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The Archimedean Origin of Modern Positional Number Systems

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Abstract: A symbolic analysis of Archimedes's periodical number system is developed, from which a natural link emerges with the modern positional number systems with zero. After the publication of Fibonacci's Liber Abaci, the decimal Indo-Arabic positional system was the basis of the algorithmic and algebraic trend of modern mathematics, but even if zero plays a crucial role in algebra and mathematical analysis, zeroless positional systems show the same capability of producing efficient arithmetical algorithms based on operation tables over digits. The crucial role of digits is assessed, by considering a representation of numbers based on strings in lexicographic order. A new algorithm for the determination of decimal periods is presented by remarking on the cruciality of this topic in number theory. Periods of ordinal numbers and enumerations of recursive enumerability are shortly recalled. Concluding remarks are formulated about the deep relationship between numbers and information, which shed new light on a red line passing through the whole history of mathematics.

Keywords: Archimedean periodical system; lexicographic number representation; fraction decimal representation; transfinite ordinals; Turing computable numbers

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1. Preamble

A synthesis of the main results of this paper is given here, which can help in recognizing strong connections along all the parts presented in this paper.

This paper shows that the root of positional systems can be found in the third century BC in a system based on orders and periods. A circled notation is introduced that shows clearly the periodical basis of the notion of zero.

The "Base Representation Theorem" is proved as a direct consequence of periods. This is a further confirmation of the previous result.

The cruciality of periods in decimal fraction representation is emphasized, and the "Concatenation Theorem" is given, from which the correctness of decimal representations of fractions is easily proved. In particular, the correctness of 1/997 is proved, with 166 decimal digits.

A strong connection is shown between Cantor's ordinals and a generalized notion of the Archimedean Enumerative System.

Enumerative systems and their periodical generations link naturally Archimedean number representation with ordinals and Turing computable numbers.

2. Arenarius's System

In *The Sand Reckoner*, entitled "Arenarius" in the Latin tradition, Archimedes of Syracuse (III century BC) considers the problem of giving an evaluation of the size of the universe, according to Aristarchus of Samos's model, by counting the number of sand grains filling that universe.

For this reason, the great mathematician introduces a systematic method for representing numbers of an unlimited size, based on orders and periods. In modern terms, such a notion could be defined as an "Enumeration System", where linguistic or symbolic expressions denoting numbers, that is, numerals, are generated in a totally ordinate manner, in such a way that each numeral is different from those previously generated and greater than all of them (creativity and order), and the rules of generation can always be applied for producing a new numeral after any already generated numeral (infinity). This idea is completely new, because all the numerals of the ancient languages reach a biggest number. After that, it is possible, of course, to provide expressions for bigger numbers, such as "the double of …" or the "…plus one", but no systematic and efficient way was available for going up in the succession of numbers. The method given by Archimedes is very simple [1,2]. He starts from a finite set of initial numerals, the words of Ionic tradition for numbers from 1 to 10^8 (the *double myriad* $\mathcal{M}$, where one Myriad is equal to 10^4). The ordinate list of numerals

$$1, 2, \ \ldots \ \mathcal{M}$$

is called the first order by Archimedes. Then, the second order is the progression of numerals

$$\mathcal{M}, 2\mathcal{M}, \ \ldots \ \mathcal{M}^2.$$

Going on, in the same way, the last $\mathcal{M}$th order is

$$\mathcal{M}^{\mathcal{M}-1}, 2\mathcal{M}^{\mathcal{M}-1} \ \ldots \ \mathcal{M}^{\mathcal{M}}.$$

The given $\mathcal{M}$ orders determine the first period. The second period continues the same rule of generation, where $\mathcal{M}$ is replaced by $\mathcal{M}^{\mathcal{M}}$. It is clear that this method is a recurrent method where all the numbers are denoted by means of expressions constructed by the numerals of the first order, and, of course, there is no limit in this process. In fact, after the first period, the second one can be generated, terminating with $(\mathcal{M}^{\mathcal{M}})^{\mathcal{M}}$, and so on for any following period.

The first two periods, using modern exponential notation, are given below.

First Period

$1, 2, 3, 4, 5, 6, 7, 8, 9, \mathcal{M}$

$\mathcal{M}, 2\mathcal{M}, 3\mathcal{M}, \ \ldots \ \mathcal{M}^2$

$\mathcal{M}^2, 2\mathcal{M}^2, 3\mathcal{M}^2, \ \ldots \ \mathcal{M}^3$

………

$\mathcal{M}^{\mathcal{M}-1}, 2\mathcal{M}^{\mathcal{M}-1}, 3\mathcal{M}^{\mathcal{M}-1}, \ \ldots \ \mathcal{M}^{\mathcal{M}}$

Second Period

$\mathcal{M}^{\mathcal{M}}, 2\mathcal{M}^{\mathcal{M}} \ 3\mathcal{M}^{\mathcal{M}}, \ \ldots \ \mathcal{M}(\mathcal{M}^{\mathcal{M}})$

$\mathcal{M}(\mathcal{M}^{\mathcal{M}}), 2\mathcal{M}(\mathcal{M}^{\mathcal{M}}), 3\mathcal{M}(\mathcal{M}^{\mathcal{M}}), \ \ldots \ \mathcal{M}^2(\mathcal{M}^{\mathcal{M}})$

$\mathcal{M}^2(\mathcal{M}^{\mathcal{M}}), 2\mathcal{M}^2(\mathcal{M}^{\mathcal{M}}), 3\mathcal{M}^2(\mathcal{M}^{\mathcal{M}}), \ \ldots \ \mathcal{M}^3(\mathcal{M}^{\mathcal{M}})$

………

$(\mathcal{M}^{\mathcal{M}-1})(\mathcal{M}^{\mathcal{M}}), 2(\mathcal{M}^{\mathcal{M}-1})(\mathcal{M}^{\mathcal{M}}), 3(\mathcal{M}^{\mathcal{M}-1})(\mathcal{M}^{\mathcal{M}}), \ \ldots \ (\mathcal{M}^{\mathcal{M}})^2$

………

………

$(\mathcal{M}^{\mathcal{M}})^{\mathcal{M}-1}, 2(\mathcal{M}^{\mathcal{M}})^{\mathcal{M}-1}, 3(\mathcal{M}^{\mathcal{M}})^{\mathcal{M}-1}, \ \ldots \ (\mathcal{M}^{\mathcal{M}})^{\mathcal{M}}.$

The numbers denoted by this method are exponentials (of base $\mathcal{M}$) or multiples of exponentials. For this reason, we denoted them in the modern exponential notation. However, Archimedes does not use any symbolic notation but expresses the logic of his method in natural language (Greek) and discovers some basic properties of these numbers and in particular a rule that corresponds to the identity:

$$b^x \times b^y = b^{(x+y)}$$

a sort of anticipation of the product-sum rule of logarithms.

The basic rule of Archimedes's enumeration method is that orders are arithmetic progressions, where any order has $\mathcal{M}$ numerals, and the last numeral of any order coincides with the first numeral of the next order and with the ratio of the progression. The first period has $\mathcal{M}$ orders, the second one $\mathcal{M}^2$ orders, and so on for the following periods.

3. From Arenarius to Zeroless Decimal Systems

Now, we consider Archimedes's system with a symbolic notation adherent to Arenarius's formulation given in natural language. For this purpose, we introduce a symbol expressing the end of orders and periods, realized as a circled superscript. We use an initial order of ten numerals, denoted by the usual decimal symbols 1, 2, 3, 4, 5, 6, 7, 8, and 9, but ten is denoted by 1^o because it corresponds to the end of the first order. The first period is given completely; the second period is indicated by omitting some orders.

$1, 2, 3, 4, 5, 6, 7, 8, 9, 1^o$
$1^o1, 1^o2, 1^o3, 1^o4, 1^o5, 1^o6, 1^o7, 1^o8, 1^o9, 2^o$
$2^o1, 2^o2, 2^o3, 2^o4, 2^o5, 2^o6, 2^o7, 2^o8, 2^o9, 3^o$
$3^o1, 3^o2, 3^o3, 3^o4, 3^o5, 3^o6, 3^o7, 3^o8, 3^o9, 4^o$
$4^o1, 4^o2, 4^o3, 4^o4, 4^o5, 4^o6, 4^o7, 4^o8, 4^o9, 5^o$
$5^o1, 5^o2, 5^o3, 5^o4, 5^o5, 5^o6, 5^o7, 5^o8, 5^o9, 6^o$
$6^o1, 6^o2, 6^o3, 6^o4, 6^o5, 6^o6, 6^o7, 6^o8, 6^o9, 7^o$
$7^o1, 7^o2, 7^o3, 7^o4, 7^o5, 7^o6, 7^o7, 7^o8, 7^o9, 8^o$
$8^o1, 8^o2, 8^o3, 8^o4, 8^o5, 8^o6, 8^o7, 8^o8, 8^o9, 9^o$
$9^o1, 9^o2, 9^o3, 9^o4, 9^o5, 9^o6, 9^o7, 9^o8, 9^o9, 1^{oo}$

$1^{oo}, 1^{oo}2, 1^{oo}3, 1^{oo}4, 1^{oo}5, 1^{oo}6, 1^{oo}7, 1^{oo}8, 1^{oo}9, 1^{oo}1^o$
$1^{oo}1^o1, 1^{oo}1^o2, 1^{oo}1^o3, 1^{oo}1^o4, 1^{oo}1^o5, 1^{oo}1^o6, 1^{oo}1^o7, 1^{oo}1^o8, 1^{oo}1^o9, 1^{oo}2^o$
$1^{oo}2^o1, 1^{oo}2^o2, 1^{oo}2^o3, 1^{oo}2^o4, 1^{oo}2^o5, 1^{oo}2^o6, 1^{oo}2^o7, 1^{oo}2^o8, 1^{oo}2^o9, 1^{oo}3^o$
$1^{oo}3^o1, 1^{oo}3^o2, 1^{oo}3^o3, 1^{oo}3^o4, 1^{oo}3^o5, 1^{oo}3^o6, 1^{oo}3^o7, 1^{oo}3^o8, 1^{oo}3^o9, 1^{oo}4^o$
......

$1^{oo}9^o1, 1^{oo}9^o2, 1^{oo}9^o3, 1^{oo}9^o4, 1^{oo}9^o5, 1^{oo}9^o6, 1^{oo}9^o7, 1^{oo}9^o8, 1^{oo}9^o9, 2^{oo}$
......
......

$2^{oo}9^o1, 2^{oo}9^o2, 2^{oo}9^o3, 2^{oo}9^o4, 2^{oo}9^o5, 2^{oo}9^o6, 2^{oo}9^o7, 2^{oo}9^o8, 2^{oo}9^o9, 3^{oo}$
......
......
......

$9^{oo}9^o1, 9^{oo}9^o2, 9^{oo}9^o3, 9^{oo}9^o4, 9^{oo}9^o5, 9^{oo}9^o6, 9^{oo}9^o7, 9^{oo}9^o8, 9^{oo}9^o9, 1^{ooo}$

In the above representation, any numeral is a sequence of digits and the symbol ^{o} for indicating the end of a cycle (order or period). A number of k consecutive ^{o} determines a period of level k, also called the k-period. For example, $1^{oo}3^o6$ is the numeral of the second period at its first 1-period, at its third order, in the sixth position. In fact, periods are arranged in increasing levels, and within a period of level $k > 1$ there are ten $k-1$-periods (mono-circled digits correspond to orders). These circled numerals correspond to exponentials, but their forms resemble the linguistic expression of the periodical mechanism used by Archimedes, where small circles provide the arrangement of a cyclic generation of numerals. The translation of circled numerals in exponentials is the product of the exponential interpretation of the circled digits, where $D^o = 10^{D\cdot 10}$, $D^{oo} = 10^{D\cdot 100}$, and so on. For example, $1^{oo}3^o6$ represents $6 \cdot 10^{130}$.

The circle, which was the central topic of many of Archimedes's investigations [3], emerges in this symbolism as the basic mechanism of a counting process, by adding to the already seen properties of number enumerations (creativity, order, infinity) the property of recurrence. In fact, all numerals are represented by a finite sets of symbols that continuously recur in the generation. Expressions such as "third order of first period" and "second prime period of the second period" translate, respectively, into 3^o and 2^{oo}.

An enumeration is complete when it generates the numerals of all numbers. In a complete enumeration, a number denoted by a numeral coincides with its position in the enumeration. It is reasonable to suppose that John Wallis, who translated Arenarius, introduced in 1655 the symbol ∞ for infinity (a rotation of digit 8) because he was impressed by the enormous size of Archimedes's numbers and inspired by the Archimedean term "octad", which refers to eight consecutive powers of ten. By the way, the size of the universe was evaluated by Archimedes at the eighth order of his first period with a value around 10^{63} (assuming 10^{24} particles in a sand grain, we obtain the modern evaluation for the particles contained in our universe).

The enumeration given above can be defined as a linear ordering defined on monads, where we call the monad a circled digit, that is, a digit with a number of circles as exponents. Monads are ordered by requiring that $\alpha > \beta$ if α has a number of circles greater than β, or when they have the same number of circles, if the digit of α is greater than the digit of β $(9 > 8 > 7 > 6 > 5 > 4 > 3 > 2 > 1)$. A numeral is a sequence of monads where any monad needs to have a smaller number of circles than those on its left. Then, if ν_1, ν_2 are numerals, $\nu_1 > \nu_2$ when their leftmost monads are μ_1 and μ_2, and this satisfies $\mu_1 > \mu_2$.

The Archimedes enumeration is not complete, because it represents numbers but not all numbers of the natural succession. In fact, only exponentials or multiples of them appear. However, if we change the interpretation by considering each numeral as the successor of the previous one, then we obtain a complete enumeration. The obtained system, which we call the Decimal Archimedes System (DAS), is a zeroless system very close to the usual decimal system, which we call the 0-decimal system (0DS).

Now, we will translate the DAS into another zeroless decimal system, which we call the X-decimal system (XDS). At this end, we translate monads in strings over the alphabet of digits 1, 2, 3, 4, 5, 6, 7, 8, 9, and X:

$$1^o ==> 1\text{X}, 2^o ==> 2\text{X}, \ \ldots \ 9^o ==> 9\text{X}$$

$$1^{oo} ==> \text{XX}, 2^{oo} ==> 2\text{XX}, \ \ldots \ 9^{oo} ==> 9\text{XX}$$

and so on, for monads with greater number of circles (X, XX, ... abbreviates 1X, 1XX, ... respectively, when they occur as first monads, from the left). In this way, the first period of the DAS in the XDS becomes

$1, 2, 3, 4, 5, 6, 7, 8, 9, X$
X1, X2, X3, X4, X5, X6, X7, X8, X9, 2X
2X1, 2X2, 2X3, 2X4, 2X5, 2X6, 2X7, 2X8, 2X9, 3X
3X1, 3X2, 3X3, 3X4, 3X5, 3X6, 3X7, 3X8, 3X9, 4X
4X1, 4X2, 4X3, 4X4, 4X5, 4X6, 4X7, 4X8, 4X9, 5X
5X1, 5X2, 5X3, 5X4, 5X5, 5X6, 5X7, 5X8, 5X9, 6X
6X1, 6X2, 6X3, 6X4, 6X5, 6X6, 6X7, 6X8, 6X9, 7X
7X1, 7X2, 7X3, 7X4, 7X5, 7X6, 7X7, 7X8, 7X9, 8X
8X1, 8X2, 8X3, 8X4, 8X5, 8X6, 8X7, 8X8, 8X9, 9X
9X1, 9X2, 9X3, 9X4, 9X5, 9X6, 9X7, 9X8, 9X9, XX

For example, the XDS translation of $1^{oo}3^{o}6$ is XX3X6. The logic of the XDS enumeration is based on powers of ten: 1, X, XX, XXX, ... Any numeral is the concatenation of multiples of these powers, and their sum provides the number expressed by the numeral. It is interesting to remark that this structure resembles exactly the construction of numerals in many natural languages. However, neither the DAS nor XDS are positional systems in the strict sense of our usual decimal system with zero, but they could be better characterized as polynomial systems (where monads are monomials). Polynomial systems of number representation occur, in primitive forms, in many ancient systems and in the measurement of angles. The mathematician and astronomer Claudius Ptolomaeus (first century, author of the *Almagest*) used circled digits, and in some contexts his circle resembles zero.

Going back to DAS numerals, we could avoid putting circles to digits when in a numeral all the monads smaller than the leftmost monad occur. In fact, in that case the level of any monad corresponds to its position. For example, $1^{oo}3^{o}6$ is completely expressed by 136. But, if circles are deleted in $1^{oo}6^{o}$ and $1^{oo}6$, we obtain, in both cases, 16, which does not distinguish between the two different numerals. However, we can avoid circles if the missing monads are indicated.

Therefore, zero, which was discovered at the end of the fifth century within the Indo-Arabic mathematical tradition [4], has a natural motivation in Archimedes's periodical system, as a new digit 0 expressing the absence of any monad having a number of circles corresponding to its position (distance from the rightmost digit).

Nevertheless, the zero digit is not necessary for having a positional system, because zeroless positional systems in the sense of the 0DS can be defined. One such system is based on the strings that can be constructed over a finite sets of digits [5,6]. Let us assume the ten digits (without zero) in the order

$$1, 2, 3, 4, 5, 6, 7, 8, 9, X.$$

For each digit, the strings of two digits are generated according to the following square, where ordering is from left to the right in the columns and from the top to the bottom for the rows:

1(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
2(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
3(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
4(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
5(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
6(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
7(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
8(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
9(1, 2, 3, 4, 5, 6, 7, 8, 9, X)
X(1, 2, 3, 4, 5, 6, 7, 8, 9, X)

In general, numerals are generated by orders L_i, for $i = 1, 2, \ldots$

$$L_1 = 1, 2, 3, 4, 5, 6, 7, 8, 9, X.$$

$$L_{i+1} = 1L_i, 2L_i, \ \ldots \ XL_i$$

where numerals of L_{j+1} follow those of L_j, and for any digit D, the following equation holds:

$$DL_i = \{D\alpha | \alpha \in L_i\} \tag{1}$$

with $D\beta > D\alpha$ for any $\beta > \alpha$ in L_j, and $j > 1$. This is the structure of any enumeration system, over strings, based on orders and periods.

The ordering associated with this enumeration corresponds to the lexicographic ordering, characterized by the following conditions ($|\alpha|$ is the length of string α, and x, y are any digits):

$$|\alpha| < |\beta| ==> \alpha < \beta$$

$$\alpha < \beta ==> \alpha x < \beta y$$

$$\alpha < \beta ==> x\alpha < x\beta.$$

We call this enumeration the LXS (Lexicographic X-decimal System). The LXS is a positional system, where digits contribute to the value of the denoted number according to their positions.

A complete enumeration system based on orders and periods is an Archimedean Enumeration System (AES). An AES is monotone if its numerals are non-empty strings over a finite set of symbols, and any numeral α followed by a digit x is a numeral too, such that the number $[[\alpha]]$ denoted by α coincides with the number of orders before the order where αx occurs. Such a system has period p if its initial order has p numerals. The XDS, as well as the usual decimal positional system, the D0S, are Archimedean natural monotone enumeration systems.

The following theorem easily generalizes a well-known theorem of positional systems [7] to the natural monotone AES.

Theorem 1. *Let E be an Archimedean monotone enumeration system of period p. Then, the following recurrent equation holds in E, for any digit x:*

$$[[\alpha x]] = [[\alpha]]p + [[x]]. \tag{2}$$

from which the base representation equation follows.

Proof. From the hypotheses on E, the product $[[\alpha]]p$ represents the number of numerals before the order where αx occurs. Then, we have the asserted equation above. If we apply Equation (2) iteratively, we obtain the fundamental base representation equation of a positional number system of base $b > 1$:

$$[[a_n a_{n-1} \dots a_1]] = \sum_{i=1,n} [[a_i]] b^{i-1}.$$

□

4. The Algorithmic Value of Digits

One of the main novelties of digits is the *algorist* trend as opposed to the *abacist* approach of ancient methods of number calculation (from *abacus*). In 1585, Simon Stevin published a book in Flemish, entitled *De Thiende* (the Tenth) [8], where the algorithms for computing the four arithmetical operations are given, which correspond to the methods that are now taught in primary schools. These methods are independent from the particular basis and essentially reduce the computation of any operation to the knowledge of its results for all the pair of digits, that is, to a finite set of basic rules. The same situation arises with zeroless positional systems.

Let us consider the zeroless lexicographic systems of four digits, with the following first 16 numerals:

- 1 2 3 4 11 12 13 14 21 22 23 24 31 32 33 34

Tables 1 and 2 express the sum and multiplication for a lexicographic systems of four digits. Figure 1 is a multiplication based on Tables 1 and 2.

Table 1. Table of sum for a lexicographic system of four digits.

+	1	2	3	4
1	2	3	4	11
2	3	4	11	12
3	4	11	12	13
4	11	12	13	14

Table 2. Table of multiplication for a lexicographic system of four digits.

$\times$	1 1	2 2	3 3	4 4
1	1	2	3	4
2	2	4	12	14
3	3	12	21	24
4	4	14	24	34

For example, 32×21 in the four-digit lexicographic system is obtained from the above tables and provides the same result obtained in the usual decimal system:
$32 \to_{10} 14$
$21 \to_{10} 9$
$14 \times_{10} 9 = 126$
$32 \times_4 14 = 1332 \to_{10} 64 + 3 \times 16 + 3 \times 4 + 2 \to_{10} 126$

$$
\begin{array}{r}
32\times \\
21 = \\
\hline
3\,2 \\
1\,2\,4 \\
\hline
1\,3\,3\,2
\end{array}
$$

Figure 1. A multiplication in the lexicographic system of base 4.

In conclusion, zero is not necessary for having positional systems, even if it is essential for further developments of mathematics: in the infinitesimal analysis and in the algebraic structures. In fact, the negative enumeration, which from zero goes back in the opposite direction of the natural (positive) enumeration, giving the negative of any number, makes integers an additive group with zero as a neuter element.

In his 1585 book, Stevin introduces a notation essentially equivalent to the usual decimal notation. Using this notation, Stevin's division algorithm applied to $p : q$, with $p < q$, provides a decimal representation of type $0, x_1 x_2 x_3 \ldots$ for the fraction p/q, where x_i are decimal digits.

The following theorem is an easy consequence of the *pigeonhole principle*, where $(p_i | i > 1)$ is the succession of prime numbers (if n objects are distributed among $m < n$ cells, then there exists some cell containing more than an object).

Theorem 2. *For $i > 3$, the fraction $1/p_i$ has a decimal representation with infinite digits, obtained by the division algorithm, where a sequence of digits, called a period, repeat indefinitely, and the length of the period is surely lesser than p_i.*

In virtue of the above theorem, any fraction has a finite decimal representation or an infinite but periodical one. Therefore, an infinite decimal representation that is not periodical represents a number that is not a fraction and is called an irrational number (Greek mathematicians use the term Logoi for irrationals). In conclusion, the existence of irrational numbers follows from Stevin's representation.

The following theorem is the converse of the above theorem; that is, for any periodical decimal representation there is an equivalent fraction.

Theorem 3. *For every fraction p/q with $p, q \in \mathbb{Z}$ (set of integers) there exist $r, n, m, k \in \mathbb{Z}$ such that*

$$p/q = k + r/(10^n - 1)10^m$$

with $r < (10^n - 1)10^m$.

Both theorems above can be easily extended to any positional system with zero and base > 1. However, computing the exact periodical representation of a fraction and showing its correctness is not an easy task. After Stevin's work and Napier's formulation, in his second book on logarithms [9], a tradition of works on decimal fractions was developed in the 17th and 18th centuries [10].

Now, we show that a simple theorem can give an efficient solution for a systematic and reliable determination of the exact periodical representation of fractions. In fact, the following theorem, which can be easily proven, is the basis for efficient algorithms (extensible to any base) for computing fraction periods and for checking their correctness.

Theorem 4 (Concatenation Theorem)**.** *Stevin algorithms for the basic arithmetical operations can be "concatenated". Let us express this fact only for division and multiplication (concatenations of additions and subtractions are obvious), where Greek letters denote strings of digits of decimal representations.*

$$(1) \quad 1/q = 0, \alpha\beta$$

if $r : q = \alpha$ with remainder r and $r : q = \beta$ with remainder 1.

For some natural n,

$$(2) \quad \alpha \times (\beta\delta) = 9^n\theta$$

if, for some naturals k, j, i,

$$\alpha \times \beta = 9^k\gamma$$
$$\alpha \times \delta = \eta 9^j\theta$$
$$\gamma + \eta = 9^i$$

with $|\gamma| = |\eta| = i$ and $n = k + j + i$.

By using the theorem above, when arithmetic operations have a precision of p decimal digits, then operations can be concatenated by obtaining periodical representations of any period length, and the correctness of the obtained results can be easily proved.

Given the length limits of computer number representation, no computer can directly compute the exact decimal value of a simple fraction such as 1/19. The representation of fraction $1/p_i$ for $p_i < 100$, based on division and multiplication concatenations, is given below, where periods are indicated within brackets and stars mark periods that reach the maximum possible length.

1/2 = 0,5 = 0,4[9]

1/3 = 0,[3]

1/5 = 0,2 = 0,1[9]

1/7 = 0,[142857]*

1/11 = 0,[09]

1/13 = 0,[076923]

1/17 = 0,[0588235294117647]*

1/19 = 0,[052631578947368421]*

1/23 = 0,[0434782608695652173913]

1/29 = 0,[0344827586206896551724137931]*

1/31 = 0,[032258064516129]

1/37 = 0,[027]

1/41 = 0,[02439]

1/43 = 0,[023255813953488372093]

1/47 = 0,[0212765957446808510638297872340425531914893617]*

!/53 = 0,[0188679245283]

1/59 = 0,[0169491525423728813559322033898305084745762711864406779661]*

1/61 = 0,[016393442622950819672131147540983606557377049180327868852459]*

1/67 = 0,[014925373134328358208955223880597]

1/71 = 0,[01408450704225352112676056338028169]

1/73 = 0,[013698630136986301369863]

1/79 =0,[01265822784810126582278481]

1/83 = 0,[01204819277108433734939759036144578313253]

1/89 = 0,01123595505617977528089887640449438202247191]

1/97 = 0,[0103092783505154639175257731958762886597938144329896907216494
84536082474226804123711340206185567]*

All these representations were checked using multiplication concatenation. Moreover, they coincide with those, up to 1/67, of Johann III Bernoulli's table (1771–1773) reported in [10]. We remark that fraction periodical representations were extensively investigated by Carl Friedrich Gauss, who introduced an entire theory for their calculation [10].

As an example, the computation of 1/17 is here reported, using operations reliable up 12 digits.

$$1:17 = 0,05882352941$$

Remainder = 3

$$3:17 = 0,17647[058823$$

where the open bracket is put after the last digit of the period. Namely, digits 058823 coincide with the initial digits of the first division. Therefore, by concatenating the two divisions, according to the Concatenation Theorem, we have

$$1 : 17 = 0,0588235294117647$$

Now, we prove the correctness of the above periodical representation, by concatenating two multiplications:

$$1/17 = 588235294117647/9999999999999999$$

that is,

$$17 \times 588235294117647 = 9999999999999999$$

In fact, $17 \times 58823529 = 999999993$ and $17 \times 4117647 = 69999999$, and the concatenation of the two results, according to the Concatenation Theorem, is just 9999999999999999.

Gauss spent years computing decimal periods of prime fractions. For this purpose, he developed a theory [10] (of *indices*), which was the seed of his theory of congruences. The biggest fraction he computed was 1/997, which we computed in seconds with the following Python program, using Stevin's division algorithm going up until a remainder was obtained that was already generated. By the way, it is interesting to observe that unitary division is the essence of any division, which is always equivalent to a multiplication of the result of a unitary division.

```
def compute-period(p):
    results = []
    remainders = []
    d = 1
    q = 0
    r = 1
    while r not in remainders:
        results.append(str(q))
        remainders.append(r)
        d = r*10
        q = int(d/p)
        r = d%p
    remainders.append(r)
    results.append(str(q))
    steps = len(results)-1
    res = “”.join(results)
    res = “(“ + res[1:]+ “)”
    return steps,res,remainders
p = int(input(“Input a natural number p: “))
period = compute-period(p)
print(“Period Length: “ , period[0])
print(“Period: “ , period[1])
```

A python program computing periods.

Decimal Period of 1/997 (166 digits)
d = 0010030090270812437311935807422266800401203610832497492477432296890672016048144433299899699097291875626880641925777331995987963891675025075225677031093279839518555667 (The period was generated by the Python program above, and its correctness proof is given in Table 3 below).

Table 3. Proving that the decimal period of $1/997$ is correct: The column on the left gives, in consecutive rows, blocks of the period d of $1/997$. In each equation of the second column, the last three digits added to the first three digits of the number below provides 999; therefore, according to the Concatenation Theorem, the equations above prove that $0, d \times 997 = 0, 9^{217}$. Therefore, 9^{217} being a period, it follows that $0, [9^{217}] = 0, [9] = 1$.

$$
\begin{gathered}
00100300902 \times 997 = 999 - 9^8 - 294 \\
7081243731 \times 997 = 705 - 9^7 - 807 \\
1935807422 \times 997 = 192 - 9^7 - 734 \\
266800401 \times 997 = 265 - 9^6 - 797 \\
2036108324 \times 997 = 202 - 9^7 - 028 \\
97492477432 \times 997 = 971 - 9^8 - 704 \\
29689067201 \times 997 = 295 - 9^8 - 397 \\
60481444332 \times 997 = 602 - 9^8 - 004 \\
99899699097 \times 997 = 995 - 9^8 - 709 \\
2918756268 \times 997 = 290 - 9^7 - 196 \\
8064192577 \times 997 = 803 - 9^7 - 269 \\
7331995987 \times 997 = 730 - 9^7 - 039 \\
96389167502 \times 997 = 960 - 9^8 - 494 \\
507522567703 \times 997 = 505 - 9^9 - 891 \\
10932798395 \times 997 = 108 - 9^8 - 815 \\
18555667 \times 997 = 184 - 9^8 - 999
\end{gathered}
$$

Let us conclude the section by shortly reporting other crucial passages that are based on the diffusion in Europe of the positional representation of numbers. After the publication by Leonardo Fibonacci of *Liber Abaci*, in 1202, a long process begun of conceptual and notational development of modern mathematics [11,12].

In 1591, François Viète's book [13] appeared, where expressions with symbols for *indeterminates* appear, and *ars speciosa* is also the name of a new arithmetic perspective that is the seed of modern algebra. In 1614, John Napier introduced logarithms [9], where he provided a *synchronization* between geometrical and arithmetical progressions covering with good approximation a real interval. This passage in modern mathematics is crucial and full of practical and theoretical consequences.

The passage from symbols with numeric meanings to indeterminates of unknown numeric values, on which operations can be performed independently from their meaning, is a crucial step toward variables, which became the main tool of Cartesian geometry, where in 1637 René Descartes introduced coordinates, by reversing the Greek relationship between space and numbers [14]. From this point, the process of *arithmetization* of mathematics started, toward the foundational perspectives of the 19th and 20th centuries.

5. Enumerations in Ordinals and in Computability

Archimedes's mark is not only in the roots of modern mathematics and in infinitesimal calculus, including his introduction of the geometric representation of infinitesimals, especially in his *Method*, a book that was lost and was discovered at the beginning of the twentieth century (and lost again during the second world war, but now completely restored) [15]. In fact, the crucial role of recurrence in Archimedes's enumeration is apparent in Cantor's ordinal numbers [16] and in the theory of computability [17]. Rigorous foundations of numbers were provided by Dedekind, Frege, and Peano [18–20], but the most synthetic and expressive definition of numbers is that one given in terms of set theory, according to a construction due to John von Neumann: *a number is the set of numbers that precede it, in a number enumeration*. In this way, 0 coincides with the empty set $\emptyset$, 1 is the set containing the empty set $\{\emptyset\}$, 2 is the set $\{\emptyset, \{\emptyset\}\} = \{0, 1\}$, and so on. In this formulation, even if "a number enumeration" is mentioned, the numbers stem prescinding from any specific system of counting, in a very abstract manner, where the process of counting results in the true essence of numbers. In fact, if $e_1, e_2, \ldots$ is any enumeration and $[[e_1]], [[e_2]] \ldots$ the corresponding numbers, then $[[e_1]] = \emptyset$, $[[e_2]] = \{\emptyset\} = \{[[e_1]]\}$, $[[e_2]] = \{\emptyset, \{\emptyset\}\} = \{[[e_1]], [[e_2]]\}$, and so on, by obtaining exactly what von Neumann defined. Moreover, the theory of ordinals can be expressed in terms of enumerations of enumerations, in the same way as Archimedes's periods are generated, because the essence of a recurrent enumeration is that a number is the position where its numeral is, and this position is completely identified by the numerals that precedes it. In this way, if a name a given to an entire enumeration, this name is a sort of hypernumeral that we can imagine as the last position of its numerals. Then, let us call ω the natural enumeration

$$\omega = 0, 1, 2 \ldots$$

If we assume ω as the first infinite order, we can go further with the following orders in an analogous way using Archimedes's periodical system:

$0, 1, 2, \ldots, \omega$
$\omega + 1, \omega + 2, \ldots, 2\omega$
$2\omega + 1, 2\omega + 2 \ldots, 3\omega$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots \ , \omega^2$
$\omega^2 + 1, \omega^2 + 2 \ldots, \omega^2 + \omega$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ , \omega^3$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ , \omega^\omega$

where the name of any (infinite) enumeration is put at the end of it, and then usual symbols for ordinals are intended as names of the consecutive enumerations preceding them (enumerations of enumerations, and so on, at successive levels). It is not our intention to go into further details of such an approach to ordinals, but this short outline suggests clearly that ordinals are a natural generalization of Archimedes's periods.

It can be shown that a one-to-one correspondence can be established between the natural enumeration ω and ordinals at any exponential level ($\omega^\omega, \omega^{\omega^\omega} \ldots$), but no one-to-one correspondence exists between ω and real numbers, which are represented by infinite sequences of decimal digits (and the set of ordinals one-to-one with ω is an ordinal that is not one-to-one with ω). This crucial result, based on a famous *diagonal argument*, is the access gate to cardinal numbers and abstract sets, or Cantor Paradise, such as Hilbert defined set theory [21], within which any mathematical theory can be expressed.

In 1936, Turing published an epochal paper on computable numbers, that is, real numbers where the sequence of digits can be generated by means of a computing device, a Turing machine. Sets of numbers that can be generated by the Turing machine, as outputs of the computing process, are called Turing enumerable, or recursively enumerable, sets. However, in general, there is no Turing machine that, given a Turing enumerable set A and a number n, is able to tell, in a finite number of steps, if a does belong or does not belong to A. The sets for which this is possible are called decidable or recursive. The recursively enumerable sets for which this decision possibility does not hold are called *semidecidable*. A function is computable if and only if its graphic is recursively enumerable.

What is really surprising is that Turing proves the existence of recursively enumerable sets, by adapting Cantor's diagonal argument according to which real numbers are not one-to-one with any natural enumeration. This story tells us that an *infinity line* [22] links, along centuries, Archimedes with Cantor and Turing: these three giants follow a common idea, *counting the infinite*, according to an arithmetical perspective, to a more general set theoretic perspective, or to a computational perspective of symbolic manipulation processes, performed by machines.

A function on natural numbers is Turing computable if it is computed by some Turing machine (giving as output the image of the function in correspondence to any argument given as input). Turing machines are identified by Turing programs, which are strings, which when put in a lexicographic ordering provide an enumeration. From this, again by a diagonal argument, the following theorem can be proved.

Theorem 5. *Turing computable functions surely include partial functions, which do not give results in correspondence of some arguments, and no Turing machine can exist that can always tell, in finite time, if a Turing machine gives a result in correspondence to a given argument.*

6. Conclusions

Numbers need numerals to be expressed and manipulated, but numerals are strings, that is, linear forms of information representation, able to encode any kind of data. However, strings, when considered in a lexicographic order, represent numbers, with the empty string naturally associated with zero. Therefore, an intrinsic circularity links numbers to strings or, equivalently, numbers to information. Nevertheless, while numbers are abstract entities, independent from any physical reality, symbols and strings are necessarily based on physical realities. At the same time, their physicality, even if necessary, is not essential, in the sense that any physical support can be replaced equivalently by another one, and, similarly, any encoding of data as strings can be translated into another one. The theory of information, which together with computability is the basis of the new informational age, according to Shannon's perspective, introduced in his famous booklet of 1948 [23], discovered the possibility of measuring information independently from specific codes and from specific physical supports. In this approach, information is expressed in terms of negative logarithms of probabilities. But probabilities are pure numbers (between 0 and 1); therefore, "pure" information coincides with numbers, and, conversely, numbers coincide with pure information, because their essence abstracts from any specific system of numerals. This simple remark explains why number theoretic properties are so crucial in information processing, at many different levels, from cryptography to the theory of codes and to the algebraic and algorithmic perspectives of computer science. This means that arithmetic, the oldest mathematical theory, is strongly linked to the youngest theory of information, computation, and communication, born in the twentieth century. Then, the image of a circle, so often evoked in this paper, is a very appropriate image for the conclusion of this bird flying over the landscape of mathematics, going from Archimedes, to Fibonacci, Stevin, Viète, Napier, Descartes, Gauss, Cantor, Turing, and Shannon, just as a reminder of the great minds mentioned in our travel.

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