

Article

An Algorithm for Coloring of Picture Fuzzy Graphs Based on Strong and Weak Adjacencies, and Its Application

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Abstract: The idea of strong and weak adjacencies between vertices has been generalized into fuzzy graphs and intuitionistic fuzzy graphs (IFGs), and it is an important part of making decisions. However, one or two membership degrees are not always sufficient for making decisions on real-world problems that need an answer of types “yes, neutral, and no”. Consequently, in previous work, we generalized the concept into picture fuzzy graphs (PFGs) where each element in the PFG has membership, neutral, and non-membership degrees. Moreover, we constructed the notion of the coloring of PFGs based on strong and weak adjacencies between vertices. In this paper, we investigate some properties of the chromatic number of PFGs based on the concept of strong and weak adjacencies between vertices. According to these properties, we construct an algorithm to find the chromatic number of PFGs. The algorithm is useful when we work with large PFGs. Further, we improve the method to implement the PFG’s coloring for determining traffic signal phasing at an intersection. A case study has also been carried to evaluate the method.

Keywords: coloring; chromatic number; picture fuzzy graph; strong; weak; traffic signal phasing



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1. Introduction

The concept of fuzzy graph had been proposed by Rosenfeld [1] to handle indeterminate phenomena on vertices and relation between vertices. Therefore, the vertices and edges have membership degrees to represent the indeterminacy situation. In real-world problems, the degrees of non-membership of elements in a network are needed, for example, in situations that need an answer of types “yes” and “no”. To handle this problem, Atanassov [2] proposed an intuitionistic fuzzy set (IFS) and an intuitionistic fuzzy graph (IFG). Each element in IFG has membership and non-membership degrees. Numerous studies had been conducted on intuitionistic fuzzy graphs (IFGs), including the coloring of IFGs ([3,4]), the application of wiener index for IFGs in water pipeline network [5], interval-valued intuitionistic (S, T)-fuzzy graphs [6], and interval-valued intuitionistic fuzzy competition graphs [7].

Two categories of memberships are not always sufficient for making decisions. Therefore, Cuong [8] proposed a picture fuzzy set where each element not only had membership and non-membership degrees but also had a neutral membership degree. For instance, in an election problem, the committee must count the number of people who chose or did not choose a candidate and how many abstained (the neutral condition). Further, the concept of a picture fuzzy graph (PFG) was developed in [9], wherein the vertices and edges had membership, neutral, and non-membership degrees.

Researchers recently expanded PFGs in numerous types, such as q-rung PFGs [10], balanced PFGs [11], the application of PFGs for selecting best routes in an airlines network [12], picture fuzzy soft graphs [13], complex PFGs [14], regular PFGs [15], and so on. Numerous

studies had also been conducted on the use of PFGs in practical issues such as the application of balanced PFGs [11], decision making under picture fuzzy soft graphs [13], the implementation of regular PFGs in communication networks [15], road map design using picture fuzzy multigraphs [16], the application of PFGs in social networks [17], the shortest path algorithm in picture fuzzy digraphs [18], the site selection problem using laplacian energy of PFGs [19], the application of picture fuzzy tolerance graphs [20], the genus of PFGs [21], and multiple attribute decision-making via PFGs [22].

The theory of vertex coloring and edge coloring had been generalized in various types of fuzzy graphs. Some researchers proposed various generalizations of graph coloring such as the coloring of fuzzy graphs based on strong and weak adjacencies [23], fuzzy graph coloring based on δ -fuzzy independent vertex sets [24], the fuzzy fractional coloring of fuzzy graphs [25], the fuzzy coloring of fuzzy graphs [26], the fuzzy colouring of m-polar fuzzy graphs [27], the chromatic number and perfectness of fuzzy graphs [28], and the edge coloring of fuzzy graphs [29]. Several researchers have also proposed the coloring methods through the α -cut approach and two forms of adjacencies (strong and weak) in fuzzy graphs and IFGs, as seen in [3,4,23,30]. The fact that a PFG is an extension of an IFG inspired us to generalize the vertex coloring from IFGs into PFGs in 2021 [31]. We utilized the (α, β, δ) -cut approach to color PFGs. However, the computation of PFG's coloring through the cut was complicated since we should have used various values of (α, β, δ) when determining the cut chromatic numbers. Therefore, we need another approach for coloring the PFGs.

Strong and weak adjacencies—two different forms of adjacencies—between vertices in fuzzy graphs and IFGs are crucial in decision-making issues. Hence, we generalized strong and weak adjacencies into PFGs and proposed a concept to color PFGs based on strong and weak adjacencies [32]. When we work with PFGs with many vertices and edges, we need a computational tool to identify the strong and weak adjacencies and find the chromatic number of PFGs. In this paper, we construct an algorithm to handle the problem. In PFGs, we can classify connections between two movements (two vertices) into one of these three situations, i.e., crossing conflict, merging conflict, and non-conflict. The crowdedness of traffic flows in conflicting movements (crossing or merging conflicts) is a phenomenon that needs an answer of types “yes”, “no”, and “neutral”. The situation at an intersection is usually crowded during peak times (06.30 a.m.–08.30 a.m. and 04.00 p.m.–06.00 p.m.). However, occasionally it is not congested during non-peak hours (06.00 p.m.–06.00 a.m.) or neutral conditions about whether it is crowded or not during 08.30 a.m.–03.30 p.m. Therefore, we need a PFG to deal with this situation and propose a traffic signal phasing wherein there are no traffic flows from merging conflicts that move simultaneously at the same time. In this article, we improve the method to model traffic flows at an intersection using PFGs and to determine the traffic signal phasing. Moreover, we also evaluate the proposed method through a case study. This is a new finding in view of the application of coloring of PFGs.

The following is the structure of this paper: The first section explains an introduction, and Section 2 discusses research challenges and gaps. Section 3 presents preliminary materials. Section 4 contains the most important findings in this research, and Section 5 provides an experimental result. Finally, the conclusions are given in Section 6.

2. Research Challenges and Gaps

1. An essential idea that can be used in making decisions on various networks of real-world problems is the coloring method of fuzzy graphs and IFGs ([3,4,25,33]).
2. However, one membership degree or two membership degrees of elements in a network are not sufficient for making decision. Therefore, we initiated a coloring method of PFGs based on (α, β, δ) -cut of PFGs [31]. Since the computation of PFG's coloring through the cuts was complicated in determining the chromatic number, we propose a coloring method of PFGs based on strong and weak adjacencies between vertices in [32].

3. In this paper, we investigate the connection between the cut chromatic numbers and the chromatic number of PFGs based on strong and weak adjacencies. Moreover, we construct an algorithm to find the chromatic number of PFGs and evaluate the performance of the algorithm using Python and Matlab R2022b. The algorithm is useful when we deal with large PFGs.
4. An implementation of the coloring method of fuzzy graphs in road networks has been proposed in [33]. However, it did not consider three types of connections between traffic movements and three conditions of crowdedness of traffic flows at an intersection. Sometimes, the traffic flow is crowded during peak times, or sometimes it is not crowded, or sometimes there is a neutral condition about whether it is crowded or not at non peak-times. In this research, we improve the method to implement the PFG’s coloring for determining traffic signal phasing at an intersection and evaluate the method through a case study.

3. Preliminaries

We review some of the key ideas from this study in this part. In the beginning, we are going to discuss intuitionistic fuzzy sets (IFSs) and the construction of an IFS from a fuzzy set.

Given an ordinary finite non-empty set X and $A \subseteq X$, an Atanassov IFS on X is a set of the form $A_I = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$ wherein $\mu_A, \nu_A : X \rightarrow [0, 1]$ and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Meanwhile, a degree of hesitation (intuitionistic fuzzy index) of an element x in IFS A_I is defined as $\pi_{A_I}(x) = 1 - \mu_A(x) - \nu_A(x)$.

A method to construct an IFS from a fuzzy set is given in Proposition 1, Theorem 1, and Corollary 1, which are cited from [34,35].

Proposition 1 ([34,35]). *Let F be a mapping that is defined as $F : [0, 1]^2 \times [0, 1] \rightarrow L^*$ with $L^* = \{(x, y) | (x, y) \in [0, 1] \times [0, 1] \ \& \ x + y \leq 1\}$, and $F(x, y, \delta) = (F_\mu(x, y, \delta), F_\nu(x, y, \delta))$, where $F_\mu(x, y, \delta) = x(1 - \delta y)$ and $F_\nu(x, y, \delta) = 1 - x(1 - \delta y) - \delta y$. Let $\pi^* : L^* \rightarrow [0, 1]$ be a function defined as $\pi^*(u, v) = 1 - u - v$ for $u, v \in [0, 1]$. The mapping F satisfies the following conditions:*

1. *If $y_1 \leq y_2$, then $\pi^*(F(x, y_1, \delta)) \leq \pi^*(F(x, y_2, \delta))$ for $x \in [0, 1]$,*
2. *$F_\mu(x, y, \delta) \leq x \leq 1 - F_\nu(x, y, \delta)$ for $x \in [0, 1]$,*
3. *$F(x, 0, \delta) = (x, 1 - x)$,*
4. *$F(0, y, \delta) = (0, 1 - \delta y)$,*
5. *$F(x, y, 0) = (x, 1 - x)$,*
6. *$\pi^*(F(x, y, \delta)) = \delta y$.*

Theorem 1 ([34,35]). *Let $FS(X)$ be a set of all fuzzy set in X and $A \subseteq X$. Let $A_F = \{(x, \mu_A(x)) | x \in X\}$ be a fuzzy set in $FS(X)$, where $\mu_A : X \rightarrow [0, 1]$. Let $\pi, \zeta : X \rightarrow [0, 1]$ be two functions defined on X . The set*

$$A_I = \{(x, F(\mu_A(x), \pi(x), \zeta(x)) | x \in X\}$$

is an Atanassov IFS, where the function F is defined as in Proposition 1.

Corollary 1 ([34]). *Let π^* and F be two functions as defined in Proposition 1. If we choose $\zeta(x) = 1$ for each $x \in X$ in Theorem 1, then*

$$\pi(x) = \pi^*(F(\mu_A(x), \pi(x), 1)).$$

Under the condition in Corollary 1, we obtain an IFS:

$$A_I = \{(x, \mu_A(x)(1 - \pi(x)), 1 - (\mu_A(x)(1 - \pi(x))) - \pi(x) | x \in X\}. \tag{1}$$

The non-membership degree of IFS A_I in (1) will be used in the implementation of coloring of PFGs in Section 5.

Furthermore, the notion of a picture fuzzy set (PFS) and the construction of a PFS from an IFS are discussed.

Definition 1 ([8]). Given a universal set X and $A \subseteq X$. A set of the form $\tilde{A} = \{(v, \mu_A(v), \eta_A(v), \nu_A(v)) | v \in X\}$ is mentioned as a PFS on X , wherein $\mu_A(v) \in [0, 1]$ is a membership degree that describes the truth value of existence of element v in \tilde{A} , $\eta_A(v) \in [0, 1]$ is a NeuM degree that represents the indeterminacy degree of existence of v in \tilde{A} , and $\nu_A(v) \in [0, 1]$ is a non-membership degree that shows the falsity degree of existence of v in PFS \tilde{A} , such that $0 \leq \mu_A(v) + \eta_A(v) + \nu_A(v) \leq 1$. The value $\pi_{\tilde{A}}(v) = 1 - (\mu_A(v) + \eta_A(v) + \nu_A(v))$ is called a refusal degree of membership of v in \tilde{A} .

A method to construct a PFS from an IFS is given in Theorem 2.

Theorem 2 ([36]). If $A_I = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$ is an IFS on X and $g : [0, 1] \rightarrow [0, 1]$ is any function such that $g(0) = 0$ and $g(x) \leq x$, then

$$A_p = \{(x, P_\mu(\mu_A(x), \nu_A(x)), P_\eta(\mu_A(x), \nu_A(x)), P_\nu(\mu_A(x), \nu_A(x)))\}$$

is a PFS on X wherein the mapping $P : [0, 1] \times [0, 1] \rightarrow [0, 1]^2 \times [0, 1]$ is defined by $P(u, v) = (P_\mu(u, v), P_\eta(u, v), P_\nu(u, v))$ with $P_\mu(u, v) = u$, $P_\eta(u, v) = g(1 - u - v)$, and $P_\nu(u, v) = v$.

In other words, $P_\mu(\mu_A(x), \nu_A(x)) = \mu_A(x)$, $P_\eta(\mu_A(x), \nu_A(x)) = g(1 - \mu_A(x) - \nu_A(x))$, and $P_\nu(\mu_A(x), \nu_A(x)) = \nu_A(x)$ for $x \in X$. Further, the function g in Theorem 2 is called a neutral or refusal membership function of PFS A_p , and it will be used in Section 5.

In Definitions 2 and 3, we present ideas of an empty PFS and a universal PFS, which are cited from [37].

Definition 2 ([37]). Let $\tilde{A} = \{(v, \mu_A(v), \eta_A(v), \nu_A(v))\}$ be a PFS on X and $A \subseteq X$. The set \tilde{A} is called an empty PFS if $\mu_A(v) = 0$, $\eta_A(v) = 0$, and $\nu_A(v) = 1$ for each $v \in X$. The empty PFS is denoted by \emptyset_{pfs} .

Definition 3 ([37]). Given PFS \tilde{A} in Definition 2, the set \tilde{A} is named a universal PFS if $\mu_A(v) = 1$, $\eta_A(v) = 0$, and $\nu_A(v) = 0$ for each $v \in X$.

Additionally, we provide information on the picture fuzzy subset in Definition 4, which is quoted from [8].

Definition 4 ([8]). Let X be a universal set and $A, B \subseteq X$. Given two PFSs on X : $\tilde{A} = \{(a, \mu_A(a), \eta_A(a), \nu_A(a))\}$ and $\tilde{B} = \{(b, \mu_B(b), \eta_B(b), \nu_B(b))\}$, $a, b \in X$. The PFS \tilde{A} is mentioned as the picture fuzzy subset of \tilde{B} , denoted by $\tilde{A} \subseteq \tilde{B}$, if

$$\mu_A(v) \leq \mu_B(v), \eta_A(v) \leq \eta_B(v), \nu_A(v) \geq \nu_B(v)$$

for all $v \in X$.

The notion of PFS is used as a basis to define a PFG, as described in Definition 5.

Definition 5 ([9]). We assume that X is a universal set that contains vertices. We mention a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ as a PFG if $\tilde{V} = \{(x, \mu_1(x), \eta_1(x), \nu_1(x))\}$ is a picture fuzzy vertex set (PFVS) on X with the membership, neutral, and non-membership functions as follows: $\mu_1, \eta_1, \nu_1 : X \rightarrow [0, 1]$, in which $0 \leq \mu_1(x) + \eta_1(x) + \nu_1(x) \leq 1$ for each $x \in X$. Meanwhile, $\tilde{E} = \{(xy, \mu_2(xy), \eta_2(xy), \nu_2(xy))\}$ is a picture fuzzy edge set (PFES) on $E \subseteq X \times X$ with the mem-

bership, neutral membership, and non-membership functions as follows:
 $\mu_2, \eta_2, \nu_2 : X \times X \rightarrow [0, 1]$, such that

$$\mu_2(xy) \leq \min\{\mu_1(x), \mu_1(y)\}, \eta_2(xy) \leq \min\{\eta_1(x), \eta_1(y)\}, \nu_2(xy) \leq \max\{\nu_1(x), \nu_1(y)\},$$

and $0 \leq \mu_2(xy) + \eta_2(xy) + \nu_2(xy) \leq 1$ for each $xy \in E$.

The value of $\mu_1(x)$ is a membership degree that represents the truth value of existence of vertex x in X , $\eta_1(x)$ is NeuM degree that describes the indeterminacy degree of existence of x in X , and $\nu_1(x)$ is non-membership degree that shows the falsity degree of the existence of vertex x in X . Meanwhile, the values of $\mu_2(xy), \eta_2(xy), \nu_2(xy)$ represent a membership degree that tells the truth of adjacency of x and y , an NeuM degree that describes the indeterminacy of adjacency of x and y , and a non-membership degree that shows the falsity of adjacency between x and y , respectively.

Furthermore, we discuss the concepts of underlying graph, picture fuzzy subgraph, and a complete picture fuzzy graph (CPFG) that will be used in the next section.

Definition 6 ([38]). Given a PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ on a universal set X , an underlying graph of \tilde{G} , symbolized as $G^* = (V^*, E^*)$, is a graph wherein $\mu_1(x) > 0, \eta_1(x) > 0$, and $\nu_1(x) > 0$ for each $x \in V^*$.

Definition 7 ([17]). Let $\tilde{G} = (\tilde{V}(G), \tilde{E}(G))$ and $\tilde{H} = (\tilde{V}(H), \tilde{E}(H))$ be PFGs on a universal set X . The PFG \tilde{H} is said to be a picture fuzzy subgraph of \tilde{G} , denoted by $\tilde{H} \subseteq \tilde{G}$, if $\tilde{V}(H) \subseteq \tilde{V}(G)$ and $\tilde{E}(H) \subseteq \tilde{E}(G)$.

Definition 8 ([15]). Given PFG $\tilde{G} = (\tilde{V}, \tilde{E})$, where \tilde{V} is a PFS on universal set X . We call x and y as neighbor vertices in \tilde{G} if $\mu_2(xy) > 0, \eta_2(xy) > 0$, and $\nu_2(xy) > 0$. Meanwhile, the set $N_{\tilde{E}}(x) = \{y \in X \mid x \text{ and } y \text{ are neighbor vertices}\}$. In addition, $|N_{\tilde{E}}(v)|$ denotes the cardinality of neighbors of vertex v in \tilde{G} .

Definition 9 ([12]). Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a PFG, where \tilde{V} is a PFS on X . We mention PFG \tilde{G} as a CPFG if

$$\min\{\mu_1(u), \mu_1(v)\} = \mu_2(uv); \min\{\eta_1(u), \eta_1(v)\} = \eta_2(uv); \max\{\nu_1(u), \nu_1(v)\} = \nu_2(uv)$$

for each pair $u, v \in X$.

3.1. Strong and Weak Adjacencies between Vertices in PFGs

The terms “strong adjacency” and “weak adjacency” have been defined in an IFG by [4]. We expand upon these ideas in terms of PFGs in the previous work [32].

Definition 10 ([32]). Given a PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ where \tilde{V} is a PFS on V . The vertices $u, v \in V$ are mentioned as strongly adjacent vertices if

$$\frac{1}{2} \min\{\mu_1(u), \mu_1(v)\} \leq \mu_2(uv); \frac{1}{2} \min\{\eta_1(u), \eta_1(v)\} \leq \eta_2(uv); \frac{1}{2} \max\{\nu_1(u), \nu_1(v)\} \leq \nu_2(uv).$$

Otherwise, we mention u and v as weakly adjacent vertices.

3.2. Coloring of PFGs Based on Strong and Weak Adjacencies between Vertices

In this part, we discuss a coloring of PFGs based on strong and weak adjacencies between vertices.

Definition 11 ([32]). Given PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ where \tilde{V} is a PFS on $V = \{v_1, v_2, \dots, v_n\}$, i.e., $\tilde{V} = \{(v_i, \mu_1(v_i), \eta_1(v_i), \nu_1(v_i)) \mid v_i \in V\}$. Whereas, $\tilde{E} = \{(v_i v_j, \mu_2(v_i v_j), \eta_2(v_i v_j), \nu_2(v_i v_j)) \mid i \neq j\}$ is a PFS on $E \subseteq V \times V$. Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be a family of picture fuzzy (PF) subsets of \tilde{V} where

$$\gamma_i = \{(v_j, \mu_{\gamma_i}(v_j), \eta_{\gamma_i}(v_j), \nu_{\gamma_i}(v_j))\}$$

for $1 \leq i \leq k$, and $1 \leq j \leq n$. The family Γ is called as a k -vertex coloring of \tilde{G} if

$$1. \quad \bigcup_{i=1}^k \gamma_i = \tilde{V}, \text{ i.e.,}$$

$$\max\{\mu_{\gamma_i}(v_j)\} = \mu_1(v_j), \max\{\eta_{\gamma_i}(v_j)\} = \eta_1(v_j), \min\{\nu_{\gamma_i}(v_j)\} = \nu_1(v_j),$$

for $1 \leq i \leq k; 1 \leq j \leq n$.

$$2. \quad \gamma_i \cap \gamma_j = \emptyset_{pfs}, \forall i \neq j, \text{ i.e.,}$$

$$\min\{\mu_{\gamma_i}(v_l), \mu_{\gamma_j}(v_l)\} = 0, \min\{\eta_{\gamma_i}(v_l), \eta_{\gamma_j}(v_l)\} = 0, \max\{\nu_{\gamma_i}(v_l), \nu_{\gamma_j}(v_l)\} = 1,$$

for $1 \leq i, j \leq k; 1 \leq l \leq n$.

3. For every pair of strongly adjacent vertices $u, v \in V$:

$$\min\{\mu_{\gamma_i}(u), \mu_{\gamma_i}(v)\} = 0, \min\{\eta_{\gamma_i}(u), \eta_{\gamma_i}(v)\} = 0, \max\{\nu_{\gamma_i}(u), \nu_{\gamma_i}(v)\} = 1,$$

for $1 \leq i \leq k$. In other words, every pair of strongly adjacent vertices belongs to different PF subsets.

The minimum value k for which \tilde{G} has k -vertex coloring is referred as the chromatic number of \tilde{G} , denoted by $\chi^f(\tilde{G})$.

We describe the coloring of a PFG in Example 1 to provide a better understanding of the concept.

Example 1. Let us consider PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ in Figure 1. The set \tilde{V} is a PFVS on universal set $V = \{A, B, C, D\}$.

1. The pairs of strongly adjacent vertices are $\{A, B\}, \{A, D\}, \{B, C\}, \{B, D\}$, and $\{C, D\}$. Meanwhile, $\{A, C\}$ is the pair of weakly adjacent vertices.
2. Therefore, we obtain the PF-subset $\gamma_1 = \{(A, 0.1, 0.3, 0.1), (C, 0, 0.2, 0.1)\}$. Since B and D are strongly adjacent, $\gamma_2 = \{(B, 0.2, 0.1, 0.5)\}$, and $\gamma_3 = \{(D, 0.2, 0.3, 0.5)\}$. We obtain the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$.
3. Thus, the chromatic number of \tilde{G} is $\chi^f(\tilde{G}) = 3$.

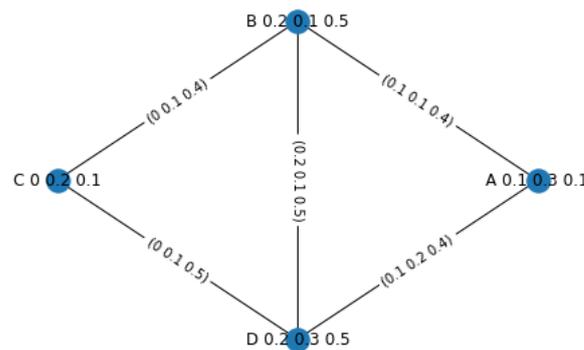


Figure 1. The picture fuzzy graph \tilde{G} for Example 1.

3.3. The Chromatic Number of PFGs Based on (α, β, δ) -Cut Coloring

In this part, we discuss the concept of (α, β, δ) -cut of PFGs and the cut chromatic number in Definition 12.

Definition 12 ([31]). Given a PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ and its underlying graph $G^*(V^*, E^*)$, a level set of \tilde{V} is defined as a set $L_{\tilde{V}} = \{\alpha | \mu_1(v) = \alpha, v \in V^*\} \cup \{\beta | \eta_1(v) = \beta, v \in V^*\} \cup \{\delta | \nu_1(v) = \delta, v \in V^*\}$, whereas a level set of \tilde{E} is set $L_{\tilde{E}} = \{\alpha | \mu_2(uv) = \alpha, uv \in E^*\} \cup \{\beta | \eta_2(uv) =$

$\beta, uv \in E^* \} \cup \{ \delta | v_2(uv) = \delta, uv \in E^* \}$. Moreover, a level set of \tilde{G} is a set $L = L_{\tilde{V}} \cup L_{\tilde{E}}$. Given $\alpha, \beta, \delta \in L$, an (α, β, δ) -cut of \tilde{G} is a crisp graph $G_{\alpha, \beta, \delta} = (V_{\alpha, \beta, \delta}, E_{\alpha, \beta, \delta})$ where

$$V_{\alpha, \beta, \delta} = \{ v \in V^* | \mu_1(v) \geq \alpha \ \& \ \eta_1(v) \geq \beta \ \& \ v_1(v) \leq \delta \}$$

and

$$E_{\alpha, \beta, \delta} = \{ uv \in E^* | \mu_2(uv) \geq \alpha \ \& \ \eta_2(uv) \geq \beta \ \& \ v_2(uv) \leq \delta \}.$$

The (α, β, δ) -cut chromatic number, denoted by $\chi_{\alpha, \beta, \delta}$, is the chromatic number obtained from crisp coloring of the cut $G_{\alpha, \beta, \delta}$.

Example 2. We present an illustration of Definition 12 for PFG in Figure 1. The level set of \tilde{V} and \tilde{E} are $L_{\tilde{V}} = \{0, 0.1, 0.2, 0.3, 0.5\}$, and $L_{\tilde{E}} = \{0, 0.1, 0.2, 0.4, 0.5\}$, respectively. Meanwhile, the level set of \tilde{G} is $L = \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$. Examples of $(0, 0.1, 0.5)$ -cut, $(0.1, 0.1, 0.5)$ -cut, and $(0.1, 0.2, 0.5)$ -cut are depicted in Figure 2. The $(0, 0.1, 0.5)$ -cut chromatic number is 3, the $(0.1, 0.1, 0.5)$ -cut chromatic number is also 3, and the $(0.1, 0.2, 0.5)$ -cut chromatic number is 2.

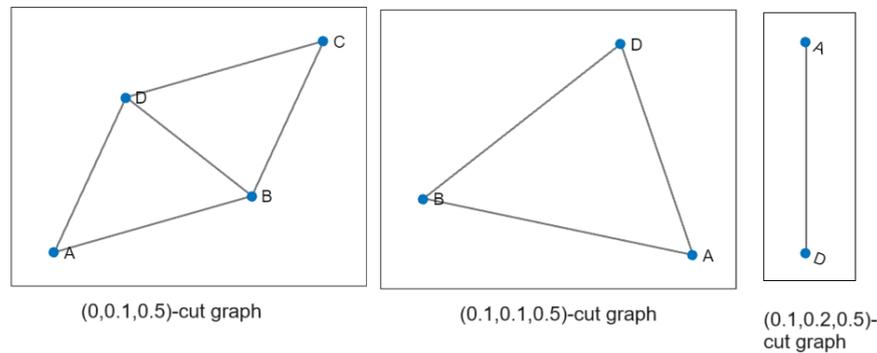


Figure 2. Several cut graphs of PFG in Figure 1.

4. Main Results

In this section, we present some properties of the chromatic number of PFGs and an algorithm to compute the chromatic number.

4.1. Some Characteristics of the Chromatic Number of PFGs

In this part, we investigate an upper bound for the chromatic number of PFGs in Theorem 3.

Theorem 3. If $\tilde{G} = (\tilde{V}, \tilde{E})$ is a PFG with the underlying graph $G^* = (V^*, E^*)$, then

$$\chi^f(\tilde{G}) \leq \max\{|N_{\tilde{E}}(x)| : x \in V^*\} + 1.$$

Proof. Let $V^* = \{x_1, x_2, \dots, x_n\}$. Assume $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ is a family of PF-subsets on \tilde{V} where $\gamma_i = \{(x_j, \mu_{\gamma_i}(x_j), \eta_{\gamma_i}(x_j), v_{\gamma_i}(x_j))\}, 1 \leq i \leq k, 1 \leq j \leq n$. Suppose that $k > \max\{|N_{\tilde{E}}(x)| : x \in V^*\} + 1$.

Based on Conditions 1–2 in Definition 11, we have:

1. $\bigcup_{i=1}^k \gamma_i = \tilde{V}$,
2. $\gamma_i \cap \gamma_j = \emptyset_{pfs}, \forall i \neq j, \text{ for } 1 \leq i, j \leq k$.

When every pair of vertices $x_i, x_j (i \neq j)$ is strongly adjacent, the vertices should be placed in different picture fuzzy (PF) subsets. According to the Condition 3 in Definition 11, we have n PF subsets $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$. Meanwhile, $\max\{|N_{\tilde{E}}(x)| : x \in V^*\} + 1 = n - 1 + 1 = n$. Thus, $\chi^f(\tilde{G}) = n = \max\{|N_{\tilde{E}}(x)| : x \in V^*\} + 1$. Otherwise, when there is a pair or weakly adjacent vertices, $\chi^f(\tilde{G}) = k < n = \max\{|N_{\tilde{E}}(x)| : x \in V^*\} + 1$. It is a contradiction. Thus, $\chi^f(\tilde{G}) = k \leq \max\{|N_{\tilde{E}}(x)| : x \in V^*\} + 1$. \square

Moreover, we investigate the connection between the chromatic number of PFGs based on strong and weak adjacencies and the (α, β, δ) -cut chromatic number in Definition 12. Firstly, we define the chromatic number of a PFG by means of its-cut chromatic number.

Definition 13. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a PFG. The chromatic number of \tilde{G} through the (α, β, δ) -cut chromatic number is defined as follows:

$$\chi(\tilde{G}) = \max\{\chi_{\alpha,\beta,\delta} | \alpha, \beta, \delta \in L\},$$

where L is the level set of \tilde{G} and $\chi_{\alpha,\beta,\delta} = \chi(G_{\alpha,\beta,\delta})$.

When $\delta = 0$, Definition 13 becomes the chromatic number of IFGs. In certain conditions, we prove that the chromatic number of the underlying graph of PFG is equal to the chromatic number concept in Definition 13.

Example 3. Let us consider Example 2. According to Definition 13, the chromatic number of \tilde{G} through the (α, β, δ) -cut is

$$\chi(\tilde{G}) = \max\{\chi_{\alpha,\beta,\delta} | \alpha, \beta, \delta \in L\} = 3.$$

Theorem 4. Given a PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ with the underlying graph $G^* = (V^*, E^*)$. If $\alpha_1 = \min\{\alpha | \alpha \in L\}$, $\beta_1 = \min\{\beta | \beta \in L\}$, and $\delta_1 = \max\{\delta | \delta \in L\}$, then the chromatic number of \tilde{G} :

$$\chi(\tilde{G}) = \chi(G^*).$$

Proof. Since $\alpha_1 = \min\{\alpha | \alpha \in L\}$, $\beta_1 = \min\{\beta | \beta \in L\}$, and $\delta_1 = \max\{\delta | \delta \in L\}$, we have $\chi(G_{\alpha_1,\beta_1,\delta_1}) = \max\{\chi_{\alpha,\beta,\delta} | \alpha, \beta, \delta \in L\}$.

Further, all vertices and edges of \tilde{G} become elements of the crisp graph $G_{\alpha_1,\beta_1,\delta_1}$. This implies $G_{\alpha_1,\beta_1,\delta_1} = G^*$ and $\chi(G^*) = \chi(G_{\alpha_1,\beta_1,\delta_1}) = \max\{\chi_{\alpha,\beta,\delta} | \alpha, \beta, \delta \in L\} = \chi(\tilde{G})$. \square

Theorem 5. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a PFG with the underlying graph $G^* = (V^*, E^*)$. If all edges in \tilde{E} connect strongly adjacent vertices, then

$$\chi^f(\tilde{G}) = \chi(G^*).$$

Proof. Assume that $\chi^f(\tilde{G}) = k$. According to Definition 11, we have a family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of PF-subsets of \tilde{V} such that it satisfies 3 conditions in Definition 11:

1. $\bigcup_{i=1}^k \gamma_i = \tilde{V}$,
2. $\gamma_i \cap \gamma_j = \emptyset_{pfs}$, for $1 \leq i, j \leq k$.
3. Based on the third condition in Definition 11, we have $\gamma_i = \{(v_l, \mu_1(v_l), \eta_1(v_l), \nu_1(v_l)) \cup \{(v_m, \mu_1(v_m), \eta_1(v_m), \nu_1(v_m))\}$ where $\mu_2(v_l v_m) = 0$, $\eta_2(v_l v_m) = 0$, and $\nu_2(v_l v_m) = 1$ (since all edges in \tilde{E} connect strongly adjacent vertices). This shows that each γ_i becomes a crisp independent vertex set in G^* for $1 \leq i \leq k$.

Hence, the family Γ becomes a partition of V^* into k -independent vertex set (crisp set). Therefore, $\chi^f(\tilde{G}) = k = \chi(G^*)$. \square

According to Theorem 5, we obtain the following corollary.

Corollary 2. If $\tilde{G} = (\tilde{V}, \tilde{E})$ is a complete picture fuzzy graph (CPFG) with n vertices, then $\chi^f(\tilde{G}) = n$.

4.2. An Algorithm for Finding the Chromatic Number of PFGs

We create an algorithm to compute the chromatic number of PFGs with the exception for CPFG. We use the assumption that not all edges in \tilde{E} connect strongly adjacent vertices.

Algorithm 1 can also be used for the coloring of IFGs when the inputs are the intuitionistic fuzzy vertex set and the intuitionistic fuzzy edge set.

Algorithm 1 To find the chromatic number of PFGs

Input: The PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ with the elements:

- Vertex set $V = \{v_i\}, i = 1, 2, \dots, n$, and edge set $E = \{e_l\}, l = 1, 2, \dots, m$.
- Degree of vertices $W_V = \{w_{v_i}\}$, with $w_{v_i} = (\mu_1(v_i), \eta_1(v_i), \nu_1(v_i))$.
- Degree of edges $W_E = \{w_{e_l}\}$, where $w_{e_l} = (\mu_2(e_l), \eta_2(e_l), \nu_2(e_l)), e_l = v_i v_j, i \neq j \in \{1, \dots, n\}$.

Output: The chromatic number: $\chi^f(\tilde{G})$

```

1: for h = 1 to n - 1 do
2:   for j = 1 to n - h do
3:     Check for all pairs  $Q_h = [v_h v_{j+h}]$ 
4:     if  $\frac{1}{2} \min(\mu_1(v_h), \mu_1(v_{j+h})) \leq \mu_2(v_h v_{j+h}), \frac{1}{2} \min(\eta_1(v_h), \eta_1(v_{j+h})) \leq \eta_2(v_h v_{j+h})$ , and
 $\frac{1}{2} \max(\nu_1(v_h), \nu_1(v_{j+h})) \leq \nu_2(v_h v_{j+h})$  then
5:       "vh and vj+h are strongly adjacent"
6:     else
7:       "vh and vj+h are weakly adjacent"
8:       Assign  $C_h = [\{v_h, v_{h+r}\}]$  for  $r \in \{1, 2, \dots, n - h\}$ 
9:     end if
10:   end for
11: end for
12: if All pairs in  $Q_h$  are strongly adjacent vertices then
13:    $C_h = \emptyset$ 
14:   if  $|V| = 2$  then
15:     Create two picture fuzzy (PF) subsets  $\gamma_{11} = \{v_1\}, \gamma_{12} = \{v_2\}$  and get  $\chi^f(\tilde{G}) = 2$ .
16:   end if
17: else
18:   Go to Step 20
19: end if
20: if  $C_1 \neq \emptyset$  then
21:   Initialization  $\gamma_{11} = C_1$ 
22:   sc1 = number of elements of  $C_1$ 
23:   for i = 1 to sc1-1 do
24:     if  $\{v_{i+1}, v_{i+2}\} \in C_2$  then
25:       Assign  $\gamma_{11} = \text{union}(C_1, \{v_{i+2}\})$ 
26:     end if
27:   end for
28:   if elements of  $\gamma_{11}$  are not elements of  $C_2$  then
29:     Assign  $\gamma_{12} = C_2$ 
30:   else
31:      $\gamma_{12} = C_3$ 
32:   end if
33:   Repeat the process in Steps 28–32 to create the picture fuzzy (PF) subsets  $\gamma_{11}, \gamma_{12}, \dots, \gamma_{1,k_1}$  such that
 $\gamma_{11} \cup \gamma_{12} \cup \dots \cup \gamma_{1,k_1} = V, \gamma_i \cap \gamma_j = \emptyset$ , for  $1 \leq i, j \leq k_1$ , and every pair of strongly adjacent vertices belongs
to different  $\gamma_{1i}$  for  $1 \leq i \leq k_1$ .
34:   if There is only one PF subset  $\gamma_{11}$  then
35:     Stop the process, and go to Step 47
36:   end if
37: else
38:   Initialization  $\gamma_{11} = C_2$ 
39:   Do the same process in Steps 22–33 to get the PF-subsets  $\gamma_{21}, \gamma_{22}, \dots, \gamma_{2,k_2}$ 
40: end if
41: Do the same process in Steps 21–33 to get the PF subsets  $\gamma_{s1}, \gamma_{s2}, \dots, \gamma_{s,k_s}$  with initialization  $\gamma_{s1} = C_s$  and
 $s < n - 1$ .
42: if  $k_s < k_{s-1}$  then
43:   Stop the process, and go to Step 47
44: else
45:   Back to Step 41
46: end if
47: Choose  $k = k_{\min} = \min\{k_1, k_2, \dots, k_s\}$  and obtain the family  $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$  where  $\gamma_1 = \gamma_{k_{\min}1}, \gamma_2 =$ 
 $\gamma_{k_{\min}2}, \dots, \gamma_k = \gamma_{k_{\min}k}$ .
48: Obtain the chromatic number:  $\chi^f(\tilde{G}) = k$ .

```

We show that Algorithm 1 gives the chromatic number $\chi^f(\tilde{G}) = k$ for $k \leq n$ and proves the correctness of the algorithm by using mathematical induction on the cardinality $|V|$ as follows.

Base step: for $|V| = 2$ and $V = \{v_1, v_2\}$.

If the two vertices in V are strongly adjacent vertices, then Steps 1–11 will produce $C_1 = \emptyset$. Further, Steps 12–16 give $\gamma_{11} = \{(v_1, \mu_1(v_1), \eta_1(v_1), \nu_1(v_1))\}$, $\gamma_{12} = \{(v_2, \mu_1(v_2), \eta_1(v_2), \nu_1(v_2))\}$, and the chromatic number $\chi^f(\tilde{G}) = 2$. If the two vertices in V are weakly adjacent vertices, then Steps 1–11 produce $C_1 = \{v_1, v_2\}$. Further, Steps 17–36 give PF-subset $\gamma_1 = \{(v_1, \mu_1(v_1), \eta_1(v_1), \nu_1(v_1)), (v_2, \mu_1(v_2), \eta_1(v_2), \nu_1(v_2))\}$. Finally, Steps 47–48 give the family $\Gamma = \{\gamma_1\}$ and $\chi^f(\tilde{G}) = 1$. Thus, the chromatic number $\chi^f(\tilde{G}) = k \leq 2$. The base step is satisfied.

Inductive step: Assume that Algorithm 1 is correct for cardinality $|V| = n$. Steps 1–11 produce the set of weakly adjacent vertices and the set of strongly adjacent vertices. Furthermore, Steps 20–41 give the PF-subsets:

$$\gamma_{11}, \gamma_{12}, \dots, \gamma_{1,k1}; \gamma_{21}, \gamma_{22}, \dots, \gamma_{2,k2}; \dots; \gamma_{s1}, \gamma_{s2}, \dots, \gamma_{s,ks}.$$

In Steps 42–48, we obtain $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$, where $k = k_{min} = \min\{k1, k2, \dots, ks\}$ and $\gamma_1 = \gamma_{k_{min}1}, \gamma_2 = \gamma_{k_{min}2}, \dots, \gamma_k = \gamma_{k_{min}k}$. The chromatic number $\chi^f(\tilde{G}) = k \leq n$.

We prove that Algorithm 1 is correct for PFG \tilde{G} with cardinality $|V| = n + 1$.

Assume that there is an edge $v_n v_{n+1}$. Based on the assumption in the inductive step, we have the family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$. Without a loss of generality, vertex v_n is an element of the PF subset γ_k .

If $\{v_n, v_{n+1}\}$ is a pair of weakly adjacent vertices, then vertex v_{n+1} could be an element of γ_k . Otherwise, when $\{v_n, v_{n+1}\}$ is a pair of strongly adjacent vertices, vertex v_{n+1} could be an element of γ_1 , or γ_2 , or \dots , γ_{k-1} , and hence we obtain the chromatic number $\chi^f(\tilde{G}) = k$. According to Theorem 3, the chromatic number $\chi^f(\tilde{G}) = k \leq \max\{|N_{\tilde{E}}(v_i)|; v_i \in V\} + 1 = n + 1$. Thus, the inductive step is true and Algorithm 1 is correct.

Examples 4 and 5 show the determination of the chromatic number of PFGs by employing Algorithm 1, and the performance of the algorithm is evaluated using Python and Matlab R2022b.

Example 4. Given PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ in Figure 3 with picture fuzzy vertex set $\tilde{V} = \{(A, 0.3, 0.2, 0.4), (B, 0.3, 0.2, 0.2), (C, 0.4, 0.2, 0.3), (D, 0.3, 0.2, 0.3), (E, 0.5, 0.2, 0.1)\}$ and picture fuzzy edge set $\tilde{E} = \{(AB, 0.1, 0.2, 0.4), (AC, 0.2, 0.2, 0.4), (BC, 0.3, 0.2, 0.3), (BD, 0.3, 0.2, 0.3), (CD, 0.3, 0.2, 0.3), (CE, 0.2, 0.2, 0.3), (DE, 0.3, 0.2, 0.3)\}$.

The output of Algorithm 1 in determining the chromatic number of \tilde{G} is presented in Figure 4.

1. In Steps 1–11, we obtain the sets of the pairs of weakly adjacent vertices, i.e., $C_1 = \{\{A, B\}, \{A, D\}, \{A, E\}\}$, $C_2 = \{\{B, E\}\}$, $C_3 = \emptyset$, $C_4 = \emptyset$.
2. In Steps 17–41, we obtain the PF-subsets $\gamma_{11} = \{(A, 0.3, 0.2, 0.4), (B, 0.3, 0.2, 0.2)\}$, $\gamma_{12} = \{(C, 0.4, 0.2, 0.3)\}$, $\gamma_{13} = \{(D, 0.3, 0.2, 0.3)\}$, $\gamma_{14} = \{(E, 0.5, 0.2, 0.1)\}$, with the initialization $\gamma_{11} = C_1(1) = \{A, B\}$.

Other PF-subsets are as follows:

$$\gamma_{21} = \{(B, 0.3, 0.2, 0.2), (E, 0.5, 0.2, 0.1)\}, \gamma_{22} = \{(A, 0.3, 0.2, 0.4), (D, 0.3, 0.2, 0.3)\}, \text{ and } \gamma_{23} = \{(C, 0.4, 0.2, 0.3)\}, \text{ with the initialization } \gamma_{21} = C_2 = \{B, E\}.$$

3. In Steps 42–48, since $k_2 = 3 < k_1 = 4$ then stop. We choose $\gamma_1 = \gamma_{21}, \gamma_2 = \gamma_{22}, \gamma_3 = \gamma_{23}$, and obtain the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Thus, the chromatic number $\chi^f(\tilde{G}) = 3$.

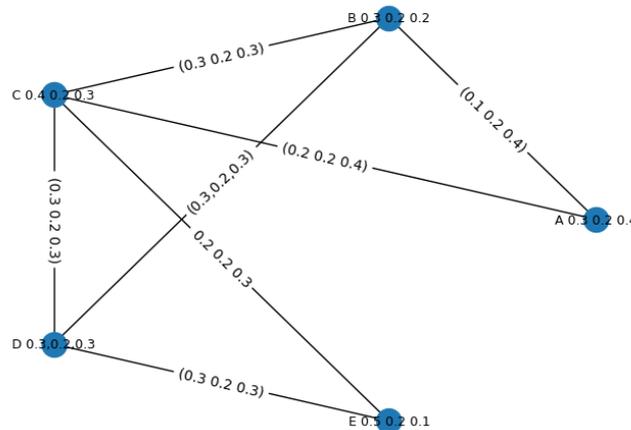


Figure 3. The picture fuzzy graph \tilde{G} for Example 4.

```

V= ['A', 'B', 'C', 'D', 'E']
Q1c= {'B', 'A'}
Vertices in Q1c are weakly adjacent
Q1a= {'A'}
Q1b= {'C'}
Q1a and Q1b are strongly adjacent
Q1c= {'A', 'D'}
Vertices in Q1c are weakly adjacent
Q1c= {'A', 'E'}
Vertices in Q1c are weakly adjacent
C1= [{'B', 'A'}, {'A', 'D'}, {'A', 'E'}]
Q2a= {'B'}
Q2b= {'C'}
Q2a and Q2b are strongly adjacent
Q2a= {'B'}
Q2b= {'D'}
Q2a and Q2b are strongly adjacent
Q2c= {'B', 'E'}
Vertices in Q2c are weakly adjacent
C2= [{'B', 'E'}]
Q3a= {'C'}
Q3b= {'D'}
Q3a and Q3b are strongly adjacent
Q3a= {'C'}
Q3b= {'E'}
Q3a and Q3b are strongly adjacent
C3= []
Q4a= {'D'}
Q4b= {'E'}
Q4a and Q4b are strongly adjacent
C4= []
gamma1= {'A', 'D'}
False
gamma2= {'B', 'E'}
V1= ['C']
gamma3= {'C'}
Gamma= ({'A', 'D'}, {'B', 'E'}, {'C'})
Chromatic_number= 3
    
```

Figure 4. The output of Algorithm 1 for finding $\chi^f(\tilde{G})$ in Figure 3.

Example 5. Let us consider PFG $\tilde{G} = (\tilde{V}, \tilde{E})$ in Figure 5, where \tilde{V} is a PFS on $V = \{WE, WS, SN, SE, NW, NS, EN, EW\}$.

The output of Algorithm 1 for PFG \tilde{G} in Figure 5 is shown in Figure 6. We obtain the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and the chromatic number $\chi^f(\tilde{G}) = 4$.

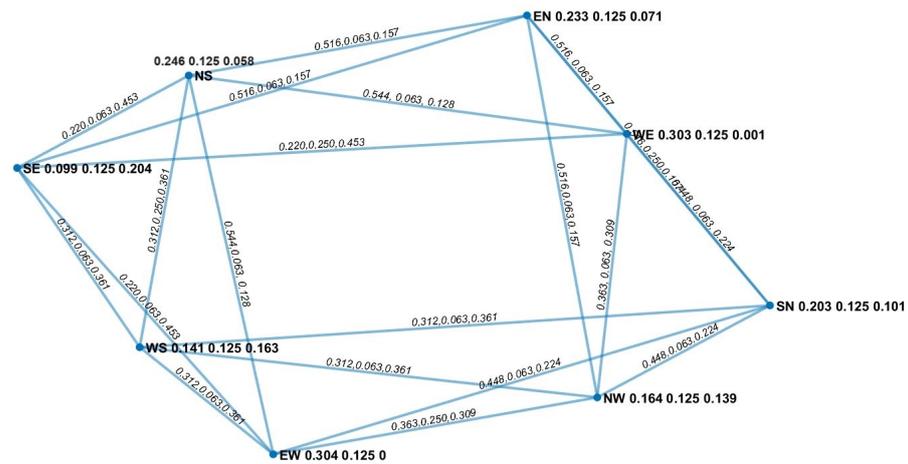


Figure 5. The PFG \tilde{G} for Example 5.

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Determining the sets of strongly adjacent
and weakly adjacent vertices
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
StrongAdj1 =          C31 =
'WE SN'              'SN SE'
'WE SE'              'SN NS'
'WE NW'
'WE NS'
'WE EN'              StrongAdj4 =
C11 =                'SE NS'
'WE WS'              'SE EN'
'WE EW'              'SE EW'

StrongAdj2 =          C41 = 'SE NW'
'WS SN'              StrongAdj5 =
'WS SE'              'NW EN'
'WS NW'              'NW EW'
'WS NS'
'WS EW'              C51 = 'NW NS'
C21 =                StrongAdj6 =
'WS EN'              'NS EN'
'NS EW'
StrongAdj3 =          C61 = []
'SN NW'              C71 = 'EN EW'
'SN EN'              StrongAdj7 = []
'SN EW'

Chrom_number = 4
Family_Gamma =
      Gamma1          Gamma2
-----
{'WE'}  {'WS'}  {'WE'}  {'EW'}
{'SN'}  {'SE'}  {'SN'}  {'NS'}
{'NW'}  {'NS'}  {'NW'}  {'SE'}
{'EN'}  {'EW'}  {'WS'}  {'EN'}
    
```

Figure 6. The output of Algorithm 1 for finding $\chi^f(\tilde{G})$ in Figure 5.

5. Experimental Result

In this section, we discuss an implementation of Algorithm 1 in determining traffic signal phasing at an intersection. A phase is defined as any traffic light display that has its own timings, and it determines how a specific vehicle or pedestrian will move, whereas the term phase set refers to “any distinct combination of concurrent vehicle or pedestrian phases”. Conflicting phases are those that cannot both have green indicators at the same time [39]. There are two types of conflicts between traffic movements at an intersection. The first type is crossing conflict, which is a collision that occurs when two separate directions of traffic try to cross paths at one spot. The second type is merging conflict, i.e., a conflict that happens when vehicles from multiple lanes or directions merge into a single lane traveling in a single direction [40].

Traffic flow is “the number of traffic elements passing an undisturbed point upstream in the approach per unit of time”. It is measured by the number of vehicles per hour or the passenger car unit (pcu) per hour [40]. In this research, the traffic flow data are presented in pcu per hour, where the conversion factors are as follows: 0.2 for motor cycle (MC); 1 for

light vehicle (LV), including “passenger cars, pick-up, and micro buses”; and 1.3 for heavy vehicle (HV), including “two or three-axle trucks and buses”.

5.1. The Method to Model Traffic Flows at an Intersection Using PFGs

We visualize traffic movements from different directions as vertices and connect two vertices with an edge if the two movements are in conflict (crossing conflict or merging conflict). Hence, we obtain the data of vertex and edge sets V and E , respectively. The degrees of edges and vertices show the following circumstances:

- A vertex’s membership degree ($\mu_1(x)$) indicates the possibility of the crowdedness of traffic flow on the movement x at the intersection. A vertex’s non-membership degree ($\nu_1(x)$) shows whether the flow of traffic on the movement x is likely to be free of congestion. The NeuM degree of a vertex ($\eta_1(x)$) indicates the possibility of an unknown circumstance about the crowdedness of traffic flow on x . We obtain a PFVS $\tilde{V} = \{(x, \mu_1(x), \eta_1(x), \nu_1(x))\}$.
- Traffic flow on an edge that connects two vertices (traffic movements) is determined through the minimum of traffic flows on both movements. In addition, the membership degree of any edge uv in \tilde{E} , that is, $\mu_2(uv)$, indicates the possibility of the crowdedness of traffic flows on conflicting movements uv . On the contrary, the non-membership degree $\nu_2(uv)$ shows the possibility of the non-crowdedness of traffic flows on uv . The NeuM degree $\eta_2(uv)$ represents the possibility of the unknown condition of the crowdedness of traffic flows on uv . We obtain a PFES $\tilde{E} = \{(uv, \mu_2(uv), \eta_2(uv), \nu_2(uv))\}$.

The degrees of vertices and edges are determined as follows:

- The membership degree $\mu_1(x)$ is calculated through triangular or trapezoidal membership functions, whereas the non-membership degree is calculated through Equation (1) in Corollary 1, i.e., $\mu_1(x) = \mu_1(f_x)(1 - \pi(x))$ and $\nu_1(x) = 1 - \mu_1(x)(1 - \pi(x)) - \pi(x)$, $x \in V$ by choosing $\pi(x) = \text{mean}\{\mu_1(x)|x \in V\}$, where f_x stands for traffic flow on movement x .
- The NeuM degree $\eta_1(x)$ is determined through the function g in Theorem 2:

$$\eta_1(x) = g(1 - \mu_1(x) - \nu_1(x))$$

where $g(x) = \frac{1 - \mu_1(x) - \nu_1(x)}{a}$ for $x \in V$ and $a = \sum_{x \in V} (1 - \mu_1(x) - \nu_1(x))$ ([35]). The NeuM degree of each edge $\eta_2(uv)$ is defined similarly.

- The membership degree $\mu_2(uv)$ and non-membership degree $\nu_2(uv)$ are calculated through formulas in Corollary 1: $\mu_2(uv) = \mu_2(uv)(1 - \pi(uv))$, $uv \in E$, where $\mu_2(uv) = \mu_1(\min\{f_u, f_v\})$, $\nu_2(uv) = 1 - \mu_2(uv)(1 - \pi(uv)) - \pi(uv)$, $uv \in E$ by choosing $\pi(uv) = \min\{\mu_2(uv)|uv \in E\}$.

5.2. Case Study

We take a case study at an intersection in the Special Region of Yogyakarta, Indonesia, i.e., the Pingit intersection. The location of the intersection is depicted in Figure 7 (right side). Tentara Pelajar street is in the Southern (S) direction, Diponegoro street is to the Eastern (E) direction, Magelang street is to the Northern (N) direction, and Kyai Mojo street is to the Western (W) direction.

A sketch of the intersection is also given in Figure 7 (left side). We collect the data of traffic flows on 25–27 January 2023 in the morning (06.30–07.30 a.m.) and in the evening (16.30–17.30 p.m.). There are 12 traffic movements in the intersection, i.e., WN, WE, WS, SN, SW, SE, NE, NW, NS, EN, EW, and ES. This means that the vertex set V contains 12 vertices.

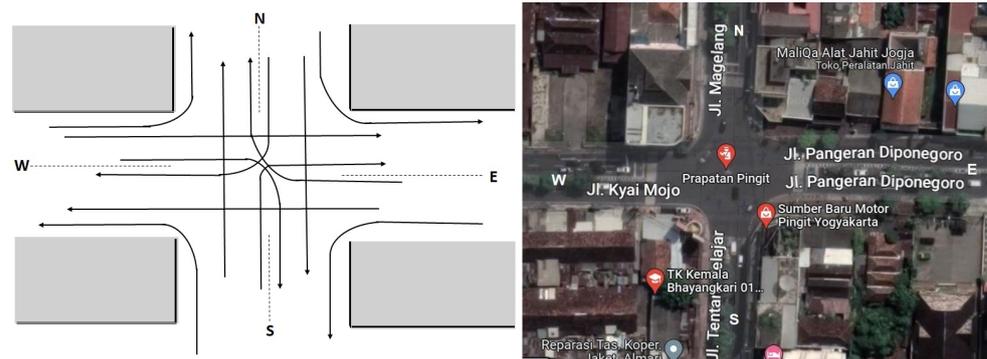


Figure 7. Location of the Pingit intersection in the Special Region of Yogyakarta.

The next step is to transform the traffic flow data into PFVS \tilde{V} , wherein the membership degree of each element is determined via triangular and trapezoidal membership functions in Figure 8.

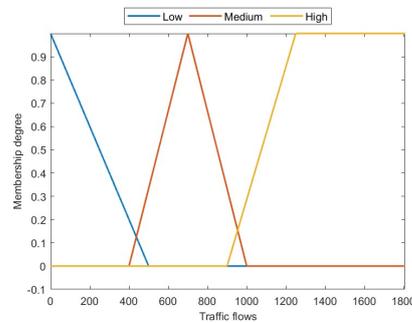


Figure 8. The function to calculate membership degree in the PFVS \tilde{V} .

The PFVS \tilde{V} of traffic flows in the intersection is displayed in Table 1.

Table 1. The PFVS \tilde{V} of traffic flow data in the Pingit intersection.

Traffic Movements (Vertices)	Traffic Flows	Clusters	Degree of Vertices
WN	1805	High	(0.383, 0.083, 0)
WE	605	Medium	(0.262, 0.083, 0.121)
WS	100	Low	(0.306, 0.083, 0.077)
SN	354	Low	(0.112, 0.083, 0.271)
SW	32	Low	(0.359, 0.083, 0.024)
SE	458	Low	(0.032, 0.083, 0.351)
NE	530	Medium	(0.166, 0.083, 0.217)
NW	831	Medium	(0.216, 0.083, 0.167)
NS	838	Medium	(0.207, 0.083, 0.176)
EN	337	Low	(0.125, 0.083, 0.258)
EW	1210	High	(0.339, 0.083, 0.044)
ES	742	Medium	(0.329, 0.083, 0.054)

It is shown in Table 1 that traffic flows on WN and EW have a high possibility of being crowded compared to traffic flows on other movements. Conversely, the traffic flow on SW has a lower possibility of being crowded.

Further, the picture fuzzy edge sets from crossing and merging conflicts in the Pingit intersection are shown in Tables 2 and 3, respectively. We observe that most of the edges in both tables connect strongly adjacent vertices. Moreover, the traffic flows on edges WS SN, WS SE, WS NW, WS EW, SW NW, SW and EW have a high possibility of being crowded.

Table 2. The PFES \tilde{E} of crossing conflict in the Pingit intersection

Crossing Conflict (Edges)	Degree of Conflict (Degree of Edges)	Crossing Conflict (Edges)	Degree of Conflict (Degree of Edges)
WE SN	(0.2675, 0.0625, 0.6485)	SN NW	(0.2675, 0.0625, 0.6485)
WE NW	(0.6259, 0.0625, 0.290)	SN EW	(0.2675, 0.0625, 0.6485)
WE NS	(0.6259, 0.0625, 0.290)	SE NS	(0.0769, 0.0625, 0.8390)
WE EN	(0.2986, 0.0625, 0.6174)	SE EN	(0.2986, 0.0625, 0.6174)
WS SN	(0.7328, 0.0625, 0.1832)	SE EW	(0.0769, 0.0625, 0.839)
WS SE	(0.7328, 0.0625, 0.1832)	NW EN	(0.2986, 0.0625, 0.6174)
WS NW	(0.7328, 0.0625, 0.1832)	NS EN	(0.2986, 0.0625, 0.6174)
WS EW	(0.7328, 0.0625, 0.1832)	NS EW	(0.4946, 0.0625, 0.4214)

Table 3. The PFES \tilde{E} of merging conflict.

Merging Conflict (Edges)	Degree of Conflict (Degree of Edges)	Merging Conflict (Edges)	Degree of Conflict (Degree of Edges)
WN SN	(0.2675, 0.083, 0.6485)	SN EN	(0.2986, 0.083, 0.6174)
WN EN	(0.2986, 0.083, 0.6174)	SW NW	(0.8574, 0.083, 0.0586)
WE SE	(0.0769, 0.083, 0.839)	SW EW	(0.8574, 0.083, 0.0586)
WE NE	(0.3969, 0.083, 0.519)	SE NE	(0.0769, 0.083, 0.839)
WS NS	(0.7328, 0.083, 0.1832)	EW NW	(0.516, 0.083, 0.3999)
WS ES	(0.7328, 0.083, 0.1832)	NS ES	(0.7877, 0.083, 0.1282)

The PFG model $\tilde{G} = (\tilde{V}, \tilde{E})$ of traffic flow data is depicted in Figure 9.

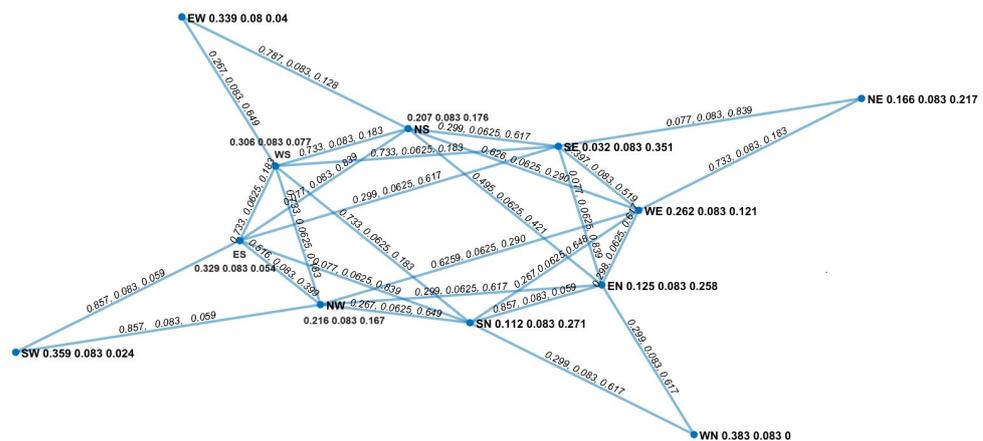


Figure 9. The PFG model of traffic flows in the Pingit intersection.

We obtain the chromatic number $\chi(\tilde{G}) = 4$ through Algorithm 1, and it has been evaluated in Matlab R2022b, where the output is depicted in Figure 10. The traffic flows can be arranged in 4 phases, and the patterns of traffic signal phasing are presented in Table 4.

```

Determining the sets of strongly adjacent
and weakly adjacent vertices
StrongAdj1 =      StrongAdj4 =      StrongAdj8 =
'WN SN'          'SN NW'          'NW EN'
'WN EN'          'SN EN'          'NW EW'
C11 =            C41 =            C81 =
'WN WE'          'SN SW'          'NW NS'
'WN WS'          'SN SE'          'NW ES'
'WN SW'          'SN NE'          StrongAdj9 =
'WN SE'          'SN NS'          'NS EN'
'WN NE'          'SN ES'          'NS EW'
'WN NW'          StrongAdj5 =      'NS ES'
'WN NS'          'SW EW'          C91 = []
'WN EW'          C51 =            StrongAdj10=[]
'WN ES'          'SW SE'          C101 =
StrongAdj2 =      'SW NE'          'EN EW'
'WE SN'          'SW NW'          'EN ES'
'WE SE'          'SW NS'          StrongAdj11 = []
'WE NE'          'SW EN'          C111 = 'EW ES'
'WE NW'          'SW ES'
'WE NS'
'WE EN'
C21 =            StrongAdj6 =
'WE WS'          'SE NE'
'WE SW'          'SE NS'
'WE EW'          'SE EN'
'WE ES'          'SE EW'
StrongAdj3 =      C61 =
'WS SN'          'SE NW'
'WS SE'          'SE ES'
'WS NW'          StrongAdj7 = []
'WS NS'          C71 =
'WS EW'          'NE NW'
'WS ES'          'NE NS'
C31 =            'NE EN'
'WS SW'          'NE EW'
'WS NE'          'NE ES'
'WS EN'
Gamma1 =
'WN'            'WE'            'WS'
'SN'            'SW'            'SE'
'NE'            'NW'            'NS'
'EN'            'ES'            'EW'
Gamma2 =
'WE'            'WS'            'SW'            '0'
'SN'            'SE'            'ES'            '0'
'WN'            'NE'            'NW'            'NS'
'EN'            'EW'            '0'            '0'
Gamma3 =
'WS'            'SW'            'EN'
'SE'            'NW'            'ES'
'SN'            'NE'            'NS'
'EW'            'WE'            'WN'
chrom_number_G = 4
    
```

Figure 10. Determination of the chromatic number of PFG model \tilde{G} in Figure 9.

Table 4. The traffic signal phasing that can be implemented at the Pingit intersection.

Patterns	Phase 1	Phase 2	Phase 3	Phase 4
1	WN, WE, WS	SE, SN, SW	NE, NW, NS	EN, EW, ES
2	WE, WS, SW	SN, SE, ES	WN, NE, NW, NS	EN, EW
3	WS, SW, EN	SE, NW, ES	SN, NE, NS	EW, WE, WN

5.3. Comparison to the Fuzzy Graph Coloring Method

In this part, we compare the result in Table 4 with a traffic signal phasing obtained from the fuzzy graph coloring method based on δ -fuzzy independent vertices ($\delta \in [0, 1]$) as given in [33]. The fuzzy graph model of traffic flows in the Pingit intersection is depicted in Figure 11.

For $\delta = 0.8857$, the sets $\{SW, EW, NW\}$, $\{ES, WS, NS\}$, and $\{SE, WE, NE\}$ are the sets of δ -fuzzy independent vertices since $\mu(uv) \leq \delta$ for each pair u, v in the above sets. Therefore, we obtain 4-phase scheduling as follows:

$$(i)SW, EW, NW, WN; \quad (ii)SN, NS, ES; \quad (iii)EN, WS, NE; \quad (iv)WE, SE. \quad (2)$$

We observe that some pairs of vertices in (2) are elements of merging conflict in Table 3. Hence, the traffic signal phasing obtained from PFG coloring in Table 4 is safer than the signal phasing from the fuzzy graph coloring in (2) since there are no traffic flows from merging conflict that move simultaneously at the same phase.

Abbreviations

The following abbreviations are used in this manuscript:

IFS	Intuitionistic fuzzy set
IFG	Intuitionistic fuzzy graph
PFS	Picture fuzzy set
PF	Picture fuzzy
PFVS	Picture fuzzy vertex set
PFES	Picture fuzzy edge set
PFG	Picture fuzzy graph
NeuM	Neutral membership
CPFG	Complete picture fuzzy graph

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