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Higher-Order Curvatures of Plane and Space Parametrized Curves

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Abstract: We start by introducing and studying two sequences of curvatures provided by the higher-order derivatives of the usual Frenet equation of a given plane curve C. These curvatures are expressed by a recurrence starting with the pair $(0, -k)$ where k is the classical curvature function of C. Moreover, for the space curves, we succeed in introducing three recurrent sequences of curvatures starting with the triple $(-k, 0, \tau)$. Some kinds of helices of a higher order are defined.

Keywords: plane (space) parametrized curve; higher-order curvatures; helices

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1. Introduction

The role played by the curvature function k in the geometry of plane curves is well-known. The early history of this notion can be found in the excellent survey [1] and recently we deformed it to a new curvature, called *hyperbolic*, in the paper [2].

The aim of the present note is to develop a series of curvature functions starting from k. In fact, we obtain two recurrent sequences of curvature functions having as initial terms the zero function and the given k, respectively. The n-pair of curvatures is provided by the derivative of order n of the usual Frenet equation in which we use as main tool the (almost) complex structure J of the plane $\mathbb{R}^2$.

Our approach is contained in the second section, following a review of the Euclidean theory of plane curves; for more specialized topics of this classical theory we recommend the very recent book [3]. We use a matrix version of this theory since J, being a linear endomorphism of $\mathbb{R}^2$, is already expressed as a matrix. The main result of Section 2 expresses the recurrence relation between the higher-order curvatures, and then we derive the first four terms of both sequences. A constant interest consists in the vanishing cases of these new curvatures and we include some computational examples as well.

The third section collects some possible further studies. More precisely, we formulate some open problems concerning higher-order curvatures, e.g., to study higher-order versions of elastic curves.

The last section present the matrix version of the Frenet equation for a space curve $C \subset \mathbb{R}^3$ based on some specific subgroups of $SO(3)$. Unfortunately, the derivative of this Frenet equation involves some other matrices as the initial ones, and then the three-dimensional extension of the second section seems impossible in this procedure. However, we succeed in introducing three recurrent sequences of curvatures starting with the triple $(-k, 0, \tau)$, by gradually deriving the middle Frenet equation. In both cases of plane and space curves, these higher-order curvatures give the possibility to define higher-order helices.

2. Preliminaries on Plane Curves

Let us fix an open interval $I \subseteq \mathbb{R}$ and consider $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)) = x(t)\bar{i} + y(t)\bar{j}, \quad \|r'(t)\| > 0, \quad t \in I. \tag{1}$$

The ambient setting $\mathbb{R}^2$ is a Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = u^1v^1 + u^2v^2, \quad u = (u^1, u^2) \in \mathbb{R}^2, v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2 = \mathbb{C}$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi(u) = i \cdot u = i \cdot (u^1 + iu^2), \quad i = \sqrt{-1}. \tag{3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, r)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = (u^1_0 \cos t - u^2_0 \sin t, u^1_0 \sin t + u^2_0 \cos t) = R(t) \cdot \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix}, \quad t \in \mathbb{R}, \\ r = \|u_0\| = \|(u^1_0, u^2_0)\|, \quad R(t) := \begin{pmatrix} \cos t & -\sin\, t \\ \sin t & \cos\, t \end{pmatrix} \in SO(2) = S^1 \end{cases} \tag{4}$$

and since the rotations $R(t)$ are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2 = |dz|^2$, it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions, i.e., multiples of the square norm: $f_\alpha(x, y) = \alpha(x^2 + y^2)$, $\alpha \in \mathbb{R}$. For an arbitrary vector field $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$, its Lie bracket with ξ is:

$$[X, \xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y}, \tag{5}$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ commutes with *the radial* (or Euler) vector field $E(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, which is also a linear complete vector field having as integral curves the homotheties $\gamma^E_{u_0}(t) = e^t u_0$ for all $t \in \mathbb{R}$.

The Frenet apparatus of the curve C is provided by ([4]):

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|} \in S^1, \quad N(t) = i \cdot T(t) = \frac{1}{\|r'(t)\|}(-y'(t), x'(t)) \in S^1, \\ k(t) = \frac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \frac{1}{\|r'(t)\|^3}\langle r''(t), ir'(t)\rangle = \frac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{6}$$

Hence, if C is naturally parametrized (or parametrized by arc length), i.e., $\|r'(s)\| = 1$ for all $s \in I = (0, L \leq +\infty)$, then $r''(s) = k(s)ir'(s)$. In a complex approach based on $z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$, we have:

$$\begin{cases} k(t) = \frac{1}{|z'(t)|^3} Im(\bar{z}'(t) \cdot z''(t)) = \frac{1}{|z'(t)|} Im\left(\frac{z''(t)}{z'(t)}\right) = \frac{1}{|z'(t)|} Im\left[\frac{d}{dt}(\ln z'(t))\right] \in \mathbb{R}, \\ Re(\bar{z}'(t) \cdot z''(t)) = \frac{1}{2}\frac{d}{dt}\|r'(t)\|^2, \quad f_\alpha(z) = \alpha|z|^2. \end{cases} \tag{7}$$

The multiplication with the complex unit i corresponds to the rotation $J := R\left(\frac{\pi}{2}\right)$; we have also:

$$\frac{d}{dt}R(t) = R\left(t + \frac{\pi}{2}\right) = R(t)R\left(\frac{\pi}{2}\right) = R\left(\frac{\pi}{2}\right)R(t), \tag{8}$$

and the Frenet equations can be unified by means of the column matrix $\mathcal{F}(t) = \begin{pmatrix} T \\ N \end{pmatrix}(t)$ as:

$$\frac{d}{dt}\mathcal{F}(t) = \|r'(t)\|k(t)R\left(-\frac{\pi}{2}\right)\mathcal{F}(t) = -\|r'(t)\|k(t)J\mathcal{F}(t). \tag{9}$$

It is an amazing fact that if the general rotation $R(t)$ belongs to the Lie group $SO(2) = S^1$, its particular values $R\left(\pm\frac{\pi}{2}\right)$ are elements of its Lie algebra $so(2)$ of skew-symmetric 2×2 matrices. In fact, $\{J = R\left(\frac{\pi}{2}\right)\}$ is exactly the basis of $so(2)$.

3. Higher-Order Curvatures from the Iteration of the Frenet Equation

This short note defines new curvature functions for C assumed to be parametrized by arc length; hence, we have a smooth function $\theta : I \to \mathbb{R}$ such that $r'(s) = e^{i\theta(s)}$. It follows that the curvature is generated by this *structural angle function* through:

$$k(s) = \theta'(s). \tag{10}$$

Let us fix also an integer $n \in \mathbb{N}^* := \{1, 2, 3, ...\}$

Definition 1. *The n-order Frenet equation of C is:*

$$\frac{d^n}{ds^n}\mathcal{F}(s) = K_p^n(s)\mathcal{F}(s) + K_o^n(s)J\mathcal{F}(s). \tag{11}$$

with K_p^n the parallel n-curvature and K_o^n the orthogonal n-curvature of C.

We note that the new curvature functions $K_{p;o}^n$ are smooth and from (9) it results the first ones are $K_p^1 = 0$ and $K_o^1 = -k$. More precisely, these curvatures are obtained through a recurrence process expressed by the following theorem.

Theorem 1. *The higher-order curvatures are provided by the following recurrence relation in which I_2 is the 2×2 unit matrix:*

$$\begin{pmatrix} K_p^{n+1} \\ K_o^{n+1} \end{pmatrix} = \frac{d}{ds}\begin{pmatrix} K_p^n \\ K_o^n \end{pmatrix} - kJ\begin{pmatrix} K_p^n \\ K_0^n \end{pmatrix} = \left(I_2\frac{d}{ds} - kJ\right)\begin{pmatrix} K_p^n \\ K_0^n \end{pmatrix}. \tag{12}$$

Working in the complex algebra $\mathbb{C}$ with the sequence of smooth functions $z_n(s) = K_p^n(s) + iK_o^n(s)$, the recurrence $z_{n+1} = z_n' - kiz_n$ follows.

This immediately yields the first two higher-order curvatures:

$$K_p^2 = -k^2 \leq 0, \quad K_o^2 = -k' = -\theta'', \tag{13}$$

$$K_p^3 = -3kk', \quad K_o^3 = k^3 - k'' = (\theta')^3 - \theta''' \tag{14}$$

and then the $o2$-inflection points of C (i.e., the zeros of K_o^2) are exactly its vertices. Note that the unit circle S^1 has an almost periodicity of order three: $K_p^3 = 0 = K_p^1$, $K_0^3 = 1 = -K_0^1$.

Example 1. *Since the particular case of a vanishing curvature deserves attention, we note that $K_o^3 = 0$ if and only if $k(s) = \frac{\sqrt{2}}{s}$ for $s \in (0, +\infty)$; equivalently, $\theta(s) = \sqrt{2}\ln s$. It results in the curve:*

$$C : r(s) = \frac{s}{3}\left(\cos(\sqrt{2}\ln s) + \sqrt{2}\sin(\sqrt{2}\ln s), \sin(\sqrt{2}\ln s) - \sqrt{2}\cos(\sqrt{2}\ln s)\right) \tag{15}$$

and the reparametrization $s = e^{\frac{t}{\sqrt{2}}}$ yields:

$$C : r(t) = \frac{e^{\frac{t}{\sqrt{2}}}}{3}(\cos t + \sqrt{2}\sin t, \sin t - \sqrt{2}\cos t), \quad r'(t) = \frac{1}{\sqrt{2}}e^{\left(\frac{1}{\sqrt{2}}+i\right)t}. \tag{16}$$

We note in passing that the minimal polynomial of the complex number $\frac{1}{\sqrt{2}} + i$ is $P(X) = 4X^4 + 4X^2 + 9$. Recall that a logarithmic spiral has the polar parametrization $\rho_{R,\alpha}(t) = Re^{\alpha t}$ with $R > 0$ and $\alpha > 0$ as its generators. It follows that the derivative curve C' is the logarithmic spiral with $R = \alpha = \frac{1}{\sqrt{2}}$.

Remark 1. *More generally, due to the presence of the third derivative of* θ *in the expression of* K^3_o*, we recall its Schwarzian derivative* ([5])*:*

$$S_\theta = \frac{\theta'''}{\theta'} - \frac{3}{2}\left(\frac{\theta''}{\theta'}\right)^2 = \frac{\theta'''}{k} - \frac{3}{2}\left(\frac{k'}{k}\right)^2 \tag{17}$$

and then we obtain a new formula for K^3_0*:*

$$K^3_o = k\left(k^2 - \frac{3}{2}\left(\frac{k'}{k}\right)^2 - S_\theta\right). \tag{18}$$

The next pair of curvatures is:

$$K^4_p = -4kk'' - 3(k')^2 + k^4, \quad K^4_o = 6k^2k' - k''' = k'\left(6k^2 - \frac{3}{2}\left(\frac{k''}{k'}\right)^2 - S_k\right) \tag{19}$$

and then the problem of a vanishing K^4_o becomes the following example.

Example 2. *We have* $K^4_o = 0$ *if and only if* $k(s) = \frac{1}{s}$*, having also* $S_k = 0$*; equivalently,* $\theta = \ln s$ *which yields the curve:*

$$C : r(s) = \frac{s}{2}(\sin(\ln s) + \cos(\ln s), \sin(\ln s) - \cos(\ln s)) \tag{20}$$

and the reparametrization $s = e^t$ *gives the curve:*

$$C : r(t) = \frac{e^t}{2}(\sin t + \cos t, \sin t - \cos t), \quad r'(t) = e^{(1+i)t}. \tag{21}$$

Now, the derivative curve C' *is the logarithmic spiral with* $R = \alpha = 1$*.*

Example 3. *For the well-known catenary curve:*

$$C : r(s) = (\ln(s + \sqrt{1+s^2}), \sqrt{1+s^2}). \quad \theta(s) = \arctan s \tag{22}$$

we have:

$$k(s) = \frac{1}{1+s^2} > 0, \quad k'(s) = -\frac{2s}{(1+s^2)^2} \leq 0, \quad k''(s) = \frac{2(3s^2-1)}{(1+s^2)^2} \tag{23}$$

and then:

$$K^3_p(s) = \frac{6s}{(1+s^2)^3} \geq 0, \quad K^3_o(s) = \frac{3(1-2s^2)}{(1+s^2)^3}. \tag{24}$$

There exists only one o3-inflection point $P \in C$ *corresponding to* $s = \frac{1}{\sqrt{2}}$ *which gives the point* $P\left(\frac{\ln(2+\sqrt{3})}{2}, \sqrt{\frac{3}{2}}\right) \in C$*.*

Example 4. *Fix* $\alpha \in \mathbb{R}^*$*. The* α*-clothoid, also called the Cornu spiral, is the curve:*

$$C : r(s) = \left(\int_0^s \cos\frac{\alpha u^2}{2}du, \int_0^s \sin\frac{\alpha u^2}{2}du\right), \tag{25}$$

having the linear curvature $k(s) = \alpha s$*. Then,*

$$\begin{cases} K^2_p(s) = -\alpha^2 s^2 \leq 0, \quad K^2_o = -\alpha^2 < 0, \\ K^3_p(s) = -3\alpha^2 s, K^3_o(s) = \alpha^3 s^3, K^4_p(s) = \alpha^2(\alpha^2 s^4 - 3), K^4_o(s) = 6\alpha^3 s^2 \end{cases} \tag{26}$$

and there exists a p4-inflection point $s = \frac{\sqrt[4]{3}}{\sqrt{|\alpha|}}$.

4. Further Studies

3.1 An important tool in dynamics described by curves is the Fermi–Walker derivative. Let $\mathcal{X}_C$ be the set of vector fields along the curve C. Then, the Fermi–Walker derivative is the map ([6]) $\nabla_C^{FW} : \mathcal{X}_C \to \mathcal{X}_C$:

$$\nabla_C^{FW}(X) := \frac{d}{dt}X + \|r'(\cdot)\|k[\langle X, N\rangle T - \langle X, T\rangle N] = \frac{d}{dt}X + \|r'(\cdot)\|k[X^\flat(N)T - X^\flat(T)N] \tag{27}$$

with $X^\flat$ the differential 1-form dual to X with respect to the Euclidean metric. In a matrix form, we can express this as follows:

$$\nabla_C^{FW}(\cdot) = \frac{d}{dt} - \|r'\|k \begin{vmatrix} (\cdot)^\flat(T) & (\cdot)^\flat(N) \\ T & N \end{vmatrix} = \frac{d}{dt} + \|r'\|k \begin{vmatrix} T & (\cdot)^\flat(T) \\ N & (\cdot)^\flat(N) \end{vmatrix}. \tag{28}$$

For example, the Frenet frame is Fermi–Walker parallel:

$$\nabla_C^{FW}(\mathcal{F}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{29}$$

Using the new functions, we define two sequences of Fermi–Walker derivatives for naturally parametrized plane curves:

$$\begin{cases} \nabla_{C,p}^{FW,n}(X) := \frac{d}{ds}X + K_p^n[\langle X, N\rangle T - \langle X, T\rangle N], \\ \nabla_{C,o}^{FW,n}(X) := \frac{d}{ds}X + K_o^n[\langle X, N\rangle T - \langle X, T\rangle N]. \end{cases} \tag{30}$$

A first open problem is to study the role of these derivatives in the geometry of C.

3.2 We can connect our study with some previous papers in the geometry of space curves. More precisely, we associate to our plane curve C a sequence $\tilde{C}_n$ of curves in $\mathbb{R}^3$ having the function $(-K_o^n)$ as the curvature $\tilde{k}_n$ and the function K_p^n as the torsion $\tilde{\tau}_n$. A second open problem is to recast in this way the sequence of space curves introduced in [7].

Concerning this approach, we can introduce a class of special plane curves as follows:

Definition 2. *Let $\omega \in \mathbb{R}^*$ be a fixed number. We call the curve $C \subset \mathbb{R}^2$ as being an ω-helix of order $n \in \mathbb{N}^*$ if the following two conditions hold: (i) the torsion $\tilde{\tau}_n = K_p^n$ is different from zero; (ii) the Lancret function of $\tilde{C}_n$, namely, $L_n(C) := \frac{\tilde{k}_n}{\tilde{\tau}_n} = -\frac{K_0^n}{K_p^n}$, is equal to ω.*

We note that the Lancret invariant of slant curves in three-dimensional warped products was expressed in the paper [8]. For the present approach it follows that

$$L_2(C) = -\frac{k'}{k^2} = \left(\frac{1}{k}\right)', \quad L_3(C) = \frac{k^3 - k''}{3kk'} \tag{31}$$

and then we have the following proposition.

Proposition 1. *The curve $C \subset \mathbb{R}^2$ is an ω-helix of order two if and only if $k(s) = \frac{1}{\omega s + \omega_0}$ with $\omega_0 \in \mathbb{R}$ an arbitrary constant; equivalently, $\theta(s) = \frac{1}{\omega}\ln(\omega s + \omega_0)$. The curve C is an ω-helix of order three if and only if $k(s) = \frac{\mathcal{C}_\pm(\omega)}{s}$ with $\mathcal{C}_\pm(\omega) := \frac{-3\omega \pm \sqrt{9\omega^2+8}}{2}$.*

Hence, a third open problem is to study/obtain higher-order helices. We note that the equation $L_3(C) = \omega$ is somehow similar to equation (26) from [9].

3.3 A fourth open problem is to study/find higher-order elastic curves by following the approach of [10].

5. The Case of Three-Dimensional Curves

Suppose now that the given curve is a space one, $C \subset \mathbb{R}^3$, having the curvature function $k = k(s)$ and the torsion function $\tau = \tau(s)$. Consider the matrices:

$$\begin{cases} R_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin\, t \\ 0 & \sin t & \cos\, t \end{pmatrix} \in \{1\} \times SO(2), \\ R_2(t) = \begin{pmatrix} \cos t & 0 & \sin\, t \\ 0 & 1 & 0 \\ \sin t & 0 & -\cos t \end{pmatrix} \in O^-(3) := O(3) \setminus SO(3) = \{\Gamma \in O(3); \det\Gamma = -1\}, \\ R_3(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(2) \times \{1\}. \end{cases} \tag{32}$$

The Frenet frame

$$\mathcal{F}(s) := \begin{pmatrix} T \\ N \\ B \end{pmatrix}(s) \tag{33}$$

satisfies the matrix Frenet equation

$$\frac{d}{ds}\mathcal{F}(s) = -[\tau(s)R_1'(0) + k(s)R_3'(0)]\mathcal{F}(s), \tag{34}$$

which recalls the expression of the associated Darboux vector field:

$$\Omega(s) := \tau(s)T(s) + k(s)B(s). \tag{35}$$

Deriving (34) we get:

$$\frac{d^2}{ds^2}\mathcal{F} = [-\tau' R_1'(0) - k' R_3'(0) + (\tau R_1'(0) + kR_3'(0))^2]\mathcal{F}. \tag{36}$$

We have:

$$[R_1'(0)]^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \{0\} \times SO(2), [R_3'(0)]^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in SO(2) \times \{0\}, \tag{37}$$

$$R_1'(0)R_3'(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, R_3'(0)R_1'(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R_1'(0)R_3'(0) + R_3'(0)R_1'(0) = R_2'(0). \tag{38}$$

The occurrence of $R_2'(0)$ in expression (36) and the fact that $O^-(3)$ is not a subgroup in $O(3)$ make the attempt to find some higher-order curvatures very difficult in the present manner, so we look for an alternative way.

More precisely, we consider the middle Frenet equation $N'(s) = -k(s)T(s) + \tau(s)B(s)$ as the first rung of a ladder:

$$\frac{d^n}{ds^n}N = K_T^n T + K_N^n N + K_B^n B, \quad n \geq 1, \quad K_T^1 = -k, \quad K_N^1 = 0, \quad K_B^1 = \tau \tag{39}$$

and then we get immediately the following theorem.

Theorem 2. *The recurrence relation of the curvature's triple (K_T, K_N, K_B) is:*

$$\begin{pmatrix} K_T^{n+1} \\ K_N^{n+1} \\ K_B^{n+1} \end{pmatrix} = \frac{d}{ds}\begin{pmatrix} K_T^n \\ K_N^n \\ K_B^n \end{pmatrix} - \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} K_T^n \\ K_N^n \\ K_B^n \end{pmatrix}. \tag{40}$$

Therefore, we have the first two higher-order triples:

$$\begin{cases} (K_T^2, K_N^2, K_B^2) = (-k', -(k^2+\tau^2), \tau'), \\ (K_T^3, K_N^3, K_B^3) = (k(k^2+\tau^2) - k'', -3(kk' + \tau\tau'), \tau'' - \tau(k^2+\tau^2)). \end{cases} \quad (41)$$

Remark 2. *(i) If C reduces to a plane curve, then $\tau = 0$ implies $K_T^1 = -k = K_o^1$, $K_T^2 = K_o^2$ and $K_T^3 = K_o^3$, which shows that this approach is the correct one. Moreover, $K_p^1 = K_N^1$, $K_p^2 = K_N^2$ and $K_p^3 = K_N^3$.*

(ii) Suppose that C is a Mannheim curve; then, according to relation (2) of [7], we have $k = \lambda(k^2+\tau^2)$ for a nonzero constant λ. Then, the curvatures (41) become:

$$(K_T^2, K_N^2, K_B^2) = \left(-k', -\frac{k}{\lambda}, \tau'\right), \quad (K_T^3, K_N^3, K_B^3) = \left(\lambda k^2 - k'', -\frac{3k'}{2\lambda}, \tau'' - \lambda k\tau\right). \quad (42)$$

Definition 3. *Let $\omega \in \mathbb{R}^*$ be a fixed number. We call the curve $C \subset \mathbb{R}^3$ an ω-helix of order $n \in \mathbb{N}^*$, if the following two conditions hold: (i) the n-tangential curvature K_T^n is different from zero; (ii) the n-Lancret function of C, namely, $Lancret_n(C) := -\frac{K_B^n}{K_T^n}$, is equal to ω.*

Example 5. *The circular helix $C_{a,b} : r(t) = (a\cos t, a\sin t, bt)$ with the parameters $a, b \in (0, +\infty)$ is a well-known 1-helix and $t = \frac{s}{\sqrt{a^2+b^2}}$. It is not a 2-helix, but it is 3-helix with $Lancret_3(C_{a,b}) = Lancret_1(C_{a,b}) = \frac{b}{a}$.*

Finally we remark that the first derivatives of the Darboux vector field also satisfy the following interesting relations:

$$\Omega' = \tau' T + k' B, \quad \Omega'' = \tau'' T + (\tau' k - k'\tau)N + k'' B, \quad (43)$$

$$\Omega''' = \tau''' T + 2(\tau'' k - k''\tau)N + k''' B. \quad (44)$$

6. Conclusions

The main idea of the present study was that behind the usual curvature, such as the Euclidean geometry invariant of plane curves, there were a lot of secondary curvatures, which described the higher-order derivatives of the position vector $r = r(s)$ of a given plane (or space) curve. These higher-order curvatures are useful for a Taylor series of the vectorial function $r(s)$.

Moreover, using these curvatures, we introduced new classes of interesting curves as natural generalizations of helices or elastic curves, and there is hope to develop some possible practical technical or design applications.

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