F_{z_{\ell-2},z_{\ell-1}}(t), F_{z_{\ell-2},y_{\ell-1}}(t), F_{z_{\ell-1},y_\ell}(t), \\ &F_{z_{\ell-2},y_\ell}(2t), F_{z_{\ell-1},y_{\ell-1}}(2t)\}. \end{aligned}$$

Now,

$$F_{z_{\ell-2},y_{\ell-1}}(t) = F_{z_{\ell-2},z_{\ell-1}}(t), \qquad F_{z_{\ell-1},y_{\ell-1}}(2t) = 1,$$

and by (4),

$$F_{z_{\ell-2},y_\ell}(2t) \ge \min\{F_{z_{\ell-2},z_{\ell-1}}(t), F_{z_{\ell-1},y_\ell}(t)\}.$$

Since $z_{\ell-1} = y_{\ell-1}$, it follows that

$$F_{y_{\ell-1},y_\ell}(\varphi(t)) \ge \min\{F_{z_{\ell-2},z_{\ell-1}}(t), F_{y_{\ell-1},y_\ell}(t)\}, \qquad t > 0. \tag{18}$$

If for some $t > 0$,

$$F_{y_{\ell-1},y_\ell}(t) < F_{z_{\ell-2},z_{\ell-1}}(t),$$

then (18) gives

$$F_{y_{\ell-1},y_\ell}(\varphi(t)) \geq F_{y_{\ell-1},y_\ell}(t).$$

Since $y_{\ell-1} \in K$, $y_\ell \notin K$ and $\varphi(t) < t$, this contradicts the strict increase in $F_{y_{\ell-1},y_\ell}$. Therefore,

$$F_{y_{\ell-1},y_\ell}(t) \geq F_{z_{\ell-2},z_{\ell-1}}(t), \qquad t > 0. \tag{19}$$

Now, for $n \geq 0$ and $t > 0$, define

$$m_n(t) = \inf\{F_{z_k,z_\ell}(t) : k, \ell \geq n\}. \tag{20}$$

We claim that

$$m_n(\varphi(t)) \geq m_{n-2}(t), \qquad n \geq 2, \quad t > 0. \tag{21}$$

Fix $k, \ell \geq n$ and put

$$M = m_{n-2}(t).$$

Suppose first that $y_k \in K$ and $y_\ell \in K$. Then,

$$z_k = y_k = f(x_{k-1}), \qquad z_\ell = y_\ell = f(x_{\ell-1}).$$

Applying (7) to x_{k-1} and $x_{\ell-1}$, we obtain

$$\begin{aligned} F_{z_k,z_\ell}(\varphi(t)) \geq \min\{&F_{z_{k-1},z_{\ell-1}}(t), F_{z_{k-1},z_k}(t), F_{z_{\ell-1},z_\ell}(t), \\ &F_{z_{k-1},z_\ell}(2t), F_{z_{\ell-1},z_k}(2t)\}. \end{aligned}$$

All indices occurring here are at least $n-2$. Hence, by the definition of M and the monotonicity of the distribution functions,

$$F_{z_k,z_\ell}(\varphi(t)) \geq M.$$

Next, suppose that $y_\ell \notin K$. We first establish an auxiliary estimate. If $r \geq n-1$ and $y_r \in K$, then

$$F_{z_r,y_\ell}(\varphi(t)) \geq M. \tag{22}$$

Indeed,

$$z_r = y_r = f(x_{r-1}).$$

Applying (7) to x_{r-1} and $x_{\ell-1}$, the first two terms on its right-hand side are bounded below by M. Moreover, (19) gives

$$F_{z_{\ell-1},y_\ell}(t) \geq F_{z_{\ell-2},z_{\ell-1}}(t) \geq M.$$

Also, by (4),

$$F_{z_{r-1},y_\ell}(2t) \geq \min\{F_{z_{r-1},z_r}(t), F_{z_r,y_\ell}(t)\} \geq \min\{M, F_{z_r,y_\ell}(t)\},$$

while

$$F_{z_{\ell-1},z_r}(2t) \geq M.$$

Consequently,

$$F_{z_r,y_\ell}(\varphi(t)) \geq \min\{M, F_{z_r,y_\ell}(t)\}.$$

If $z_r = y_\ell$, then (22) is immediate. Otherwise, if

$$F_{z_r,y_\ell}(t) < M,$$

then

$$F_{z_r,y_\ell}(\varphi(t)) \geq F_{z_r,y_\ell}(t),$$

which contradicts the strict increase in F_{z_r,y_ℓ}, since $\varphi(t) < t$. Thus, (22) holds.

Now suppose that $y_k \in K$ and $y_\ell \notin K$. Then,

$$z_k = y_k, \qquad z_\ell = S(z_{\ell-1}, y_\ell, \theta_\ell).$$

Since $z_{\ell-1} \in K$ and $y_\ell \notin K$, we have $z_{\ell-1} \neq y_\ell$. Hence, (14) gives

$$F_{z_k,z_\ell}(\varphi(t)) \geq \min\{F_{z_k,z_{\ell-1}}(\varphi(t)), F_{z_k,y_\ell}(\varphi(t))\}.$$

The second term is bounded below by M by (22). Since $y_{\ell-1} \in K$, we have

$$z_{\ell-1} = y_{\ell-1} = f(x_{\ell-2}),$$

and the first case calculation, applied to x_{k-1} and $x_{\ell-2}$, gives

$$F_{z_k,z_{\ell-1}}(\varphi(t)) \geq M.$$

Therefore,

$$F_{z_k,z_\ell}(\varphi(t)) \geq M.$$

The case $y_k \notin K$ and $y_\ell \in K$ follows by symmetry.

Finally, suppose that

$$y_k \notin K, \qquad y_\ell \notin K.$$

Then,

$$y_{k-1}, y_{\ell-1} \in K, \qquad z_{k-1} = y_{k-1}, \qquad z_{\ell-1} = y_{\ell-1}.$$

Applying (14) twice yields

$$\begin{aligned} F_{z_k,z_\ell}(\varphi(t)) \geq \min\{&F_{z_{k-1},z_{\ell-1}}(\varphi(t)), F_{z_{k-1},y_\ell}(\varphi(t)), \\ &F_{y_k,z_{\ell-1}}(\varphi(t)), F_{y_k,y_\ell}(\varphi(t))\}. \end{aligned}$$

The first term is bounded below by M by the first case calculation, applied to x_{k-2} and $x_{\ell-2}$. The second term is bounded below by M by (22), with $r = k-1$ and by symmetry, the third term is bounded below by M.

For the last term, apply (7) to x_{k-1} and $x_{\ell-1}$. We obtain

$$\begin{aligned} F_{y_k,y_\ell}(\varphi(t)) \geq \min\{&F_{z_{k-1},z_{\ell-1}}(t), F_{z_{k-1},y_k}(t), F_{z_{\ell-1},y_\ell}(t), \\ &F_{z_{k-1},y_\ell}(2t), F_{z_{\ell-1},y_k}(2t)\}. \end{aligned}$$

The first term is at least M. By (19),

$$F_{z_{k-1},y_k}(t) \geq F_{z_{k-2},z_{k-1}}(t) \geq M,$$

and similarly

$$F_{z_{\ell-1},y_\ell}(t) \geq F_{z_{\ell-2},z_{\ell-1}}(t) \geq M.$$

Furthermore, (22) and monotonicity give

$$F_{z_{k-1},y_\ell}(2t) \geq M, \qquad F_{z_{\ell-1},y_k}(2t) \geq M.$$

Consequently,

$$F_{y_k,y_\ell}(\varphi(t)) \geq M.$$

Hence,

$$F_{z_k,z_\ell}(\varphi(t)) \geq M.$$

Since $k, \ell \geq n$ were arbitrary, (21) follows.

Let $s > 0$ and $\lambda \in (0,1)$. By the probabilistic boundedness of K, there exists $t_0 > 0$ such that

$$F_{u,v}(t_0) > 1 - \frac{\lambda}{2}, \qquad u, v \in K.$$

Since $z_k, z_\ell \in K$ for all $k, \ell \geq 0$, we have

$$m_0(t_0) \geq 1 - \frac{\lambda}{2} > 1 - \lambda.$$

By (5), choose r such that

$$\varphi^r(t_0) < s.$$

Iterating (21) gives

$$m_{2r}(\varphi^r(t_0)) \geq m_0(t_0) > 1 - \lambda.$$

Since m_{2r} is nondecreasing,

$$m_{2r}(s) > 1 - \lambda.$$

Therefore,

$$F_{z_k, z_\ell}(s) > 1 - \lambda, \qquad k, \ell \geq 2r,$$

and hence (z_n) is Cauchy. By completeness of X, there exists $z \in X$ such that $z_n \longrightarrow z$. Since $z_n \in K$ for every n and K is closed, $z \in K$. Moreover, $z_n = g(x_n) \in g(K)$ for every n, and $g(K)$ is closed. Hence $z \in g(K)$, so there exists $\omega \in K$ such that $g(\omega) = z$. By (17), no two consecutive terms of (y_n) lie outside K. Hence, there are infinitely many indices $n \geq 1$ for which $y_n \in K$. Choose a strictly increasing sequence (n_j) of such indices. Then, by (16),

$$z_{n_j} = y_{n_j} = f(x_{n_j - 1}) \longrightarrow z,$$

while

$$z_{n_j - 1} = g(x_{n_j - 1}) \longrightarrow z.$$

We now prove that $f(\omega) = z$. Suppose, on the contrary, that $f(\omega) \neq z$, and fix $t > 0$. Set

$$a = F_{z, f(\omega)}(t).$$

Since $z \neq f(\omega)$ and $F_{z, f(\omega)}$ is strictly increasing on $(0, +\infty)$, we have $a < 1$. Choose $\lambda \in (0,1)$ such that

$$a < 1 - \lambda.$$

For all sufficiently large j,

$$F_{z_{n_j - 1}, z}(t) > a, \qquad F_{z_{n_j - 1}, z_{n_j}}(t) > a, \qquad F_{z, z_{n_j}}(2t) > a.$$

Moreover, by (4),

$$F_{z_{n_j - 1}, f(\omega)}(2t) \geq \min\{F_{z_{n_j - 1}, z}(t), F_{z, f(\omega)}(t)\} \geq a.$$

Applying (7) to $x_{n_j - 1}$ and ω, and using

$$f(x_{n_j - 1}) = z_{n_j}, \qquad g(x_{n_j - 1}) = z_{n_j - 1}, \qquad g(\omega) = z,$$

we obtain, for all sufficiently large j,

$$F_{z_{n_j}, f(\omega)}(\varphi(t)) \geq a.$$

Set

$$\eta = \frac{t - \varphi(t)}{2} > 0, \qquad \tau = \varphi(t) + \eta = \frac{t + \varphi(t)}{2} < t.$$

For all sufficiently large j,

$$F_{z,z_{n_j}}(\eta) > a.$$

Hence, by (4),

$$\begin{aligned} F_{z,f(\omega)}(\tau) &= F_{z,f(\omega)}(\eta + \varphi(t)) \\ &\geq \min\{F_{z,z_{n_j}}(\eta), F_{z_{n_j},f(\omega)}(\varphi(t))\} \\ &\geq a = F_{z,f(\omega)}(t). \end{aligned}$$

This contradicts the strict increase in $F_{z,f(\omega)}$, since $\tau < t$. Therefore,

$$f(\omega) = z = g(\omega),$$

and hence $\omega \in K$ is a coincidence point of f and g.

Assume now that f and g are coincidentally commuting. Since

$$f(\omega) = g(\omega) = z \in K,$$

we obtain

$$f(z) = f(g(\omega)) = g(f(\omega)) = g(z).$$

Applying (7) to z and ω gives

$$F_{f(z),z}(\varphi(t)) \geq F_{f(z),z}(t), \qquad t > 0.$$

If $f(z) \neq z$, this contradicts the strict increase in $F_{f(z),z}$. Hence,

$$f(z) = g(z) = z.$$

Thus, z is a common fixed point of f and g.

Finally, suppose that $u \in K$ is another common fixed point. Applying (7) to z and u, we obtain

$$F_{z,u}(\varphi(t)) \geq F_{z,u}(t), \qquad t > 0.$$

If $z \neq u$, this contradicts the strict increase in $F_{z,u}$. Therefore $z = u$, and the common fixed point is unique. □

Remark 4. *The corrected strict convexity identities* (10) *are not used in Theorem* 2*. Indeed, even the* strict *version of the segment inequality is unnecessary: every appeal in the proof is to a lower bound, so* (14) *with* $\geq$ *suffices. The strict increase in* $F_{u,v}$*, for* $u \neq v$*, is still a substantive requirement. In particular, the step-type metric-induced distributions in Example* 1 *cannot be substituted directly into the present theorem because they are not strictly increasing.*

As to a possible local replacement for the global segment assumption, the proof invokes (14) *for segments involving the recursively constructed sequences, including points that may lie outside* K*, and with comparison points varying along those sequences. Consequently, the proof does not establish the conclusion when the inequality is known only in an unspecified neighborhood of each point. The argument can be localized if the inequality is valid for* all *segment endpoints and comparison points actually used in the construction, but such a condition requires additional invariance or localization assumptions. The question of a genuinely local hypothesis without further conditions remains open here.*

Proposition 4. *The geometric hypotheses of Theorem* 2 *do not imply the corrected endpoint-scaling identities* (10) *for the given convex structure, even when the strict inequality* (13) *holds whenever the endpoints differ.*

Proof. Equip $X = \mathbb{R}$ with the Menger PM-structure (11) and $T = T_{\min}$. Define

$$h(\theta) = 3\theta^2 - 2\theta^3, \qquad S_h(x, y, \theta) = h(\theta)x + (1 - h(\theta))y.$$

We verify that S_h is a convex structure in the sense of (2). The identity

$$h(\theta) - \theta = \theta(1 - \theta)(2\theta - 1)$$

shows that $0 \le h(\theta) \le \theta$ when $0 \le \theta \le 1/2$, whereas $\theta \le h(\theta) \le 1$ when $1/2 \le \theta \le 1$. For fixed x, y, z, put

$$M = \max\{\theta|x - z|, (1 - \theta)|y - z|\}, \qquad A = \theta x + (1 - \theta)y.$$

As in Lemma 1, $|A - z| \le 2M$. If $\theta \le 1/2$, then $S_h(x, y, \theta)$ lies on the interval between y and A, and $|y - z| \le 2(1 - \theta)|y - z| \le 2M$. If $\theta \ge 1/2$, it lies between A and x, and $|x - z| \le 2\theta|x - z| \le 2M$. Convexity of the absolute value therefore gives

$$|S_h(x, y, \theta) - z| \le 2M.$$

By formula (11), this is exactly the probabilistic convex-structure inequality (2); the endpoint identities are immediate from $h(0) = 0$ and $h(1) = 1$.

The paths $\theta \mapsto S_h(x, y, \theta)$ are continuous and $F_{u,v}$ is strictly increasing for $u \ne v$. Moreover, $0 < h(\theta) < 1$ for $0 < \theta < 1$, so when $x \ne y$, the point $S_h(x, y, \theta)$ lies strictly between x and y. Consequently, for every $z \in \mathbb{R}$ and $t > 0$,

$$F_{S_h(x,y,\theta),z}(t) > \min\{F_{z,x}(t), F_{z,y}(t)\}.$$

For $x = y$, the non-strict inequality (14) holds with equality. Hence, all the geometric assumptions of Theorem 2 are satisfied.

Nevertheless, take $x = 1$, $y = 0$, and $\theta = 1/4$. Then $S_h(1, 0, 1/4) = h(1/4) = 5/32$, and with $t = 1$,

$$F_{1,0}(4) = \frac{4}{5} \quad \text{while} \quad F_{S_h(1,0,1/4),0}(1) = \frac{32}{37}.$$

Thus, the first identity in (10) fails for S_h. □

Remark 5. *Proposition 4 proves independence of the endpoint-scaling identities* for the designated convex structure S_h. *It does not say that the underlying space* $\mathbb{R}$ *cannot carry a different convex structure satisfying those identities; indeed, the affine structure* (12) *does. Thus the example distinguishes the chosen convex structure from any alternative structures on the same space.*

Corollary 1. *Let* (X, F, T), S, K, *and* φ *satisfy the assumptions of Theorem 2. Let* $f : K \to X$ *satisfy*

$$\begin{aligned} F_{f(x),f(y)}(\varphi(t)) \ge \min\{&F_{x,y}(t), F_{x,f(x)}(t), F_{y,f(y)}(t), \\ &F_{x,f(y)}(2t), F_{y,f(x)}(2t)\}, \end{aligned} \tag{23}$$

for all $x, y \in K$ *and* $t > 0$. *If* $f(\partial K) \subseteq K$, *then* f *has a unique fixed point in* K.

Proof. Apply Theorem 2 with g equal to the identity mapping. The range and closedness assumptions are then automatic, and the boundary condition reduces to $f(\partial K) \subseteq K$. □

Example 2. *The mappings used in the example following Theorem 4.1 of [11] fit the corrected framework if the step-type induced probabilistic metric is replaced by* (11). *Let*

$$K = [1, 2], \qquad f(x) = \frac{6}{5}x - \frac{1}{5}, \qquad g(x) = x^2,$$

and define

$$\varphi(t) = \begin{cases} \dfrac{t}{1+t}, & 0 < t < \frac{1}{2}, \\ \dfrac{2}{3}t, & t \geq \frac{1}{2}. \end{cases}$$

Then, φ is continuous, nondecreasing, and $\varphi(t) < t$, while $\varphi(t) \geq \frac{2}{3}t$. For $x, y \in [1, 2]$,

$$|f(x) - f(y)| = \frac{6}{5}|x - y|, \qquad |g(x) - g(y)| = (x + y)|x - y|,$$

and therefore

$$\frac{\varphi(t)}{t}|g(x) - g(y)| \geq \frac{4}{3}|x - y| > \frac{6}{5}|x - y|.$$

A direct comparison in (11) gives

$$F_{f(x),f(y)}(\varphi(t)) \geq F_{g(x),g(y)}(t),$$

so (7) is satisfied. The range and boundary conditions are the same as in the published example, and the pair is coincidentally commuting at its coincidence point $x = 1$. Hence Theorem 2 gives the unique common fixed point 1.

Example 3. *Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ and define, for $u, v \in X$,*

$$F_{u,v}(t) = \begin{cases} 0, & t \leq 0, \\ \dfrac{t}{t + \|u - v\|_\infty}, & t > 0. \end{cases} \qquad T = T_{\min}.$$

The metric argument in Lemma 1 works with any norm in place of $|\cdot|$, so this is a complete Menger PM-space. Equip it with the affine convex structure $S(u, v, \theta) = \theta u + (1 - \theta)v$. The triangle inequality gives (2), and

$$\|S(u, v, \theta) - w\|_\infty \leq \theta\|u - w\|_\infty + (1 - \theta)\|v - w\|_\infty \leq \max\{\|u - w\|_\infty, \|v - w\|_\infty\}.$$

Consequently, (14) holds. The paths are continuous and every nontrivial distance distribution is strictly increasing. Nevertheless, the corresponding strict *segment inequality fails. Indeed, for*

$$u = (-1, 0), \quad v = (1, 0), \quad \theta = \tfrac{1}{2}, \quad w = (0, 2),$$

all three distances from w to u, v and $S(u, v, 1/2)$ equal 2; hence, equality holds in (14). The unique-point condition in the corrected strict convexity definition also does not hold for this structure: both $p = (0, 0)$ and $q = (0, 1/2)$ are at distance 1 from each of u, v when $\theta = 1/2$.

For a genuine non-self application, put $e_1 = (1, 0)$,

$$K = \{se_1 : 1 \leq s \leq 2\}, \qquad f(se_1) = \left(\tfrac{6}{5}s - \tfrac{1}{5}\right)e_1, \qquad g(se_1) = s^2 e_1,$$

and take the continuous control function

$$\varphi(t) = \begin{cases} t/(1 + t), & 0 < t < 1/2, \\ 2t/3, & t \geq 1/2. \end{cases}$$

Then, $\varphi(t) < t$ and $\varphi(t)/t \geq 2/3$ for every $t > 0$. For $s, r \in [1, 2]$,

$$\|f(se_1) - f(re_1)\|_\infty = \tfrac{6}{5}|s - r| \leq \tfrac{\varphi(t)}{t}(s + r)|s - r| = \tfrac{\varphi(t)}{t}\|g(se_1) - g(re_1)\|_\infty.$$

Therefore, for the chosen probabilistic metric,

$$F_{f(se_1),f(re_1)}(\varphi(t)) \geq F_{g(se_1),g(re_1)}(t),$$

which implies (7). *Here, K is nonempty, closed and probabilistically bounded. Its interior in X is empty, so $\partial K = K$. Furthermore, $g(K) = [1,4]e_1$ is closed and contains both ∂K and $f(K) \cap K$. If $g(se_1) \in \partial K$, then $1 \leq s \leq \sqrt{2}$, so $f(se_1) \in K$. Finally, the unique coincidence point is e_1: $s^2 = (6s-1)/5$ reduces to $(s-1)(5s-1) = 0$ and only $s = 1$ belongs to $[1,2]$. The maps commute at this point and have the unique common fixed point e_1. In particular, $f(2e_1) = \frac{11}{5}e_1 \notin K$, so f is genuinely non-self.*

6. Concluding Remarks

The generalized nonlinear contraction and boundary/range requirements of [11] admit a common fixed point theorem without the endpoint-scaling identities associated with strict convexity. The proof uses continuous convex paths, the weak segment inequality (14) and strictly increasing nontrivial distance distributions. Proposition 4 demonstrates that these hypotheses do not implicitly restore the endpoint-scaling identities for the chosen convex structure. Example 3 additionally shows that the theorem applies to a non-strictly convex normed space where the strict segment inequality fails, with genuinely non-self mappings. The defects in the published geometric setup are logically separate: one concerns the printed definition, another concerns a universal strict inequality and a third concerns its step-distribution example. The corrected fixed point theorem and the two independence examples provide a nonvacuous alternative under explicit geometric assumptions.

The proof also explains the role of each assumption: path continuity supports the boundary construction, the weak segment inequality controls the iterative estimates and strict increase in the distance distributions yields the final coincidence and uniqueness arguments. The rational probabilistic model establishes that the assumptions have non-trivial instances, while the two examples show their independence from stronger forms of convexity. The framework may be relevant to constrained nonlinear operator equations and to approximation schemes in probabilistic metric spaces, provided the required contractive, geometric and range conditions can be verified. Concrete applications and constructive convergence results remain to be investigated.

Possible directions for subsequent work include determining whether a localized segment inequality suffices under suitable invariant set assumptions, studying related nonlinear contractive pairs with other continuous t-norms, and developing constructive approximation or application results for non-self mappings in probabilistic settings. These questions lie beyond the hypotheses and conclusions proved in the present paper.

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