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Containment Control of First-Order Multi-Agent Systems under PI Coordination Protocol †

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Abstract: This paper investigates the containment control problem of discrete-time first-order multi-agent system composed of multiple leaders and followers, and we propose a proportional-integral (PI) coordination control protocol. Assume that each follower has a directed path to one leader, and we consider several cases according to different topologies composed of the followers. Under the general directed topology that has a spanning tree, the frequency-domain analysis method is used to obtain the sufficient convergence condition for the followers achieving the containment-rendezvous that all the followers reach an agreement value in the convex hull formed by the leaders. Specially, a less conservative sufficient condition is obtained for the followers under symmetric and connected topology. Furthermore, it is proved that our proposed protocol drives the followers with unconnected topology to converge to the convex hull of the leaders. Numerical examples show the correctness of the theoretical results.

Keywords: containment control problem; first-order multi-agent systems; PI coordination control; containment-rendezvous

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1. Introduction

Distributed coordination control of multi-agent systems has been an active research area for its widespread potential engineering applications including cooperative surveillance, sensor networks, spacecraft formation flying, etc. The most fundamental problem studied in coordination control is consensus problem [1–11] which means that each agent reaches an agreement based on the information from its relative agents. According to the different number of leaders in the multi-agent system, the consensus problem is generally divided into leaderless case [12,13], single-leader–follower case [14–16], and multiple-leader–follower case.

The containment control problem [17–24] is considered as a special multiple-leader–follower consensus problem for multi-agent system with multiple leaders and followers, and it requires all followers to converge into the convex hull spanned by leaders. Many valuable results on the containment control problem of first-order multi-agent systems, which are investigated in this paper, have been obtained in recent decades. Basic results for realizing containment of continuous-time first-order multi-agent systems with stationary leaders have been given by Liu in [25], and the convergence conditions under fixed and switching topologies are dependent on the topology structure. Wang and their colleagues [26] investigated the containment problem of first-order multi-agent system with communication noises, and designed a time-varying gain to reduce noises. A PD-type control protocol was introduced and a parameter condition was given to guarantee the containment achievement under input delays in Rong's work [27]. Mu and partners[28] provided necessary and sufficient criteria for containment convergence if the communication data rates are limited. Miao and colleagues [29] applied the event-triggered scheme into containment control algorithms, and the convergence problem with single time delay and multiple time delays were discussed, respectively. Considering the discrete-time first-order multi-agent systems. The authors in [30] adopted the asynchronous containment control protocol and got sufficient conditions for reaching containment. Additionally, containment problem of multi-agent systems with second-order dynamics and general linear dynamics has also attracted extensive attention [31–34].

According to aforementioned works, we note that general containment control protocols make each follower finally converge to different point in the convex hull spanned by leaders. General containment problem only focuses on the containment and ignores the consensus, which is called rendezvous in this paper, of the followers. If we take both containment and consensus into consideration, a different control protocol is required, and we define this problem as a containment-rendezvous problem in this paper. Actually, the average-tracking problem of reference signals is a typical containment-rendezvous problem, if we regard the reference signals as leaders. Shan and Liu [35] considered the average-tracking problem of first-order multi-agent systems with unmatched reference signals, and used the frequency-domain analysis method, which will be adopted in this paper, to get the convergence condition. On the basis of average-consensus algorithm, Chung [36] designed a containment-rendezvous algorithm that made all followers track one point in the convex hull of first-order agents, but the results depended on the balance property of communication topology.

Inspired by above works, this paper focuses on the containment-rendezvous problem of discrete-time first-order multi-agent systems under general communication topology. On the basis of the average-consensus algorithm, a proportional-integral (PI) coordination control protocol is proposed for reaching containment-rendezvous. We first analyze the protocol for the followers under general connected topology, and obtain a sufficient convergence condition. Besides, we consider the followers with symmetric and connected topology and get a less conservative convergence condition which has been simply discussed in our initial work [37]. Furthermore, we investigate the followers under unconnected topology that is regarded as a union of several connected parts, and obtain the convergence condition according to that of general connected topology.

This paper is organized as follows. We give some basic concepts about graph theory, the multi-agent systems, and the coordination control protocol in Section 2. The convergence of the protocol under general connected topology, symmetric and connected topology and unconnected topology of the followers is proved in Section 3. In Sections 4 and 5, the numerical simulations and the conclusion are presented, respectively.

2. Problem Formulation

2.1. Graph Theory

Consider a multi-agent system with n agents is denoted by a graph $G(V,E)$, where $V = \{1,2,\ldots,n\}$ and $E \subseteq V \times V$ stand for the vertex set and edge set, respectively. An edge $(i,j) \in E$ represents that agent j is able to access information of agent i and means that vertex i is a neighbor of vertex j. If agent j has no neighbor, it is called a leader, otherwise, it is a follower. The index set is denoted as $N_j = \{i \in V : (i,j) \in E, i \neq j\}$. A directed path from i to j is a sequence of edges in a graph of the form $(i,h_0),(h_1,h_2),\ldots,(h_k,j)$, where $h_k \in V$. The adjacency matrix is a nonnegative matrix $A = [a_{ij}] \in R^{n\times n}$ defined as $a_{ji} > 0$ if $(i,j) \in E$, and $a_{ji} = 0$ otherwise. Furthermore, self edges are not allowed in this paper, i.e., $a_{ii} = 0$. The Laplacian matrix is defined as $L = [l_{ij}] \in R^{n\times n}$, where $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$.

A directed graph is called a directed tree if each node in graph has exactly one parent except for one node which is called the root, and the root has directed paths to each other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph has a spanning tree if there exists a directed spanning tree as a subset of the directed graph.

2.2. Agents' Dynamics and Coordination Protocol

Investigate a discrete-time multi-agent system consisting of m leaders and $n-m$ followers labelled by $1,\ldots,m$ and $m+1,\ldots,n$, respectively. The dynamic model of agent i is given by

$$\begin{aligned} &x_i(k+1) = x_i(k), i = 1,\ldots,m, \\ &x_i(k+1) = x_i(k) + u_i(k), i = m+1,\ldots,n, \end{aligned} \tag{1}$$

where $x_i(k) \in R^p$ and $u_i(k) \in R^p$ are the state and control input of agent i, respectively.

According to the PI control strategy, we use the following PI coordination control algorithm for the first-order agents,

$$\begin{aligned} &u_i(k) = \gamma_1 \sum_{j=1}^{n} a_{ij}(x_j(k) - x_i(k)) + \gamma_2 r_i(k), \\ &r_i(k+1) = r_i(k) + \sum_{j=m+1}^{n} a_{ij}(x_j(k) - x_i(k)), \\ &i = m+1,\ldots,n, \end{aligned} \tag{2}$$

where a_{ij} is the (i,j) entry of the adjacency matrix A, $r_i(k)$ represents the integral term, γ_1 and γ_2 are positive gain parameters to be decided for the proportional term and integral term, respectively.

The states of leaders (1) remain static since the inputs of leaders are always zero, so we only investigate the followers' dynamics here. With the protocol (2), the dynamics of follower i are rewritten as

$$\begin{aligned} &x_i(k+1) = x_i(k) + \gamma_1 \sum_{j=1}^{n} a_{ij}(x_j(k) - x_i(k)) + \gamma_2 r_i(k), \\ &r_i(k+1) = r_i(k) + \sum_{j=m+1}^{n} a_{ij}(x_j(k) - x_i(k)), \\ &i = m+1,\ldots,n. \end{aligned} \tag{3}$$

In order to analyze the convergence performance of system (3), we introduce two topologies, one of which is named as leader–follower topology composed of the leaders and followers, and the other one is named as follower topology composed of the followers. The Laplacian matrix of leader–follower topology is L given by

$$L = \begin{bmatrix} 0_{m\times m} & 0_{m\times(n-m)} \\ L_1 & L_2 \end{bmatrix},$$

where L_1 represents the topology between leaders and followers, and L_2 represents the topology among followers. Meanwhile, the Laplacian matrix corresponding to the follower topology is L_F formulated as

$$L_F = L_2 - D,$$

where the diagonal matrix D is defined as

$$D = diag\{\sum_{j=1}^{m} a_{m+1,j},\ldots,\sum_{j=1}^{m} a_{n,j}\}.$$

Generally, we need the basic assumption on the leader–follower topology of system (3) as follows.

Assumption 1. *For each of the followers in the leader–follower topology, there is at least one leader that has a directed path to the follower.*

On the basis of Assumption 1, we make further assumptions on the follower topology as follows.

Assumption 2. *The follower topology has a directed spanning tree.*

Assumption 3. *The edges of the follower topology are bidirectional, i.e., Laplacian matrix L_F is symmetric, and it has a directed spanning tree.*

Then we have the following lemma.

Lemma 1 ([38])**.** *The Laplacian matrix L_F of the follower topology has a simple eigenvalue 0 with the corresponding eigenvector $[1, 1, \ldots, 1]^T$ and all the other eigenvalues have positive real parts if and only if the topology satisfies Assumption 2.*

3. Main Results

System (3) is reformulated in a vector form as

$$\begin{aligned} X_F(k+1) =&((I_{n-m} - \gamma_1 L_2) \otimes I_p) X_F(k) \\ &- \gamma_1 (L_1 \otimes I_p) X_L(k) + \gamma_2 R(k), \\ R(k+1) =&R(k) - (L_F \otimes I_p) X_F(k), \end{aligned} \tag{4}$$

where $X_L(k) = [x_1^T(k), \ldots, x_m^T(k)]^T$, $X_F(k) = [x_{m+1}^T(k), \ldots, x_n^T(k)]^T$, $R(k) = [r_{m+1}^T(k), \ldots, r_n^T(k)]^T$.

Define $\hat{R}(k) = R(k) - \frac{\gamma_1}{\gamma_2}(L_1 \otimes I_p) X_L(k)$, and we get

$$\begin{aligned} X_F(k+1) &= (I_{n-m} - \gamma_1 L_2 \otimes I_p) X_F(k) + \gamma_2 \hat{R}(k), \\ \hat{R}(k+1) &= \hat{R}(k) - (L_F \otimes I_p) X_F(k). \end{aligned} \tag{5}$$

Let $Y(k) = [X_F^T(k), \hat{R}^T(k)]^T$, and the system (5) is expressed in a compact form as

$$Y(k+1) = (\begin{bmatrix} I_{n-m} - \gamma_1 L_1 & \gamma_2 I_{n-m} \\ -L_F & I_{n-m} \end{bmatrix} \otimes I_p) Y(k). \tag{6}$$

To continue the convergence analysis of system (6), some useful lemmas are listed firstly.

Lemma 2 ([39])**.** *Let $P(z)$ be a polynomial of order two with complex coefficients in the form of $P(z) = z^2 + (p_1 + \mathrm{j} q_1) z + p_2 + \mathrm{j} q_2$, where j is the imaginary unit. The polynomial $P(z)$ has all its zeros in the open left half of the z-complex plane if and only if $p_1 > 0$ and $p_1^2 p_2 + p_1 q_1 q_2 - q_2^2 > 0$.*

Lemma 3 ([35])**.** *If Q_4 is invertible,* $\det \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} = \det(Q_4) \det(Q_1 - Q_2 Q_4^{-1} Q_3)$.

Lemma 4 ([40])**.** *Let $Q \in C^{n \times n}$, $Q = Q^* \geq 0$ and $T = diag\{t_i, t_i \in C\}$, then*

$$\lambda(QT) \in \rho(Q) Co(0 \cup \{t_i\}),$$

where $\lambda(\cdot)$ denotes matrix eigenvalue, $\rho(\cdot)$ denotes matrix spectral radius and $Co(\cdot)$ denotes the convex hull.

Next, we will obtain the convergence conditions of the system (6) according to different follower topologies.

3.1. General Directed Follower Topology

Theorem 1. *Consider the multi-agent system (3) with leader–follower and follower topologies satisfying Assumptions 1 and 2, respectively. With condition $\sum_{i=m+1}^{n} q_i r_i(0) = 0$, all the followers reach containment-rendezvous asymptotically that the followers converge to an agreement value in the convex hull spanned by the leaders, if γ_1 and γ_2 satisfy*

$$\gamma_1 - \gamma_2 > 0,$$

$$(\gamma_1 - \gamma_2)^2(\gamma_2 - 2\gamma_1 + \frac{4Re(\kappa_i)}{|\kappa_i|^2}) - \frac{4\gamma_2 Im(\kappa_i)^2}{|\kappa_i|^4} > 0,$$

for $i = m+1, \dots, n$, and

$$|\lambda(\gamma_2 D \Theta^{-1}(e^{j\omega}))| < 1 \tag{7}$$

hold with $\omega \in (0, \pi]$, where $\Theta(e^{j\omega}) = (e^{j\omega} - 1)^2 I + \gamma_1(e^{j\omega} - 1)(L_F + D) + \gamma_2(L_F + D)$ and $\kappa_i, i = m+1, \dots, n$ represent the eigenvalues of the matrix $L_F + D$.

Proof. According to the properties of the Kronecker product, we set the agents' state dimension as $p = 1$ in the following proof, and system (6) is written as

$$\hat{Y}(k+1) = \begin{bmatrix} I_{n-m} - \gamma_1(L_F + D) & \gamma_2 I_{n-m} \\ -L_F & I_{n-m} \end{bmatrix} \hat{Y}(k). \tag{8}$$

Meanwhile, we divide the proof into two steps including convergence analysis and the analysis of final rendezvous state.

Step 1: To analyze the convergence performance of system (8), we investigate the characteristic equation of (8) as follows,

$$\det(zI - \begin{bmatrix} I - \gamma_1(L_F + D) & \gamma_2 I \\ -L_F & I \end{bmatrix}) = 0, \tag{9}$$

and it can be reformulated from Lemma 3 as

$$\det((z-1)^2 I + \gamma_1(z-1)(L_F + D) + \gamma_2 L_F) = 0. \tag{10}$$

Evidently, it is obtained from Lemma 1 that the Equation (10) has a root at $z = 1$. Before analyzing Equation (10), we first pay attention to the following equation

$$\det((z-1)^2 I + \gamma_1(z-1)(L_F + D) + \gamma_2(L_F + D)) = 0,. \tag{11}$$

Obviously, (11) is equivalent to

$$z^2 + (\gamma_1\kappa_i - 2)z + (\gamma_2 - \gamma_1)\kappa_i + 1 = 0, i = m+1, \dots, n. \tag{12}$$

Instead of studying Equation (12) directly, we apply the bilinear transformation $s = \frac{z+1}{z-1}$ to it and get

$$\gamma_2\kappa_i s^2 + 2(\gamma_1 - \gamma_2)\kappa_i s + (\gamma_2 - 2\gamma_1)\kappa_i + 4 = 0. \tag{13}$$

Thus, Equation (12) has all roots within the unit circle, if and only if Equation (13) has all roots in the open left half complex plane. Then, we reformulate Equation (13) as

$$s^2 + (p_1 + jq_1)s + p_{2i} + jq_{2i} = 0, i = m+1, \dots, n, \tag{14}$$

where $p_1 = \frac{2(\gamma_1 - \gamma_2)}{\gamma_2}$, $q_1 = 0$, $p_{2i} = \frac{\gamma_2 - 2\gamma_1}{\gamma_2} + \frac{4Re(\kappa_i)}{\gamma_2|\kappa_i|^2}$, $q_{2i} = -\frac{4Im(\kappa_i)}{\gamma_2|\kappa_i|^2}$.

Based on Lemma 2, the polynomial has all zeros in the open left half complex plane if and only if the gains γ_1 and γ_2 satisfy

$$\begin{aligned} &p_1 > 0, \\ &p_1^2 p_{2i} + p_1 q_1 q_{2i} - q_{2i}^2 > 0, i = m+1, \ldots, n. \end{aligned} \tag{15}$$

which is equivalent to

$$\begin{aligned} &\gamma_1 - \gamma_2 > 0, \\ &(\gamma_1 - \gamma_2)^2 (\gamma_2 - 2\gamma_1 + \frac{4Re(\kappa_i)}{|\kappa_i|^2}) - \frac{4\gamma_2 Im^2(\kappa_i)}{|\kappa_i|^4} > 0, \\ &i = m+1 \ldots, n. \end{aligned} \tag{16}$$

Thus, the roots of Equation (11) all lie inside the unit circle if and only if there exist gain parameters γ_1 and γ_2 satisfying condition (16).

Back to Equation (10), we reformulate it as

$$\det(I - \gamma_2 D \Theta^{-1}(z)) = 0, \tag{17}$$

where $\Theta(z) = (z-1)^2 I + \gamma_1 (z-1)(L_F + D) + \gamma_2 (L_F + D)$. Under condition (16), it is obvious that if condition $|\lambda(\gamma_2 D \Theta^{-1}(e^{j\omega}))| < 1$ holds with $\omega \in (0, \pi]$, the roots of Equation (17) lie inside the unit circle except for one root at $z = 1$. Thus, the characteristic Equation (9) has all roots within or on the unit circle and we have finally proved the asymptotic convergence of the system under Assumption 1, i.e.

$$\lim_{k \to \infty} \hat{Y}(k) = \lim_{k \to \infty} [X_F^T(k), \hat{R}^T(k)]^T = [X_d^T, \hat{R}_d^T]^T, \tag{18}$$

where $X_d \in R^{n-m}$ and $\hat{R}_d \in R^{n-m}$ are constant vectors.

Step 2: We will prove that all followers reach an agreement value in convex hull spanned by leaders.

According to Equation (4), we obtain with $p = 1$

$$\lim_{k \to \infty} L_F X_F(k) = 0. \tag{19}$$

Under Assumption 1,we get from Lemma 1 and Equation (19)

$$\lim_{k \to \infty} x_i(k) = x_d, i = m+1, \ldots, n, \tag{20}$$

where $x_d \in R$ is a constant. From Equation (20), it is clear that followers converge to the same value. Taking the z transformation of Equation (4) with $p = 1$, we get

$$\begin{aligned} zX_F(z) - zX_F(0) =& (I - \gamma_1 L_2) X_F(z) + \gamma_2 R(z) \\ & - \frac{z}{z-1} \gamma_1 L_1 X_L. \\ zR(z) - zR(0) =& R(z) - L_F X_F(z) \end{aligned} \tag{21}$$

Re-express Equation (21) as

$$\begin{aligned} X_F(z) =& [(z-1)I + \gamma_1 L_2 + \gamma_2 \frac{L_F}{z-1}]^{-1} \\ & [zX_F(0) + \frac{z}{z-1}\gamma_2 R(0) - \frac{z}{z-1}\gamma_1 L_1 X_L] \\ =& [(z-1)I + \gamma_1 D + (\gamma_1 + \frac{\gamma_2}{z-1}) L_F]^{-1} \\ & [zX_F(0) + \frac{z}{z-1}\gamma_2 R(0) - \frac{z}{z-1}\gamma_1 L_1 X_L]. \end{aligned} \tag{22}$$

Since the convergence of the system has been proved, using the final value theorem yields

$$\begin{aligned}\lim_{z\to 1}(z-1)X_F(z) =&[(z-1)I+\gamma_1 D+(\gamma_1+\frac{\gamma_2}{z-1})L_F]^{-1}\\ &[z(z-1)X_F(0)+z\gamma_2 R(0)-z\gamma_1 L_1 X_L]\\ =&x_d[1,1,\dots,1]^T.\end{aligned} \tag{23}$$

Letting $W(z)=(z-1)X_F(z)$, we have $\lim_{z\to 1} W(z)=x_d[1,1,\dots,1]^T$ and

$$\begin{aligned}&[(z-1)I+\gamma_1 D+(\gamma_1+\frac{\gamma_2}{z-1})L_F]W(z)\\ &=[z(z-1)X_F(0)+\gamma_2 zR(0)-z\gamma_1 L_1 X_L].\end{aligned} \tag{24}$$

Multiplying $Q=[q_{m+1},q_{m+2},\dots,q_n]$ on both sides of (24), we get

$$\begin{aligned}&Q[(z-1)I+\gamma_1 D]W(z)\\ &=Q[z(z-1)X_F(0)+\gamma_2 zR(0)-z\gamma_1 L_1 X_L],\end{aligned} \tag{25}$$

where Q is the left eigenvector of L_F corresponding to eigenvalue 0. Then, we take the limit of Equation (25) as $z\to 1$ and have

$$Q\gamma_1 Dx_d[1,1,\dots,1]^T=Q[\gamma_2 R(0)-\gamma_1 L_1 X_L]. \tag{26}$$

Equation (26) is rewritten as

$$\begin{aligned}\gamma_1 x_d\sum_{i=m+1}^{n}\sum_{j=1}^{m} q_i a_{ij} =&\gamma_1\sum_{i=m+1}^{n}\sum_{j=1}^{m} q_i a_{ij} x_j\\ &+\gamma_2\sum_{i=m+1}^{n} q_i r_i(0),\end{aligned} \tag{27}$$

and we finally get

$$x_d=\frac{\sum_{i=m+1}^{n}\sum_{j=1}^{m} q_i a_{ij} x_j}{\sum_{i=m+1}^{n}\sum_{j=1}^{m} q_i a_{ij}}, \tag{28}$$

with $\sum_{i=m+1}^{n} q_i r_i(0)=0$. Apparently, x_d is in the convex hull formed by leaders.

When $p\neq 1$, the proof is the same while the final state x_d is a constant vector instead of a constant. Hence, we can draw the conclusion that all followers will eventually reach an agreement value in the convex hull spanned by leaders under our proposed protocol. Theorem 1 is proved. □

3.2. Symmetric Follower Topology

By means of some existing results [40,41], we present the convergence condition, which is less conservative than (7), of system (3) under symmetric follower topology.

Theorem 2. *The leader–follower and follower topologies of multi-agent system (3) satisfy Assumptions 1 and 3, respectively. With $\sum_{i=m+1}^{n} r_i(0)=0$, all the followers converge to an agreement value $\frac{\sum_{i=m+1}^{n}\sum_{j=1}^{m} a_{ij}x_j}{\sum_{i=m+1}^{n}\sum_{j=1}^{m} a_{ij}}$ that lies in convex hull spanned by the leaders, if γ_1 and γ_2 satisfy*

$$\gamma_2\leq\gamma_1\leq\frac{2}{\theta_i}, \tag{29}$$

and

$$\rho(L_F)(\gamma_2-2\gamma_1)-2\gamma_1\theta_i+4>0,$$

for $i=m+1,\dots,n$, where $\theta_i=\sum_{j=1}^{m} a_{ij}$

Proof. We have already known that the characteristic Equation (9) has a root as $z = 1$, so we only consider the situation that $z \neq 1$. From Lemma 3, Equation (9) equals to

$$\det((z-1)((z-1)I + \gamma_1 D) + (\gamma_1(z-1) + \gamma_2)L_F) = 0. \tag{30}$$

When $z \neq 1$, (30) is reformulated as

$$\det(I + G(z)L_F) = 0, \tag{31}$$

where

$$G(z) = \begin{bmatrix} \frac{\gamma_1(z-1)+\gamma_2}{(z-1)(z-1+\gamma_1\theta_{m+1})} & & \\ & \ddots & \\ & & \frac{\gamma_1(z-1)+\gamma_2}{(z-1)(z-1+\gamma_1\theta_n)} \end{bmatrix},$$

and $\theta_i = \sum_{j=1}^{m} a_{ij}, i = m+1, \ldots, n$ is the (i,i) entry of the diagonal matrix D.

Let $F(z) = \det(I + G(z)L_F)$. According to the generalized Nyquist stability criterion [41], the zeros of $F(z)$ are in the unit circle when $G(z)$ does not have poles out of the unit circle, if $\lambda(G(e^{j\omega})L_F)$ does not enclose the point $(-1, j0)$ for $\omega \in [-\pi, \pi]$. Because of the symmetry of the Laplacian matrix L_F, it follows from Lemma 4 that

$$\lambda(G(e^{j\omega})L_F) \in \rho(L_F)Co(0 \cup g_i(e^{j\omega})), i = m+1, \ldots, n, \tag{32}$$

where

$$g_i(e^{j\omega}) = \frac{\gamma_1(e^{j\omega} - 1) + \gamma_2}{(e^{j\omega} - 1)(e^{j\omega} - 1 + \gamma_1\theta_i)}. \tag{33}$$

For calculating conveniently, we assume that $\gamma_2 \leq \gamma_1$. In order to ensure that all the poles of $G(e^{j\omega})$ are within or on the unit circle, we have

$$0 \leq \gamma_1\theta_i \leq 2. \tag{34}$$

Equation (33) is rewritten as

$$g_i(e^{j\omega}) = \frac{\gamma_1 \cos\omega + \gamma_2 - \gamma_1 + j\gamma_1 \sin\omega}{a + jb}, \tag{35}$$

where $a = 2\cos^2\omega + (\gamma_1\theta_i - 2)\cos\omega - \gamma_1\theta_i$ and $b = \sin\omega(2\cos\omega + \gamma_1\theta_i - 2)$.

To analyze the intersections of $g_i(e^{j\omega})$ on the real axis, we get

$$\begin{aligned} Im(g_i(e^{j\omega})) =& (2(\gamma_1 - \gamma_2)\cos\omega + (\gamma_1 - \gamma_2)(\gamma_1\theta_i - 2) \\ & - \gamma_1^2\theta_i)\sin\omega = 0. \end{aligned} \tag{36}$$

For $\omega \in (0, \pi]$, Equation (36) has only one solution $\omega = \pi$ and

$$g_i(e^{j\pi}) = \frac{\gamma_2 - 2\gamma_1}{4 - 2\gamma_1\theta_i}. \tag{37}$$

It is evident that $\rho(L_F)Co(0 \cup g_i(e^{j\omega}))$ does not enclose the point $(-1, j0)$, if all the points $(\rho(L_F)g_i(e^{j\pi}), j0)$ are on the right side of the point $(-1, j0)$ for $i = m+1, \ldots, n$, i.e.,

$$\rho(L_F)\frac{\gamma_2 - 2\gamma_1}{4 - 2\gamma_1\theta_i} > -1. \tag{38}$$

Hence, $\lambda(G(e^{j\omega})L_F)$ does not enclose the point $(-1, j0)$ and $G(z)$ has no poles out of the unit circle, if we choose the gain parameters γ_1 and γ_2 satisfying $\gamma_2 \leq \gamma_1 \leq \frac{2}{\theta_i}$, and $\rho(L_F)(\gamma_2 - 2\gamma_1) - 2\gamma_1\theta_i + 4 > 0$, for $i = m+1, \ldots, n$. Thus, $F(z)$ has all zeros within the unit circle, which means that the roots of characteristic Equation (9) are within or on the unit circle. Hence, the convergence of the system is proved.

The analysis of final rendezvous state is omitted here for it is almost same as the proof of Theorem 1. The only difference is that the left eigenvector of the symmetric Laplacian Matrix L_F corresponding to eigenvalue 0 becomes $[1, \ldots, 1]$, and the final value is

$$x_d = \frac{\sum_{i=m+1}^{n} \sum_{j=1}^{m} a_{ij} x_j}{\sum_{i=m+1}^{n} \sum_{j=1}^{m} a_{ij}}. \tag{39}$$

Theorem 2 is proved. □

3.3. Unconnected Follower Topology

Considering the follower topology is unconnected, i.e., it has no spanning tree, the containment-rendezvous cannot be achieved by our proposed PI coordination control protocol. In this case, we can divide the follower topology into several connected parts, and obtain the convergence conditions based on the results in Theorem 1.

In order to analyze the convergence behavior, we divide the unconnected follower topology into N connected parts, each of which has a spanning tree or has only one agent, and we labelled the parts as $1, \ldots, N$. For each part, the state and integral vectors of followers are defined as $X_{F_l}(k)$ and $R_l(k), l = 1, 2, \cdots, N$. Hence, the dynamic models of followers are formulated as

$$\begin{aligned} X_{F_l}(k+1) =& (I_{n-m} - \gamma_1 L_{2_l} \otimes I_p) X_{F_l}(k) \\ & - \gamma_1 (L_{1_l} \otimes I_p) X_L(k) + \gamma_2 R_l(k), \\ R_l(k+1) =& R_l(k) - (L_{F_l} \otimes I_p) X_{F_l}(k), l = 1, \ldots, N. \end{aligned} \tag{40}$$

where L_{1_l}, L_{2_l} and L_{F_l} are same as the definitions of above L_1, L_2 and L_F, respectively.

For each part l, the characteristic equation is given by

$$\det((z-1)^2 I + \gamma_1 (z-1)(L_{F_l} + D_l) + \gamma_2 L_{F_l}) = 0,$$

where D_l is same as the definition of above D.

Similar to Theorem 1, we take into account the following equation

$$\det((z-1)^2 I + \gamma_1 (z-1)(L_{F_l} + D_l) + \gamma_2 (L_{F_l} + D_l)) = 0. \tag{41}$$

Then, in the light of Theorem 1, we come to the following results.

Theorem 3. *The leader–follower topology of multi-agent system (3) satisfies Assumption 1 and the follower topology is unconnected. All the $n - m$ followers converge to the convex hull spanned by the m leaders, if for all N parts, the roots of Equation (41) lie in the unit circle, and*

$$|\lambda(\gamma_2 D_l \Theta_l^{-1}(e^{j\omega}))| < 1$$

hold with $\omega \in (0, \pi]$, where $\Theta(e^{j\omega}) = (e^{j\omega} - 1)^2 I + \gamma_1 (e^{j\omega} - 1)(L_{F_l} + D_l) + \gamma_2 (L_{F_l} + D_l)$.

Proof. Divide the unconnected follower topology into N connected parts (40), and the state and the integral term of followers are expressed as $X_F(k) = [X_{F_1}(k), \ldots, X_{F_N}(k)]^T$ and $R(k) = [R_1(k), \ldots, R_N(k)]^T$. It is evident the the dynamics (40) of each part have completely same form as (4).

According to the proof of Theorem 1, the followers in one part reach the containment-rendezvous under condition in Theorem 3. In each part. the followers reach an agreement value in convex hull spanned by the leaders. Evidently, the convex hull spanned by the leaders in each part must be contained in the convex hull composed of all the leaders in the system. Hence, all followers converge to the convex hull composed of all the leaders. □

4. Simulations

In this section, we consider a first-order multi-agent system including 6 leaders and 5 followers labelled by $1, \ldots, 6$ and $7, \ldots, 11$, respectively. The initial states of agent i are set to $x_1(0) = [4,4]^T$, $x_2(0) = [5,2]^T$, $x_3(0) = [6,7]^T$, $x_4(0) = [7,1]^T$, $x_5(0) = [9,3]^T$, $x_6(0) = [10,8]^T$, $x_7(0) = [5,-1]^T$, $x_8(0) = [9,2]^T$, $x_9(0) = [2,6]^T$, $x_{10}(0) = [8,10]^T$ and $x_{11}(0) = [10,5]^T$. All the leader–follower topologies considered in this section satisfy Assumption 1. The weight of each edge between a leader and a follower is set to 1 and the weight of each edge among followers is shown in the pictures. The simulation results with different follower topology of the agents are exhibited as follows.

General Topology. The general follower topology satisfying Assumptions 1 and 2 is shown in Figure 1.

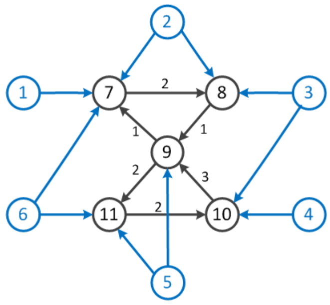

Figure 1. General topology of the agents.

Select the gain parameters as $\gamma_1 = 0.2$ and $\gamma_2 = 0.05$ satisfying the conditions in Theorem 1, and all followers reach an agreement value in the convex hull spanned by leaders (see Figures 2 and 3).

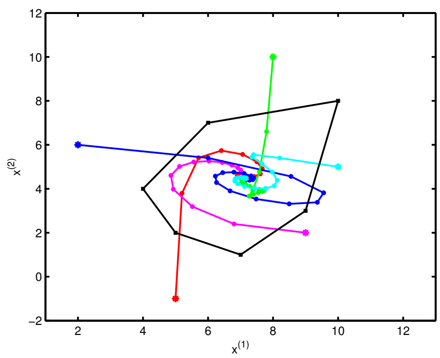

Figure 2. Trajectories of agents with general topology.

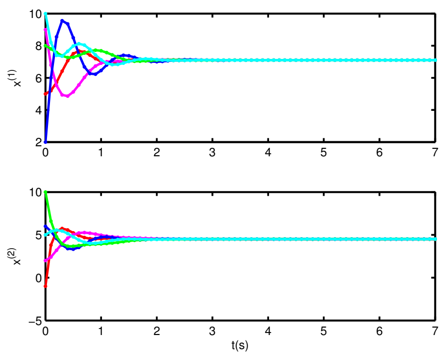

Figure 3. States of followers with general topology.

Symmetric Topology. The symmetric follower topology satisfying Assumptions 1 and 3 is shown in Figure 4.

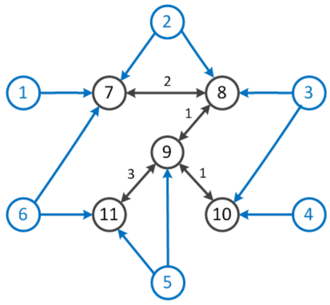

Figure 4. Symmetric topology of agents.

Since the condition is easier to calculate, we are able to give a more specific range of the parameters. Under the given topology, we have $\gamma_1 \leq 0.67$ and we choose γ_1 as 0.2 here. With the condition $\gamma_1 = 0.2$, we then have $\gamma_2 > 0.1$. Finally, we choose $\gamma_1 = 0.2$ and $\gamma_2 = 0.11$ to guarantee that the conditions in Theorem 2 hold. Then, all followers reach the containment-rendezvous asymptotically (see Figures 5 and 6).

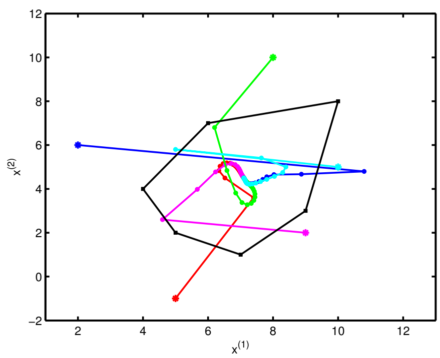

Figure 5. Trajectories of agents with symmetric topology.

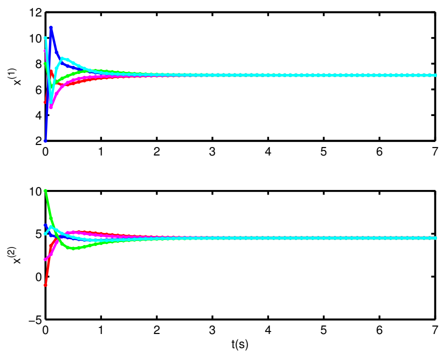

Figure 6. States of followers with symmetric topology.

Unconnected Topology. The unconnected follower topology shown in Figure 7 satisfies Assumption 1. It is evident that the topology can be divided into two connected follower topologies as shown in Figure 8.

The gain parameters are set as $\gamma_1 = 0.2$ and $\gamma_2 = 0.1$ satisfying the requirement in Theorem 3. It is seen from Figures 9 and 10, all followers are divided into two group and followers in each group reach own agreement value in the convex hull spanned by leaders.

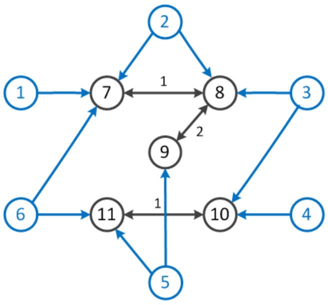

Figure 7. Unconnected topology of the agents

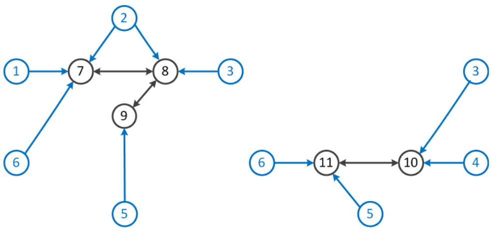

Figure 8. Two connected parts in the unconnected topology.

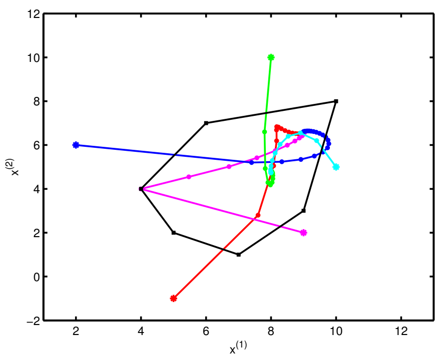

Figure 9. Trajectories of agents with unconnected topology.

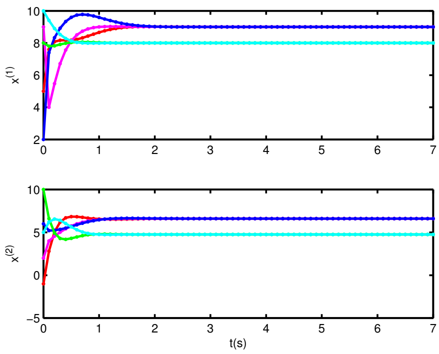

Figure 10. States of followers with unconnected topology.

5. Conclusions

In this paper, containment-rendezvous problem of discrete-time first-order multi-agent systems is analyzed. The proposed control protocol includes a proportional term and an integral term. The proportional term ensures the realization of the containment, and the integral term guarantees the rendezvous. According to the frequency-domain analysis and numerical example, the effectiveness of our proposed protocol under the general connected follower topology is proved. For the symmetric and connected follower topology, a simpler convergence condition is presented. The containment control problem under unconnected follower topology is further discussed. Notably, the unconnected follower topology can be divided into several connected ones, so the followers still converge to the containment formed by all the leaders. Since the work in our paper is only a theoretic research and do not consider the trajectory of the agents, compared with some practical works [42–44], we will continue to study this question in a more practical way in our future work.

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References

1. Olfati-Saber, R.; Murray, R.M. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Automat. Control* **2004**, *49*, 1520–1533. [CrossRef]
2. Olfati-Saber, R. Distributed Kalman filter with embedded consensus filters. In Proceedings of the 44th IEEE Conference on Decision and Control, Seville, Spain, 15 December 2005.
3. Freeman, R.A.; Yang, P.; Lynch, K.M. Stability and convergence properties of dynamic average consensus estimators. In Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, CA, USA, 13–15 December 2006; pp. 338–343.
4. Olfati-Saber, R.; Fax, J.A.; Murray, R.M. Consensus and cooperation in networked multi-agent systems. *Proc. IEEE* **2007**, *95*, 215–233. [CrossRef]
5. Ren, W.; Beard, R.W. *Distributed Consensus in Multi-Vehicle Cooperative Control*; Springer: London, UK, 2008; Volume 27, pp. 71–82.
6. Liu, D.J.; Liu, C.L. Consensus problem of discrete-time second-order multi-agent network with communication delays. In Proceedings of the 2009 Third International Symposium on Intelligent Information Technology Application, Nanchang, China, 21–22 November 2009; pp. 340–344.
7. Li, S.; Guo, Y. Distributed consensus filter on directed switching graphs. *Int. J. Robust Nonlinear Control* **2015**, *25*, 2019–2040. [CrossRef]
8. Rezaei, M.H.; Kabiri, M.; Menhaj, M.B. Adaptive consensus for high-order unknown nonlinear multi-agent systems with unknown control directions and switching topologies. *Inf. Sci.* **2018**, *459*, 224–237. [CrossRef]
9. Rezaei, M.H.; Menhaj, M.B. Adaptive output stationary average consensus for heterogeneous unknown linear multi-agent systems. *IET Control Theory Appl.* **2018**, *12*, 847–856. [CrossRef]
10. Liu, C.L.; Gu, X.Y.; Shan, L. Average-consensus tracking of ramp inputs via second-order multi-agent systems. In Proceedings of the 2019 Chinese Control And Decision Conference (CCDC), Nanchang, China, 3–5 June 2019; pp. 3442–3447.
11. Zheng, M.; Liu, C.L.; Liu, F. Average-consensus tracking of sensor network via distributed coordination control of heterogeneous multi-agent systems. *IEEE Control Syst. Lett.* **2019**, *3*, 132–137. [CrossRef]
12. Ren, W.; Beard, R.W.; Atkins, E.M. A survey of consensus problems in multi-agent coordination. In Proceedings of the 2005 American Control Conference, Portland, OR, USA, 8–10 June 2005; pp. 1859–1864.
13. Rezaei, M.H.; Menhaj, M.B. Stationary average consensus protocol for a class of heterogeneous high-order multi-agent systems with application for aircraft. *Int. J. Syst. Sci.* **2018**, *49*, 284–298. [CrossRef]
14. Ren, W.; Beard, R.W. Consensus seeking in multiagent systems under dynamically changing interaction topologies *IEEE Trans. Autom. Control* **2005**, *50*, 1859–1864.
15. Jiang, W.; Wen, G.; Meng, Y.; Rahmani, A. Distributed adaptive time-varying formation tracking for linear multi-agent systems: A dynamic output approach. In Proceedings of the 2017 36th Chinese Control Conference (CCC), Dalian, China, 26–28 July 2017; pp. 8571–8576.
16. Jiang, W.; Peng, Z.; Rahmani, A.; Hu, W.; Wen, G. Distributed consensus of linear MASs with an unknown leader via a predictive extended state observer considering input delay and disturbances. *Neurocomputing* **2018**, *315*, 465–475. [CrossRef]
17. Cao, Y.C.; Ren, W.; Egerstedt, M. Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks. *Automatica* **2012**, *48*, 1586–1597. [CrossRef]
18. Li, B.; Cheng, Z.Q.; Liu, Z.X.; Zhang, Q. Containment control of discrete-time multi-agent systems with multiple stationary leaders and time-delays. In Proceedings of the 2015 34th Chinese Control Conference(CCC), Hangzhou, China, 28–30 July 2015; pp. 7062–7066.
19. Hua, Y.Z.; Dong, X.W.; Han, L.; Li, Q.D.; Zhang, R. Formation-containment tracking for general linear multi-agent systems with a tracking-leader of unknown control input. *Syst. Control Lett.* **2018**, *122*, 67–76. [CrossRef]

20. Jiang, W. Fully Distributed Time-Varying Formation and Containment Control for Multi-Agent/Multi-Robot Systems. Ph.D. Thesis, Ecole Centrale de Lille, Villeneuve-d'Ascq, France, 2018.
21. Wang, F.Y.; Liu, Z.X.; Chen, Z.Q. Containment control for second-order nonlinear multi-agent systems with aperiodically intermittent position measurements. *J. Frankl. Inst.* **2019**, *356*, 8706–8725. [CrossRef]
22. Shi, L.; Xiao, Y.; Shao, J.L.; Zheng, W.X. Containment control of asynchronous discrete-time general linear multiagent systems with arbitrary network topology. *IEEE Trans. Cybern.* **2020**, *50*, 2546–2556. [CrossRef]
23. Ding, Y.; Ren, W. Sampled-data containment control for double-integrator agents with dynamic leaders with nonzero inputs. *Syst. Control Lett.* **2020**, *139*, 104673. [CrossRef]
24. Wang, F.Y.; Ni, Y.H.; Liu, Z.X. Containment control for general second-order multiagent systems with switched dynamics. *IEEE Trans. Autom. Control* **2020**, *50*, 550–560. [CrossRef]
25. Liu, H.Y.; Xie, G.M.; Wang, L. Necessary and sufficient conditions for containment control of networked multi-agent systems. *Automatica* **2012**, *48*, 1415–1422. [CrossRef]
26. Wang, Y.P.; Cheng, L.; Hou, Z.G.; Min, T.; Wang, M. Containment control of multi-agent systems in a noisy communication environment. *Automatica* **2014**, *50*, 1922–1928. [CrossRef]
27. Rong, L.N.; Shen, H. Distributed PD-type protocol based containment control of multi-agent systems with input delays. *J. Frankl. Inst.* **2015**, *352*, 3600–3611. [CrossRef]
28. Mu, X.W.; Liu, K. Containment control of single-integrator network with limited communication data rate. *IEEE Trans. Autom. Control* **2016**, *61*, 2232–2238. [CrossRef]
29. Miao, G.Y.; Cao, J.D.; Alsaedi, A.; Alsaadif, F.E. Event-triggered containment control for multi-agent systems with constant time delays. *J. Frankl. Inst.* **2017**, *354*, 6956–6977. [CrossRef]
30. Shao, J.L.; Shi, L.; Gong, L.S. Analysis of asynchronous containment control problem for discrete-time multi-agent systems. In Proceedings of the 2017 IEEE 2nd Advanced Information Technology, Electronic and Automation Control Conference (IAEAC), Chongqing, China, 25–26 March 2017; pp. 247–251.
31. Han, T.; Li, J.; Guan, Z.H.; Cai, C.X.; Zhang, D.X.; He, D.X. Containment control of multi-agent systems via a disturbance observer-based approach. *J. Frankl. Inst.* **2019**, *35*, 2919–2933. [CrossRef]
32. Liu, H.Y.; Chong, L.; Tan, M.; Hou, Z.G. Containment control of continuous-time linear multi-agent systems with aperiodic sampling. *Automatica* **2015**, *57*, 78–84. [CrossRef]
33. Liu, T.F.; Qi, L.; Jiang, Z.P. Distributed containment control of multi-agent systems with velocity and acceleration saturations. *Automatic* **2020**, *117*, 108992. [CrossRef]
34. Wang, D.; Wang, D.; Wang, W. Necessary and sufficient conditions for containment control of multi-agent systems with time delay. *Automatica* **2019**, *103*, 418–423. [CrossRef]
35. Shan, L.; Liu, C.L. Average-consensus tracking of multi-agent systems with additional interconnecting agents. *J. Frankl. Inst.* **2018**, *355*, 8957–8970. [CrossRef]
36. Chung, Y.F.; Kia, S.S. Distributed dynamic containment control over a strongly connected and weight-balanced digraph. *IFAC-PapersOnLine* **2019**, *52*, 25–30. [CrossRef]
37. Huang, M.Y.; Liu, C.L. Containment-rendezvous control for first-order multi-agent systems. In Proceedings of the 2020 Chinese Automation Congress (CAC), Shanghai, China, 6–8 November 2020; pp. 4885–4890.
38. Lin, Z.Y.; Francis, B.; Maggiore, M. Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Trans. Autom. Control* **2005**, *50*, 121–127.
39. Parks, P.C.; Hahn, V. *Stability Theory*; Prentice Hall: Englewood Cliffs, NJ, USA, 1993.
40. Lestas, I.; Vinnicombe, G. Scalable robustness for consensus protocols with heterogeneous dynamics. *IFAC Proc. Vol.* **2005**, *38*, 185–190. [CrossRef]
41. Desoer, C.A.; Wang, Y.T. On the generalized Nyquist stability criterion. *IEEE Trans. Autom. Control* **1980**, *50*, 187–196. [CrossRef]
42. Olfati-Saber, R. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Trans. Autom. Control* **2006**, *51*, 401–420. [CrossRef]
43. Nguyen, T.; La, H.M.; Le, T.D.; Jafari, M. Formation control and obstacle avoidance of multiple rectangular agents with limited communication ranges. *IEEE Trans. Control Netw. Syst.* **2016**, *4*, 680–691. [CrossRef]
44. Tran, V.P.; Garratt, M.; Petersen, I.R. Switching time-invariant formation control of a collaborative multi-agent system using negative imaginary systems theory. *Control Eng. Pract.* **2020**, *95*, 104245. [CrossRef]