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Bounding the Zeros of Polynomials Using the Frobenius Companion Matrix Partitioned by the Cartesian Decomposition

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Abstract: In this work, some new inequalities for the numerical radius of block n-by-n matrices are presented. As an application, the bounding of zeros of polynomials using the Frobenius companion matrix partitioned by the Cartesian decomposition method is proved. We highlight several numerical examples showing that our approach to bounding zeros of polynomials could be very effective in comparison with the most famous results as well as some recent results presented in the field. Finally, observations, a discussion, and a conclusion regarding our proposed bound of zeros are considered. Namely, it is proved that our proposed bound is more efficient than any other bound under some conditions; this is supported with many polynomial examples explaining our choice of restrictions.

Keywords: numerical radius; operator matrix; zeros of polynomials; algorithm

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1. Introduction

The problem of finding the zeros of complex monic polynomials of the form

$$P_n(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1, \qquad z \in \mathbb{C}, n \geq 2, a_1 \neq 0,$$

is one of the most interesting, difficult, and oldest problems in mathematics. Several approaches have been taken to study this problem, with significant effort expended. See, for example [1,2], and the references therein.

One of the most common problems in this topic is determining or approximating the radius of the disk containing these zeros. In modern mathematics, the numerical range of a given Hilbert space is a very important concept related to the problem of bounding zeros of complex polynomials. The numerical range in matrix analysis (finite dimensional Hilbert space) is equivalent to the Rayleigh quotient for a given complex Hermitian matrix C. It can be easily shown that, for a given matrix C, the Rayleigh quotient reaches its smallest and largest eigenvalues of C. In terms of the field of values of a complex $n \times n$ matrix, one of the well-known tools used in this problem is the Frobenius Companion matrix $C(P_n)$, which corresponds to the polynomial P_n. Therefore, inequalities for the numerical radius of such matrices would be very helpful in this direction. See [3,4].

The spectrum of a non-singular matrix C is the set of all its eigenvalues. Because the spectral set is contained in the numerical range, the radius of the spectral set is less than (or equal to) the radius of the numerical range. The problem with dealing with a spectral radius is that it is not as easy to deal with as a numerical radius. Because of that, in the last two decades, the problem of bounding the numerical radius has caught the attention of many researchers. It is worth mentioning that when an operator (matrix) C is a normal operator, the spectral radius is the same as the numerical radius.

We continue our investigation of the numerical radius of Hilbert space operators in this article by establishing new inequalities for $n \times n$ operator matrices for general Hilbert space operators. Our approach depends mainly on using the Cartesian decomposition of a general Hilbert space operator. To the best of our knowledge, no known algorithm has been employed to constrain the zeros of the polynomial $P_n(z)$ using the Frobenius companion matrix partitioned using the Cartesian decomposition method, and this is one of the main originalities of the study. In particular, we remark that applying the numerical radius inequalities to the companion matrix presents a partition difficulty because the primary diagonal in the reported results necessitates a square sub-matrix. To that end, we show that the standard approaches for splitting the companion matrix, considered by many authors in the literature, are ineffective. As applications, several bounds for the zeros of complex polynomials are introduced. Various numerical examples show that our results are far superior to most of the well-known bounds found in the literature.

The rest of the article is structured as follows: Section 2 presents the mathematical background of the study. Section 3 describes the numerical radius inequalities of the matrix operator. Applications for bounding zeros of polynomials are given in Section 4. The numerical radius of the real and imaginary parts of the considered matrix is determined in Section 5. Observations and discussion of the study are provided in Section 6. Concluding remarks are given in Section 7.

2. Mathematical Background

Some preliminaries and notations are now introduced for the purposes of the study. Let $\mathscr{H}$ be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators from $\mathscr{H}$ into itself. When $\mathscr{H} = \mathbb{C}^n$, we identify $\mathscr{B}(\mathscr{H})$ with the algebra $\mathscr{M}_n$ of n-by-n complex matrices. For a bounded linear operator T on a Hilbert space $\mathscr{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to \langle Tx, x \rangle$ associated with the operator. More precisely, we can set

$$W(T) = \{\langle Tx, x \rangle : x \in \mathscr{H}, \|x\| = 1\}.$$

In addition, the numerical radius is defined to be

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

The spectral radius of an operator T is defined to be

$$r(T) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}.$$

We recall that the usual operator norm of an operator T is defined to be

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\}.$$

Several numerical radius type inequalities improving and refining the inequality

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\| \qquad (T \in \mathscr{B}(\mathscr{H}))$$

have been recently obtained by many other authors. See, for example, refs. [5–10]. Four important facts concerning the numerical radius inequalities of $n \times n$ operator matrices are obtained by different authors, which are grouped together as follows:

Let $\mathbf{S} = [S_{ij}] \in \mathscr{B}(\bigoplus_{i=1}^{n} \mathscr{H}_i)$ such that $S_{ij} \in \mathscr{B}(\mathscr{H}_j, \mathscr{H}_i)$. Then, Refs [5,9,10]

$$w(\mathbf{S}) \leq \begin{cases} w\left(\left[t_{kj}^{(1)}\right]\right), \\ w\left(\left[t_{kj}^{(2)}\right]\right), \\ w\left(\left[t_{kj}^{(3)}\right]\right), \\ w\left(\left[t_{kj}^{(4)}\right]\right), \end{cases};$$

where

$$t_{kj}^{(1)} = w\left(\left[\left\|S_{kj}\right\|\right]\right); \qquad t_{kj}^{(2)} = \begin{cases} \frac{1}{2}\left(\left\|S_{kj}\right\| + \left\|S_{kj}^2\right\|^{1/2}\right), & k = j \\ \left\|S_{kj}\right\|, & k \neq j \end{cases};$$

$$t_{kj}^{(3)} = \begin{cases} w\left(S_{kj}\right), & k = j \\ \left\|S_{kj}\right\|, & k \neq j \end{cases}; \qquad t_{kj}^{(4)} = \begin{cases} w\left(S_{kj}\right), & k = j \\ w\begin{pmatrix} 0 & S_{kj} \\ S_{jk} & 0 \end{pmatrix}, & k \neq j \end{cases}.$$

Clearly, the third and fourth bounds above are gentle refinements of the first and second bounds; therefore, both $w\left(\left[t_{kj}^{(3)}\right]\right)$ and $w\left(\left[t_{kj}^{(4)}\right]\right)$ give better upper estimates for the numerical radius of $\mathbf{S} = \left[S_{ij}\right]$.

Let T_n be the tridiagonal Toeplitz matrix denoted by $T_n = \mathrm{tridiag}(b, a, c)$, i.e.,

$$T_n := \begin{bmatrix} a & c & 0 & \cdots & 0 \\ b & a & c & \ddots & \vdots \\ 0 & b & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c \\ 0 & \cdots & 0 & b & a \end{bmatrix}_{n \times n}, \qquad n \geq 2.$$

It is well known that the eigenvalues of T_n are given by [11] as

$$\lambda_k = a + 2\sqrt{|bc|}\cos\left(\frac{k\pi}{n+1}\right), \qquad k = 1, 2, \cdots, n$$

and they have the polar form

$$\lambda_k = a + 2\sqrt{|bc|\mathrm{e}^{i(\theta+\phi)/2}\cos\left(\frac{k\pi}{n+1}\right)}, \qquad k = 1, 2, \cdots, n,$$

where $\theta = \arg(b)$ and $\phi = \arg(c)$. In the case of $bc \neq 0$, T_n has n simple eigenvalues, and all of them lie in the closed segment

$$S_{n,\lambda} = \left\{a + t\mathrm{e}^{i(\theta+\phi)/2} : t \in \mathbb{R}, |t| \leq 2\sqrt{|bc|}\cos\left(\frac{\pi}{n+1}\right)\right\} \subset \mathbb{C}.$$

The eigenvalues are located symmetrically with respect to a. Thus, the spectral radius of T_n is given by

$$r(T_n) = \max\left\{\left|a + 2\sqrt{|bc|}\mathrm{e}^{i(\theta+\phi)/2}\cos\left(\frac{\pi}{n+1}\right)\right|, \left|a + 2\sqrt{|bc|}\mathrm{e}^{i(\theta+\phi)/2}\cos\left(\frac{n\pi}{n+1}\right)\right|\right\}. \tag{1}$$

Furthermore, if $bc \neq 0$, then the eigenvectors $x_k = [x_{1,n}, x_{2,n}, \cdots, x_{k,n}]^T$ associated with the eigenvalue λ_k of T_n are given in the form $x_{k,j} = \left(\frac{c}{b}\right)^{k/2}\sin\left(\frac{kj\pi}{n+1}\right), k, j = 1, 2, \cdots, n$. For a comprehensive study of Toeplitz matrices, the reader may refer to the interesting book [12].

The following result is of great interest in the results presented next [11].

Lemma 1. *The tridiagonal Toeplitz matrix* $T_n = \mathrm{tridiag}(b, a, c)$ *is normal (i.e.,* $T_n^* T_n = T_n T_n^*$*) if and only if* $|b| = |c|$.

Lemma 2. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and* $\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be an operator matrix with $A \in \mathscr{B}(\mathscr{H}_1)$, $B \in \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$, $C \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ and $D \in \mathscr{B}(\mathscr{H}_2)$. Then,*

$$\omega(\mathbf{T}) \leq \frac{1}{2}\left(\omega(A) + \omega(D) + \sqrt{(\omega(A) - \omega(D))^2 + (\|B\| + \|C\|)^2}\right).$$

Lemma 3. *If $S := \left[s_{kj}\right] \in \mathscr{M}_n(\mathbb{C})$, then,*

$$\omega(S) \leq \omega\left(\left[\left|s_{kj}\right|\right]\right) = \frac{1}{2} r\left(\left[\left|s_{kj}\right| + \left[\left|s_{kj}\right|\right]\right]\right).$$

Let $A \in \mathscr{B}(\mathscr{H})$, then,

$$|\langle Ax, y\rangle|^2 \leq \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle, \qquad 0 \leq \alpha \leq 1,$$

for any vectors $x, y \in \mathscr{H}$, where $|A| = (A^*A)^{1/2}$. This inequality is well known as the mixed Schwarz inequality, which was introduced in [13] and generalized later in [14].

The following result presents the Cartesian decomposition of the mixed Schwarz inequality [8].

Lemma 4. *Let $A \in \mathscr{B}(\mathscr{H})$ be with the Cartesian decomposition $A = P + iQ$. If f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, then,*

$$|\langle Ax, y\rangle| \leq \|f(|P|)x\| \|g(|P|)y\| + \|f(|Q|)x\| \|g(|Q|)y\|$$

for all $x, y \in \mathscr{H}$.

3. Numerical Radius Inequalities of $m \times m$ Matrix Operator

Let us now turn our attention to numerical radius inequalities of the $m \times m$ matrix operator, starting with the following theorem:

Theorem 1. *Let $\mathbf{A} = \left[A_{kj}\right] \in \oplus_m \mathscr{M}_n(\mathbb{C})$ be an $m \times m$ operator, such that $P_{kj} + iQ_{kj}$ is the corresponding Cartesian decomposition of A_{kj}. If f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, then,*

$$w(\mathbf{A}) \leq w^{1/2}\left(\left[c_{kj}\right]\right), \tag{2}$$

where

$$c_{kj} = m \cdot \begin{cases} w\left(P_{kk}^2 + Q_{kk}^2\right), & j = k \\ \frac{1}{4}\left\|f^2\left(\left|P_{kj}\right|\right) + g^2\left(\left|P_{kj}\right|\right) + f^2\left(\left|Q_{kj}\right|\right) + g^2\left(\left|Q_{kj}\right|\right)\right\|^2, & j \neq k. \end{cases}$$

Proof. Let $x = [x_1, x_2, \cdots, x_m]^T \in \bigoplus_{\ell=1}^{m} \mathscr{M}_n(\mathbb{C})$ with $\|x\| = 1$; then, we have

$$\begin{aligned}
&\frac{1}{m^2}|\langle \mathbf{A}x,x\rangle|^2 = \frac{1}{m^2}\left|\sum_{k,j=1}^{m}\left\langle A_{kj}x_j,x_k\right\rangle\right|^2 \le \frac{1}{m}\sum_{k,j=1}^{m}\left|\left\langle A_{kj}x_j,x_k\right\rangle\right|^2 \quad \text{(by Jensen's inequality)}\\
&= \frac{1}{m}\sum_{k=1}^{m}|\langle A_{kk}x_k,x_k\rangle|^2 + \frac{1}{m}\sum_{\substack{j=1\\ j\neq k}}^{m}\left|\left\langle A_{kj}x_j,x_k\right\rangle\right|^2\\
&\le \frac{1}{m}\sum_{k=1}^{m}\left(\langle P_{kk}x_k,x_k\rangle^2 + \langle Q_{kk}x_k,x_k\rangle^2\right) + \frac{1}{m}\sum_{\substack{j=1\\ j\neq k}}^{m}\left[\left\langle f^2\left(\left|P_{kj}\right|\right)x_j,x_k\right\rangle^{\frac{1}{2}}\left\langle g^2\left(\left|P_{kj}\right|\right)x_j,x_k\right\rangle^{\frac{1}{2}}\right.\\
&\qquad\qquad \left. + \left\langle f^2\left(\left|Q_{kj}\right|\right)x_j,x_k\right\rangle^{\frac{1}{2}}\left\langle g^2\left(\left|Q_{kj}\right|\right)x_j,x_k\right\rangle^{\frac{1}{2}}\right]^2\\
&\le \frac{1}{m}\sum_{k=1}^{m}\left(\|P_{kk}x_k\|^2 + \|Q_{kk}x_k\|^2\right)\\
&\quad + \frac{1}{4m}\sum_{\substack{j=1\\ j\neq k}}^{m}\left\{\left\langle\left[f^2\left(\left|P_{kj}\right|\right) + g^2\left(\left|P_{kj}\right|\right)\right]x_j,x_k\right\rangle + \left\langle\left[f^2\left(\left|Q_{kj}\right|\right) + g^2\left(\left|Q_{kj}\right|\right)\right]x_j,x_k\right\rangle\right\}^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m}\sum_{k=1}^{m}\left(\left\langle P_{kk}^2x_k,x_k\right\rangle + \left\langle Q_{kk}^2x_k,x_k\right\rangle\right)\\
&\quad + \frac{1}{4m}\sum_{\substack{j=1\\ j\neq k}}^{m}\left\|f^2\left(\left|P_{kj}\right|\right) + g^2\left(\left|P_{kj}\right|\right) + f^2\left(\left|Q_{kj}\right|\right) + g^2\left(\left|Q_{kj}\right|\right)\right\|^2\|x_j\|^2\|x_k\|^2\\
&\le \frac{1}{m}\sum_{k=1}^{m}\left\langle\left(P_{kk}^2 + Q_{kk}^2\right)x_k,x_k\right\rangle\\
&\quad + \frac{1}{4m}\sum_{\substack{j=1\\ j\neq k}}^{m}\left\|f^2\left(\left|P_{kj}\right|\right) + g^2\left(\left|P_{kj}\right|\right) + f^2\left(\left|Q_{kj}\right|\right) + g^2\left(\left|Q_{kj}\right|\right)\right\|^2\|x_j\|\|x_k\|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left(\text{because } \|x_j\|^2 \le \|x_j\|,\ \forall j\right)\\
&= \frac{1}{m}\sum_{k=1}^{m}\left\|P_{kk}^2 + Q_{kk}^2\right\|\|x_k\|^2\\
&\quad + \frac{1}{4m}\sum_{\substack{j=1\\ j\neq k}}^{m}\left\|f^2\left(\left|P_{kj}\right|\right) + g^2\left(\left|P_{kj}\right|\right) + f^2\left(\left|Q_{kj}\right|\right) + g^2\left(\left|Q_{kj}\right|\right)\right\|\|x_j\|\|x_k\|\\
&= \left\langle\left[c_{kj}\right]y,y\right\rangle,
\end{aligned}$$

where $y = [\|x_1\|, \|x_2\|, \cdots, \|x_m\|]^T$. Taking the supremum over $x \in \bigoplus_m \mathscr{M}_n$, we obtain the desired result. □

Particularly, we are interested in the following 2×2 cases:

Corollary 1. *If* $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$ *is in* $\mathscr{M}_n \bigoplus \mathscr{M}_n$*, then,*

$$\begin{aligned}
&\omega\left(\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\right)\\
&\le \sqrt{\omega\left(P_{11}^2 + Q_{11}^2\right) + \omega\left(P_{22}^2 + Q_{22}^2\right) + \sqrt{\left(\omega\left(P_{11}^2 + Q_{11}^2\right) - \omega\left(P_{22}^2 + Q_{22}^2\right)\right)^2 + N^2}}, \qquad (3)
\end{aligned}$$

where $N = \||P_{12}| + |Q_{12}|\| + \||P_{21}| + |Q_{21}|\|$.

Proof. From Theorem 1 and by employing Lemma 3, we have

$$
\begin{aligned}
&\omega\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \\
&\leq \sqrt{2}\cdot\omega^{\frac{1}{2}}\left(\begin{bmatrix} \omega(P_{11}^2+Q_{11}^2) & \||P_{12}|+|Q_{12}|\| \\ \||P_{21}|+|Q_{21}|\| & \omega(P_{22}^2+Q_{22}^2) \end{bmatrix}\right) \\
&= \sqrt{2}\cdot r^{\frac{1}{2}}\left(\begin{bmatrix} \omega(P_{11}^2+Q_{11}^2) & \frac{\||P_{12}|+|Q_{12}|\|+\||P_{21}|+|Q_{21}|\|}{2} \\ \frac{\||P_{12}|+|Q_{12}|\|+\||P_{21}|+|Q_{21}|\|}{2} & \omega(P_{22}^2+Q_{22}^2) \end{bmatrix}\right) \\
&= \sqrt{\omega(P_{11}^2+Q_{11}^2)+\omega(P_{22}^2+Q_{22}^2)+\sqrt{\left(\omega(P_{11}^2+Q_{11}^2)-\omega(P_{22}^2+Q_{22}^2)\right)^2+N^2}},
\end{aligned}
$$

which proves the result. □

Corollary 2. *If* $\mathbf{A}=\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ *in* $\mathscr{M}_n \oplus \mathscr{M}_n$*, then,*

$$
\omega\left(\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}\right) \leq \max\left(\omega\left(P_{11}^2+Q_{11}^2\right),\omega\left(P_{22}^2+Q_{22}^2\right)\right),
$$

where $P_{kk}+iQ_{kk}$ *is the Cartesian decomposition of* A_{kk}*.*

Proof. Setting $A_{12}=0=A_{21}$ in Corollary 1 and using the fact that $\omega\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leq \max(\omega(A),\omega(D))$, see [15]. □

4. Applications for Bounding Zeros of Polynomials

One of the most interesting and useful applications of the numerical radius inequalities is to bound zeros of complex polynomials using a suitable partition of the well-known Frobenius companion matrix. Let

$$
p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1, \tag{4}
$$

be any polynomial with $a_1 \neq 0$. The general corresponding companion matrix is defined as

$$
C(p) := \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{5}
$$

It is well known that the eigenvalues of $C(p)$ are exactly the zeros of $p(z)$, see [3], p. 316.

Based on some numerical radius estimations of $C(p)$, several authors paid serious attention to finding various upper bounds of the zeros of $p(z)$. Some famous upper bounds are listed as follows:

If λ is a zero of $p(z)$, then,

1. Cauchy [3] obtained the following upper bound:

$$
|\lambda| \leq 1+\max\{|a_k| : k=1,2,,\cdots,n\}. \tag{6}
$$

2. Carmichael and Mason [3] provided the following estimate:

$$
|\lambda| \leq \sqrt{1+\sum_{k=1}^{n}|a_k|^2}. \tag{7}
$$

3. Montel [3] proved the following estimate:

$$|\lambda| \le \max\left\{1, \sum_{k=1}^{n} |a_k|\right\}. \tag{8}$$

4. Fujii and Kubo [16] have shown that

$$|\lambda| \le \cos\left(\frac{\pi}{n+1}\right) + \frac{1}{2}\left(|a_n| + \sum_{k=1}^{n} |a_k|^2\right). \tag{9}$$

5. Abdurakhmanov [17] provided the following estimate:

$$|\lambda| \le \frac{1}{2}\left(|a_n| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(|a_n| - \cos\left(\frac{\pi}{n}\right)\right)^2 + \left(1 + \sum_{k=1}^{n-1} |a_k|^2\right)^2}\right). \tag{10}$$

   It seems that Paul and Bag [18] did not notice the Abdurakhmanov result where they provided the same estimate.
6. Linden [19] provided the following estimate:

$$|\lambda| \le \frac{|a_n|}{n} + \sqrt{\frac{n-1}{n}\left(n - 1 + \sum_{k=1}^{n} |a_k|^2 - \frac{|a_n|^2}{n}\right)}. \tag{11}$$

7. Kittaneh [14] improved the Abdurakhmanov estimate by proving that

$$|\lambda| \le \frac{1}{2}\left(|a_n| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(|a_n| - \cos\left(\frac{\pi}{n}\right)\right)^2 + (|a_{n-1}| - 1)^2 + \sum_{j=1}^{n-2} |a_j|^2}\right). \tag{12}$$

8. Abu-Omar and Kittaneh [20] introduced the following estimate:

$$|\lambda| \le \frac{1}{2}\left(\frac{|a_n| + \alpha}{2} + \cos\left(\frac{\pi}{n+1}\right) + \sqrt{\left(\frac{|a_n| + \alpha}{2} - \cos\left(\frac{\pi}{n+1}\right)\right)^2 + 4\beta}\right), \tag{13}$$

   where $\alpha = \sqrt{\sum_{k=1}^{n} |a_k|^2}$ and $\beta = \sqrt{\sum_{k=1}^{n-1} |a_k|^2}$.
9. Al-Dolat et al. [21] provided the following estimate:

$$|\lambda| \le \frac{1}{2}\left(|a_n| + 2\cos\left(\frac{\pi}{n}\right) + \sqrt{t^2|a_n|^2 + \sum_{k=1}^{n-1} |a_k|^2} + \sqrt{1 + (1-t)^2|a_n|^2}\right) \tag{14}$$

   for $t \in [0,1]$. In fact, the upper bound above should be rewritten by taking 'min' over $t \in [0,1]$, which gives the best value for this estimate.

No single known algorithm has been used in the literature to bound the zeros of the polynomial $p(z)$ using the Frobenius companion matrix partitioned by the Cartesian decomposition method, as far as we are aware.

To apply the numerical radius inequalities established in the previous section to $C(p)$, we note that we have a little partition challenge in applying our obtained results because the main diagonal in the presented results requires a square sub-matrix. As a result, the usual well-known methods of partitioning the companion matrix $C(p)$ are useless, as demonstrated by the inequalities presented in the previous section, for example in (3).

The approach we propose is to consider the degree $p(z)$ in (4) to be even with a fixed integer n, such that $a_1 \neq 0$. To this end, we consider the even polynomial

$$q(z) = z^{2n} + a_{2n} z^{2n-1} + \cdots + a_2 z + a_1, \qquad n \geq 2,\ a_1 \neq 0. \tag{15}$$

Let

$$C(q) = \left[\begin{array}{c|c}
\overbrace{\begin{bmatrix} -a_{2n} & -a_{2n-1} & \cdots & -a_{n+2} & -a_{n+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \end{bmatrix}_{n\times n}}^{A_{11}} &
\overbrace{\begin{bmatrix} -a_{n} & -a_{n-1} & \cdots & -a_{2} & -a_{1} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \end{bmatrix}_{n\times n}}^{A_{12}} \\ \hline
\underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}}_{A_{21}} &
\underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}}_{A_{22}}
\end{array}\right]_{2n\times 2n} \tag{16}$$

be the corresponding companion matrix, partitioned as what it is. Constructing the Cartesian decomposition of $C(q)$, we have

$$\mathrm{Re}(C(q)) = \left[\begin{array}{c|c}
\overbrace{\begin{bmatrix} -\mathrm{Re}(a_{2n}) & \frac{-a_{2n-1}+1}{2} & \cdots & \frac{-a_{n+2}}{2} & \frac{-a_{n+1}}{2} \\ \frac{-\overline{a_{2n-1}}+1}{2} & 0 & \ddots & 0 & 0 \\ \vdots & \frac{1}{2} & \ddots & \frac{1}{2} & \vdots \\ 0 & 0 & \ddots & 0 & \frac{1}{2} \\ \frac{-\overline{a_{n+1}}}{2} & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}_{n\times n}}^{P_{11}} &
\overbrace{\begin{bmatrix} \frac{-a_{n}}{2} & \frac{-a_{n-1}}{2} & \cdots & \frac{-a_{2}}{2} & \frac{-a_{1}}{2} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}}^{P_{12}} \\ \hline
\underbrace{\begin{bmatrix} \frac{-\overline{a_{n}}}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{-\overline{a_{n-1}}}{2} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \frac{-\overline{a_{2}}}{2} & \vdots & \cdots & \cdots & \vdots \\ \frac{-\overline{a_{1}}}{2} & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}}_{P_{21}} &
\underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \vdots \\ 0 & \frac{1}{2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}_{n\times n}}_{P_{22}}
\end{array}\right]_{2n\times 2n} \tag{17}$$

and

$$\mathrm{Im}(C(q)) = \left[\begin{array}{c|c}
\overbrace{\begin{bmatrix} -\mathrm{Im}(a_{2n}) & \frac{-a_{2n-1}-1}{2i} & \cdots & \frac{-a_{n+2}}{2i} & \frac{-a_{n+1}}{2i} \\ \frac{\overline{a_{2n-1}}+1}{2i} & 0 & \ddots & 0 & 0 \\ \vdots & \frac{1}{2i} & \ddots & \frac{-1}{2i} & \vdots \\ 0 & 0 & \ddots & 0 & \frac{-1}{2i} \\ \frac{\overline{a_{n+1}}}{2i} & 0 & \cdots & \frac{1}{2i} & 0 \end{bmatrix}_{n\times n}}^{Q_{11}} &
\overbrace{\begin{bmatrix} \frac{-a_{n}}{2i} & \frac{a_{n-1}}{2i} & \cdots & \frac{a_{2}}{2i} & \frac{a_{1}}{2i} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{2i} & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}}^{Q_{12}} \\ \hline
\underbrace{\begin{bmatrix} \frac{\overline{a_{n}}}{2i} & 0 & \cdots & 0 & \frac{1}{2i} \\ \frac{\overline{a_{n-1}}}{2i} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \frac{\overline{a_{2}}}{2i} & \vdots & \cdots & \cdots & \vdots \\ \frac{\overline{a_{1}}}{2i} & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}}_{Q_{21}} &
\underbrace{\begin{bmatrix} 0 & \frac{-1}{2i} & 0 & \cdots & 0 \\ \frac{1}{2i} & 0 & \frac{-1}{2i} & 0 & \vdots \\ 0 & \frac{1}{2i} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{-1}{2i} \\ 0 & \cdots & 0 & \frac{1}{2i} & 0 \end{bmatrix}_{n\times n}}_{Q_{22}}
\end{array}\right]_{2n\times 2n}. \tag{18}$$

Hence,

$$C(q) := \mathrm{Re}(C(q)) + i\mathrm{Im}(C(q)).$$

Moreover, it is easy to observe that $A_{kj} = P_{kj} + iQ_{kj}$ for $k, j = 1, 2$. This observation may not hold true in general for operator matrices, i.e., the Cartesian decomposition of operator matrices is not equal to the Cartesian decomposition of their sub-matrices. However, because we construct a special type of $C(q)$ partition, it appears that such equality holds true only for this construction.

It is not easy to apply (3) to a general algorithm because it has the norms $\||P_{11}| + |Q_{11}|\|$ and $\||P_{22}| + |Q_{22}|\|$, which are difficult to evaluate for general $n \times n$ matrices. In fact, it will be easier as long as we have numeric entries with specific n, as explored in the presented examples below.

Table 1 illustrates that our upper bound of any zero of $q_1(z) = z^6 + \frac{5}{4}z^5 + \frac{4}{3}z^4 + z^3 + 2z^2 + 3z + 4$, obtained by (3), is much better among all given upper bounds listed in Table 1.

Table 1. Comparisons of upper bounds based on the considered polynomial $q_1(z)$.

Mathematician	Upper Bound
Cauchy (6)	5
Carmichael and Mason (7)	5.860057831
Montel (8)	12.58333333
Fujii and Kubo (9)	18.19610776
Abdurakhmanov (10)	17.44802607
Linden (11)	5.845408848
Kittaneh (12)	4.040959271
Abu-omar and Kittaneh (13)	4.916052295
Al-Dolat et al. (14)	4.867955746
Corollary 1	**3.941508802**

We note that the same approach can be used for polynomials of odd degrees with zero absolute terms; for example, in (15), assume that n is odd ≥ 5 and $a_1 = 0$. Then, $p(z)$ can be written as

$$p(z) = z\left(z^{n-1} + a_n z^{n-2} + \cdots + a_2\right) = zp_1(z),$$

where $p_1(z)$ is an even polynomial of degree ≥ 4. Because $z = 0$ is a zero for $p(z)$, it must belong to the disk containing the zeros of $p_1(z)$. Hence, in this case we have $\omega(C(p)) = \omega(C(p_1))$. We leave the details to the interested reader.

5. Numerical Radius of Real and Imaginary Parts of $C(p)$

The numerical radius of real and imaginary parts of $C(p)$ is now investigated. To this end, let $T \in \mathscr{M}_n(\mathbb{C})$ with the Cartesian decomposition $T = P + iQ$. Then,

$$W(T) \subseteq W(P) + W(Q).$$

Hence,

$$\sigma(T) \subseteq [\lambda_{\min}(P), \lambda_{\max}(P)] \times [\lambda_{\min}(Q), \lambda_{\max}(Q)].$$

In case of the companion matrix $C(p)$, it follows that all the zeros of $p(z)$ in (4) are located in the rectangle

$$[\lambda_{\min}(\mathrm{Re}[C(p)]), \lambda_{\max}(\mathrm{Re}[C(p)])] \times [\lambda_{\min}(\mathrm{Im}[C(p)]), \lambda_{\max}(\mathrm{Im}[C(p)])]. \tag{19}$$

In [14], Kittaneh provided an explicit formula for the characteristic polynomial of $\mathrm{Re}[C(p)]$, which is given as

$$p_{\mathrm{Real}}(z) := (z + \mathrm{Re}(a_n)) \prod_{j=1}^{n-1} \left(z - \cos\left(\frac{j\pi}{n}\right) \right) - \sum_{j=1}^{n-1} \left(\prod_{k \neq j}^{n-1} \left(z - \cos\left(\frac{k\pi}{n}\right) \right) \right) |v_j|^2,$$

where

$$v_j = \frac{1}{\sqrt{2n}} \left[(1 - \bar{a}_{n-1}) \sin\left(\frac{j\pi}{n}\right) - \sum_{k=2}^{n-1} \bar{a}_{n-k} \sin\left(\frac{kj\pi}{n}\right) \right].$$

In the same work [14], an explicit rectangle that contains the rectangle (19), and thus that contains all the zeros of $p(z)$, is obtained in the result below.

Theorem 2 ([14], Kittaneh)**.** *If z is any zero of $p(z) = z^n + c_n z^{n-1} + \cdots + c_2 z + c_1$, $(c_1 \neq 0)$, then z belongs to the rectangle $[-c, c] \times [-d, d]$, where*

$$c = \frac{1}{2} \left[|\mathrm{Re}(c_n)| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(|\mathrm{Re}(c_n)| - \cos\left(\frac{\pi}{n}\right) \right)^2 + |c_{n-1} - 1|^2 + \sum_{k=1}^{n-2} |c_k|^2} \right]$$

and

$$d = \frac{1}{2} \left[|\mathrm{Im}(c_n)| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(|\mathrm{Im}(c_n)| - \cos\left(\frac{\pi}{n}\right) \right)^2 + |c_{n-1} - 1|^2 + \sum_{k=1}^{n-2} |c_k|^2} \right].$$

Based on the results obtained in this work, in what follows, we provide another possible rectangle. In order to establish our result, we need the following lemma (see [21]):

Lemma 5. *Let $A, B, C, D \in \mathscr{B}(\mathscr{H})$ and $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then,*

$$\omega(T) \leq \frac{1}{2} \left(\omega(A) + \omega(D) + \sqrt{(\omega(A) - \omega(D))^2 + (\omega(B + C) + \omega(B - C))^2} \right).$$

Now we can give our explicit rectangle, which contains the rectangle (19) and thus all of the zeros of $p(z)$.

Theorem 3. *Let $q(z)$ be any even complex polynomial as given in (15) whose degree is ≥ 4. If z is any zero of $q(z)$, then z belongs to the rectangle $[-s, s] \times [-t, t]$, where*

$$\begin{aligned} s := {} & \frac{1}{4} \left(|\mathrm{Re}(a_{2n})| + \cos\left(\frac{\pi}{n}\right) + F \right) + \frac{1}{2} \cos\left(\frac{\pi}{n+1}\right) \\ & + \frac{1}{2} \sqrt{ \left(\frac{1}{2} \left(|\mathrm{Re}(a_{2n})| + \cos\left(\frac{\pi}{n}\right) + F \right) - \cos\left(\frac{\pi}{n+1}\right) \right)^2 + \left(\frac{|\mathrm{Re}(a_n)| + G + |\mathrm{Im}(a_n)| + H}{2} \right)^2 } \end{aligned}$$

and

$$\begin{aligned} t := {} & \frac{1}{4} \left(|\mathrm{Im}(a_{2n})| + \cos\left(\frac{\pi}{n}\right) + J \right) + \frac{1}{2} \cos\left(\frac{\pi}{n+1}\right) \\ & + \frac{1}{2} \sqrt{ \left(\frac{1}{2} \left(|\mathrm{Im}(a_{2n})| + \cos\left(\frac{\pi}{n}\right) + J \right) - \cos\left(\frac{\pi}{n+1}\right) \right)^2 + \left(\frac{|\mathrm{Re}(a_n)| + G + |\mathrm{Im}(a_n)| + H}{2} \right)^2 }, \end{aligned}$$

where

$$F := \sqrt{\left(|\mathrm{Re}(a_{2n})| - \cos\left(\frac{\pi}{n}\right)\right)^2 + |1 - a_{2n-1}|^2 + \sum_{k=n+1}^{2n-2} |a_k|^2},$$

$$G := \sqrt{|\mathrm{Re}(a_n)|^2 + |1 - a_1|^2 + \sum_{k=2}^{n-1} |a_k|^2},$$

$$H := \sqrt{|\mathrm{Im}(a_n)|^2 + |1 + a_1|^2 + \sum_{k=2}^{n-1} |a_k|^2},$$

and

$$J := \sqrt{\left(|\mathrm{Im}(a_{2n})| - \cos\left(\frac{\pi}{n}\right)\right)^2 + |1 + a_{2n-1}|^2 + \sum_{k=n+1}^{2n-2} |a_k|^2}.$$

Proof. Employing Lemma 5, on the real part of $C(q)$ (17), which is obtained in the previous section, by setting $A = P_{11}$, $B = P_{12}$, $C = P_{21}$ and $D = P_{22}$, it is enough to show that

$$\begin{aligned} &\omega(\mathrm{Re}[C(q)]) \\ &\leq \frac{1}{2}\left(\omega(P_{11}) + \omega(P_{22}) + \sqrt{(\omega(P_{11}) - \omega(P_{22}))^2 + (\omega(P_{12} + P_{21}) + \omega(P_{12} - P_{21}))^2}\right). \end{aligned} \tag{20}$$

Let us simplify that. Indeed, we have

$$P_{11} = \left[\begin{array}{c|ccccc} -\mathrm{Re}(a_{2n}) & \frac{-a_{2n-1}+1}{2} & \cdots & & \frac{-a_{n+2}}{2} & \frac{-a_{n+1}}{2} \\ \hline & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{-\overline{a_{2n-1}}+1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \vdots \\ \vdots & 0 & \frac{1}{2} & \ddots & \ddots & 0 \\ \frac{-\overline{a_{n+2}}}{2} & \vdots & \ddots & \ddots & \ddots & \frac{1}{2} \\ \frac{-\overline{a_{n+1}}}{2} & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{array}\right],$$

which can be written as

$$P_{11} = \begin{bmatrix} -\mathrm{Re}(a_{2n}) & u^* \\ u & T_{n-1} \end{bmatrix},$$

where $x := \left[\frac{-\overline{a_{2n-1}}+1}{2}, \cdots, \frac{-\overline{a_{n+2}}}{2}, \frac{-\overline{a_{n+1}}}{2}\right]$. Thus, by employing Lemmas 2 and 3, we have

$$\begin{aligned} \omega(P_{11}) &= \omega\left(\begin{bmatrix} |\mathrm{Re}(a_{2n})| & x^* \\ x & T_{n-1} \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} \omega(|\mathrm{Re}(a_{2n})|) & \|x\| \\ \|x\| & \omega(T_{n-1}) \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} \omega(|\mathrm{Re}(a_{2n})|) & \|x\| \\ \|x\| & \omega(T_{n-1}) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(|\mathrm{Re}(a_{2n})| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(|\mathrm{Re}(a_{2n})| - \cos\left(\frac{\pi}{n}\right)\right)^2 + 4\|x\|^2}\right) \\ &= \frac{1}{2}\left(|\mathrm{Re}(a_{2n})| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(|\mathrm{Re}(a_{2n})| - \cos\left(\frac{\pi}{n}\right)\right)^2 + |1 - a_{2n-1}|^2 + \sum_{k=n+1}^{2n-2} |a_k|^2}\right). \end{aligned} \tag{21}$$

On the other hand, we have

$$P_{12} + P_{21} = \begin{bmatrix} \frac{-a_n}{2} & \frac{-a_{n-1}}{2} & \cdots & \frac{-a_2}{2} & \frac{-a_1}{2} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-\overline{a_n}}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{-\overline{a_{n-1}}}{2} & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \frac{-\overline{a_2}}{2} & \vdots & \cdots & \cdots & \vdots \\ \frac{-\overline{a_1}}{2} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|cccc} -\frac{a_n+\overline{a_n}}{2} & -\frac{a_{n-1}}{2} & \cdots & -\frac{a_2}{2} & -\frac{a_1}{2}+\frac{1}{2} \\ \hline -\frac{\overline{a_{n-1}}}{2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ -\frac{\overline{a_2}}{2} & 0 & \cdots & \cdots & \vdots \\ -\frac{\overline{a_1}}{2}+\frac{1}{2} & 0 & 0 & \cdots & 0 \end{array}\right]_{n\times n}.$$

Let $v := \left[\frac{-\overline{a_{n-1}}}{2}, \cdots, \frac{-\overline{a_2}}{2}, \frac{-\overline{a_1}+1}{2}\right]^T$. So that, by Lemmas 2 and 3, we have

$$\begin{aligned} \omega(P_{12}+P_{21}) &= \omega\left(\begin{bmatrix} |\mathrm{Re}(a_n)| & v^* \\ v & 0 \end{bmatrix}\right) \le \omega\left(\begin{bmatrix} w(|\mathrm{Re}(a_n)|) & \|v\| \\ \|v\| & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} w(|\mathrm{Re}(a_n)|) & \|v\| \\ \|v\| & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\left(|\mathrm{Re}(a_n)| + \sqrt{|\mathrm{Re}(a_n)|^2 + 4\|v\|^2}\right) \\ &= \frac{1}{2}\left(|\mathrm{Re}(a_n)| + \sqrt{|\mathrm{Re}(a_n)|^2 + |1-a_1|^2 + \sum_{k=2}^{n-1}|a_k|^2}\right). \end{aligned} \tag{22}$$

Similarly, we have

$$P_{12} - P_{21} = \left[\begin{array}{c|cccc} -\frac{a_n-\overline{a_n}}{2} & -\frac{a_{n-1}}{2} & \cdots & -\frac{a_2}{2} & -\frac{a_1}{2}-\frac{1}{2} \\ \hline \frac{\overline{a_{n-1}}}{2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ \frac{\overline{a_2}}{2} & 0 & \cdots & \cdots & \vdots \\ \frac{\overline{a_1}}{2}+\frac{1}{2} & 0 & 0 & \cdots & 0 \end{array}\right]_{n\times n}.$$

Let $u := \left[\frac{\overline{a_{n-1}}}{2}, \cdots, \frac{\overline{a_2}}{2}, \frac{\overline{a_1}+1}{2}\right]^T$.
So that, by Lemmas 2 and 3, we have

$$\begin{aligned} \omega(P_{12}-P_{21}) &= \omega\left(\begin{bmatrix} |-i\mathrm{Im}(a_n)| & u^* \\ u & 0 \end{bmatrix}\right) \le \omega\left(\begin{bmatrix} w(|\mathrm{Im}(a_n)|) & \|u\| \\ \|u\| & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} w(|\mathrm{Im}(a_n)|) & \|u\| \\ \|u\| & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\left(|\mathrm{Im}(a_n)| + \sqrt{|\mathrm{Im}(a_n)|^2 + 4\|u\|^2}\right) \\ &= \frac{1}{2}\left(|\mathrm{Im}(a_n)| + \sqrt{|\mathrm{Im}(a_n)|^2 + |1+a_1|^2 + \sum_{k=2}^{n-1}|a_k|^2}\right). \end{aligned} \tag{23}$$

In addition, by (1) and Lemma 1, we have

$$\omega(P_{22}) = \cos\left(\frac{\pi}{n+1}\right). \tag{24}$$

Combining all above inequalities and equalities (21)–(24) in (20), and following the same steps for $\omega(\mathrm{Im}[C(q)])$, we obtain the required result in Theorem 3. Thus, the proof of Theorem 3 is established. □

The example in Table 2 shows that our estimated rectangle given in Theorem 3 might be better than the one given in Theorem 2.

Example 1. *Consider $q_2(z) = z^6 + 2iz^5 + 4iz^4 + \frac{1}{4}z + \frac{1}{16}$; then, the real and imaginary parts of the zeros of $q_2(z)$ are bounded as obtained in Table 2.*

Table 2. Comparisons of upper bounds based on the considered polynomial $q_2(z)$.

Result	Upper Bound of $\lvert\mathrm{Re}(\lambda)\rvert$	Upper bound of $\lvert\mathrm{Im}(\lambda)\rvert$
Kittaneh [14]	3.999737494	3.576384821
Theorem 3	2.476786336	2.585204772

Remark 1. *Several particular cases of Theorem 3, which are of great interest, could be deduced. Among others, we note the following cases:*

- $a_{2n} = a_n = 0$, $a_{2n-1} = \pm 1$ *and* $a_1 = 1$.
- $a_{2n} = a_n = 0$, $a_{2n-1} = \pm 1$ *and* $a_k = 0$ *for all* $k = 2, 3, \cdots, n-1$. *In particular, take* $a_1 = 1$.
- $a_{2n} = a_n = 0$, $a_{2n-1} = \pm 1$ *and* $a_k = 0$ *for all* $k = n+1, n+2, \cdots, 2n-2$. *In particular, take* $a_1 = 1$.

Theorem 4. *Under the assumptions of Theorem 3, we have*

$$\omega(C(q)) \leq \frac{1}{2}\left(L + \cos\left(\frac{\pi}{n+1}\right) + \sqrt{\left(L - \cos\left(\frac{\pi}{n+1}\right)\right)^2 + (D_1 + D_2)^2}\right), \tag{25}$$

where

$$L := \frac{1}{2}\left(\sqrt{\sum_{k=n+2}^{2n} |a_k|^2} + \sqrt{\sum_{k=n+2}^{2n} |a_k|^2 + (|a_{n+1}| + 1)^2}\right),$$

$$D_1 = \frac{1}{2}\left(|a_n| + \sqrt{|a_n|^2 + |1 - a_1|^2 + \sum_{k=2}^{n-1} |a_k|^2}\right)$$

and

$$D_2 = \frac{1}{2}\left(|a_n| + \sqrt{|a_n|^2 + |1 + a_1|^2 + \sum_{k=2}^{n-1} |a_k|^2}\right),$$

Proof. Applying (20) to $C(q)$ given in (16) by setting $A = A_{11}$, $B = A_{12}$, $C = A_{21}$ and $D = A_{22}$. So that, we obtain

$$\omega(C(q)) \leq \frac{1}{2}\left(\omega(A_{11}) + \omega(A_{22}) + \sqrt{(\omega(A_{11}) - \omega(A_{22}))^2 + (\omega(A_{12} + A_{21}) + \omega(A_{12} - A_{21}))^2}\right). \tag{26}$$

Let us observe that

$$A_{11} = \left[\begin{array}{cccc|c} -a_{2n} & -a_{2n-1} & \cdots & -a_{n+2} & -a_{n+1} \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & 0 & \vdots \\ 0 & \cdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]_{n\times n}.$$

Let $d := [-a_{2n} \cdots -a_{n+2}]$. So that, by Lemmas 2 and 3, we have

$$\begin{aligned}\omega(A_{11}) &= \omega\left(\begin{bmatrix} d & -a_{n+1} \\ I & 0 \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} \|d\| & |a_{n+1}| \\ \|I\| & \|0\| \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} \|d\| & \frac{|a_{n+1}|+1}{2} \\ \frac{|a_{n+1}|+1}{2} & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\sqrt{\sum_{k=n+2}^{2n} |a_k|^2} + \sqrt{\sum_{k=n+2}^{2n} |a_k|^2 + (|a_{n+1}|+1)^2}\right). \end{aligned} \tag{27}$$

Now, let $b := [-a_{n-1}, \cdots, -a_2, 1-a_1]$. So that, by Lemma 3, we have

$$A_{12} + A_{21} = \left[\begin{array}{c|cccc} -a_n & -a_{n-1} & \cdots & -a_2 & 1-a_1 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right]_{n\times n},$$

and thus

$$\begin{aligned}\omega(A_{12}+A_{21}) &= \omega\left(\begin{bmatrix} |-a_n| & b \\ 0 & 0 \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} w(|a_n|) & \|b\| \\ 0 & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} w(|a_n|) & \frac{1}{2}\|b\| \\ \frac{1}{2}\|b\| & 0 \end{bmatrix}\right) = \frac{1}{2}\left(|a_n| + \sqrt{|a_n|^2 + \|b\|^2}\right) \\ &= \frac{1}{2}\left(|a_n| + \sqrt{|a_n|^2 + |1-a_1|^2 + \sum_{k=2}^{n-1} |a_k|^2}\right). \end{aligned} \tag{28}$$

Similarly, let $z := [a_{n-1}, \cdots, a_2, 1+a_1]$. So that, by Lemma 3, we have

$$A_{12} - A_{21} = \left[\begin{array}{c|cccc} a_n & a_{n-1} & \cdots & a_2 & 1+a_1 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right]_{n\times n}.$$

So that

$$\begin{aligned}\omega(A_{12}-A_{21}) &= \omega\left(\begin{bmatrix} |a_n| & b \\ 0 & 0 \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} w(|a_n|) & \|z\| \\ 0 & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} w(|a_n|) & \frac{1}{2}\|z\| \\ \frac{1}{2}\|z\| & 0 \end{bmatrix}\right) = \frac{1}{2}\left(|a_n| + \sqrt{|a_n|^2 + \|z\|^2}\right) \\ &= \frac{1}{2}\left(|a_n| + \sqrt{|a_n|^2 + |1+a_1|^2 + \sum_{k=2}^{n-1} |a_k|^2}\right). \end{aligned} \tag{29}$$

Because $\omega(A_{12} - A_{21}) = \cos(\frac{\pi}{n+1})$, by substituting (27)–(29) into (26), we obtain the required result in (25). $\square$

The following example illustrates that the upper bound given in Theorem 4 is better than some famous and recent upper bounds obtained in the literature. Any zero of $q_3(z) = z^6 + \frac{1}{2}z^5 + \frac{1}{16}z^2 + 1$ is bounded by any values given in Table 3 and shows that our presented results could be much better than all compared upper bounds listed in Table 2.

Table 3. Comparisons of upper bounds based on the considered polynomial $q_3(z)$.

Mathematician	Upper Bound
Cauchy (6)	2
Carmichael and Mason (7)	1.501301519
Montel (8)	1.5625
Fujii and Kubo (9)	1.777921993
Abdurakhmanov (10)	1.701542875
Linden (11)	2.350962955
Kittaneh (12)	1.455651176
Abu-omar and Kittaneh (13)	1.857439836
Al-Dolat et al. (14)	2.147748325
Theorem 4	**1.307548659**

Corollary 3. *Under the assumptions of Theorem 4. If $a_k = 0$ for all $k = 2, 3, \cdots, n$ and $a_1 = 1$ (or $a_1 = -1$), then, we have*

$$\omega(C(q)) \leq \frac{1}{2}\left(L + \cos\left(\frac{\pi}{n+1}\right) + \sqrt{\left(L - \cos\left(\frac{\pi}{n+1}\right)\right)^2 + 1}\right). \tag{30}$$

where L is defined in Theorem 4.

Proof. From (20), we have $\omega(A_{12} + A_{21}) = 0$ and $\omega(A_{12} - A_{21}) = 1$, if $a_1 = 1$. Thus, we have

$$(\omega(A_{12} + A_{21}) + \omega(A_{12} - A_{21}))^2 = 1.$$

In addition, we always have $\omega(A_{22}) = \cos(\frac{\pi}{n+1})$. Employing (20), we obtain the desired result in (30). $\square$

Remark 2. *It is convenient to note that* (25) *can be rewritten as*

$$\omega(C(q)) \leq \frac{1}{2}\left(\omega(A_{11}) + \cos\left(\frac{\pi}{n+1}\right) + \sqrt{\left(\omega(A_{11}) - \cos\left(\frac{\pi}{n+1}\right)\right)^2 + (D_1 + D_2)^2}\right).$$

Moreover, if $a_{n+1} \neq 0$ in Theorem 4, one can replace the upper bound of $\omega(A_{11})$ by any other upper bound established in the literature. Indeed, the choice could be as minimal as possible, and this improves our result in (25)*.*

We end this work by giving a new upper bound for the numerical radius of the companion matrix that represents the polynomial $g(z) = z^n + c_n z^{n-1} + c_{n-1} z^{n-2} + \cdots + c_2 z + c_1$ (with $c_1 \neq 0$) of any degree ≥ 2.

Our upper bound is exactly the number L defined in Theorem 4. Namely, $c_n = a_{2n}, c_{n-1} = a_{2n-1}, \cdots, c_2 = a_{n+2}, c_1 = a_{n+1}$. If c_k's are all reals such that $|c_k| \underset{\neq}{<} 1$ and $|c_{k+1}| \underset{\neq}{>} |c_k|$ $(\forall k = 1, 2, \cdots, n-1)$, then, we have

$$\omega(C(g)) \leq \frac{1}{2}\left(\sqrt{\sum_{k=2}^{n}|c_k|^2} + \sqrt{\sum_{k=2}^{n}|c_k|^2 + (|c_1|+1)^2}\right) := \mathbf{MW}, \tag{31}$$

provided that $\sum_{k=2}^{n}|c_k| \geq \frac{2}{3}$. **Otherwise**, the result is still valid even we do not have these assumption(s), i.e., if $|c_k| \geq 1$ for some $k \neq 1$, then (31) it remains always true for both real and complex coefficients. The analysis of the proof is mentioned in the proof of Theorem 4, as stated for $\omega(A_{11})$, under the assumption that $c_1 = a_{n+1} \neq 0$.

6. Observations and Discussion Regarding MW

Let us now discuss the findings of our study on the applied side. First, consider $q_4(z) = z^6 + \frac{1}{4}z^5 + \frac{1}{9}z^4 + \frac{1}{16}z^3 + \frac{1}{25}z^2 + \frac{1}{36}z + \frac{1}{49}$; we find that the largest zero has modulus $= \mathbf{0.5447544053}$. Table 4 shows that our result **MW** is pretty close to the exact modulus and it is much better than all other upper bounds.

Table 4. Comparisons of upper bounds based on the considered polynomial $q_4(z)$.

Mathematician	Upper Bound
Cauchy (6)	1.25
Carmichael and Mason (7)	1.039971167
Montel (8)	1
Fujii and Kubo (9)	1.066738881
Abdurakhmanov (10)	1.198213950
Linden (11)	2.091031073
Kittaneh (12)	1.152835774
Abu-omar and Kittaneh (13)	1.072449189
Al-Dolat et al. (14)	1.573586825
Theorem 4	1.219108946
MW	**0.6721175730**

Another example shows the efficiency of our result (31), consider $q_5(z) = z^6 + \frac{1}{3}z^4 + \frac{1}{4}z^3 + \frac{1}{9}z^2 + \frac{1}{100}$; we find that the largest zero has modulus $= \mathbf{0.7419983061}$. Table 5 shows that our result **MW** is very close to the exact modulus.

Table 5. Comparisons of upper bounds based on the considered polynomial $q_5(z)$.

Mathematician	Upper Bound
Cauchy (6)	1.333333333
Carmichael and Mason (7)	1.089062344
Montel (8)	1
Fujii and Kubo (9)	0.9939972629
Abdurakhmanov (10)	1.167303296
Linden (11)	2.078873251
Kittaneh (12)	1.325435041
Abu-omar and Kittaneh (13)	1.299097566
Al-Dolat et al. (14)	1.581696908
Theorem 4	1.351458429
MW	**0.7647166222**

After careful considerations and investigations, for almost all cases of c_k's (with $|c_k| < 1$), we discovered that (31) is very effective as long as $|c_k| < 1$, besides the other mentioned assumptions. The following example explains why we chose the presented conditions in (31), and thus it solidifies and supports the reason for our selection. Let

$$h_1(z) = z^6 + \frac{1}{6}z^4 + \frac{1}{5}z^2 + \frac{1}{4}.$$

The real coefficients of $h_1(z)$ do not respect our conditions in (31). So, the largest zero has modulus $= 0.8120242973$, but the upper bound in (31) $= 0.7685824855$, which is an incorrect upper bound, implying that (31) does not apply arbitrary for any c_k's unless we have some restriction(s). To ensure the correctness—and after long extrapolation by testing many cases—we established the investigated assumptions in (31) for all polynomials with real coefficients.

In this regard, it is worth noting that the assumptions about c_k's, whenever $|c_k| < 1$, are only valid for real coefficients. However, if some of c_k's is complex, then it is not true that we cannot apply (31). For example, the polynomial

$$h_2(z) = z^6 + \left(\frac{1}{4} + i\frac{1}{4}\right)z^5 + \frac{i}{9}z^4 + \frac{i}{16}z^3 + \frac{1}{25}z^2 + \frac{1}{36}z + \frac{1}{49},$$

refuting three assumptions of (31). Namely, we have that some c_k's is complex, with $\sum_{k=2}^{n} |c_k| = 0.5949422794 \leq \frac{2}{3}$, and $|c_3| < |c_2|$. At the same time, we find that the largest zero has a modulus $= 0.6408240287$. However, the upper bound in (31) $= 0.7337440145$, which means that (31) still can be applied under other conditions as long as $|c_k| < 1$. We are not able to determine the sufficient condition for complex coefficients in the case that $|c_k| < 1$.

Furthermore, the polynomial

$$h_3(z) = z^6 + \frac{1}{4}z^4 + \frac{1}{3}z^2 + \frac{1}{4},$$

is a good example, showing that the conditions $|c_{k+1}| < |c_k|$ and $\sum_{k=2}^{n} |c_k| = 0.5833333333 \leq \frac{2}{3}$ are not necessary but they are sufficient. In other words, the restricted conditions in (31) do not mean that there are no other examples that violate these conditions. Indeed, the largest zero of $h_3(z)$ has modulus $= 0.8310538215$; however, the upper bound in (31) $= 0.8671411790$. Therefore, (31) can be applied even if we do not have our restrictions.

In this matter, we would say our restricted assumptions in (31) are very sufficient and hold correctly over all reals as long as we have the mentioned assumptions in (31); however, they are not necessary, in general. We leave it up to the interested reader to investigate the sufficiency and necessity conditions in the case of $|c_k| < 1$. When some of $|c_k| > 1$, $k \neq 1$, (31) is always valid, regardless of what types of coefficients we have, their arrangement, or their sum. Last but not least, the upper bound (31) has a very high impact efficiency, especially when all $|c_k|$'s are < 1.

7. Conclusions

In this article, inequalities of the numerical radius of Hilbert space operators are introduced. For the first time and to the best of our knowledge, the Cartesian decomposition is used in applications to find upper bounds for the numerical radius of the disk containing zeros of real polynomials. As we showed, we found a new method (bound) to find the upper bounds through the Cartesian decomposition with some sufficient restrictions, and we showed with examples that our method is much better than the other bounds that were established earlier in the literature. The constraints we have assumed are not necessary but are quite sufficient to prove that the zeros of real polynomials lie within a given disk. As for complex polynomials, there is no condition for them. Our method is valid for all these types of polynomials, as obtained in $h_2(z)$. It remains to point out that the restrictions placed on our method are not alone, and we have shown with an example that our method can work without these restrictions, as obtained in $h_3(z)$. For further details and more information, we recommend the reader refer to Section 6. Moreover, we would like to mention that, based on the results obtained in this work, all the zeros of a complex polynomial are located in a new possible rectangle (Theorem 3, other than Kittaneh rectangle Theorem 2), provided with an example. This shows that our proposed rectangle is better (in some cases) than the one proposed in Theorem 2.

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