algorithms

MDPI

Article

Using Markov Random Field and Analytic Hierarchy Process to Account for Interdependent Criteria

Jih-Jeng Huang [1] and Chin-Yi Chen [2,*]

[1] Department of Computer Science & Information Management, SooChow University, No. 56 Kueiyang Street, Section 1, Taipei 100, Taiwan; jjhuang@scu.edu.tw
[2] Department of Business Administration, Chung Yuan Christian University, No. 200 Chung Pei Road, Chung Li District, Taoyuan 320, Taiwan
* Correspondence: iris@cycu.edu.tw

Abstract: The Analytic Hierarchy Process (AHP) has been a widely used multi-criteria decision-making (MCDM) method since the 1980s because of its simplicity and rationality. However, the conventional AHP assumes criteria independence, which is not always accurate in realistic scenarios where interdependencies between criteria exist. Several methods have been proposed to relax the postulation of the independent criteria in the AHP, e.g., the Analytic Network Process (ANP). However, these methods usually need a number of pairwise comparison matrices (PCMs) and make it hard to apply to a complicated and large-scale problem. This paper presents a groundbreaking approach to address this issue by incorporating discrete Markov Random Fields (MRFs) into the AHP framework. Our method enhances decision making by effectively and sensibly capturing interdependencies among criteria, reflecting actual weights. Moreover, we showcase a numerical example to illustrate the proposed method and compare the results with the conventional AHP and Fuzzy Cognitive Map (FCM). The findings highlight our method's ability to influence global priority values and the ranking of alternatives when considering interdependencies between criteria. These results suggest that the introduced method provides a flexible and adaptable framework for modeling interdependencies between criteria, ultimately leading to more accurate and reliable decision-making outcomes.

Keywords: Analytic Hierarchy Process (AHP); Markov Random Fields (MRFs); multi-criteria decision-making; criteria interdependencies; pairwise comparison matrices

Citation: Huang, J.-J.; Chen, C.-Y. Using Markov Random Field and Analytic Hierarchy Process to Account for Interdependent Criteria. *Algorithms* **2024**, *17*, 1. https://doi.org/10.3390/a17010001

Academic Editor: Roberto Montemanni

Received: 2 December 2023
Revised: 17 December 2023
Accepted: 18 December 2023
Published: 19 December 2023

1. Introduction

The Analytic Hierarchy Process (AHP) has gained widespread recognition as a versatile multi-criteria decision-making method with successful applications across diverse fields such as finance, healthcare, and manufacturing [1,2]. The essence of AHP lies in its ability to decompose complex decision-making scenarios into a hierarchy of goals, criteria, and alternatives, enabling a systematic and structured approach to decision analysis [3,4]. This hierarchical structure, paired with a methodical pairwise comparison process, allows for a quantifiable evaluation of options against multiple criteria, making it an invaluable tool in strategic planning and resource allocation.

Despite its widespread acceptance and versatility, AHP's conventional application rests on a pivotal assumption: the independence of decision criteria [5]. In numerous real-world scenarios, however, decision criteria often exhibit varying degrees of interdependence, leading to complexities that the traditional AHP model may not adequately capture. Such interdependencies can stem from economic, environmental, social, or technical factors, and their oversight can lead to inaccurate prioritization, ultimately affecting the reliability and validity of decision outcomes [6]. This limitation of AHP is not merely theoretical but has practical implications in fields where decision criteria are intrinsically interlinked, such as in environmental policy formulation, healthcare management, and urban planning.

In response to this challenge, the research community has explored and developed numerous methodologies to account for criteria interdependence in MCDM. Prominent among these are the Analytic Network Process (ANP), a natural extension of AHP proposed by Saaty himself, which explicitly considers interdependencies and feedback loops among decision elements ([7]). Similarly, the Decision-Making Trial and Evaluation Laboratory (DEMATEL) method has been instrumental in visualizing and analyzing complex causal relationships among criteria [8]. Further enriching the landscape are Fuzzy Cognitive Maps (FCMs), which combine elements of fuzzy logic and neural networks to model systems where human judgment and imprecision play a significant role [9]. These developments, coupled with integrated fuzzy MCDM approaches such as Fuzzy Analytic Hierarchy Process (FAHP) and Fuzzy Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), have significantly advanced the field [10].

Moreover, hybrid MCDM methodologies have emerged, such as the integration of ANP with Fuzzy ELECTRE methods, offering nuanced and sophisticated decision-making frameworks [11]. These innovative approaches have been applied to various domains, ranging from resource allocation to strategic planning, highlighting their versatility and effectiveness. Ref. [12] provided a comprehensive review of methods considering criteria dependencies, including ANP, DEMATEL, and Fuzzy Gray Relational Analysis (Fuzzy GRA). Ref. [13] proposed a method for selecting interdependent information system projects, employing ANP to evaluate interdependencies and Goal Programming (GP) for project selection. These methods, while robust, often rely on elaborate pairwise comparison matrices (PCMs), which can become cumbersome and computationally intensive, particularly in scenarios involving a large number of criteria and alternatives.

This research gap has led us to explore the potential of discrete Markov Random Fields (MRFs) as an innovative means to model criteria interdependencies within the AHP framework. MRFs, a cornerstone in the field of probabilistic graphical models, offer a robust and versatile tool for capturing dependencies in a structured manner. Widely employed in areas such as computer vision, natural language processing, and social network analysis, MRFs provide a mathematical framework for representing complex relational structures in a probabilistic context [14]. By embedding MRFs into the AHP, we propose a methodology that not only maintains the simplicity and intuitiveness of AHP but also effectively captures the nuances of interdependent decision criteria. This integration paves the way for a more accurate and reliable decision-making process, especially in contexts where the interplay of criteria significantly influences outcomes.

To illustrate the efficacy of our approach, we present a numerical example that compares the proposed MRF-AHP method with the traditional AHP and Fuzzy Cognitive Map (FCM) approaches. This comparison is pivotal in demonstrating how the incorporation of MRFs into the AHP framework can lead to significant shifts in criteria weighting and prioritization, reflecting a more realistic representation of interdependencies in decision-making processes. The results showcase a notable ranking reversal in the proposed method, underscoring its capability to more effectively account for interdependencies compared to the conventional AHP model.

The remainder of this paper is structured as follows: Section 2 presents a concise review of existing methods for addressing criteria interdependencies, the AHP, and discrete MRFs. Section 3 introduces our proposed method, seamlessly incorporating MRFs into the AHP framework. Section 4 showcases the effectiveness of our approach through a case study. Finally, Section 5 concludes the paper and highlights potential avenues for future research.

2. Literature Review

This section provides a comprehensive overview of the advancements in MCDM methods for interdependent criteria, with a focus on the AHP and MRFs, and addresses the need for integrating these methodologies.

2.1. Methods for Interdependent Criteria

The growing complexity of decision-making problems has necessitated the advancement of MCDM methods that take into account interdependencies among criteria. Ref. [9] offered an extensive analysis of MCDM methods that model the interactions among criteria, stressing the importance of considering preference, criterion, and utility interactions. A prominent MCDM method that tackles interdependencies is the ANP, which is introduced by [7]. This method expands the AHP by addressing interdependencies and feedback loops among criteria and alternatives, providing a structured approach to managing intricate decision-making problems.

Nonetheless, the ANP has certain drawbacks, including significant computational complexity and potential inconsistencies in pairwise comparisons [15]. In addition, the massive use of subjective judgments in the ANP can lead to bias, impacting the overall trustworthiness of the results. Researchers have proposed hybrid models that combine various MCDM methods to overcome these challenges. For example, ref. [10] suggested a model that merges the DEMATEL method with ANP, while ref. [16] presented an integrated hybrid model that combines a modified TOPSIS with a PGP model for supplier selection with interdependent criteria. These hybrid models addressed the limitations of traditional MCDM methods concerning interdependencies and showcased their effectiveness through real-world applications [10,16].

Despite the benefits of these methods in handling interdependencies, some challenges remain. For instance, the integration of DEMATEL within the ANP framework can lead to inconsistencies and ambiguity for decision-makers due to the absence of a cohesive approach [11]. Moreover, hybrid models, such as the FCM-AHP approach proposed by [17], introduced extra complexity and dependence on expert opinions, which can result in subjectivity and reduced practical usability. In their recent publication, ref. [18] presented a novel FCM-based MCDM method for addressing complex decision-making problems.

The FCM approach, introduced by [19], builds upon the cognitive maps concept [20] by incorporating fuzzy degrees of interrelationships between concepts. FCMs illustrate the influence of vertices, referred to as concepts, on one another through cause–effect relationships, which are quantified and typically normalized to the $[-1, 1]$ interval.

Let $w_{ij} \in [-1, 1]$ be the degree of influence from the ith concept, C_i, to the jth concept, C_j, where the sign signifies a positive or negative influence. A value of -1 denotes a full negative impact, while 1 signifies a full positive impact. The influence of concept x can be calculated by the following equation:

$$x_i(t+1) = f(\sum_{j=1, j\neq i}^{n} x_j(t) w_{j,i}) \tag{1}$$

where n represents the number of concepts and $f(\cdot)$ is the transfer function, which compresses the multiplication result into a specific range, e.g., [0, 1] or $[-1, 1]$. Commonly, bivalent, trivalent, and sigmoid functions are used in the FCM. Ref. [21] proposed a modified version of Kosko's FCM that accounts for the previous value of each concept, thereby examining the self-loop effect. Consequently, Equation (1) can be altered and expanded as follows:

$$x_i(t+1) = f(\sum_{j=1, j\neq i}^{n} x_j(t) w_{j,i} + x_i(t)) \tag{2}$$

Another FCM variant used for rescale inference is presented below:

$$x_i(t+1) = f(\sum_{j=1, j\neq i}^{n} (2x_j(t) - 1) w_{j,i} + (2x_i(t) - 1)) \tag{3}$$

In an FCM, the presence of positive and negative influences suggests that the centrality of a vertex within a graph should consider two opposing forces to determine the final centrality.

Ref. [12] highlighted the need for ongoing research in MCDM methods that consider the dependency between criteria and evaluate their advantages and disadvantages in various decision-making contexts. Recognizing the limitations of the ANP method and other hybrid models, our study aims to enhance the AHP by incorporating interdependency between criteria using the MCM approach. By integrating MCM into the AHP framework, we aim to develop a more user-friendly method for addressing interdependencies in decision-making problems. The incorporation of MCM will allow decision-makers to effectively assess the relationships between criteria while minimizing the complexity and expert input demanded by the ANP and other hybrid models.

2.2. AHP

AHP was introduced by [1] as a decision-making methodology for complex, multi-criteria problems. AHP is based on pairwise comparisons in a hierarchical structure, which simplifies decision making by breaking it down into smaller, more manageable parts [4]. The mathematical model of AHP involves constructing the pairwise comparison matrix (PCM), calculating the weights of criteria or alternatives, and computing the consistency ratio [1,22]. Since its introduction, AHP has seen significant development in both theory and application. Researchers have focused on improving the mathematical foundations, extending their applicability, and integrating it with other methods to address various decision-making problems.

To quantify the PCMs, ref. [1] introduced a scale ranging from 1 to 9, where 1 represents equal importance, and 9 represents extreme importance. The decision-maker assigns values from this scale to express the relative importance of one element over another. This results in a reciprocal PCM, which is used to calculate the eigenvector corresponding to the largest eigenvalue. This eigenvector represents the normalized weights of the criteria or alternatives [1], which can be represented as solving the following eigenvalue problem:

$$\boldsymbol{A}\boldsymbol{w} = \lambda_{\max}\boldsymbol{w} \tag{4}$$

where $\boldsymbol{A}$ is the PCM, $\boldsymbol{w}$ denotes the weight vector of criteria, and $\lambda_{\max}$ denotes the maximum eigenvalue.

AHP's methodology also accounts for the consistency of the decision-maker's judgments. Ref. [1] proposed the consistency ratio (CR), which compares the consistency index (CI) to the random index (RI)—an average index derived from randomly generated matrices. A CR less than or equal to 0.10 is generally considered acceptable, indicating that the decision-maker's judgments exhibit a reasonable level of consistency. CR and CI can be formulated as follows:

$$CR = \frac{CI}{RI} = \frac{\lambda_{\max} - n}{n-1} / RI \tag{5}$$

where n denotes the number of criteria; and RI (random index) is used to estimate the average consistency of a randomly generated pairwise comparison matrix and can refer to Table 1.

Table 1. RI table.

Number of Criteria	Random Index
3	0.58
4	0.90
5	1.12
6	1.24
7	1.32
8	1.41

In terms of theoretical foundations, consistency and inconsistency measures have been proposed to evaluate the reliability of pairwise comparisons, such as the consistency ratio (CR) [1] and the geometric consistency index (GCI) by [23]. In recent research, ref. [24]

addressed the critical issue of maintaining consistency, especially ordinal consistency, in AHP-based decision-making models. This study proposes optimization models to achieve transitive preferences and to solve individual consistency and group consensus problems in AHP. It offers a methodological advancement by formulating constraints for ordinal consistency and developing mixed integer linear optimization models to ensure both ordinal and cardinal consistencies in group decision-making settings.

Alternative weight derivation methods have been developed, including the row geometric mean method (RGMM) by [25], the logarithmic least squares method (LLSM) by [26], and the least square priorities method (LSM) by [27]. Additionally, group decision making has been incorporated into AHP through aggregation methods like the weighted arithmetic mean (WAM) by [28] and the geometric aggregation operator (GAO) by [29]. Regarding practical applications, AHP has been used in supply chain management for supplier selection and evaluation, incorporating green supplier evaluation by [30] and assessing supply chain resilience by [31]. In the field of environmental decision making, AHP has been applied to prioritize climate change adaptation strategies by [32] and renewable energy source prioritization in Turkey [33].

Recent developments in AHP and pairwise comparisons are well illustrated in the selected papers. Ref. [34] study streamlined AHP for complex scenarios, showcasing advancements in methodology simplification. Ref. [35] presented a novel, data-driven fuzzy AHP method, exemplifying the integration of fuzzy logic in environmental decision making. Ref. [36] contribute by evaluating various scaling methods in AHP, enhancing decision accuracy. These papers, together with [37] research on rank reversals in AHP, offer a comprehensive insight into the latest trends and methodologies in AHP and pairwise comparisons.

However, a key assumption of AHP is that criteria and alternatives are independent, which may not hold true in real-world decision-making scenarios due to interdependencies between criteria [5,6]. To address this issue, several approaches have been proposed in the literature, such as the ANP by [7], fuzzy AHP by [38], and hybrid methods that combine AHP with Bayesian networks by [33] or FCMs by [39]. However, these approaches can be challenging to implement or require extensive data, highlighting the need for more efficient and intuitive methods to address criteria interdependencies in AHP.

2.3. MRF

MRFs are a class of probabilistic graphical models that capture statistical relationships between random variables through an undirected graph [40]. MRFs have gained attention in various fields due to their ability to model complex dependencies and provide an elegant mathematical framework for modeling and reasoning under uncertainty. MRFs have been used in computer vision, natural language processing, and social network analysis, among other domains [41–43]. Several extensions and variations of MRFs have been proposed, such as CRFs by [44], Gaussian MRFs by [45], and factor graphs by [46]. Despite the success of MRFs in modeling complex dependencies, challenges remain in scalability, handling large-scale and high-dimensional problems, and developing more robust and flexible MRF models [44,47].

The mathematical model of a discrete MRF is defined by an undirected graph $G = (V, E)$, where V is a set of nodes representing random variables, and E is a set of edges denoting the pairwise interactions between the variables. A discrete MRF is associated with a set of potential functions $\varphi_j(w_{C_j})$ that capture the strength of the relationship between the variables [40]. The joint probability distribution of the random variables in an MRF, defined by the Gibbs distribution, is given by:

$$P(X = x) = \frac{1}{Z} exp\left(-\sum_{j=1}^{J} \varphi_j(w_{C_j})\right), \tag{6}$$

where C_j denotes the set of maximal cliques, X represents the set of random variables, x is a specific assignment of values to these variables, w_{C_j} denotes the values of the variables in clique C_j, and Z is the normalization constant known as the partition function.

3. MRF-Based AHP

By incorporating discrete MRFs into the AHP, we aim to address the gap of accounting for interdependency between criteria. This integration ultimately enhances the decision-making process and leads to more accurate and reliable results.

Step 1: Model the interdependencies between criteria using MRFs

To begin with, we must establish the MRF's structure to illustrate the criteria's interconnections. An undirected graph can represent these interdependencies, where nodes denote the criteria and edges indicate direct connections between them. For example, we have an adjacency matrix of a graph with four criteria:

$$\boldsymbol{A} = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The adjacency indicates that, for example, C_1 is interdependent with C_2 and C_3.

Step 2: Pairwise comparisons and local priorities

Next, we need to obtain the local priorities of the criteria and alternatives based on pairwise comparisons. For this, we will use the traditional AHP approach. Decision-makers will provide pairwise comparison matrices for each criterion with respect to the goal and for each alternative with respect to each criterion.

Step 3: Calculate joint probability distribution using MRFs

With the MRF structure and local priorities in place, we can now calculate the joint probability distribution over all criteria using MRFs. This is completed by combining the local probability distributions (obtained from the pairwise comparisons) with the pairwise potentials between connected criteria, which capture the strength of the interdependencies. The joint probability distribution will provide the overall priorities of the criteria, accounting for their interdependencies. For example, if we have initial weights derived from the AHP as $\boldsymbol{p} = [0.2, 0.3, 0.3, 0.2]$, and the interdependencies between criteria can be quantified by expert opinion as:

$$\boldsymbol{W} = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{bmatrix} 0 & 0.3 & 0.2 & 0 \\ 0.3 & 0 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0 & 0.3 \\ 0 & 0.4 & 0.3 & 0 \end{bmatrix}$$

Note that the pairwise potentials matrix $\boldsymbol{W}$ is derived to represent the strength of the relationships between criteria in the MRF framework. This matrix is a key component in calculating the joint probability distribution of criteria, accounting for their interdependencies. To obtain $\boldsymbol{W}$, expert opinions or data-driven methods can be utilized to quantify the degree of dependency between criteria pairs. Each entry $\boldsymbol{W}_{ij}$ in the matrix reflects the strength of the relationship between criteria i and j. This process involves both subjective judgment and quantitative analysis to ensure that the pairwise potentials accurately represent the interdependencies within the decision-making context. In this paper, we simply assume that the potential matrix is quantified by the expert and the objective potential matrix could be explored by the future research.

Then, we derive the joint probability distribution of each criterion as follows. Following our assumption above, let us denote the initial priorities of the criteria as $\boldsymbol{P}$ = [p_1, p_2, p_3, p_4] and the pairwise potentials matrix as $\boldsymbol{W}$, where W_{ij} represents the strength of the relationship between criteria i and j. The goal is to estimate the adjusted priorities $\boldsymbol{P'} = [p_1{}', p_2{}', p_3{}', p_4{}']$ through Gibbs sampling.

The algorithm starts with the initial priority $\boldsymbol{P}$ and proceeds as follows:

1. For each criterion, i, identify the incoming and outgoing pairwise potentials. Incoming edges are the rows in the $\boldsymbol{W}$ matrix with a positive value in the ith column ($W_{ij} > 0, \forall j$), and outgoing edges are the columns in the $\boldsymbol{W}$ matrix with a positive value in the ith row ($W_{ij} > 0, \forall i$).
2. For each criterion, i, update its priority $p_i{}'$ based on the sum of the incoming and outgoing pairwise potentials multiplied by their corresponding priorities. Mathematically, this can be represented as:

$$p'_i = \sum W_{ij} p_j + \sum W_{ji} p_j, \forall i \neq j. \tag{7}$$

3. Normalize the updated priorities to ensure they sum up to 1:

$$p'(t+1) = p'(t) / \sum p'_i(t) \tag{8}$$

4. Repeat steps 1–3 for a specified number of iterations (e.g., 10,000). Store the priorities obtained in each iteration.
5. Calculate the mean of the priorities across all iterations to obtain the adjusted priorities $\boldsymbol{P'} = [p_1{}', p_2{}', p_3{}', p_4{}']$. The adjusted priorities are derived from the Gibbs sampling as $\boldsymbol{p'} = [0.200, 0.292, 0.230, 0.278]$.

Step 4: Compute the global priorities of the alternatives

Using the overall priorities of the criteria obtained from the MRF model, we can now compute the global priorities of the alternatives by aggregating the local priorities of the alternatives with respect to each criterion, taking into account the overall priorities of the criteria.

$$\text{Adjusted priority} = (\frac{1}{N}) \times \sum_{t=1}^{N} p'(t) \tag{9}$$

Note that the method for obtaining adjusted priorities via the mean of priorities across all iterations, instead of just using the last iteration, is chosen to ensure stability and robustness in the decision-making process. This approach mitigates the risk of outliers or anomalies in any single iteration heavily influencing the final decision. Using the mean of all iterations provides a more representative and balanced view of the priorities, reflecting a consensus over the entire process. This method, while more complex than ANP, aims to accurately capture the nuances of interdependencies in criteria, offering a comprehensive and reliable decision-making framework.

Step 5: Select the best alternative

Finally, we can select the best alternative based on the information in the decision table, as shown in Table 2, and choose the highest global priority.

Table 2. Decision table for the best alternative.

Decision Table	C_1	C_2	C_3	C_4	Total
A_1	0.5	0.4	0.6	0.3	0.4383
A_2	0.3	0.1	0.2	0.4	0.2464
A_3	0.2	0.5	0.2	0.3	0.3153
Adjusted weights	0.2000	0.292	0.230	0.278	1

Next, we give a numerical example to demonstrate the proposed method with different graph models and compare the results with the AHP and FCM.

4. Numerical Example

Consider a decision problem that involves choosing the most suitable location for a new store from three alternatives: A1 (City Center), A2 (Suburbs), and A3 (Industrial Zone). The decision is based on four criteria: C1 (Foot Traffic), C2 (Rent Cost), C3 (Proximity to Competitors), and C4 (Accessibility). We begin by constructing pairwise comparison matrices for each criterion in relation to the goal and for each alternative with respect to each criterion. Let us assume the decision-makers provided the following matrices (using the standard AHP 1–9 scale):

Criteria	C1	C2	C3	C4
C1	1	1/3	1	1/5
C2	3	1	3	1/3
C3	1	1/3	1	1/5
C4	5	3	5	1

Alternatives matrices for each criterion:

C1	A1	A2	A3	C2	A1	A2	A3
A1	1	3	1/3	A1	1	1/5	5
A2	1/3	1	1/5	A2	5	1	1/5
A3	3	5	1	A3	1/5	5	1
C3	A1	A2	A3	C4	A1	A2	A3
A1	1	5	1/5	A1	1	6	1/6
A2	1/5	1	1/2	A2	1/6	1	1/3
A3	5	2	1	A3	6	3	1

After calculating the eigenvectors and normalizing them, we obtain local priorities for criteria and alternatives as shown in Table 3:

Table 3. Decision table for the numerical example.

Decision Table	**A1**	**A2**	**A3**	**Priorities**
C1	0.16	0.68	0.16	0.15
C2	0.16	0.68	0.16	0.40
C3	0.49	0.16	0.35	0.15
C4	0.65	0.16	0.19	0.30

With the given interdependencies, we define the structure of the MRF as shown in Figure 1. We can represent the interdependencies with an undirected graph, where nodes represent the criteria and edges represent the direct connections between criteria. First, let us assume the dependency and pairwise potentials between connected criteria are provided by expert opinion as follows:

The pairwise potentials between criteria can be represented as (0.1, 0.3, 0.5, 0.7, 0.9) to indicate the degree of dependency between criteria as (very slightly dependent, slightly dependent, moderately dependent, highly dependent, very highly dependent). Take the MRF (1) for example; we can formulate the pairwise potential matrix as:

$$\boldsymbol{W} = \begin{bmatrix} 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0.9 & 0 & 0 & 0.7 \\ 0 & 0.5 & 0.7 & 0 \end{bmatrix}$$

We combine the local probability distributions (obtained from the pairwise comparisons) with the pairwise potentials between connected criteria to compute the joint probability distribution over all criteria using MRFs. This provides the criteria's overall priorities, considering their interdependencies, as shown in Table 4. The joint probability distribution results in the following adjusted criteria priorities, which can be derived from Gibbs sampling.

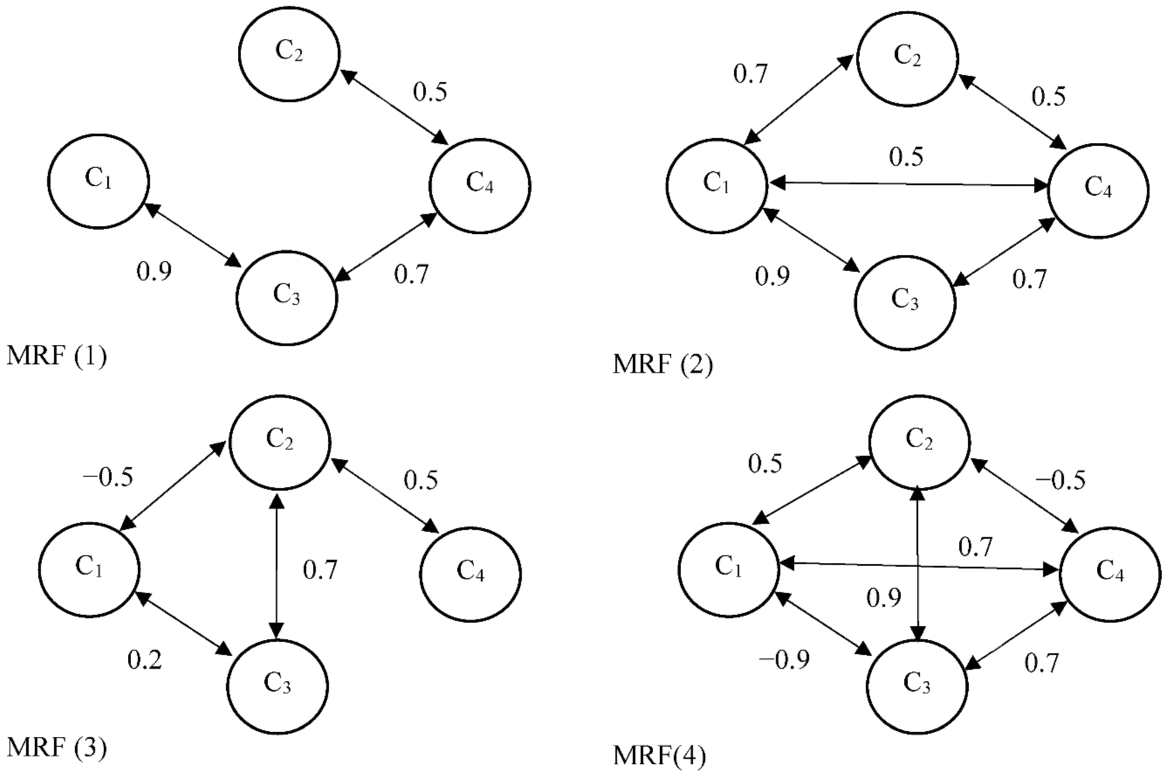

Figure 1. The dependency and pairwise potential between criteria.

Table 4. Weights for different methods.

Weights	C_1	C_2	C_3	C_4
AHP	0.150	0.400	0.150	0.300
MRF-adjusted weights (1)	0.273	0.109	0.360	0.258
FCM-adjusted weights (1)	0.244	0.217	0.280	0.259
MRF-adjusted weights (2)	0.291	0.200	0.256	0.253
FCM-adjusted weights (2)	0.267	0.231	0.250	0.253
MRF-adjusted weights (3)	0.063	0.343	0.324	0.270
FCM-adjusted weights (3)	0.192	0.272	0.281	0.254
MRF-adjusted weights (4)	0.213	0.256	0.285	0.246
FCM-adjusted weights (4)	0.223	0.261	0.254	0.261

The iteration of the Gibbs sampling can refer to Figure 2 and ensure the convergence of the final priorities.

Using the overall priorities of the criteria obtained from the MRF model, we can calculate the global priorities of the alternatives. This is accomplished by aggregating the local priorities of the alternatives concerning each criterion, considering the overall priorities of the criteria. With the new adjusted criteria priorities, the ranking of the alternatives has changed. The MRF-adjusted AHP now ranks the alternatives as B, A, and C, while the original AHP ranked them as A, B, and C, as demonstrated in Table 5.

This indicates that considering the interdependencies between the criteria with the adjusted criteria priorities affects the overall ranking of the alternatives. In this numerical example, we have demonstrated how discrete MRF can be incorporated into the AHP framework to account for the interdependencies between criteria using concrete values. The proposed method enhances decision making and can be applied to various real-world decision-making problems where criteria interdependencies play a crucial role.

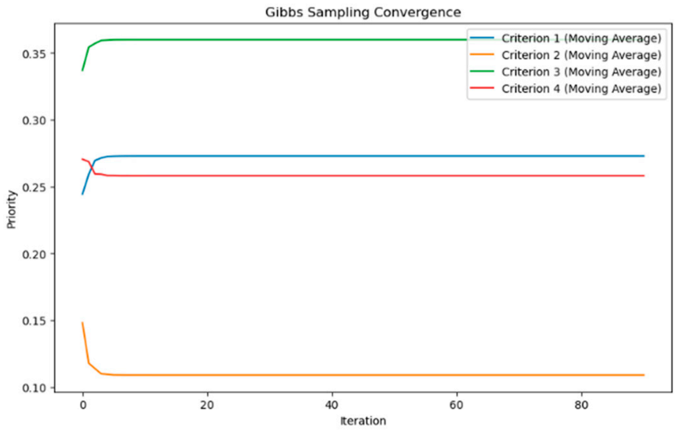

Figure 2. The convergence of criteria priorities.

Table 5. Comparison between the proposed method and the AHP.

Scores	A	B	C	Rank
AHP	0.3270	0.3260	0.3470	C, A, B
MRF-adjusted weights (1)	0.2617	0.3948	0.3435	B, C, A
FCM-adjusted weights (1)	0.2700	0.3840	0.3460	B, C, A
MRF-adjusted weights (2)	0.2644	0.3920	0.3436	B, C, A
FCM-adjusted weights (2)	0.2718	0.3816	0.3466	B, C, A
MRF-adjusted weights (3)	0.2605	0.3978	0.3417	B, C, A
FCM-adjusted weights (3)	0.2680	0.3860	0.3460	B, C, A
MRF-adjusted weights (4)	0.2626	0.3922	0.3452	B, C, A
FCM-adjusted weights (4)	0.2695	0.3815	0.3490	B, C, A

5. Discussion

This paper introduces an innovative approach to address the issue of interdependency among criteria in the AHP by incorporating discrete MRFs. The traditional AHP assumes that criteria are independent, which may not always be the case in real-world decision-making scenarios. Overlooking these interdependencies can result in misleading or suboptimal decisions. Our proposed method aims to enhance the decision-making process by capturing and representing the interdependencies between criteria more efficiently and intuitively.

In the numerical example presented, we showcased the application of the MRF-adjusted AHP in an MCDM scenario. The example demonstrated how criteria interdependencies could be considered using pairwise potentials, representing the strength of dependencies between criteria. The findings underlined the discrepancies in global priority values and the potential shifts in alternative rankings when interdependencies are considered. Furthermore, both the MRF-adjusted AHP and FCM methods were employed to address criteria interdependencies. The results revealed that the overall ranking of alternatives differed when compared to the conventional AHP, which assumes criteria independence. This underscores the significance of accounting for criteria interdependencies in real-world decision-making situations. Although both methods effectively capture interdependencies, the MRF-adjusted AHP provides a more intuitive and straightforward representation through pairwise potentials, simplifying the decision-makers' understanding and interpretation of the results. The two methods exhibit distinct applications and data requirements. Moreover, the FCM method requires setting multiple parameters to obtain accurate results, which is not necessary for the proposed method.

Compared to other approaches that address criteria interdependence in the AHP, such as ANP and Bayesian networks, the MRF-adjusted AHP offers several advantages. First, it provides a more intuitive and straightforward representation of criteria interdependencies through the pairwise potentials matrix, which decision-makers can easily understand and interpret. In contrast to ANP, the MRF-adjusted AHP simplifies the process of comparing the criteria's relative weights. Next, the proposed method can account for both positive and negative interdependencies between criteria to apply them to more complicated problems. In addition, decision-makers can effortlessly alter the dependency strength between criteria by adjusting the pairwise potentials matrix, enabling sensitivity analysis and scenario exploration. Furthermore, the MRF-adjusted AHP is built on the well-established foundation of the original AHP, ensuring compatibility with existing AHP-based tools and methodologies. If interdependencies between criteria are disregarded, the proposed method reverts to the conventional AHP.

6. Conclusions

This paper presents a significant advancement in MCDM by introducing a novel method that integrates discrete MRFs into the AHP. Our approach successfully addresses a notable limitation in traditional AHP by considering the interdependencies among criteria, which is an aspect that is often overlooked in conventional methods. By incorporating MRFs, we not only extend the applicability of AHP to more complex decision-making scenarios but also enhance the accuracy and reliability of the outcomes. The MRF-adjusted AHP method demonstrates how global priority values and alternative rankings can shift when interdependencies are considered, offering insights into the dynamic nature of decision-making processes. The real-world application showcased in this paper illustrates the method's practicality and its potential to lead to more informed and effective decisions. The introduction of this method marks a significant contribution to the field, opening avenues for further research and application in various complex decision-making contexts. Our findings suggest that the MRF-adjusted AHP is not just an improvement over existing methods but a transformative step toward more nuanced and realistic decision-making models.

Author Contributions: Conceptualization, J.-J.H.; methodology, J.-J.H.; writing—original draft preparation, J.-J.H. and C.-Y.C.; writing—review and editing, J.-J.H. and C.-Y.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data available on request from the authors.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Saaty, T.L. *The Analytic Hierarchy Process*; McGraw Hill: New York, NY, USA, 1980; Volume 70.
2. Vaidya, O.S.; Kumar, S. Analytic hierarchy process: An overview of applications. *Eur. J. Oper. Res.* **2006**, *169*, 1–29. [CrossRef]
3. Saaty, T.L. Decision making with the analytic hierarchy process. *Int. J. Serv. Sci.* **2008**, *1*, 83–98. [CrossRef]
4. Ishizaka, A.; Labib, A. Review of the main developments in the analytic hierarchy process. *Expert Syst. Appl.* **2011**, *38*, 14336–14345. [CrossRef]
5. Mikhailov, L. Deriving priorities from fuzzy pairwise comparison judgements. *Fuzzy Sets Syst.* **2003**, *134*, 365–385. [CrossRef]
6. Tzeng, G.H.; Huang, J.J. *Multiple Attribute Decision Making: Methods and Applications*; CRC Press: Boca Raton, FL, USA, 2011.
7. Saaty, T.L. *Decision Making with Dependence and Feedback: The Analytic Network Process*; RWS Publications: Pittsburgh, PA, USA, 1996.
8. Gabus, A.; Fontela, E. *World Problems, an Invitation to Further Thought within the Framework of DEMA^{TEL}*; Battelle Geneva Research Center: Geneva, Switzerland, 1972; Volume 1, pp. 12–14.

9. Mandic, K.; Bobar, V.; Delibašić, B. Modeling Interactions Among Criteria in MCDM Methods: A Review. In Proceedings of the Decision Support Systems V–Big Data Analytics for Decision Making, Belgrade, Serbia, 27–29 May 2015; Delibašić, B., Hernández, J.E., Papathanasiou, J., Dargam, F., Zaraté, P., Ribeiro, R., Liu, S., Linden, I., Eds.; Springer International Publishing: Cham, Switzerland, 2015; pp. 98–109.
10. Yang, J.L.; Chiu, H.N.; Tzeng, G.-H.; Yeh, R.H. Vendor selection by integrated fuzzy MCDM techniques with independent and interdependent relationships. *Inf. Sci.* **2008**, *178*, 4166–4183. [CrossRef]
11. Gölcük, İ.; Baykasoğlu, A. An analysis of DEMATEL approaches for criteria interaction handling within ANP. *Expert Syst. Appl.* **2016**, *46*, 346–366. [CrossRef]
12. Öztürk, Z.K. A review of multi criteria decision making with dependency between criteria. *Multi-Criteria Decis. Mak.* **2006**, *5*, 19–29.
13. Lee, J.W.; Kim, S.H. Using analytic network process and goal programming for interdependent information system project selection. *Comput. Oper. Res.* **2000**, *27*, 367–382. [CrossRef]
14. Liliana, D.Y.; Basaruddin, C. A review on conditional random fields as a sequential classifier in machine learning. In Proceedings of the 2017 International Conference on Electrical Engineering and Computer Science (ICECOS), Palembang, Indonesia, 22–23 August 2017; pp. 143–148.
15. Hülle, J.; Kaspar, R.; Möller, K. Analytic network process–An overview of applications in research and practice. *Int. J. Oper. Res.* **2013**, *16*, 172–213. [CrossRef]
16. Kasirian, M.N.; Yusuff, R.M. An integration of a hybrid modified TOPSIS with a PGP model for the supplier selection with interdependent criteria. *Int. J. Prod. Res.* **2013**, *51*, 1037–1054. [CrossRef]
17. López, C.; Ishizaka, A. A hybrid FCM-AHP approach to predict impacts of offshore outsourcing location decisions on supply chain resilience. *J. Bus. Res.* **2019**, *103*, 495–507. [CrossRef]
18. Ketipi, M.K.; Karakasis, E.G.; Koulouriotis, D.E.; Emiris, D.M. Multi-criteria decision making using fuzzy cognitive maps–preliminary results. *Procedia Manuf.* **2020**, *51*, 1305–1310. [CrossRef]
19. Kosko, B. Fuzzy cognitive maps. *Int. J. Man-Mach. Stud.* **1986**, *24*, 65–75. [CrossRef]
20. Axelrod, R. *Structure of Decision: The Cognitive Maps of Political Elites*; Princeton University Press: Princeton, NJ, USA, 2015; ISBN 978-1-4008-7195-7.
21. Stylios, C.D.; Groumpos, P.P. Mathematical formulation of fuzzy cognitive maps. In Proceedings of the 7th Mediterranean Conference on Control and Automation, Mediterranean Control Association. Nicosia, Cyprus, 28–30 June 1999; Volume 2014, pp. 2251–2261.
22. Saaty, T.L. How to make a decision: The Analytic Hierarchy Process. *Eur. J. Oper. Res.* **1990**, *48*, 9–26. [CrossRef]
23. Alonso, J.A.; Lamata, M.T. Consistency in the analytic hierarchy process: A new approach. *Int. J. Unc. Fuzz. Knowl. Based Syst.* **2006**, *14*, 445–459. [CrossRef]
24. Wu, Z.; Tu, J. Managing transitivity and consistency of preferences in AHP group decision making based on minimum modifications. *Inf. Fusion* **2021**, *67*, 125–135. [CrossRef]
25. Crawford, G.; Williams, C. A note on the analysis of subjective judgment matrices. *J. Math. Psychol.* **1985**, *29*, 387–405. [CrossRef]
26. Harker, P.T.; Vargas, L.G. The theory of ratio scale estimation: Saaty's Analytic Hierarchy Process. *Manag. Sci.* **1987**, *33*, 1383–1403. [CrossRef]
27. Barzilai, J. Notes on the Analytic Hierarchy Process. In Proceedings of the NSF Design and Manufacturing Research Conference, Tampa, FL, USA, 7–10 January 2001; pp. 1–6.
28. Saaty, T.L.; Vargas, L.G. *Models, Methods, Concepts & Applications of the Analytic Hierarchy Process*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2001.
29. Krejčí, J.; Stoklasa, J. Aggregation in the Analytic Hierarchy Process: Why weighted geometric mean should be used instead of weighted arithmetic mean. *Expert Syst. Appl.* **2018**, *114*, 97–106. [CrossRef]
30. Ho, W.; Xu, X.; Dey, P.K. Multi-criteria decision making approaches for supplier evaluation and selection: A literature review. *Eur. J. Oper. Res.* **2010**, *202*, 16–24. [CrossRef]
31. Wu, J.; Barnes, D. A literature review of decision-making models and approaches for partner selection in agile supply chains. *J. Purch. Supply Manag.* **2011**, *17*, 256–274. [CrossRef]
32. Huang, I.B.; Keisler, J.; Linkov, I. Multi-criteria decision analysis in environmental sciences: Ten years of applications and trends. *Sci. Total Environ.* **2011**, *409*, 3578–3594. [CrossRef]
33. Kabak, Ö.; Dağdeviren, M. A combined fuzzy AHP-Bayesian network approach for supplier selection. *Int. J. Prod. Res.* **2014**, *52*, 2087–2102.
34. Duleba, S. Introduction and comparative analysis of the multi-level parsimonious AHP methodology in a public transport development decision problem. *J. Oper. Res. Soc.* **2022**, *73*, 230–243. [CrossRef]
35. Cengiz, L.D.; Ercanoglu, M. A novel data-driven approach to pairwise comparisons in AHP using fuzzy relations and matrices for landslide susceptibility assessments. *Environ. Earth Sci.* **2022**, *81*, 222. [CrossRef]
36. Cavallo, B.; Ishizaka, A. Evaluating scales for pairwise comparisons. *Ann. Oper. Res.* **2023**, *325*, 951–965. [CrossRef]
37. Tu, J.; Wu, Z. Analytic hierarchy process rank reversals: Causes and solutions. *Ann. Oper. Res.* **2023**. [CrossRef]
38. Chang, D.Y. Applications of the extent analysis method on fuzzy AHP. *Eur. J. Oper. Res.* **1996**, *95*, 649–655. [CrossRef]

39. Tseng, M.L. Application of ANP and DEMA^{TEL} to evaluate the decision-making of municipal solid waste management in Metro Manila. *Environ. Monit. Assess.* **2009**, *156*, 181–197. [CrossRef]
40. Kindermann, R.; Snell, J.L. *Markov Random Fields and Their Applications*; American Mathematical Society: Providence, RI, USA, 1980.
41. Li, S.Z. *Markov Random Field Modeling in Image Analysis*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2009; ISBN 978-1-84800-279-1.
42. Szeliski, R.; Zabih, R.; Scharstein, D.; Veksler, O.; Kolmogorov, V.; Agarwala, A.; Tappen, M.; Rother, C. A Comparative Study of Energy Minimization Methods for Markov Random Fields with Smoothness-Based Priors. *IEEE Trans. Pattern Anal. Mach. Intell.* **2008**, *30*, 1068–1080. [CrossRef] [PubMed]
43. Nowicki, K.; Snijders, T.A.B. Estimation and Prediction for Stochastic Blockstructures. *J. Am. Stat. Assoc.* **2001**, *96*, 1077–1087. [CrossRef]
44. Lafferty, J.; McCallum, A.; Pereira, F. Conditional Random Fields: Probabilistic Models for Segmenting and Labeling Sequence Data. In Proceedings of the Eighteenth International Conference on Machine Learning, Departmental Papers (CIS), San Francisco, CA, USA, 28 June–1 July 2001.
45. Cressie, N.A. *Statistics for Spatial Data*; John Willy and Sons. Inc.: New York, NY, USA, 1993; p. 800.
46. Loeliger, H.A.; Kschischang, F.R.; Frey, B.J. Factor graphs and the sum-product algorithm. *IEEE Trans. Inf. Theory* **2001**, *47*, 498–519.
47. Wainwright, M.J.; Jordan, M.I. *Graphical Models, Exponential Families, and Variational Inference*; Foundations and Trends®in Machine Learning: New York, NY, USA, 2008.