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Existence and Mittag–Leffler Stability for the Solution of a Fuzzy Fractional System with Application of Laplace Transforms to Solve Fractional Differential Systems

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Abstract: This study explores the existence and Mittag–Leffler stability of solutions for fuzzy fractional systems that include Caputo derivatives and ordinary derivatives with non-local conditions using the Schauder fixed-point theorem. Following this, we employ the Laplace transform method and numerical techniques to create iterative methods for obtaining exact and approximate solutions.

Keywords: Caputo derivative; fuzzy; fractional equation; Mittag–Leffler; Laplace transform



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1. Introduction

In the study of differential equations with uncertainty, it is essential to model approximate or imprecise quantities. Fuzzy numbers provide a natural and powerful framework for capturing such uncertainties. Before introducing the fractional differential system, we briefly define the space of fuzzy numbers $\mathbb{R}_F$, which will serve as the ambient space for our solutions.

1.1. Definition of Fuzzy Numbers $\mathbb{R}_F$

A fuzzy number is a fuzzy subset $\varphi : \mathbb{R} \rightarrow [0, 1]$ satisfying the following properties:

1. Normality: There exists $z_0 \in \mathbb{R}$ such that $\varphi(z_0) = 1$.
2. Convexity: For all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \geq \min\{\varphi(x), \varphi(y)\}.$$

3. Upper Semi-Continuity: The function φ is upper semi-continuous on $\mathbb{R}$.
4. Compact Support: The support of φ ,

$$\text{supp}(\varphi) := \{x \in \mathbb{R} : \varphi(x) > 0\},$$

is a compact subset of $\mathbb{R}$.

The collection of all fuzzy numbers on $\mathbb{R}$ is denoted by $\mathbb{R}_F$. For each $\varphi \in \mathbb{R}_F$, the r -level set (also called the r -cut) is defined for each $r \in (0, 1]$ as

$$[\varphi]_r := \{x \in \mathbb{R} : \varphi(x) \geq r\}.$$

Each r -level set is a closed interval:

$$[\varphi]_r = [\underline{\varphi}_r, \overline{\varphi}_r],$$

where $\underline{\varphi}_r, \overline{\varphi}_r : [0, 1] \rightarrow \mathbb{R}$ are the lower and upper endpoints, respectively, satisfying the following:

- $\underline{\varphi}_r$ is a bounded, non-decreasing, left-continuous function;
- $\overline{\varphi}_r$ is a bounded, non-increasing, left-continuous function;
- $\underline{\varphi}_r \leq \overline{\varphi}_r$ for all $r \in [0, 1]$.

Fractional differential problems in fuzzy spaces were studied by Agarwal et al. [1], who presented a concept for solving fractional differential equations, which is associated with uncertainty, by considering the Hukuhara difference and the fractional Riemann–Liouville derivative. The authors in [2,3] discussed interval-valued functions and considered existence and uniqueness for

$${}_{\mathbb{F}}D_0^l \phi(\xi) = \gamma(\xi, \phi(\xi)),$$

with initial conditions (ICs)

$$\lim_{\xi \rightarrow 0+} \xi^{1-l} \phi(\xi) = \phi_0,$$

where $\phi \in C([0, t], \mathbb{R}_{\mathbb{F}}) \cap L^1([0, t], \mathbb{R}_{\mathbb{F}})$, $0 < l < 1$, ${}_{\mathbb{F}}D_0^l$ is the fractional derivative of order l , $\gamma : [0, t] \rightarrow \mathbb{R}_{\mathbb{F}}$ and $\gamma \in C([0, t], \mathbb{R}_{\mathbb{F}})$. Alikhani and Bahrami in [4] studied

$$\phi(\xi) = \gamma_1(\xi) + \frac{1}{\Gamma(l)} \int_0^{\xi} (\xi - z)^{l-1} \gamma_2(z, \phi(z), (T\phi)(z)) dz,$$

where $0 < l < 1$, $\xi \in J = (0, T]$, $\gamma_1 \in C(J, \mathbb{R}_{\mathbb{F}})$, $\gamma_2 \in C(J \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}})$ and

$$(T\phi)(\xi) = \int_0^{\xi} y(\xi, z) \phi(z) dz,$$

where y is continuous, and they established the existence of a solution for the following problem, which involves a Riemann–Liouville derivative of order $0 < l < 1$:

$${}_{\mathbb{F}}D_0^l \phi(\xi) = \gamma(\xi, \phi(\xi), (T\phi)(\xi)),$$

with (IC)

$$\lim_{\xi \rightarrow 0+} \xi^{1-l} \phi(\xi) = \phi_0.$$

Numerical solutions for interval-valued fractional equations was considered in [5]. Also, for solving fuzzy fractional equations with Laplace transforms we can refer [6,7], Mfadel et al., in [8], studied the Mittag–Leffler stability for the problem

$$\begin{cases} {}_{\mathbb{F}}^C D_{0+}^l \phi(\xi) = \gamma(\xi, \phi(\xi)), \\ \phi(0) = \phi_0, \end{cases}$$

where $\xi \in J = [0, +\infty]$, $\gamma : J \times \mathbb{R}_{\mathbb{F}} \rightarrow \mathbb{R}_{\mathbb{F}}$ is a fuzzy continuous function in ξ and locally Lipschitz in ϕ , and ${}_{\mathbb{F}}^C D_{0+}^l$ is the Caputo fractional derivative of $\phi(\xi)$ of order $0 < l < 1$.

Agarwal et al. [9] extended the Schauder fixed-point theorem to semilinear Banach spaces and considered the existence of solutions to fuzzy FDEs under H-differentiability. Long et al., in [10], established existence results for the following non-local fuzzy fractional system:

$$\begin{cases} {}^c_{gH}D^l\phi(\xi) = \gamma_1(\xi, \phi(\xi), \psi(\xi)), \\ {}^c_{gH}D^l\psi(\xi) = \gamma_2(\xi, \phi(\xi), \psi(\xi)). \end{cases}$$

For further results concerning fuzzy sets, numbers, calculus, and interval calculus, we refer the reader to [11–13], for fuzzy partial differential equations and fuzzy fractional partial differential equations [14–16], for numerical methods in fuzzy integration and numerical solutions of fuzzy differential equations [17–19], and for real applications of fractional calculus [20–23].

Motivated by these works, we study the following fractional system:

$$\begin{cases} {}^c_{gH}D^l\phi(\xi) = \gamma_1(\xi, \phi(\xi), \psi(\xi)) + \phi'(\xi), \\ {}^c_{gH}D^l\psi(\xi) = \gamma_2(\xi, \phi(\xi), \psi(\xi)) + \psi'(\xi), \end{cases} \quad (1)$$

with non-local conditions

$$\begin{cases} \phi(0) \oplus \sum_{j \in E_1} \alpha_j \odot (\phi(\xi_j)) \oplus {}^{RL}_{\mathbb{F}}I_{0+}^{l-1}\phi(\xi_j) \\ \quad = \sum_{j \in E_2} \alpha_j \odot (\phi(\xi_j)) \oplus {}^{RL}_{\mathbb{F}}I_{0+}^{l-1}\phi(\xi_j), \\ \psi(0) \oplus \sum_{j \in F_1} \beta_j \odot (\psi(\xi_j)) \oplus {}^{RL}_{\mathbb{F}}I_{0+}^{l-1}\psi(\xi_j) \\ \quad = \sum_{j \in F_2} \beta_j \odot (\psi(\xi_j)) \oplus {}^{RL}_{\mathbb{F}}I_{0+}^{l-1}\psi(\xi_j), \end{cases} \quad (2)$$

where $\mathbb{R}_{\mathbb{F}}$ is the real fuzzy space and the following apply:

- (1) $\phi, \psi : [0, 1] \rightarrow \mathbb{R}_{\mathbb{F}}$ are two fuzzy differentiable functions.
- (2) The functions $\gamma_1, \gamma_2 : J \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \rightarrow \mathbb{R}_{\mathbb{F}}$ are in $L^1(J, \mathbb{R}_{\mathbb{F}})$.
- (3) E_1 and E_2 are disjoint sets and their union is $\{1, \dots, n\}$, and F_1, F_2 are disjoint sets and their union is $\{1, \dots, n\}$.
- (4) $0 < \xi_1 \leq \xi_2 \leq \dots \leq \xi_n < 1$.
- (5) $0 < \alpha_j, \beta_j \in \mathbb{R}$ for $j \in \{1, \dots, n\}$ and

$$\sum_{j \in E_2} \alpha_j \xi_j^l - \sum_{j \in E_1} \alpha_j \xi_j^l > 0,$$

$$\sum_{j \in F_2} \beta_j \xi_j^l - \sum_{j \in F_1} \beta_j \xi_j^l > 0,$$

such that

$$c = (1 + \sum_{j \in E_1} \alpha_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1}) - \sum_{j \in E_2} \alpha_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1}))^{-1},$$

$$d = (1 + \sum_{j \in F_1} \beta_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1}) - \sum_{j \in F_2} \beta_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1}))^{-1}$$

are positive real numbers. We also study the stability of the mentioned system in the sense of Mittag–Leffler, with the condition

$$\begin{cases} \phi(0) = \phi_0, \\ \psi(0) = \psi_0. \end{cases} \quad (3)$$

1.2. Mittag–Leffler Function, Laplace Transform, and Adomian Decomposition Method

The Mittag–Leffler function, originally introduced as a generalization of the exponential function, plays a fundamental role in the theory of fractional differential equations. It is typically defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathbb{C},$$

where $\Gamma(\cdot)$ denotes the classical Gamma function. The Mittag–Leffler function naturally appears in the solutions of fractional-order differential and integral equations, providing an explicit representation of memory and hereditary properties inherent in such models. Recent studies have extended and applied the Mittag–Leffler framework to various contexts, including new fractional operators [24], Ulam–Hyers–Mittag–Leffler stability [25], multivariate solutions [26], improved discrete kernels [27], and numerical methods such as the Mittag–Leffler–Galerkin approach [28].

The fuzzy Laplace transform extends the classical Laplace transform to fuzzy-valued functions. Let $f : [0, \infty) \rightarrow \mathbb{R}_F$ be a fuzzy-valued function. The fuzzy Laplace transform of f , denoted by $\mathcal{L}[f](s)$, is defined by

$$\mathcal{L}[f](s) = \int_0^{\infty} e^{-st} f(t) dt,$$

Concurrently, fuzzy Laplace transform methods have been increasingly employed to solve fuzzy fractional differential equations [29,30], and specially, the second model market equilibrium in fractional fuzzy economic systems, process medical imaging data [31], and establish foundational techniques for fuzzy differential problems [32]. These contributions underline the relevance and growing importance of operational methods in analyzing and approximating fuzzy fractional systems, providing the context and motivation for the present study.

The Adomian Decomposition Method (ADM) is a well-known analytical approach for solving linear and nonlinear differential equations. In the fuzzy setting, ADM has been extended to handle fuzzy fractional differential equations, providing a powerful technique that avoids discretization and linearization. Nelson [33] applied ADM to Hilfer-type fuzzy fractional differential equations, demonstrating its effectiveness in handling generalized fractional models. Saeed and Pachpatte introduced a modified fuzzy ADM for solving time-fuzzy fractional partial differential equations with initial and boundary conditions [34], and further expanded its usage to a broader class of fuzzy fractional PDEs in [35]. These studies underline the robustness and flexibility of ADM in the analysis of fuzzy fractional systems, supporting its use in our own framework.

Finally, for a review of Caputo’s fractional derivative estimation and its application in solving the problem of solute transport in rivers, we refer to [36,37] respectively.

1.3. Achievements of the Article

The main target of this paper is to investigate the existence and Mittag–Leffler stability of solutions for fuzzy fractional differential systems with non-local conditions. One of the key advantages of our approach is the use of the Schauder fixed-point theorem combined with operational techniques, which allows us to handle the fuzzy fractional setting rigorously. Furthermore, we develop an iterative numerical scheme alongside the Laplace transform to obtain approximate solutions. These results contribute both to the theoretical analysis and practical computation of fuzzy fractional models with memory effects and uncertainty.

2. Preliminaries

In this section, we give some basic definitions and results about fuzzy numbers as well as fractional derivatives and integrals which are needed in the article.

Definition 1. Let $\mathbb{R}$ be the set of real numbers and $C = \{(z, \phi_C(z)) : z \in \mathbb{R}\}$ be the fuzzy set on $\mathbb{R}$ where $\phi_C(z) \in [0, 1]$, that is,

$$\phi_C : \mathbb{R} \rightarrow [0, 1]$$

$$z \mapsto \phi_C(z).$$

Definition 2. A fuzzy set C with its (membership) function $\phi : \mathbb{R} \rightarrow [0, 1]$ is a fuzzy number if it is normal (there exists $z_0 \in \mathbb{R}$, such that $\phi(z_0) = 1$), fuzzy convex, or upper semi-continuous, and the closure of its r -level $[\phi]_0 = \text{i.e., } \{x \in \mathbb{R} : \phi(x) > 0\}$ is a compact set. The space of all real fuzzy numbers is denoted by $\mathbb{R}_F$.

Remark 1 ([11]). For any $\phi \in \mathbb{R}_F$, we have the lower and upper functions $\underline{\phi}_r, \bar{\phi}_r$ that satisfy

$$\phi_r = [\underline{\phi}_r, \bar{\phi}_r],$$

for any $r \in [0, 1]$, where functions $\underline{\phi}_r, \bar{\phi}_r : [0, 1] \rightarrow \mathbb{R}$ are bounded (non-decreasing and non-increasing, respectively), with a left continuous function in $(0, 1]$ and a right continuous function at 0. Also, we have $\underline{\phi}_r \leq \bar{\phi}_r$.

Definition 3. The fuzzy metric $d_\infty : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}^+ \cup \{0\}$ is defined by

$$d_\infty(\phi, \psi) = \sup_{0 \leq r \leq 1} \max\{|\underline{\phi}_r - \underline{\psi}_r|, |\bar{\phi}_r - \bar{\psi}_r|\} = \sup_{0 \leq r \leq 1} d_H([\phi_r], [\psi_r]),$$

where d_H is the classical Hausdorff distance between real intervals.

Proposition 1 ([11]). Suppose that $\phi, \psi, \eta, \vartheta \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$. Then, we have the following results:

- (1) $d_\infty(\phi \ominus \psi, \eta \ominus \vartheta) \leq d_\infty(\phi, \eta) + d_\infty(\psi, \vartheta)$, and the H -differences $\phi \ominus \psi, \eta \ominus \vartheta$ exist.
- (2) $d_\infty(\phi \oplus \psi, \eta \oplus \vartheta) \leq d_\infty(\phi, \eta) + d_\infty(\psi, \vartheta)$.
- (3) $d_\infty(\lambda \odot \phi, \lambda \odot \psi) = |\lambda| d_\infty(\phi, \psi)$.
- (4) $d_\infty(\phi \oplus \eta, \psi \oplus \eta) \leq d_\infty(\phi, \psi)$.

Definition 4. Let $\xi_0 \in J$, $h \in \mathbb{R}$ such that $\xi_0 + h \in J$ and $f \in C(J, \mathbb{R}_F)$. If there exists $f'(\xi_0) \in \mathbb{R}_F$ such that

$$f'(\xi_0) = \lim_{h \rightarrow 0} \frac{1}{h} \otimes [f(\xi_0 + h) \ominus_{gH} f(\xi_0)],$$

then we say that f is gH -differentiable at ξ_0 . Let $C^1(J, \mathbb{R}_F)$ be the space of continuously gH -differentiable functions on J .

Suppose that $f \in C(J, \mathbb{R}_F)$ is gH -differentiable at $\xi_0 \in J$, $[f(\xi)]_\alpha = [f_\alpha^-(\xi), f_\alpha^+(\xi)]$ for all $0 \leq \alpha \leq 1$, $\xi \in J$. Then, we say that f is (i)- gH differentiable at ξ_0 if $[f'(\xi_0)]_\alpha = [(f_\alpha^-)'(\xi_0), (f_\alpha^+)'(\xi_0)]$, for all $\alpha \in [0, 1]$.

Remark 2. The space $C^1(J, \mathbb{R}_F)$ denotes the set of fuzzy-valued functions on J that are continuous and generalized-Hukuhara-differentiable, with continuous derivatives in the fuzzy sense. Equipped with the metric d_∞ defined in Definition 3, the space $C^1(J, \mathbb{R}_F)$ is a complete metric space, ensuring the applicability of fixed-point theorems and other analytical tools.

Definition 5.

- (1) The subset C of $\mathbb{R}_{\mathbb{F}}$ is compact-supported if there is a compact set $U \in \mathbb{R}$ such that U consists of all sets $[\phi]_0$, where $\phi \in C$.
- (2) The subset C of $\mathbb{R}_{\mathbb{F}}$ is level-equicontinuous in $r_0 \in [0, 1]$ if, for all $r \in [0, 1]$, we have

$$\lim_{r \rightarrow r_0} d_H([\phi]_r, [\phi]_{r_0}) = 0,$$

where $\phi \in C$ is a fuzzy-valued function.

- (3) The continuous function $\gamma : [a, b] \times \mathbb{R}_{\mathbb{F}} \rightarrow \mathbb{R}_{\mathbb{F}}$ is compact if, for every $a \leq c, b \leq d$ and $C \in \mathbb{R}_{\mathbb{F}}$ (C is bounded), the set $\gamma([c, d] \times C)$ is relatively compact in $\mathbb{R}_{\mathbb{F}}$.

Lemma 1 ([9]). Let C be a compact-supported subset of $\mathbb{R}_{\mathbb{F}}$. Then, C is relatively compact if and only if C is level-equicontinuous on $[0, 1]$.

Let us denote the space of all continuous fuzzy-valued functions by $C([a, b], \mathbb{R}_{\mathbb{F}})$ and the space of all non-empty, closed, bounded, and convex subset of $C([a, b], \mathbb{R}_{\mathbb{F}})$ by G .

Theorem 1 ([9]). Let X be a non-empty, closed, convex, bounded subset of $C([a, b], \mathbb{R}_{\mathbb{F}})$, and let $T : X \rightarrow X$ be a continuous compact mapping. Then, there exists at least one point $x^* \in X$ with $T(x^*) = x^*$.

Definition 6. Let $\lambda > 0$. On the space $C([a, b], \mathbb{R}_{\mathbb{F}})$, we define the supremum metric $H_J(\phi, \bar{\phi}) = \sup_{\xi \in J} (d_{\infty}(\phi(\xi), \bar{\phi}(\xi)))$ and also define the weight metric

$$\tilde{H}_J(\phi, \bar{\phi}) = \sup_{\xi \in J} \left\{ e^{-\lambda(\xi - a)} d_{\infty}(\phi(\xi), \bar{\phi}(\xi)) \right\}.$$

Definition 7. Let $\sigma \in (0, 1]$ and let $H_{\sigma}(\phi, \bar{\phi}) = \max \left\{ H_{[0, \sigma]}(\phi, \bar{\phi}), \tilde{H}_{[\sigma, 1]}(\phi, \bar{\phi}) \right\}$. Denote $\mathbb{X} = (C([a, b], \mathbb{R}_{\mathbb{F}}), C([a, b], \mathbb{R}_{\mathbb{F}}))$ and let $\tau : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+^2$ be $\tau((\phi, \psi), (\bar{\phi}, \bar{\psi})) = (H_{\sigma}(\phi, \bar{\phi}), H_{\sigma}(\psi, \bar{\psi}))$.

Proposition 2 ([14]). The metric spaces $(C(J, \mathbb{R}_{\mathbb{F}}), H_J)$, $(C(J, \mathbb{R}_{\mathbb{F}}), \tilde{H}_J)$, $(C(J, \mathbb{R}_{\mathbb{F}}), H_{\sigma})$, and $(\mathbb{X}, \rho)$ are complete.

Definition 8 ([11]).

- (1) Let $l \in (0, 1]$, $\phi \in C(J, \mathbb{R}_{\mathbb{F}})$ and $[\phi(\xi)]_r = [\underline{\phi}_r(\xi), \bar{\phi}_r(\xi)]$, $r \in [0, 1]$. The Riemann–Liouville fuzzy fractional integral is defined as

$${}_{\mathbb{F}}^{RL} I_{0+}^l \phi(\xi) = \frac{1}{\Gamma(l)} \odot \int_0^{\xi} (\xi - z)^{l-1} \odot \phi(z) dz, \quad \xi \in [a, b],$$

and for the r -level of ${}_{\mathbb{F}}^{RL} I_{0+}^l \phi(\xi)$, we have

$$[{}_{\mathbb{F}}^{RL} I_{0+}^l \phi(\xi)]_r = [{}_{\mathbb{F}}^{RL} I_{0+}^l \underline{\phi}_r(\xi), {}_{\mathbb{F}}^{RL} I_{0+}^l \bar{\phi}_r(\xi)], \quad \xi \in J,$$

where $[\phi]_r = [\underline{\phi}_r, \bar{\phi}_r]$.

- (2) Let $l \in [0, 1]$ and $\phi \in C^1([a, b], \mathbb{R}_{\mathbb{F}})$. The formula for the Caputo fractional gH -derivative of order l is

$${}_{gH}^c D^l \phi(\xi) = {}_{\mathbb{F}}^{RL} I_{0+}^{1-l} \phi'(\xi).$$

Remark 3. Throughout this paper, the subscript F in operators such as ${}_{\mathbb{F}}^{RL} I_{0+}^{\ell}$ and $D_{\mathbb{F}}^{\ell}$ indicates that the integration or differentiation is performed in the fuzzy-valued sense, based on the general-

ized Hukuhara difference. When we refer to g_H -differentiability, we mean generalized Hukuhara differentiability, which extends the classical derivative concept to fuzzy-valued functions.

Definition 9. Let $\gamma(x)$ be fuzzy continuous and assume that $\gamma(z) \odot e^{-sz} \in L^1([0, +\infty), \mathbb{R}_{\mathbb{F}})$. Then, the Laplace transform of γ is defined as

$$\mathbb{L}\{\gamma(z)\} = \int_0^{+\infty} \gamma(z) \odot e^{-sz} dz,$$

and according to the classical Laplace transform, we have

$$[\mathbb{L}\{\gamma(z)\}]_r = [l[\underline{\gamma}_r(z)], l[\overline{\gamma}_r(z)]] ,$$

where

$$[l\underline{\gamma}_r(z)]_r = \int_0^{+\infty} \underline{\gamma}(z) \odot e^{-sz} dz \text{ and } [l\overline{\gamma}_r(z)]_r = \int_0^{+\infty} \overline{\gamma}(z) \odot e^{-sz} dz.$$

Theorem 2 ([7]). Let $\gamma'(z)$ be fuzzy integrable on $[0, +\infty)$; then, we have

$$\mathbb{L}\{\gamma'(z)\} = s \odot \mathbb{L}\{\gamma(z)\} \ominus \gamma(0).$$

Proposition 3 ([8]). For the Caputo fractional derivative, we have

$$\mathbb{L}\left\{{}^c D_{0+}^l \gamma(\xi)\right\} = s^l \mathbb{L}\{\gamma(s)\} - \sum_{j=0}^{n-1} s^{l-j-1} \gamma^{(j)}(0).$$

If $0 < l < 1$, then we have $\mathbb{L}\left\{{}^c D_{0+}^l \gamma(\xi)\right\} = s^l \mathbb{L}\{\gamma(s)\} - s^{l-1} \gamma(0)$; for the convolution of two fuzzy functions γ_1 and γ_2 , we have the formula

$$\mathbb{L}\{\gamma_1(\xi) * \gamma_2(\xi)\} = \mathbb{L}\left\{\int_0^\xi \gamma_1(z) \odot \gamma_2(\xi - z) dz\right\} = \mathbb{L}\{\gamma_1(\xi)\} \odot \mathbb{L}\{\gamma_2(\xi)\};$$

and for the Mittag-Leffler function in two parameters $(\theta_1, \theta_2) \in \mathbb{C}$, we have

$$\mathbb{L}\left\{E_{\xi^{\theta_2-1}\theta_1, \theta_2}(-\lambda \xi^{\theta_1})\right\} = \left\{\sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\theta_1 j + \theta_2)}\right\} = \frac{s^{\theta_1} - \theta_2}{s^{\theta_1} + \lambda}, \quad \operatorname{Re}(s) > |\lambda^{\frac{1}{\theta_1}}|.$$

3. The Non-Local Problem for Fractional Differential Systems

In this section, we investigate the existence of a solution for systems (1) and (2).

Remark 4. We denote the following for simplicity in writing in our calculations:

$$A_1 = 1 + \frac{2c}{\Gamma(l)} \sum_{j=1}^n \alpha_j, \quad B_1 = 1 + \frac{2d}{\Gamma(l)} \sum_{j=1}^n \beta_j,$$

$$I^t[\gamma](\phi, \psi) = \frac{1}{\Gamma(l)} \odot \int_0^\xi (\xi - z)^{l-1} \odot \gamma(z, \phi(z), \psi(z)) dz,$$

$$G_n^1(\phi, \psi) = \left(\sum_{j \in E_2} c\alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi) \ominus \sum_{j \in E_1} c\alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi)\right),$$

$$G_n^2(\phi, \psi) = \left(\sum_{j \in F_2} d\beta_j \odot I^{\xi_j}[\gamma_2](\phi, \psi) \ominus \sum_{j \in F_1} d\beta_j \odot I^{\xi_j}[\gamma_2](\phi, \psi)\right).$$

Lemma 2. Assume that $\phi, \psi \in C^1([a, b], \mathbb{R}_{\mathbb{F}})$ are solutions of systems (1) and (2). Then, (ϕ, ψ) satisfy the following system provided that ϕ, ψ are (i) - gH differentiable on $[a, b]$, and

$$\begin{cases} \phi(\xi) \oplus {}^{\mathbb{R}L}_{\mathbb{F}} I_{0+}^{l-1} \phi(\xi) = \frac{1+\xi^{l-1}}{\Gamma(l)} G_n^1(\phi, \psi) \oplus I^{\xi}[\gamma_1](\phi, \psi), \\ \psi(\xi) \oplus {}^{\mathbb{R}L}_{\mathbb{F}} I_{0+}^{l-1} \psi(\xi) = \frac{1+\xi^{l-1}}{\Gamma(l)} G_n^2(\phi, \psi) \oplus I^{\xi}[\gamma_2](\phi, \psi). \end{cases} \quad (4)$$

Proof. If $\phi \in C^1([a, b], \mathbb{R}_{\mathbb{F}})$ is (i)-gH differentiable, then by taking the integral operator ${}^{\mathbb{R}L}_F I_{0+}^l$ on both sides of the equation ${}^C_{gH} D^l \phi(\xi) = \gamma_2(\xi, \phi(\xi), \psi(\xi)) + \phi'(\xi)$, we have

$$\int_0^{\xi} \phi'(z) dz = {}^{\mathbb{R}L}_F I_{0+}^l (\gamma_1(\xi, \phi(\xi), \psi(\xi)) + \phi'(\xi)),$$

which is equal to

$$\phi(\xi) \ominus \phi(0) = \frac{1}{\Gamma(l)} \odot \int_0^{\xi} (\xi - z)^{l-1} \odot \gamma_1(z, \phi(z), \psi(z)) dz \oplus \frac{1}{\Gamma(l)} \odot \int_0^{\xi} (\xi - z)^{l-1} \odot \phi'(z) dz,$$

and $\frac{1}{\Gamma(l)} \odot \int_0^{\xi} (\xi - z)^{l-1} \odot \phi'(z) dz$ is equal to

$$\begin{aligned} \frac{1}{\Gamma(l)} \odot \int_0^{\xi} (\xi - z)^{l-1} \odot \phi'(z) dz &= \left[\frac{1}{\Gamma(l)} \odot (\xi - z)^{l-1} \odot \phi(z) \right]_0^{\xi} \ominus \frac{l-1}{\Gamma(l)} \int_0^{\xi} (\xi - z)^{l-2} \odot \phi(z) dz, \\ &= \frac{\xi^{l-1}}{\Gamma(l)} \phi(0) \ominus {}^{\mathbb{R}L}_{\mathbb{F}} I_{0+}^{l-1} \phi(\xi). \end{aligned}$$

Thus,

$$\phi(\xi) = \frac{\phi(0)}{\Gamma(l)} (1 + \xi^{l-1}) \oplus I^{\xi}[\gamma_1](\phi, \psi) \ominus {}^{\mathbb{R}L}_{\mathbb{F}} I_{0+}^{l-1} \phi(\xi). \quad (5)$$

Then, for ξ_j $j = 1 \cdots n$, $0 < \xi_1 \leq \xi_2 \leq \cdots \leq \xi_n < 1$, we have

$$\phi(\xi_j) = \frac{\phi(0)}{\Gamma(l)} (1 + \xi_j^{l-1}) \oplus I^{\xi_j}[\gamma_1](\phi, \psi) \ominus {}^{\mathbb{R}L}_{\mathbb{F}} I_{0+}^{l-1} \phi(\xi_j), \quad (6)$$

and substituting (6) into the first equation of (1), we have

$$\begin{aligned} \phi(0) \oplus \sum_{j \in E_1} \alpha_j \odot \left(\frac{\phi(0)}{\Gamma(l)} (1 + \xi_j^{l-1}) \oplus I^{t_k}[\gamma_1](\phi, \psi) \right) \\ = \sum_{j \in E_2} \alpha_j \odot \left(\frac{\phi(0)}{\Gamma(l)} (1 + \xi_j^{l-1}) \oplus I^{\xi_j}[\gamma_1](\phi, \psi) \right). \end{aligned}$$

Since

$$1 + \sum_{j \in E_1} \alpha_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1}) - \sum_{j \in E_2} \alpha_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1}) > 0,$$

we have

$$\begin{aligned} (1 + \sum_{j \in E_1} \alpha_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1}) - \sum_{j \in E_2} \alpha_j \frac{1}{\Gamma(l)} (1 + \xi_j^{l-1})) \phi(0) \\ = \sum_{j \in E_1} \alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi) \ominus \sum_{j \in E_2} \alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi), \end{aligned}$$

and also

$$\phi(0) = c \left(\sum_{j \in E_1} \alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi) \ominus \sum_{j \in E_2} \alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi) \right) = G_n^1(\phi, \psi). \quad (7)$$

Combining (5) and (7), we obtain the first equation in system (4). A similar argument results in the other equation in system (4). $\square$

Lemma 3 ([10]). Suppose that $\gamma_1, \gamma_2 \in C([a, b] \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}}, \mathbb{R}_{\mathbb{F}})$ satisfy the following conditions:

$$d_{\infty}(\gamma_1(\xi, \phi(\xi), \psi(\xi)), \tilde{0}) \leq \begin{cases} b_1 d_{\infty}(\phi(\xi), \tilde{0}) + \tilde{b}_1 D_{\infty}(v(\xi), \tilde{0}) + d_1, & 0 \leq \xi \leq \xi_n, \\ c_1 d_{\infty}(\phi(\xi), \tilde{0}) + \tilde{c}_1 D_{\infty}(\psi(\xi), \tilde{0}) + d_2, & \xi_n \leq \xi \leq 1, \end{cases}$$

and

$$d_{\infty}(\gamma_2(\xi, \phi(\xi), \psi(\xi)), \tilde{0}) \leq \begin{cases} b_{11} d_{\infty}(\phi(\xi), \tilde{0}) + \tilde{b}_{11} d_{\infty}(\psi(\xi), \tilde{0}) + d_{11}, & 0 \leq \xi \leq \xi_n \\ c_{11} d_{\infty}(\phi(\xi), \tilde{0}) + \tilde{c}_{11} d_{\infty}(\psi(\xi), \tilde{0}) + d_{21}, & \xi_n \leq \xi \leq 1 \end{cases},$$

where $b_1, \tilde{b}_1, b_{11}, \tilde{b}_{11}, c_1, \tilde{c}_1, c_{11}, \tilde{c}_{11}, d_1, d_2, d_{11}$, and d_{21} are positive real numbers satisfying the condition

$$|\det(N)| > 0,$$

where

$$N = \frac{1}{\Gamma(l+1)} \begin{pmatrix} 1 - (b_1 A_1 + c_1) & -\tilde{b}_1 A_1 - \tilde{c}_1 \\ -b_{11} B_1 - c_{11} & 1 - (\tilde{b}_{11} B_1 + \tilde{c}_{11}) \end{pmatrix}.$$

Remark 5 ([10]). If γ_1 satisfies Lemma 3, then we have

$$d_{\infty}(I^{\xi}[\gamma_1](\phi, \psi), \tilde{0}) \leq \frac{1}{\Gamma(l+1)} [b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1],$$

$$d_{\infty}(G_n^1(\phi, \psi), \tilde{0}) \leq \frac{c}{\Gamma(l+1)} \sum_{j=1}^n \alpha_j [b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1].$$

Remark 6. We define a vector-valued integral operator $T = (T_1, T_2)$ by

$$T_1(\phi, \psi)(\xi) = \frac{1}{\Gamma(l)} (1 + \xi^{l-1}) G_n^1(\phi, \psi) \oplus I^{\xi}[\gamma_1](\phi, \psi) \ominus {}^{RL}_{\mathbb{F}} I_{0+}^{l-1} \phi(\xi),$$

$$T_2(\phi, \psi)(\xi) = \frac{1}{\Gamma(l)} (1 + \xi^{l-1}) G_n^2(\phi, \psi) \oplus I^{\xi}[\gamma_2](\phi, \psi) \ominus {}^{RL}_{\mathbb{F}} I_{0+}^{l-1} \psi(\xi).$$

Note: A fixed point of T is a solution of systems (1) and (2).

Lemma 4. If Lemma 3 holds, then

$$H_{[0, \xi_n]}(T_1(\phi, \psi), \tilde{0}) \leq \frac{A_1}{\Gamma(l+1)} ((b_1 + \frac{l}{A_1}) H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1),$$

$$\begin{aligned} \tilde{H}_{[\xi_n, 1]}(T_1(\phi, \psi), \tilde{0}) &\leq \frac{A_1}{\Gamma(l+1)} (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) \\ &+ ((c_1 - l) \tilde{H}_{[\xi_n, 1]}(\phi(\xi), \tilde{0}) + \tilde{c}_1 \tilde{H}_{[\xi_n, 1]}(\psi(\xi), \tilde{0}) + d_2) \frac{1}{\Gamma(l+1)} \end{aligned}$$

hold for all $(\phi, \psi) \in C([a, b], \mathbb{R}_{\mathbb{F}})$.

Proof. If $0 < \xi \leq \xi_n$, then from Remark 5, we have

$$\begin{aligned}
 & d_{\infty}(T_1(\phi, \psi)(\xi), \tilde{0}(\xi)) \\
 &= d_{\infty}(G_n^1(\phi, \psi) \oplus I^{\xi}[\gamma_1](\phi, \psi) \ominus {}_{\mathbb{F}}^{RL}I_{0+}^{l-1}\phi(\xi), \tilde{0}(\xi)) \\
 &\leq \frac{1}{\Gamma(l+1)} \left(\frac{c}{\Gamma(l)} (1 + \xi^{l-1}) \sum_{k=1}^m \alpha_j + 1 \right) (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) \\
 &\quad + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) + H_{[0, \xi_n]}({}_{\mathbb{F}}^{RL}I_{0+}^{l-1}\phi(\xi), \tilde{0}(\xi)) \\
 &\leq \frac{1}{\Gamma(l+1)} \left(\frac{2c}{\Gamma(l)} \sum_{j=1}^n \alpha_j + 1 \right) (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) \\
 &\quad + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) + \frac{1}{\Gamma(l)} H_{[0, \xi_n]}(\phi(\xi), \tilde{0}(\xi)) \\
 &= \frac{A_1}{\Gamma(l+1)} (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) + \frac{1}{\Gamma(l)} H_{[0, \xi_n]}(\phi(\xi), \tilde{0}(\xi)) \\
 &= \frac{A_1}{\Gamma(l+1)} \left((b_1 + \frac{l}{A_1}) H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1 \right), \tilde{0}(\xi).
 \end{aligned}$$

Taking the supremum on the last inequality when $\xi \in [0, \xi_n]$, we obtain the first inequality as desired. Now, for $\xi \in [\xi_n, 1]$, we have

$$\begin{aligned}
 & d_{\infty}(T_1(\phi, \psi)(\xi), \tilde{0}(\xi)) \\
 &= d_{\infty}(G_n^1(\phi, \psi) \oplus I^{\xi}[\gamma_2](\phi, \psi) \ominus {}_{\mathbb{F}}^{RL}I_{0+}^{l-1}\phi(\xi), \tilde{0}(\xi)) \\
 &\leq d_{\infty}((\sum_{j \in E_2} c \alpha_j \odot I^{t_j}[\gamma_1](\phi, \psi) \ominus \sum_{j \in E_1} c \alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi)) \\
 &\quad \oplus I^{\xi}[\gamma_1](\phi, \psi) \ominus {}_{\mathbb{F}}^{RL}I_{0+}^{l-1}\phi(\xi), \tilde{0}(\xi)) \\
 &\leq c \sum_{j=1}^n \alpha_j d_{\infty}(I^{\xi_j}[f](\phi, \psi), \tilde{0}) + d_{\infty}(I^{\xi_n}[\gamma_1](\phi, \psi), \tilde{0}) \\
 &\quad + \frac{1}{\Gamma(l)} \int_{\xi_n}^{\xi} (\xi - z)^{l-1} d_{\infty}(\gamma_1(z, \phi(z), \psi(z)), \tilde{0}(z)) dz + \frac{1}{\Gamma(l-1)} \int_{\xi_n}^{\xi} (\xi - z)^{l-2} d_{\infty}(\phi(z), \tilde{0}) dz \\
 &\leq a \sum_{j=1}^n \alpha_j \left(\frac{1}{\Gamma(l+1)} [b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1] \right. \\
 &\quad + \frac{1}{\Gamma(l+1)} [b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1] \tilde{0}(\xi) \\
 &\quad + \frac{1}{\Gamma(l)} \int_{\xi_n}^{\xi} (\xi - z)^{l-1} [c_1 d_{\infty}(\phi(z), \tilde{0}) + \tilde{c}_1 d_{\infty}(\psi(z), \tilde{0}) + d_2] dz \\
 &\quad + \frac{1}{\Gamma(l-1)} \int_{\xi_n}^{\xi} (\xi - z)^{l-2} d_{\infty}(\phi(z), \tilde{0}) dz \\
 &= \frac{A_1}{\Gamma(l+1)} (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) + (c_1 H_{[\xi_n, 1]}(\phi(\xi), \tilde{0}) \\
 &\quad + \tilde{c}_1 H_{[\xi_n, 1]}(\psi(\xi), \tilde{0}) + d_2) \frac{(\xi - \xi_n)^l}{\Gamma(l+1)} - \frac{(\xi - \xi_n)^{l-1}}{\Gamma(l)} H_{[\xi_n, 1]}(\phi(\xi), \tilde{0}) \\
 &= \frac{A_1}{\Gamma(l+1)} (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) + ((c_1 - \frac{l}{(\xi - \xi_n)}) H_{[\xi_n, 1]}(\phi(\xi), \tilde{0}) \\
 &\quad + \tilde{c}_1 H_{[\xi_n, 1]}(\psi(\xi), \tilde{0}) + d_2) \frac{(\xi - \xi_n)^l}{\Gamma(l+1)} \\
 &\leq \frac{A_1}{\Gamma(l+1)} (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) \\
 &\quad + ((c_1 - l) H_{[\xi_n, 1]}(\phi(\xi), \tilde{0}) + \tilde{c}_1 H_{[\xi_n, 1]}(\psi(\xi), \tilde{0}) + d_2) \frac{1}{\Gamma(l+1)}.
 \end{aligned}$$

Multiplying both sides of the last inequality by $e^{-\lambda \xi}$ and then taking supremum over $\xi \in [\xi_n, 1]$, we obtain the second inequality. $\square$

Lemma 5 ([10]). Suppose $\gamma_1 : [0, 1] \times X \rightarrow \mathbb{R}_{\mathbb{F}}$ is compact, where X is a non-empty, closed, convex, and totally bounded subset of $\mathbb{X}$ (here, $\mathbb{X}$ is defined in Definition 7). Then, we have the following:

- (1) $T_1(X) \subset C([a, b], \mathbb{R}_{\mathbb{F}})$ is equicontinuous;
- (2) $T_1(X)(\xi)$ is level-equicontinuous on $[0, 1]$.

Theorem 3. Suppose γ_1, γ_2 are compact and Lemma 3 holds. Then, the system (1)–(2) has a solution.

Proof. According to Lemma 4, we have

$$\begin{aligned} H_{[0, \xi_n]}(T_1(\phi, \psi), \tilde{0}) &\leq \frac{A_1}{\Gamma(l+1)} ((b_1 + \frac{q}{A_1}) H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1), \\ \tilde{H}_{[\xi_n, 1]}(T_1(\phi, \psi), \tilde{0}) &\leq \frac{A_1}{\Gamma(l+1)} (b_1 H_{[0, \xi_n]}(\phi(\xi), \tilde{0}) + \tilde{b}_1 H_{[0, \xi_n]}(\psi(\xi), \tilde{0}) + d_1) \\ &\quad + ((c_1 - l) \tilde{H}_{[\xi_n, 1]}(\phi(\xi), \tilde{0}) + \tilde{c}_1 \tilde{H}_{[\xi_n, 1]}(\psi(\xi), \tilde{0}) + d_2) \frac{1}{\Gamma(l+1)}. \end{aligned}$$

Then,

$$\begin{aligned} &H_{\xi_n}(T_1(\phi, \psi), \tilde{0}) \\ &= \max \left\{ H_{[0, \xi_n]}(T_1(\phi, \psi), \tilde{0}), \tilde{H}_{[\xi_n, 1]}(T_1(\phi, \psi), \tilde{0}) \right\} \\ &\leq \frac{1}{\Gamma(l+1)} (b_1 A_1 + l + c_1 - l) H_{\xi_n}(\phi, \tilde{0}) + \frac{1}{\Gamma(l+1)} ((\tilde{b}_1 A_1 + \tilde{c}_1) H_{\xi_n}(\psi, \tilde{0}) + d_1 A_1 + d_2) \\ &= \frac{1}{\Gamma(l+1)} (b_1 A_1 + c_1) H_{\xi_n}(\phi, \tilde{0}) + \frac{1}{\Gamma(l+1)} ((\tilde{b}_1 A_1 + \tilde{c}_1) H_{\xi_n}(\psi, \tilde{0}) + d_1 A_1 + d_2), \end{aligned}$$

and also

$$\begin{aligned} H_{\xi_n}(T_2(\phi, \psi), \tilde{0}) &\leq \frac{1}{\Gamma(l+1)} (b_{11} B_1 + c_{11}) H_{\xi_n}(\phi, \tilde{0}) \\ &\quad + \frac{1}{\Gamma(l+1)} ((\tilde{b}_{11} B_1 + \tilde{c}_{11}) H_{\xi_n}(\psi, \tilde{0}) + d_{11} B_1 + d_{21}); \end{aligned}$$

therefore, we have

$$\begin{pmatrix} H_{\xi_n}(T_1(\phi, \psi), \tilde{0}) \\ H_{\xi_n}(T_2(\phi, \psi), \tilde{0}) \end{pmatrix} \leq M \begin{pmatrix} H_{\xi_n}(\phi, \tilde{0}) \\ H_{\xi_n}(\psi, \tilde{0}) \end{pmatrix} + \frac{1}{\Gamma(l+1)} \begin{pmatrix} A_1 d_1 + d_2 \\ B_1 d_{11} + d_{21} \end{pmatrix}, \quad (8)$$

where

$$M = \frac{1}{\Gamma(l+1)} \begin{pmatrix} b_1 A_1 + c_1 & \tilde{b}_1 A_1 + \tilde{c}_1 \\ b_{11} B_1 + c_{11} & \tilde{b}_{11} B_1 + \tilde{c}_{11} \end{pmatrix},$$

and from Lemma 3, the matrix $N = \frac{1}{\Gamma(l+1)} I - M$ is invertible and its inverse $(\frac{1}{\Gamma(l+1)} I - M)^{-1}$ has non-negative elements. Therefore, there exists $p_1, p_2 \geq 0$ such that

$$\left(\frac{1}{\Gamma(l+1)} I - M \right)^{-1} \frac{1}{\Gamma(l+1)} \begin{pmatrix} d_1 A_1 + d_2 \\ d_{11} B_1 + d_{21} \end{pmatrix} \leq \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

and this is equal to

$$M \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \frac{1}{\Gamma(l+1)} \begin{pmatrix} d_1 A_1 + d_2 \\ d_{11} B_1 + d_{21} \end{pmatrix} \leq \frac{1}{\Gamma(l+1)} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \quad (9)$$

Set $X = \{(\phi, \psi) \in \mathbb{X} : H_{\xi_n}(\phi, \tilde{0}) \leq R_1, H_{\xi_n}(\psi, \tilde{0}) \leq p_2\}$. From (3.5), (3.6), we have $T(\phi, \psi) \in X$. We note that for every $z \in T_1(X)$, there exists $(\phi, \psi) \in X$ such that $z(\xi) = T_1(\phi, \psi)(\xi)$ for all $\xi \in [a, b]$. Then,

$$\begin{aligned} [z(\xi)]_0 &= \left(\sum_{j \in E_2} c\alpha_j \odot I^{\xi_j}[\gamma_1](\phi, \psi) \ominus \sum_{j \in E_1} c\alpha_k \odot I^{\xi_j}[\gamma_1](\phi, \psi) \right) \oplus I^{\xi}[\gamma_1](\phi, \psi) \ominus {}^{RL}I_{0+}^{l-1}\phi(\xi)_0, \\ &= \left(\sum_{j \in J_2} c\alpha_j \odot [I^{\xi_j}[\gamma_1](\phi, \psi)]_0 \ominus \sum_{j \in J_1} c\alpha_j \odot [I^{\xi_j}[\gamma_1](\phi, \psi)]_0 \right) \oplus [I^{\xi}[\gamma_1](\phi, \psi)]_0 \ominus [{}^{RL}I_{0+}^{l-1}\phi(\xi)]_0. \end{aligned}$$

Since $f([a, b] \times X)$ is relatively compact and $\gamma_1([a, b] \times X)$ is compact-supported and level-equicontinuous on $[0, 1]$, there exists a compact set $U \subset \mathbb{R}$, such that $[\gamma_1(\xi, \phi(\xi), \psi(\xi))]_0 \subseteq U$ for all $(\xi, \phi, \psi) \in [a, b] \times X$, and

$$\begin{aligned} [z(\xi)]_0 &\subseteq \frac{c}{\Gamma(l)} \left(\sum_{j \in E_2} \alpha_j \left(\int_0^{\xi_j} (\xi_j - z)^{l-1} dz \right) [\gamma_1(z, \phi(z), \psi(z))]_0 \right) \\ &\quad \ominus \frac{c}{\Gamma(l)} \left(\sum_{j \in E_1} \alpha_j \left(\int_0^{\xi_j} (\xi_j - z)^{l-1} dz \right) [\gamma_1(z, \phi(z), \psi(z))]_0 \right) \\ &\quad \oplus \frac{1}{\Gamma(l)} \left(\int_0^{\xi} (\xi - z)^{l-1} dz \right) [\gamma_1(z, \phi(z), \psi(z))]_0 \ominus \frac{1}{\Gamma(l-1)} \left(\int_0^{\xi} (\xi - z)^{l-2} dz \right) [\phi(z)]_0 \\ &\subseteq \frac{c}{\Gamma(l+1)} \left(\sum_{j \in J_2} \alpha_j \xi_j^l - \sum_{j \in J_1} \alpha_j \xi_j^l \right) U + \frac{1}{\Gamma(l+1)} U + \frac{l}{\Gamma(l+1)} \xi^{l-1} \overline{[\phi(z)]_0}, \end{aligned}$$

and since $\overline{[\phi(z)]_0}$ is compact, there is a compact set $U_0 \subseteq \mathbb{R}$ such that $[z(\xi)]_0 \subseteq U_0, \forall z \in T_1(X)$. This proves that $T_1(X)(\xi)$ is compact-supported. From Lemma 5 and also the Ascoli–Arzela theorem, we have that $T_1(X)$ is relative compact. We can apply a similar argument for $T_2(X)$. Thus, the operator $T(X) = (T_1(X), T_2(X))$ is relatively compact on $\mathbb{X}$. Next, let $(\phi_m, \psi_m), (\phi, \psi) \in X$ and let $(\phi_m, \psi_m) \rightarrow (\phi, \psi)$ in $\mathbb{X}$. Now,

$$\begin{aligned} &d_{\infty}(T_1(\phi_m, \psi_m), (\xi), T_1(\phi, \psi)(\xi)) \\ &\leq \frac{1}{\Gamma(l)} (1 + \xi^{l-1}) d_{\infty}(G_n^1(\phi_m, \psi_m), G_n^1(\phi, \psi)) + d_{\infty}(I^{\xi}[\gamma_1](\phi_m, \psi_m), I^{\xi}[\gamma_1](\phi, \psi)), \end{aligned}$$

so T_1 is continuous. We can apply a similar argument for T_2 so that, as a result, T is continuous. Finally, using Theorem 1, we deduce that there is a point (ϕ^*, ψ^*) in X such that $T(\phi^*, \psi^*) = (\phi^*, \psi^*)$. This point is the solution of (1)–(2), and so, the proof is complete. $\square$

4. Stability Result for the Fractional System

Definition 10 ([8]). Suppose that (ϕ, ψ) is a solution of systems (1)–(3). Then, the solution (ϕ, ψ) is Mittag–Leffler-stable if

$$\begin{cases} d_{\infty}(\phi(\xi), \tilde{0}) \leq (m(\phi_0)E_p(-\lambda\xi^l))^b, \\ d_{\infty}(\psi(\xi), \tilde{0}) \leq (m(\psi_0)E_p(-\lambda\xi^l))^b, \end{cases}$$

where $0 < p < 1, \lambda \geq 0, b > 0, m(0) = 0$, and $m(\phi) \geq 0, m(\psi) \geq 0$ and $m(\phi), m(\psi)$ are locally Lipschitz on ϕ, ψ with Lipschitz constant m_0 . Motivated by [8], we present our stability result.

Theorem 4. Let $\gamma(\xi, \phi(\xi), \psi(\xi)) : [0, +\infty) \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \rightarrow \mathbb{R}$ be a continuously differentiable function with $\gamma(\xi, \phi(\xi), \psi(\xi))$ as a locally Lipschitz function with respect to ϕ and ψ , such that

$$\gamma(\xi, \phi(\xi), \psi(\xi)) + \phi'(\xi) \leq c_2 d_{+\infty}^{ab}(\phi(\xi), \tilde{0}),$$

or

$$c_1 d_{+\infty}^a(\phi(\xi), \tilde{0}) \leq \gamma(\xi, \phi(\xi), \psi(\xi)) - \frac{c_3}{c_2} z(\xi, \phi(\xi), \psi(\xi)),$$

or

$${}_{{}_gH}^c D_{0+}^l(\gamma(\xi, \phi(\xi), \psi(\xi))) \leq -c_3 d_{+\infty}^{ab}(\phi(\xi), \tilde{0}),$$

where

$$z(\xi, \phi(\xi), \psi(\xi)) = \xi^{1-l} E_{l,l}(\frac{-c_3}{c_2} \xi^l) * \max_{\xi \in \mathbb{R}_+} \{\phi(\xi), \psi(\xi)\},$$

and also assume the same conditions for ψ . Here, $\xi \geq 0$, $0 < l < 1$, and $c_1, c_2, c_3, a, b > 0$. Then, the solution of (1)–(3) is Mittag–Leffler-stable.

Proof. From the above, we have

$${}_{{}_gH}^c D_{0+}^l(\gamma(\xi, \phi(\xi), \psi(\xi))) \leq \frac{-c_3}{c_2} (\gamma(\xi, \phi(\xi), \psi(\xi)) + \phi'(\xi)).$$

Then, we can find $h(\xi) \geq 0$, such that

$${}_{{}_gH}^c D_{0+}^l(\gamma(\xi, \phi(\xi), \psi(\xi))) + h(\xi) = \frac{-c_3}{c_2} (\gamma(\xi, \phi(\xi), \psi(\xi)) + \phi'(\xi)).$$

Taking the Laplace transform on both sides of the above equation, we have

$$\begin{aligned} & x^l \mathbb{L}\{\gamma(\xi, \phi(\xi), \psi(\xi))\} - z^{l-1}(\gamma(0, \phi(0), \psi(0))) + \mathbb{L}\{h(\xi)\} \\ &= \frac{-c_3}{c_2} \mathbb{L}\{\gamma(\xi, \phi(\xi), \psi(\xi))\} + \frac{-c_3}{c_2} \mathbb{L}\{\phi'(\xi)\}, \end{aligned}$$

and it follows that

$$\mathbb{L}\{\gamma(\xi, \phi(\xi), \psi(\xi))\} = \frac{z^{l-1}(\gamma(0, \phi(0), \psi(0))) - \mathbb{L}\{h(\xi)\} + \frac{c_3}{c_2} (z \mathbb{L}\{\phi(\xi)\} - \phi(0))}{z^l + \frac{c_3}{c_2}}.$$

Now, by taking the inverse Laplace transform on both sides of the last equation, we have

$$\gamma(\xi, \phi(\xi), \psi(\xi)) = E_l(\frac{-c_3}{c_2} \xi^l) \gamma(0, \phi(0), \psi(0)) + (-h(\xi) + \frac{c_3}{c_2} \phi(\xi)) * (\xi^{1-l} E_{l,l}(\frac{-c_3}{c_2} \xi^l)),$$

or

$$\begin{aligned} & \gamma(\xi, \phi(\xi), \psi(\xi)) - \frac{c_3}{c_2} \xi^{1-l} E_{l,l}(\frac{-c_3}{c_2} \xi^l) * \phi(\xi) \\ &= E_l(\frac{-c_3}{c_2} \xi^l) \gamma(0, \phi(0), \psi(0)) - h(\xi) * (\xi^{1-l} E_{l,l}(\frac{-c_3}{c_2} \xi^l)). \end{aligned}$$

Since $h(\xi)$ and $E_{l,l}(\frac{-c_3}{c_2} \xi^l)$ are positive functions and $z(\xi, \phi(\xi), \psi(\xi)) \geq \frac{c_3}{c_2} \xi^{1-l} E_{l,l}(\frac{-c_3}{c_2} \xi^l) * \phi(\xi)$, then we have

$$\gamma(\xi, \phi(\xi), \psi(\xi)) - \frac{c_3}{c_2} z(\xi, \phi(\xi), \psi(\xi)) \leq E_l(\frac{-c_3}{c_2} \xi^l) \gamma(0, \phi(0), \psi(0)).$$

Using the conditions on ϕ , we have

$$d_{\infty}(\phi(\xi), \tilde{0}) \leq [\frac{\gamma(0, \phi(0), \psi(0))}{c_1} E_l(\frac{-c_3}{c_2} \xi^l)]^{\frac{1}{a}}.$$

Also, using the same argument, we have

$$d_{\infty}(\psi(\xi), \bar{0}) \leq \left[\frac{\gamma(0, \phi(0), \psi(0))}{c_1} E_l\left(\frac{-c_3}{c_2} \xi^l\right) \right]^{\frac{1}{a}}.$$

Since $\gamma(\xi, \phi(\xi), \psi(\xi))$ is locally Lipschitz with respect to ϕ, ψ , we conclude that $m(\gamma) = \frac{\gamma(0, \phi(0), \psi(0))}{c_1}$ is a Lipschitz constant with respect to $\phi(0)$, $\psi(0)$, and $m(0) = 0$. This implies the stability of our system. $\square$

5. Application of Laplace Transforms to Solve Fractional Differential Systems

In this section, we consider systems (1)–(3) using the Laplace transform. Suppose the r -level of ϕ, ψ are $[\phi]_r = [\underline{\phi}_r, \bar{\phi}_r]$ and $[\psi]_r = [\underline{\psi}_r, \bar{\psi}_r]$. Now, we write the system and its initial condition with respect to its r -level, and we have

$$\begin{cases} {}^c_{gH}D^l[\underline{\phi}_r(\xi), \bar{\phi}_r(\xi)] = [\gamma_1(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi)) + \underline{\phi}'_r(\xi), \gamma_1(\xi, \bar{\phi}_r(\xi), \bar{\psi}_r(\xi)) + \bar{\phi}'_r(\xi)], \\ {}^c_{gH}D^l[\underline{\psi}_r(\xi), \bar{\psi}_r(\xi)] = [\gamma_2(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi)) + \underline{\psi}'_r(\xi), \gamma_2(\xi, \bar{\phi}_r(\xi), \bar{\psi}_r(\xi)) + \bar{\psi}'_r(\xi)], \end{cases}$$

and

$$\begin{cases} [\underline{\phi}_r(0), \bar{\phi}_r(0)] = [\underline{\phi}_{0r}, \bar{\phi}_{0r}], \\ [\underline{\psi}_r(0), \bar{\psi}_r(0)] = [\underline{\psi}_{0r}, \bar{\psi}_{0r}], \end{cases}$$

which are equivalent to the systems

$$\begin{cases} {}^c_{gH}D^l\underline{\phi}_r(\xi) = \gamma_1(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi)) + \underline{\phi}'_r(\xi), \\ {}^c_{gH}D^l\underline{\psi}_r(\xi) = \gamma_2(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi)) + \underline{\psi}'_r(\xi), \end{cases}$$

with the conditions

$$\begin{cases} \underline{\phi}_r(0) = \underline{\phi}_{0r}, \\ \underline{\psi}_r(0) = \underline{\psi}_{0r}, \end{cases}$$

and

$$\begin{cases} {}^c_{gH}D^l\bar{\phi}_r(\xi) = \gamma_1(\xi, \bar{\phi}_r(\xi), \bar{\psi}_r(\xi)) + \bar{\phi}'_r(\xi), \\ {}^c_{gH}D^l\bar{\psi}_r(\xi) = \gamma_2(\xi, \bar{\phi}_r(\xi), \bar{\psi}_r(\xi)) + \bar{\psi}'_r(\xi), \end{cases}$$

with conditions

$$\begin{cases} \bar{\phi}_r(0) = \bar{\phi}_{0r}, \\ \bar{\psi}_r(0) = \bar{\psi}_{0r}. \end{cases}$$

Solving these systems using the Laplace transform and then applying the stacking lemma, we have the integral solution of problem (4).

Example 1. Consider the system

$$\begin{cases} {}^c_{gH}D^{\frac{1}{2}}\phi(\xi) = \psi(\xi) + \phi'(\xi), \\ {}^c_{gH}D^{\frac{1}{2}}\psi(\xi) = \psi'(\xi), \end{cases}$$

with conditions

$$\begin{cases} \phi(0) = 1, \\ \psi(0) = 1. \end{cases}$$

Suppose the r -level of ϕ, ψ are $[\phi]_r = [\underline{\phi}_r, \overline{\phi}_r]$ and $[\psi]_r = [\underline{\psi}_r, \overline{\psi}_r]$. Then, we have the systems

$$\begin{cases} {}^c_{gH}D^{\frac{1}{2}}\underline{\phi}_r(\xi) = \underline{\psi}_r(\xi) + \underline{\phi}'_r(\xi), \\ {}^c_{gH}D^{\frac{1}{2}}\underline{\psi}_r(\xi) = \underline{\psi}'_r(\xi), \end{cases}$$

with conditions

$$\begin{cases} \underline{\phi}_r(0) = 1, \\ \underline{\psi}_r(0) = 1, \end{cases}$$

and

$$\begin{cases} {}^c_{gH}D^{\frac{1}{2}}\overline{\phi}_r(\xi) = \overline{\psi}_r(\xi) + \overline{\phi}'_r(\xi), \\ {}^c_{gH}D^{\frac{1}{2}}\overline{\psi}_r(\xi) = \overline{\psi}'_r(\xi), \end{cases}$$

with conditions

$$\begin{cases} \overline{\phi}_r(0) = 1, \\ \overline{\psi}_r(0) = 1, \end{cases}$$

and note that our initial conditions do not have uncertain parts. We only solve the first system. The solution of the second system is similar. Applying the Laplace transform, we have

$$\begin{cases} \mathbb{L}\{{}^c_{gH}D^{\frac{1}{2}}\underline{\phi}_r(\xi)\} = \mathbb{L}\{\underline{\psi}_r(\xi)\} + \mathbb{L}\{\underline{\phi}'_r(\xi)\}, \\ \mathbb{L}\{{}^c_{gH}D^{\frac{1}{2}}\underline{\psi}_r(\xi)\} = \mathbb{L}\{\underline{\psi}'_r(\xi)\}. \end{cases}$$

Set $\mathbb{L}\{\underline{\phi}_r(\xi)\} = F_1(z)$ and $\mathbb{L}\{\underline{\psi}_r(\xi)\} = F_2(z)$, and then we have

$$\begin{cases} s^{\frac{1}{2}}F_1(z) - z^{-\frac{1}{2}}\underline{\phi}_r(0) = F_2(z) + zF_1(z) - \underline{\phi}_r(0), \\ s^{\frac{1}{2}}F_2(z) - z^{-\frac{1}{2}}\underline{\psi}_r(0) = zF_2(z) - \underline{\psi}_r(0), \end{cases}$$

and multiplying the equations in the above system by $z^{\frac{1}{2}}$ and then applying the initial condition, we have

$$\begin{cases} zF_1(z) - 1 = F_2(z) + z^{\frac{3}{2}}F_1(z) - z^{\frac{1}{2}}, \\ zF_2(z) - 1 = z^{\frac{3}{2}}F_2(z) - z^{\frac{1}{2}}. \end{cases}$$

Solving the system, we obtain

$$F_1(z) = \frac{\sqrt{z^3} - z - 1}{z^2(\sqrt{z} - 1)}, \quad F_2(z) = \frac{1}{z},$$

and finally, using the inverse Laplace transform, we have

$$\underline{\phi}_r(\xi) = -e^{\xi}(1 - \operatorname{erf}(-\sqrt{\xi})) + 2 + \xi + 2\sqrt{\frac{\xi}{\pi}}, \quad \underline{\psi}_r(t) = 1,$$

where $\operatorname{erf}(-\sqrt{\xi})$ is the error function:

$$\operatorname{erf}(-\sqrt{\xi}) = \frac{2}{\sqrt{\pi}} \int_0^{-\sqrt{\xi}} e^{-z^2} dz.$$

In the same way we have

$$\overline{\phi}_r(\xi) = -e^{\xi}(1 - \operatorname{erf}(-\sqrt{\xi})) + 2 + \xi + 2\sqrt{\frac{\xi}{\pi}}, \quad \overline{\psi}_r(\xi) = 1.$$

Note: This transformation is mostly used when the coefficients of the unknown functions are constant.

6. Adomian Decomposition Method for Solving Fuzzy Fractional Equations

Consider systems (1)–(3). By taking ${}^{RL}_0 I_0^q$ on both sides of the system, we have

$$\begin{cases} \phi(\xi) = \phi(0) \oplus {}^{RL}_0 I_0^q \gamma_1(\xi, \phi(\xi), \psi(\xi)) \oplus {}^{RL}_0 I_0^q \phi'(\xi), \\ \psi(\xi) = \psi(0) \oplus {}^{RL}_0 I_0^q \gamma_2(\xi, \phi(\xi), \psi(\xi)) \oplus {}^{RL}_0 I_0^q \psi'(\xi). \end{cases} \quad (10)$$

Suppose that $[\phi]_r = [\underline{\phi}_r, \overline{\phi}_r]$ and $[\psi]_r = [\underline{\psi}_r, \overline{\psi}_r]$; so, for the lower functions, we have

$$\begin{cases} \underline{\phi}_r(\xi) = \underline{\phi}_0 + {}^{RL}_0 I_0^q \gamma_3(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi), \overline{\phi}_r(\xi), \overline{\psi}_r(\xi)) + {}^{RL}_0 I_0^q \underline{\phi}'_r(\xi), \\ \underline{\psi}_r(\xi) = \underline{\psi}_0 + {}^{RL}_0 I_0^q \gamma_4(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi), \overline{\phi}_r(\xi), \overline{\psi}_r(\xi)) + {}^{RL}_0 I_0^q \underline{\psi}'_r(\xi), \end{cases} \quad (11)$$

with conditions

$$\begin{cases} \underline{\phi}(0) = \underline{\phi}_0, \\ \underline{\psi}(0) = \underline{\psi}_0, \end{cases} \quad (12)$$

and for the upper functions, we have

$$\begin{cases} \overline{\phi}_r(\xi) = \overline{\phi}_0 + {}^{RL}_0 I_0^q \gamma_3(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi), \overline{\phi}_r(\xi), \overline{\psi}_r(\xi)) + {}^{RL}_0 I_0^q \overline{\phi}'_r(\xi), \\ \overline{\psi}_r(\xi) = \overline{\psi}_0 + {}^{RL}_0 I_0^q \gamma_4(\xi, \underline{\phi}_r(\xi), \underline{\psi}_r(\xi), \overline{\phi}_r(\xi), \overline{\psi}_r(\xi)) + {}^{RL}_0 I_0^q \overline{\psi}'_r(\xi), \end{cases} \quad (13)$$

with conditions

$$\begin{cases} \overline{\phi}(0) = \overline{\phi}_0, \\ \overline{\psi}(0) = \overline{\psi}_0. \end{cases} \quad (14)$$

In the ADM method, we suppose that ϕ and ψ are infinite series of differentiable functions ϕ_n and ψ_n such that

$$\underline{\phi}(\xi, r) = \sum_{n=0}^{+\infty} \underline{\phi}_n(\xi, r), \quad \underline{\psi}(\xi, r) = \sum_{n=0}^{+\infty} \underline{\psi}_n(\xi, r),$$

$$\bar{\phi}(\xi, r) = \sum_{n=0}^{+\infty} \bar{\phi}_n(\xi, r), \quad \bar{\psi}(\xi, r) = \sum_{n=0}^{+\infty} \bar{\psi}_n(\xi, r),$$

and substituting these series in (11) and (13), we have

$$\begin{aligned} \sum_{n=0}^{+\infty} \phi_n(\xi, r) &= \underline{\phi}_0 + {}^{RL}I_0^q \gamma_3(\xi, \sum_{n=0}^{+\infty} \phi_n(\xi, r), \sum_{n=0}^{+\infty} \psi_n(\xi, r), \sum_{n=0}^{+\infty} \bar{\phi}_n(\xi, r), \sum_{n=0}^{+\infty} \bar{\psi}_n(\xi, r)) \\ &\quad + {}^{RL}I_0^q \sum_{n=0}^{+\infty} \phi'_n(\xi, r), \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{+\infty} \psi_n(\xi, r) &= \underline{\psi}_0 + {}^{RL}I_0^q \gamma_4(\xi, \sum_{n=0}^{+\infty} \phi_n(\xi, r), \sum_{n=0}^{+\infty} \psi_n(\xi, r), \sum_{n=0}^{+\infty} \bar{\phi}_n(\xi, r), \sum_{n=0}^{+\infty} \bar{\psi}_n(\xi, r)) \\ &\quad + {}^{RL}I_0^q \sum_{n=0}^{+\infty} \psi'_n(\xi, r). \end{aligned}$$

Suppose that γ_3 and γ_4 are nonlinear functions and that we can express them in the following forms:

$$\begin{aligned} &\gamma_3(\xi, \phi(\xi, r), \psi(\xi, r), \bar{\phi}(\xi, r), \bar{\psi}(\xi, r)) \\ &= F_1(\xi, \phi(\xi, r)) + F_2(\xi, \psi(\xi, r)) + F_3(\xi, \phi(\xi, r)) + F_4(\xi, \psi(\xi, r)), \end{aligned}$$

and

$$\begin{aligned} &\gamma_4(\xi, \phi(\xi, r), \psi(\xi, r), \bar{\phi}(\xi, r), \bar{\psi}(\xi, r)) \\ &= G_1(\xi, \phi(\xi, r)) + G_2(\xi, \psi(\xi, r)) + G_3(\xi, \phi(\xi, r)) + G_4(\xi, \psi(\xi, r)). \end{aligned}$$

Additionally, suppose that for the function F_1 , we set $F_1 = \sum_{n=0}^{+\infty} C_{1,n}$; and $C_{1,n}$ (called the “coefficients of the Adomian polynomials”) is determined by the following formula:

$$C_{1,n} = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} F_1(\xi, \sum_{k=0}^n \phi_k(\xi, r) \lambda^k) \right]_{\lambda=0},$$

and the coefficients that specify the nonlinear functions $F_2, F_3, F_4, G_1, G_2, G_3$, and G_4 are $C_{2,n}, C_{3,n}, C_{4,n}, C_{5,n}, C_{6,n}, C_{7,n}$, and $C_{8,n}$, which are calculated in the same way. Now, substituting γ_3 and γ_4 in the above and using the Adomian coefficients, we have

$$\sum_{n=0}^{+\infty} \phi_{n+1}(\xi, r) = \underline{\phi}_0 + {}^{RL}I_0^q \sum_{n=0}^{+\infty} (C_{1,n} + C_{2,n} + C_{3,n} + C_{4,n}) + {}^{RL}I_0^q \sum_{n=0}^{+\infty} \phi'_n(\xi, r),$$

and

$$\sum_{n=0}^{+\infty} \psi_{n+1}(\xi, r) = \underline{\psi}_0 + {}^{RL}I_0^q \sum_{n=0}^{+\infty} (C_{5,n} + C_{6,n} + C_{7,n} + C_{8,n}) + {}^{RL}I_0^q \sum_{n=0}^{+\infty} \psi'_n(\xi, r).$$

We see from the initial conditions that $\phi_0(\xi) = \underline{\phi}_0$ and $\psi_0(\xi) = \underline{\psi}_0$; so, by equaling terms with the same index, we have

$$\phi_{n+1}(\xi, r) = {}^{RL}I_0^q (C_{1,n} + C_{2,n} + C_{3,n} + C_{4,n}) + {}^{RL}I_0^q \phi'_n(\xi, r),$$

and

$$\underline{\psi}_{n+1}(\xi, r) = {}^{RL}I_0^q(C_{5,m} + C_{6,m} + C_{7,m} + C_{8,m}) + {}^{RL}I_0^q \underline{\psi}'_n(\xi, r),$$

for all $n \in \mathbb{N} \cup \{0\}$. By the same argument, for upper functions, we have

$$\bar{\phi}_{n+1}(\xi, r) = {}^{RL}I_0^q(C_{1,m} + C_{2,m} + C_{3,m} + C_{4,m}) + {}^{RL}I_0^q \bar{\phi}'_n(\xi, r),$$

and

$$\bar{\psi}_{n+1}(\xi, r) = {}^{RL}I_0^q(C_{5,m} + C_{6,m} + C_{7,m} + C_{8,m}) + {}^{RL}I_0^q \bar{\psi}'_n(\xi, r),$$

and by obtaining the unknown coefficients and summing up to n steps, the n -th approximate answer is also obtained. If the functions γ_3 and γ_4 are linear, we suppose that

$$\begin{aligned} \gamma_3(\xi, \underline{\phi}(\xi, r), \underline{\psi}(\xi, r), \bar{\phi}(\xi, r), \bar{\psi}(\xi, r)) \\ = a_1(\xi) \underline{\phi}(\xi, r) + b_1(\xi) \bar{\phi}(\xi, r) + c_1(\xi) \underline{\psi}(\xi, r) + d_1(\xi) \bar{\psi}(\xi, r), \end{aligned}$$

and

$$\begin{aligned} \gamma_4(\xi, \underline{\phi}(\xi, r), \underline{\psi}(\xi, r), \bar{\phi}(\xi, r), \bar{\psi}(\xi, r)) \\ = a_2(\xi) \underline{\phi}(\xi, r) + b_2(\xi) \bar{\phi}(\xi, r) + c_2(\xi) \underline{\psi}(\xi, r) + d_2(\xi) \bar{\psi}(\xi, r). \end{aligned}$$

Substituting the series with ϕ and ψ using γ_3 and γ_4 , and finally, equaling terms with the same index, we have the following:

$$\underline{\phi}_{n+1}(\xi, r) = {}^{RL}I_0^q(a_1(\xi) \underline{\phi}(\xi, r) + b_1(\xi) \bar{\phi}(\xi, r) + c_1(\xi) \underline{\psi}(\xi, r) + d_1(\xi) \bar{\psi}(\xi, r)) + {}^{RL}I_0^q \underline{\phi}'_n(\xi, r),$$

and

$$\underline{\psi}_{n+1}(\xi, r) = {}^{RL}I_0^q(a_2(\xi) \underline{\phi}(\xi, r) + b_2(\xi) \bar{\phi}(\xi, r) + c_2(\xi) \underline{\psi}(\xi, r) + d_2(\xi) \bar{\psi}(\xi, r)) + {}^{RL}I_0^q \underline{\psi}'_n(\xi, r),$$

and for the upper functions, we have

$$\bar{\phi}_{n+1}(\xi, r) = {}^{RL}I_0^q(a_1(\xi) \underline{\phi}(\xi, r) + b_1(\xi) \bar{\phi}(\xi, r) + c_1(\xi) \underline{\psi}(\xi, r) + d_1(\xi) \bar{\psi}(\xi, r)) + {}^{RL}I_0^q \bar{\phi}'_n(\xi, r),$$

and

$$\bar{\psi}_{n+1}(\xi, r) = {}^{RL}I_0^q(a_2(\xi) \underline{\phi}(\xi, r) + b_2(\xi) \bar{\phi}(\xi, r) + c_2(\xi) \underline{\psi}(\xi, r) + d_2(\xi) \bar{\psi}(\xi, r)) + {}^{RL}I_0^q \bar{\psi}'_n(\xi, r),$$

for all $n \in \mathbb{N} \cup \{0\}$.

Example 2. Consider the following system:

$$\begin{cases} {}^c_{\mathbb{F}}D_0^{1/2} \phi(\xi) = \phi(\xi) \oplus \psi(\xi) \oplus \phi'(\xi), \\ {}^c_{\mathbb{F}}D_0^{1/2} \psi(\xi) = \psi(\xi) \ominus \phi(\xi) \oplus \psi'(\xi), \end{cases}$$

with initial conditions

$$\begin{cases} [\underline{\phi}_0, \bar{\phi}_0]_r = [r-1, 1-r], \\ [\underline{\psi}_0, \bar{\psi}_0]_r = [r-1, 1-r], \end{cases}$$

and note that this system does not contain nonlinear functions γ_1 and γ_2 , so all the Adomian coefficients are zero. Now, we write this system as two systems with lower and upper functions, and then we take ${}^{RL}I_0^q$ of both systems and use the ADM method to obtain the following iterative systems:

$$\begin{cases} \underline{\phi}_{n+1}(\xi, r) = {}^{RL}I_0^q(\underline{\phi}_n(\xi, r) + \underline{\psi}_n(\xi, r) + \underline{\phi}'_n(\xi, r)), \\ \underline{\psi}_{n+1}(\xi, r) = {}^{RL}I_0^q(\underline{\psi}_n(\xi, r) - \bar{\phi}_n(\xi, r) + \underline{\psi}'_n(\xi, r)), \end{cases}$$

with (IC)

$$\begin{cases} \underline{\phi}_0(\xi, r) = r - 1, \\ \underline{\psi}_0(\xi, r) = r - 1, \end{cases}$$

and the system

$$\begin{cases} \bar{\phi}_{n+1}(\xi, r) = {}^{RL}I_0^q(\bar{\phi}_n(\xi, r) + \bar{\psi}_n(\xi, r) + \bar{\phi}'_n(\xi, r)), \\ \bar{\psi}_{n+1}(\xi, r) = {}^{RL}I_0^q(\bar{\psi}_n(\xi, r) - \underline{\phi}_n(\xi, r) + \bar{\psi}'_n(\xi, r)), \end{cases}$$

with (IC)

$$\begin{cases} \bar{\phi}_0(\xi, r) = 1 - r, \\ \bar{\psi}_0(\xi, r) = 1 - r. \end{cases}$$

Putting $n = 0, 1, \dots$, solving both systems, and finally, summing all functions, we have

$$\begin{aligned} \underline{\phi}(\xi, r) = \underline{\psi}(\xi, r) = r - 1 + \frac{4}{\sqrt{\pi}}(r - 1)\xi^{\frac{1}{2}} + (4(r - 1)\xi + 2(r - 1)) \\ + \left(\frac{32}{3\sqrt{\pi}}(r - 1)\xi^{\frac{3}{2}} + \frac{16}{\sqrt{\pi}}(r - 1)\xi^{\frac{1}{2}}\right) \\ + (16(r - 1)\frac{\xi^2}{2} + 8(r - 1)(2\xi + 1)) + \dots, \end{aligned}$$

and

$$\begin{aligned} \bar{\phi}(\xi, r) = \bar{\psi}(\xi, r) = 1 - r + \frac{4}{\sqrt{\pi}}(1 - r)\xi^{\frac{1}{2}} + (4(1 - r)\xi + 2(1 - r)) \\ + \left(\frac{32}{3\sqrt{\pi}}(1 - r)\xi^{\frac{3}{2}} + \frac{16}{\sqrt{\pi}}(1 - r)\xi^{\frac{1}{2}}\right) \\ + (16(1 - r)\frac{\xi^2}{2} + 8(1 - r)(2\xi + 1)) + \dots. \end{aligned}$$

Example 3. Consider the system

$$\begin{cases} {}^c_{\mathbb{F}}D_0^{1/2}\phi(\xi) = (\frac{2}{\sqrt{\pi}}\sqrt{\xi} - 1) \oplus \phi'(\xi), \\ {}^c_{\mathbb{F}}D_0^{1/2}\psi(\xi) = \xi^2 \ominus \phi^2(\xi) \oplus \psi'(\xi), \end{cases}$$

where $\phi(\xi) \geq \tilde{0}$, with initial conditions

$$\begin{cases} \phi(0) = 0, \\ \psi(0) = c = (0, 1, 1.5), \end{cases}$$

where c is a triangular fuzzy number. In this system, there is a nonlinear expression $\phi^2(\xi)$, so Adomian polynomials are used. Like the previous example, we have lower and upper systems as follows:

$$\begin{cases} \underline{\phi}_{n+1}(\xi, r) = {}^{RL}I_0^{\frac{1}{2}} \underline{\phi}'_n(\xi, r), \\ \underline{\psi}_{n+1}(\xi, r) = -{}^{RL}I_0^{\frac{1}{2}} A_n(\xi, r) + \underline{\psi}'_n(\xi, r), \end{cases}$$

with (IC)

$$\begin{cases} \underline{\phi}_0(\xi, r) = {}^{RL}I_0^{\frac{1}{2}} \left(\frac{2}{\sqrt{\pi}} \sqrt{\xi} - 1 \right), \\ \underline{\psi}_0(\xi, r) = r + {}^{RL}I_0^{\frac{1}{2}} (\xi^2), \end{cases}$$

and

$$\begin{cases} \bar{\phi}_{n+1}(\xi, r) = {}^{RL}I_0^{\frac{1}{2}} \bar{\phi}'_n(\xi, r), \\ \bar{\psi}_{n+1}(\xi, r) = -{}^{RL}I_0^{\frac{1}{2}} B_n(\xi, r) + \bar{\psi}'_n(\xi, r), \end{cases}$$

with (IC)

$$\begin{cases} \bar{\phi}_0(\xi, r) = {}^{RL}I_0^{\frac{1}{2}} \left(\frac{2}{\sqrt{\pi}} \sqrt{\xi} - 1 \right), \\ \bar{\psi}_0(\xi, r) = 1.5 - 0.5r + {}^{RL}I_0^{\frac{1}{2}} (\xi^2), \end{cases}$$

where $A_n(\xi, r)$ and $B_n(\xi, r)$ are the Adomian polynomials for functions $\bar{\phi}_n^2(\xi, r)$ and $\phi_n^2(\xi, r)$, respectively. Next, putting $n = 0, 1 \dots$, solving both systems, and finally, summing all functions, we have

$$\underline{\phi}(\xi, r) = \bar{\phi}(\xi, r) = \left(\xi - \frac{2}{\sqrt{\pi}} \xi^{\frac{1}{2}} \right) + \left(\frac{2}{\sqrt{\pi}} \xi^{\frac{1}{2}} - 1 \right) + 1 + 0 + 0 + \dots = \xi,$$

and

$$\begin{aligned} \underline{\psi}(\xi, r) = & r + \left(\frac{16}{15\sqrt{\pi}} \xi^{\frac{5}{2}} \right) + \left(-\frac{16}{15\sqrt{\pi}} \xi^{\frac{5}{2}} + \frac{5}{2} \xi^2 - \frac{16}{3\pi\sqrt{\pi}} \xi^{\frac{3}{2}} \right) \\ & + \left(-\frac{5}{2} \xi^2 + \left(7 + \frac{8}{\pi} \right) \frac{4}{3\sqrt{\pi}} \xi^{\frac{3}{2}} - \left(\frac{4}{\pi} + 2 \right) \xi \right) + \left(\left(-7 - \frac{4}{\pi} \right) \frac{4}{3\sqrt{\pi}} \xi^{\frac{3}{2}} \right. \\ & \left. + 11\xi + \frac{8}{\pi} \xi - \left(3 + \frac{4}{\pi} \right) \frac{2}{\sqrt{\pi}} \xi^{\frac{1}{2}} \right) + \dots, \end{aligned}$$

and

$$\begin{aligned} \bar{\psi}(\xi, r) = & (1.5 - 0.5r) + \left(\frac{16}{15\sqrt{\pi}} \xi^{\frac{5}{2}} \right) + \left(-\frac{16}{15\sqrt{\pi}} \xi^{\frac{5}{2}} + \frac{5}{2} \xi^2 - \frac{16}{3\pi\sqrt{\pi}} \xi^{\frac{3}{2}} \right) \\ & + \left(-\frac{5}{2} \xi^2 + \left(7 + \frac{8}{\pi} \right) \frac{4}{3\sqrt{\pi}} \xi^{\frac{3}{2}} - \left(\frac{4}{\pi} + 2 \right) \xi \right) + \left(\left(-7 - \frac{4}{\pi} \right) \frac{4}{3\sqrt{\pi}} \xi^{\frac{3}{2}} \right. \\ & \left. + 11\xi + \frac{8}{\pi} \xi - \left(3 + \frac{4}{\pi} \right) \frac{2}{\sqrt{\pi}} \xi^{\frac{1}{2}} \right) + \dots. \end{aligned}$$

Note that while summing, each term is simplified by the previous one, so after some steps, all terms will be removed and we only have the constant phrase r and $1.5 - 0.5r$ for $\underline{\phi}$ and $\bar{\psi}$, respectively. Also, the Adomian coefficients for the function $N(\phi) = \phi^2$ are as follows:

$$\begin{aligned} A_0(\xi, r) &= B_0(\xi, r) = \phi_0^2(\xi, r), \\ A_1(\xi, r) &= B_1(\xi, r) = 2\phi_0(\xi, r)\phi_1(\xi, r), \\ A_2(\xi, r) &= B_2(\xi, r) = 2\phi_0(\xi, r)\phi_2(\xi, r) + \phi_1^2(\xi, r), \\ A_3(\xi, r) &= B_3(\xi, r) = 2\phi_0(\xi, r)\phi_3(\xi, r) + 2\phi_1(\xi, r)\phi_2(\xi, r), \\ &\dots \end{aligned}$$

7. Numerical Methods to Solve Fuzzy Fractional Equations

In this section, we present two methods to solve (1)–(3) numerically. First, we partition the closed interval $[0, 1]$ with N points $0 = \xi_0 < \xi_1 < \xi_2 \cdots < \xi_N = 1$, where $\xi_n = n \frac{1}{N} + \xi_0, n = 0 \cdots N$. Then, we rewrite the system as

$$\begin{cases} {}^c_{gH}D^l \phi(\xi_{n+1}) = \gamma_1(\xi_{n+1}, \phi_{n+1}(\xi), \psi(\xi_{n+1})) \oplus \phi'(\xi_{n+1}), \\ {}^c_{gH}D^l \psi(\xi_{n+1}) = \gamma_2(\xi_{n+1}, \phi(\xi_{n+1}), \psi(\xi_{n+1})) \oplus \psi'(\xi_{n+1}). \end{cases} \quad (15)$$

To estimate $\phi'(\xi_n)$ and $\psi'(\xi_n)$, we use the forward finite difference formula

$$\begin{aligned} \phi'(\xi_n) &\approx \frac{\phi(\xi_{n+1}) - \phi(\xi_n)}{h}, \quad n = 0 \cdots N-1 \\ \psi'(\xi_n) &\approx \frac{\psi(\xi_{n+1}) - \psi(\xi_n)}{h}, \quad n = 0 \cdots N-1, \end{aligned}$$

where $h = \frac{1}{N}$. Then, we write lower and upper systems by using the r -level of ϕ and ψ as follows:

$$\begin{cases} {}^cD^l \underline{\phi}(\xi_{n+1}, r) = \gamma_3(\xi_{n+1}, \underline{\phi}(\xi_{n+1}, r), \underline{\psi}(\xi_{n+1}, r), \bar{\phi}(\xi_{n+1}, r), \bar{\psi}(\xi_{n+1}, r)) \\ \quad + N.(\underline{\phi}(\xi_{n+1}, r) - \bar{\phi}(\xi_n, r)), \\ {}^cD^l \underline{\psi}(\xi_{n+1}, r) = \gamma_4(\xi_{n+1}, \underline{\phi}(\xi_{n+1}, r), \underline{\psi}(\xi_{n+1}, r), \bar{\phi}(\xi_{n+1}, r), \bar{\psi}(\xi_{n+1}, r)) \\ \quad + N.(\underline{\psi}(\xi_{n+1}, r) - \bar{\psi}(\xi_n, r)), \end{cases}$$

and

$$\begin{cases} {}^cD^l \bar{\phi}(\xi_{n+1}, r) = \gamma_3(\xi_{n+1}, \underline{\phi}(\xi_{n+1}, r), \underline{\psi}(\xi_{n+1}, r), \bar{\phi}(\xi_{n+1}, r), \bar{\psi}(\xi_{n+1}, r)) \\ \quad + N.(\bar{\phi}(\xi_{n+1}, r) - \underline{\phi}(\xi_n, r)), \\ {}^cD^l \bar{\psi}(\xi_{n+1}, r) = \gamma_4(\xi_{n+1}, \underline{\phi}(\xi_{n+1}, r), \underline{\psi}(\xi_{n+1}, r), \bar{\phi}(\xi_{n+1}, r), \bar{\psi}(\xi_{n+1}, r)) \\ \quad + N.(\bar{\psi}(\xi_{n+1}, r) - \underline{\psi}(\xi_n, r)). \end{cases}$$

Next, we use the finite difference method for the Caputo derivative, and the formula is as follows [37]:

$${}^cD^l \phi(\xi_{n+1}, r) = \begin{cases} \frac{h^{-l}}{\Gamma(2-l)} (\phi(\xi_{n+1}, r) - \phi(\xi_n, r)) \\ \quad + \frac{h^{-l}}{\Gamma(2-l)} \sum_{j=1}^n ((j+1)^{1-l} - j^{1-l}) (\phi(\xi_{n+1-j}, r) - \phi(\xi_{n-j}, r)), \quad n \geq 1 \\ \frac{h^{-l}}{\Gamma(2-l)} (\phi(\xi_1) - \phi(\xi_0)), \quad n = 0 \end{cases} \quad (16)$$

which is valid for both upper and lower functions; for more details about the finite difference formula, we refer the reader to [37]. Next, substitute (16) in the above systems and we have two iterative systems subject to $u(\xi_{n+1})$ and $u(\xi_n)$. Finally, solving these systems gives us a numerical approximate for (15).

For our second method, we apply the following estimate to ${}^cD^l \phi(\xi_{n+1}, r)$, which is valid for both upper and lower functions (for more details, see [36]):

$${}^cD^l \phi(\xi_{n+1}, r) \approx \frac{1}{\Gamma(2-l)} \phi'(\xi_0, r) (\xi_{n+1} - \xi_0)^{1-l}. \quad (17)$$

We now give a numerical example.

Example 4. Consider the following system:

$$\begin{cases} {}^c_{gH}D_0^{1/2}\phi(\xi) = (\frac{2}{\sqrt{\pi}}\sqrt{\xi} - 1) \oplus \phi'(\xi), \\ {}^c_{gH}D_0^{1/2}\psi(\xi) = \xi^2 \ominus \phi^2(\xi) \oplus \psi'(\xi), \end{cases}$$

where $\phi(\xi) \geq \tilde{0}$, and the initial conditions

$$\begin{cases} \phi(0) = 0 \quad \phi'(0) = 1, \\ \psi(0) = c = (0, 1, 1.5) \quad \psi'(0) = 0. \end{cases}$$

(1) Using the finite difference of Formula (16), we have the following iterative systems for both lower and upper functions:

$$\begin{cases} \underline{\phi}(\xi_{n+1}, r) = \frac{1}{\frac{1}{h} - \frac{h^{-l}}{\Gamma(2-l)}} \left(\frac{-2}{\sqrt{\pi}} \sqrt{\xi_{n+1}} + 1 + \frac{\phi(\xi_n, r)}{h} \right. \\ \left. + \frac{h^{-l}}{\Gamma(2-l)} (-\underline{\phi}(\xi_n, r) + \sum_{j=1}^n ((j+1)^{1-l} - j^{1-l}) (\underline{\phi}(\xi_{n-j+1}, r) - \underline{\phi}(\xi_{n-j}, r)) \right), \\ \underline{\psi}(\xi_{n+1}, r) = \frac{1}{\frac{1}{h} - \frac{h^{-l}}{\Gamma(2-l)}} \left(-\xi_{n+1}^2 + \underline{\phi}^2(\xi_{n+1}, r) + \frac{\psi(\xi_n, r)}{h} \right. \\ \left. + \frac{h^{-l}}{\Gamma(2-l)} (-\underline{\psi}(\xi_n, r) + \sum_{j=1}^n ((j+1)^{1-l} - j^{1-l}) (\underline{\psi}(\xi_{n-j+1}, r) - \underline{\psi}(\xi_{n-j}, r)) \right), \end{cases}$$

and

$$\begin{cases} \overline{\phi}(\xi_{n+1}, r) = \frac{1}{\frac{1}{h} - \frac{h^{-l}}{\Gamma(2-l)}} \left(\frac{-2}{\sqrt{\pi}} \sqrt{\xi_{n+1}} + 1 + \frac{\overline{\phi}(\xi_n, r)}{h} \right. \\ \left. + \frac{h^{-l}}{\Gamma(2-l)} (-\overline{\phi}(\xi_n, r) + \sum_{j=1}^n ((j+1)^{1-l} - j^{1-l}) (\overline{\phi}(\xi_{n-j+1}, r) - \overline{\phi}(\xi_{n-j}, r)) \right), \\ \overline{\psi}(\xi_{n+1}, r) = \frac{1}{\frac{1}{h} - \frac{h^{-l}}{\Gamma(2-l)}} \left(-\xi_{n+1}^2 + \overline{\phi}^2(\xi_{n+1}, r) + \frac{\overline{\psi}(\xi_n, r)}{h} \right. \\ \left. + \frac{h^{-l}}{\Gamma(2-l)} (-\overline{\psi}(\xi_n, r) + \sum_{j=1}^n ((j+1)^{1-l} - j^{1-l}) (\overline{\psi}(\xi_{n-j+1}, r) - \overline{\psi}(\xi_{n-j}, r)) \right), \end{cases}$$

for $n = 0, \dots, N-1$.

(2) Using the estimate for the Caputo derivative, we also have the following systems:

$$\begin{cases} \underline{\phi}(\xi_{n+1}, r) = h \left(\frac{\xi_n^{1-l}}{\Gamma(2-l)} - \frac{2\sqrt{\xi_n}}{\sqrt{\pi}} + 1 \right) + \underline{\phi}(\xi_n, r), \\ \underline{\psi}(\xi_{n+1}, r) = h (-\xi_n^2 + \underline{\phi}^2(\xi_n, r)) + \underline{\psi}(\xi_n, r), \end{cases}$$

and

$$\begin{cases} \overline{\phi}(\xi_{n+1}, r) = h \left(\frac{\xi_n^{1-l}}{\Gamma(2-l)} - \frac{2\sqrt{\xi_n}}{\sqrt{\pi}} + 1 \right) + \overline{\phi}(\xi_n, r), \\ \overline{\psi}(\xi_{n+1}, r) = h (-\xi_n^2 + \overline{\phi}^2(\xi_n, r)) + \overline{\psi}(\xi_n, r), \end{cases}$$

for $n = 0, \dots, N-1$.

As shown in Table 1, the Finite Difference Method converges to the exact solution as N increases. Meanwhile, the Caputo Derivative Method achieves machine-precision accuracy even for small N , as demonstrated in Table 2.

Table 1. Convergence of the Finite Difference Method for $\underline{\phi}(1)$ and $\underline{\psi}(1)$ at $\xi = 1$. The exact solutions are $\underline{\phi}(\xi) = \xi$ and $\underline{\psi}(\xi) = r$.

Finite Difference Method		
N	Estimate for $\underline{\phi}(1)$	Estimate for $\underline{\psi}(1)$
10	1.206168310	$0.4436465165 + r$
50	1.016466996	$0.02520916477 + r$
100	1.005738026	$0.008464752889 + r$
1000	1.000178662	$0.0002556471280 + r$
2000	1.000062538	$0.00008970896566 + r$

Table 2. Accuracy of the Caputo Derivative Method for $\underline{\phi}(1)$ and $\underline{\psi}(1)$. The exact solutions are $\underline{\phi}(\xi) = \xi$ and $\underline{\psi}(\xi) = r$.

Caputo Derivative Method		
N	Estimate for $\underline{\phi}(1)$	Estimate for $\underline{\psi}(1)$
5	0.9999999998	$6.000000000 \times 10^{-11} + r$
10	0.9999999999	$2.000000000 \times 10^{-11} + r$
15	1.000000000	$2.207333333 \times 10^{-10} + r$
20	1.000000000	r
25	1.000000000	r

8. Conclusions and Future Works

In this paper, we considered the existence of a solution and the Mittag–Leffler stability for a fuzzy fractional system. Then, we presented analytical, approximate, and numerical methods to solve these kinds of systems. In particular, we applied the Laplace transform as an analytical method and applied the Adomian’s approximation method, which enables us to cover a wide range of nonlinear systems in comparison with the Laplace method. Concerning numerical methods, we presented the finite difference method, which is very stable and gives a good approximation for a high number of points, and the Caputo derivative approximation method, which enables us to give a more accurate approximation for a much smaller number of points. For future work, we hope to examine systems and fractional equations with partial derivatives.

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