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Algorithmic Aspects of Some Variations of Clique Transversal and Clique Independent Sets on Graphs

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Abstract: This paper studies the maximum-clique independence problem and some variations of the clique transversal problem such as the $\{k\}$-clique, maximum-clique, minus clique, signed clique, and k-fold clique transversal problems from algorithmic aspects for k-trees, suns, planar graphs, doubly chordal graphs, clique perfect graphs, total graphs, split graphs, line graphs, and dually chordal graphs. We give equations to compute the $\{k\}$-clique, minus clique, signed clique, and k-fold clique transversal numbers for suns, and show that the $\{k\}$-clique transversal problem is polynomial-time solvable for graphs whose clique transversal numbers equal their clique independence numbers. We also show the relationship between the signed and generalization clique problems and present NP-completeness results for the considered problems on k-trees with unbounded k, planar graphs, doubly chordal graphs, total graphs, split graphs, line graphs, and dually chordal graphs.

Keywords: clique independent set; clique transversal number; signed clique transversal function; minus clique transversal function; k-fold clique transversal set

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Citation: Lee, C.-M. Algorithmic Aspects of Some Variations of Clique Transversal and Clique Independent Sets on Graphs. *Algorithms* **2021**, *14*, 22. https://doi.org/10.3390/a14010022

Received: 9 December 2020
Accepted: 11 January 2021
Published: 13 January 2021

1. Introduction

Every graph $G = (V, E)$ in this paper is finite, undirected, connected, and has at most one edge between any two vertices in G. We assume that the vertex set V and edge set E of G contain n vertices and m edges. They can also be denoted by $V(G)$ and $E(G)$. A graph $G' = (V', E')$ is an *induced subgraph* of G denoted by $G[V']$ if $V' \subseteq V$ and E' contains all the edge $(x, y) \in E$ for $x, y \in V'$. Two vertices $x, y \in V$ are *adjacent* or *neighbors* if $(x, y) \in E$. The sets $N_G(x) = \{y \mid (x, y) \in E\}$ and $N_G[x] = N_G(x) \cup \{x\}$ are the *neighborhood* and *closed neighborhood* of a vertex x in G, respectively. The number $deg_G(x) = |N_G(x)|$ is the *degree* of x in G. If $deg_G(x) = k$ for every $x \in V$, then G is *k-regular*. Particularly, *cubic graphs* are an alternative name for 3-regular graphs.

A subset S of V is a *clique* if $(x, y) \in E$ for $x, y \in S$. Let Q be a clique of G. If $Q \cap Q' \neq Q$ for any other clique Q' of G, then Q is a *maximal* clique. We use $C(G)$ to represent the set $\{C \mid C$ is a maximal clique of $G\}$. A clique $S \in C(G)$ is a *maximum* clique if $|S| \geq |S'|$ for every $S' \in C(G)$. The number $\omega(G) = \max\{|S| \mid S \in C(G)\}$ is the *clique number* of G. A set $D \subseteq V$ is a *clique transversal set* (abbreviated as CTS) of G if $|C \cap D| \geq 1$ for every $C \in C(G)$. The number $\tau_C(G) = \min\{|S| \mid S$ is a CTS of $G\}$ is the *clique transversal number* of G. The *clique transversal problem* (abbreviated as CTP) is to find a minimum CTS for a graph. A set $S \subseteq C(G)$ is a *clique independent set* (abbreviated as CIS) of G if $|S| = 1$ or $|S| \geq 2$ and $C \cap C' = \emptyset$ for $C, C' \in S$. The number $\alpha_C(G) = \max\{|S| \mid S$ is a CIS of $G\}$ is the *clique independence number* of G. The *clique independence problem* (abbreviated as CIP) is to find a maximum CIS for a graph.

The CTP and the CIP have been widely studied. Some studies on the CTP and the CIP consider imposing some additional constraints on CTS or CIS, such as the *maximum-clique independence problem* (abbreviated as MCIP), the *k-fold clique transversal problem* (abbreviated as k-FCTP), and the *maximum-clique transversal problem* (abbreviated as MCTP).

Definition 1 ([1,2])**.** *Suppose that $k \in \mathbb{N}$ is fixed and G is a graph. A set $D \subseteq V(G)$ is a k-fold clique transversal set (abbreviated as k-FCTS) of G if $|C \cap D| \geq k$ for $C \in \mathcal{C}(G)$. The number $\tau_C^k(G) = min\{|S| \mid S$ is a k-FCTS of $G\}$ is the k-fold clique transversal number of G. The k-FCTP is to find a minimum k-FCTS for a graph.*

Definition 2 ([3,4])**.** *Suppose that G is a graph. A set $D \subseteq V(G)$ is a maximum-clique transversal set (abbreviated as MCTS) of G if $|C \cap D| \geq 1$ for $C \in \mathcal{C}(G)$ with $|C| = \omega(G)$. The number $\tau_M(G) = min\{|S| \mid S$ is an MCTS of $G\}$ is the maximum-clique transversal number of G. The MCTP is to find a minimum MCTS for a graph. A set $S \subseteq \mathcal{C}(G)$ is a maximum-clique independent set (abbreviated as MCIS) of G if $|C| = \omega(G)$ for $C \in S$ and $C \cap C' = \emptyset$ for $C, C' \in S$. The number $\alpha_M(G) = max\{|S| \mid S$ is an MCIS of $G\}$ is the maximum-clique independence number of G. The MCIP is to find a maximum MCIS for a graph.*

The k-FCTP on balanced graphs can be solved in polynomial time [2]. The MCTP has been studied in [3] for several well-known graph classes and the MCIP is polynomial-time solvable for any graph H with $\tau_M(H) = \alpha_M(H)$ [4]. Assume that $Y \subseteq \mathbb{R}$ and $f : X \to Y$ is a function. Let $f(X') = \sum_{x \in X} f(x)$ for $X' \subseteq X$, and let $f(X)$ be the *weight* of f. A CTS of G can be expressed as a function f whose domain is $V(G)$ and range is $\{0, 1\}$, and $f(C) \geq 1$ for $C \in \mathcal{C}(G)$. Then, f is a *clique transversal function* (abbreviated as CTF) of G and $\tau_C(G) = \min\{f(V(G)) \mid f$ is a CTF of $G\}$. Several types of CTF have been studied [4–7]. The following are examples of CTFs.

Definition 3. *Suppose that $k \in \mathbb{N}$ is fixed and G is a graph. A function f is a $\{k\}$-clique transversal function (abbreviated as $\{k\}$-CTF) of G if the domain and range of f are $V(G)$ and $\{0, 1, 2, \ldots, k\}$, respectively, and $f(C) \geq k$ for $C \in \mathcal{C}(G)$. The number $\tau_C^{\{k\}}(G) = min\{f(V(G)) \mid f$ is a $\{k\}$-CTF of $G\}$ is the $\{k\}$-clique transversal number of G. The $\{k\}$-clique transversal problem (abbreviated as $\{k\}$-CTP) is to find a minimum-weight $\{k\}$-CTF for a graph.*

Definition 4. *Suppose that G is a graph. A function f is a signed clique transversal function (abbreviated as SCTF) of G if the domain and range of f are $V(G)$ and $\{-1, 1\}$, respectively, and $f(C) \geq 1$ for $C \in \mathcal{C}(G)$. If the domain and range of f are $V(G)$ and $\{-1, 0, 1\}$, respectively, and $f(C) \geq 1$ for $C \in \mathcal{C}(G)$, then f is a minus clique transversal function (abbreviated as MCTF) of G. The number $\tau_C^s(G) = min\{f(V(G)) \mid f$ is an SCTF of $G\}$ is the signed clique transversal number of G. The minus clique transversal number of G is $\tau_C^-(G) = min\{f(V(G)) \mid f$ is an MCTF of $G\}$. The signed clique transversal problem (abbreviated as SCTP) is to find a minimum-weight SCTF for a graph. The minus clique transversal problem (abbreviated as MCTP) is to find a minimum-weight MCTF for a graph.*

Lee [4] introduced some variations of the k-FCTP, the $\{k\}$-CTP, the SCTP, and the MCTP, but those variations are dedicated to maximum cliques in a graph. The MCTP on chordal graphs is NP-complete, while the MCTP on block graphs is linear-time solvable [7]. The MCTP and SCTP are linear-time solvable for any strongly chordal graph G if a *strong elimination ordering* of G is given [5]. The SCTP is NP-complete for doubly chordal graphs [6] and planar graphs [5].

According to what we have described above, there are very few algorithmic results regarding the k-FCTP, the $\{k\}$-CTP, the SCTP, and the MCTP on graphs. This motivates us to study the complexities of the k-FCTP, the $\{k\}$-CTP, the SCTP, and the MCTP. This paper also studies the MCTP and MCIP for some graphs and investigates the relationships between different *dominating functions* and CTFs.

Definition 5. *Suppose that $k \in \mathbb{N}$ is fixed and G is a graph. A set $S \subseteq V(G)$ is a k-tuple dominating set (abbreviated as k-TDS) of G if $|S \cap N_G[x]| \geq 1$ for $x \in V(G)$. The number $\gamma_{\times k}(G) = min\{|S| \mid S$ is a k-TDS of $G\}$ is the k-tuple domination number of G. The k-tuple domination problem (abbreviated as k-TDP) is to find a minimum k-TDS for a graph.*

Notice that a *dominating set* of a graph G is a 1-TDS. The *domination number* $\gamma(G)$ of G is $\gamma_{\times 1}(G)$.

Definition 6. *Suppose that $k \in \mathbb{N}$ is fixed and G is a graph. A function f is a $\{k\}$-dominating function (abbreviated as $\{k\}$-DF) of G if the domain and range of f are $V(G)$ and $\{0, 1, 2, \ldots, k\}$, respectively, and $f(N_G[x]) \geq k$ for $x \in V(G)$. The number $\gamma_{\{k\}}(G) = min\{f(V(G)) \mid f$ is a $\{k\}$-DF of $G\}$ is the $\{k\}$-domination number of G. The $\{k\}$-domination problem (abbreviated as $\{k\}$-DP) is to find a minimum-weight $\{k\}$-DF for a graph.*

Definition 7. *Suppose that G is a graph. A function f is a signed dominating function (abbreviated as SDF) of G if the domain and range of f are $V(G)$ and $\{-1, 1\}$, respectively, and $f(N_G[x]) \geq 1$ for $x \in V(G)$. If the domain and range of f are $V(G)$ and $\{-1, 0, 1\}$, respectively, and $f(N_G[x]) \geq 1$ for $x \in V(G)$, then f is a minus dominating function (abbreviated as MDF) of G. The number $\gamma_s(G) = min\{f(V(G)) \mid f$ is an SDF of $G\}$ is the signed domination number of G. The minus domination number of G is $\gamma^-(G) = min\{f(V(G)) \mid f$ is an MDF of $G\}$. The signed domination problem (abbreviated as SDP) is to find a minimum-weight SDF for a graph. The minus domination problem (abbreviated as MDP) is to find a minimum-weight MDF for a graph.*

Our main contributions are as follows.

1. We prove in Section 2 that $\gamma^-(G) = \tau_C^-(G)$ and $\gamma_s(G) = \tau_C^s(G)$ for any sun G. We also prove that $\gamma_{\times k}(G) = \tau_C^k(G)$ and $\gamma_{\{k\}}(G) = \tau_C^{\{k\}}(G)$ for any sun G if $k > 1$.
2. We prove in Section 3 that $\tau_C^{\{k\}}(G) = k\tau_C(G)$ for any graph G with $\tau_C(G) = \alpha_C(G)$. Then, $\tau_C^{\{k\}}(G)$ is polynomial-time solvable if $\tau_C(G)$ can be computed in polynomial time. We also prove that the SCTP is a special case of *the generalized clique transversal problem* [8]. Therefore, the SCTP for a graph H can be solved in polynomial time if the generalized transversal problem for H is polynomial-time solvable.
3. We show in Section 4 that$\gamma_{\times k}(G) = \tau_C^k(G)$ and $\gamma_{\{k\}}(G) = \tau_C^{\{k\}}(G)$ for any split graph G. Furthermore, we introduce H_1-*split graphs* and prove that $\gamma^-(H) = \tau_C^-(H)$ and $\gamma_s(H) = \tau_C^s(H)$ for any H_1-split graph H. We prove the NP-completeness of SCTP for split graphs by showing that the SDP on H_1-split graphs is NP-complete.
4. We show in Section 5 that $\tau_C^{\{k\}}(G)$ for a *doubly chordal graph* G can be computed in linear time, but the k-FCTP is NP-complete for doubly chordal graphs as $k > 1$. Notice that the CTP is a special case of the k-FCTP and the $\{k\}$-CTP when $k = 1$, and thus $\tau_C(G) = \tau_C^1(G) = \tau_C^{\{1\}}(G)$ for any graph G.
5. We present other NP-completeness results in Sections 6 and 7 for k-trees with unbounded k and subclasses of total graphs, line graphs, and planar graphs. These results can refine the "borderline" between P and NP for the considered problems and graphs classes or their subclasses.

2. Suns

In this section, we give equations to compute $\tau_C^{\{k\}}(G)$, $\tau_C^k(G)$, $\tau_C^s(G)$, and $\tau_C^-(G)$ for any sun G and show that $\tau_C^{\{k\}}(G) = \gamma_{\{k\}}(G)$, $\tau_C^k(G) = \gamma_{\times k}(G)$, $\tau_C^s(G) = \gamma_s(G)$, and $\tau_C^-(G) = \gamma^-(G)$.

Let $p \in \mathbb{N}$ and G be a graph. An edge $e \in E(G)$ is a *chord* if e connects two non-consecutive vertices of a cycle in G. If C has a chord for every cycle C consisting of more than three vertices, G is a *chordal* graph. A *sun* G is a chordal graph whose vertices can be partitioned into $W = \{w_i \mid 1 \leq i \leq p\}$ and $U = \{u_i \mid 1 \leq i \leq p\}$ such that (1) W is an independent set, (2) the vertices $u_1, u_2, \ldots, u_p$ of U form a cycle, and (3) every $w_i \in W$ is adjacent to precisely two vertices u_i and u_j, where $j \equiv i + 1 \pmod{p}$. We use $S_p = (W, U, E)$ to denote a sun. Then, $|V(S_p)| = 2p$. If p is odd, S_p is an *odd* sun; otherwise, it is an *even* sun. Figure 1 shows two suns.

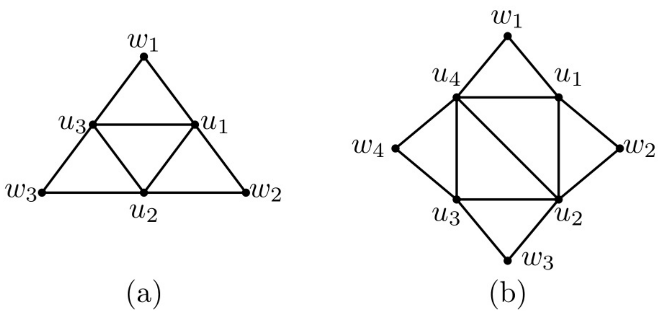

Figure 1. (**a**) The sun S_3. (**b**) A sun S_4.

Lemma 1. *For any sun $S_p = (W, U, E)$, $\tau_C^2(S_p) = p$ and $\tau_C^3(S_p) = 2p$.*

Proof. It is straightforward to see that U is a minimum 2-FCTS and $W \cup U$ is a minimum 3-FCTS of S_p. This lemma therefore holds. □

Lemma 2. *Suppose that $k \in \mathbb{N}$ and $k > 1$. Then, $\tau_C^{\{k\}}(S_p) = \lceil pk/2 \rceil$ for any sun $S_p = (W, U, E)$.*

Proof. Let $i, j \in \{1, 2, \dots p\}$ such that $j \equiv i + 1$ (mod p). Since every $w_i \in W$ is adjacent to precisely two vertices $u_i, u_j \in U$, $N_{S_p}[w_i] = \{w_i, u_i, u_j\}$ is a maximal clique of S_p. By contradiction, we can prove that there exists a minimum $\{k\}$-CTF f of S_p such that $f(w_i) = 0$ for $w_i \in W$. Since $f(N_{S_p}[w_i]) \geq k$ for $1 \leq i \leq p$, we have

$$\tau_C^{\{k\}}(S_p) = \sum_{i=1}^{p} f(u_i) = \frac{\sum_{i=1}^{p} f(N_{S_p}[w_i])}{2} \geq \frac{pk}{2}.$$

Since $\tau_C^{\{k\}}(S_p)$ is a nonnegative integer, $\tau_C^{\{k\}}(S_p) \geq \lceil pk/2 \rceil$.

We define a function $h : W \cup U \to \{0, 1, \dots, k\}$ by $h(w_i) = 0$ for every $w_i \in W$, $h(u_i) = \lceil k/2 \rceil$ for $u_i \in U$ with odd index i and $h(u_i) = \lfloor k/2 \rfloor$ for every $u_i \in U$ with even index i. Clearly, a maximal clique Q of S_n is either the closed neighborhood of some vertex in W or a set of at least three vertices in U. If $Q = N_{S_p}[w_i]$ for some $w_i \in W$, then $h(Q) = \lceil k/2 \rceil + \lfloor k/2 \rfloor = k$. Suppose that Q is a set of at least three vertices in U. Since $k \geq 2$, $h(Q) \geq 3 \cdot \lfloor k/2 \rfloor \geq k$. Therefore, h is a $\{k\}$-CTF of S_p. We show the weight of h is $\lceil pk/2 \rceil$ by considering two cases as follows.

Case 1: p is even. We have

$$h(V(S_p)) = \sum_{i=1}^{p} h(u_i) = \frac{p}{2} \cdot (\lceil k/2 \rceil + \lfloor k/2 \rfloor) = \frac{pk}{2}.$$

Case 2: p is odd. We have

$$h(V(S_p)) = \sum_{i=1}^{p} h(u_i) = \frac{(p-1)}{2} \cdot k + \lceil k/2 \rceil = \lceil pk/2 \rceil.$$

Following what we have discussed above, we know that h is a minimum $\{k\}$-CTF of S_n and thus $\tau_C^{\{k\}}(S_p) = \lceil pk/2 \rceil$. □

Lemma 3. *For any sun $S_p = (W, U, E)$, $\tau_C^-(S_p) = \tau_C^s(S_p) = 0$.*

Proof. For $1 \le i \le p$, $N_{S_p}[w_i]$ is a maximal clique of S_p. Let h be a minimum SCTF of S_p. Then, $\tau_C^s(S_p) = h(V(S_p))$. Note that $h(N_{S_p}[w_i]) \ge 1$ for $1 \le i \le p$. We have

$$h(V(S_p)) = \sum_{i=1}^{p} h(N_{S_p}[w_i]) - \sum_{i=1}^{p} h(u_i) \ge p - \sum_{i=1}^{p} h(u_i).$$

Since $\sum_{i=1}^{p} h(u_i) \le p$, we have $\tau_C^s(S_p) \ge 0$. Let f be an SCTF of S_p such that $f(u_i) = 1$ and $f(w_i) = -1$ for $1 \le i \le p$. The weight of f is 0. Then f is a minimum SCTF of S_p. Hence, $\tau_C^-(S_p) = 0$ and $\tau_C^s(S_p) = 0$. The proof for $\tau_C^-(G) = 0$ is analogous to that for $\tau_C^s(G) = 0$. □

Theorem 1 (Lee and Chang [9]). *Let S_p be a sun. Then,*

(1) $\gamma^-(S_p) = \gamma_s(S_p) = 0$;
(2) $\gamma_{\{k\}}(S_p) = \lceil pk/2 \rceil$;
(3) $\gamma_{\times 1}(S_p) = \lceil p/2 \rceil$, $\gamma_{\times 2}(S_p) = p$ *and* $\gamma_{\times 3}(S_p) = 2p$.

Corollary 1. *Let S_p be a sun. Then,*

(1) $\gamma^-(S_p) = \tau_C^-(S_p) = \gamma_s(S_p) = \tau_C^s(S_p) = 0$;
(2) $\gamma_{\{k\}}(S_p) = \tau_C^{\{k\}}(S_p) = \lceil pk/2 \rceil$ *for* $k > 1$;
(3) $\gamma_{\times 2}(S_p) = \tau_C^2(S_p) = p$ *and* $\gamma_{\times 3}(S_p) = \tau_C^3(S_p) = 2p$.

Proof. The corollary holds by Lemmas 1–3 and Corollary 1. □

3. Clique Perfect Graphs

Let $\mathcal{G}$ be the set of all induced subgraphs of G. If $\tau_C(H) = \alpha_C(H)$ for every $H \in \mathcal{G}$, then G is *clique perfect*. In this section, we study the $\{k\}$-CTP for clique perfect graphs and the SCTP for balanced graphs.

Lemma 4. *Let G be such a graph that $\tau_C(G) = \alpha_C(G)$. Then, $\tau_C^{\{k\}}(G) = k\tau_C(G)$.*

Proof. Assume that D is a minimum CTS of G. Then, $|D| = \tau_C(G)$. Let $x \in V(G)$ and let f be a function whose domain is $V(G)$ and range is $\{0, 1, \ldots, k\}$, and $f(x) = k$ if $x \in D$; otherwise, $f(x) = 0$. Clearly, f is a $\{k\}$-CTF of G. We have $\tau_C^{\{k\}}(G) \le k\tau_C(G)$.

Assume that f is a minimum-weight $\{k\}$-CTF of G. Then, $f(V(G)) = \tau_C^{\{k\}}(G)$ and $f(C) \ge k$ for every $C \in C(G)$. Let $S = \{C_1, C_2, \ldots, C_\ell\}$ be a maximum CIS of G. We know that $|S| = \ell = \alpha_C(G)$ and $\sum_{i=1}^{\ell} f(C_i) \le f(V(G))$. Therefore, $k\tau_C(G) = k\alpha_C(G) = k\ell \le \sum_{i=1}^{\ell} f(C_i) \le f(V(G)) = \tau_C^{\{k\}}(G)$. Following what we have discussed above, we know that $\tau_C^{\{k\}}(G) = k\tau_C(G)$. □

Theorem 2. *If a graph G is clique perfect, $\tau_C^{\{k\}}(G) = k\tau_C(G)$.*

Proof. Since G is clique perfect, $\tau_C(G) = \alpha_C(G)$. Hence, the theorem holds by Lemma 4. □

Corollary 2. *The $\{k\}$-CTP is polynomial-time solvable for distance-hereditary graphs, balanced graphs, strongly chordal graphs, comparability graphs, and chordal graphs without odd suns.*

Proof. Distance-hereditary graphs, balanced graphs, strongly chordal graphs, comparability graphs, and chordal graphs without odd suns are clique perfect, and the CTP can be solved in polynomial time for them [10–14]. The corollary therefore holds. □

Definition 8. *Suppose that R is a function whose domain is $C(G)$ and range is $\{0, 1, \ldots, \omega(G)\}$. If $R(C) \le |C|$ for every $C \in C(G)$, then R is a clique-size restricted function (abbreviated as*

CSRF) of G. A set $D \subseteq V(G)$ is an R-clique transversal set (abbreviated as R-CTS) of G if R is a CSRF of G and $|D \cap C| \geq R(C)$ for every $C \in C(G)$. Let $\tau_R(G) = min\{|D| \mid D$ is an R-CTS of $G\}$. The generalized clique transversal problem (abbreviated as GCTP) is to find a minimum R-CTS for a graph G with a CSRF R.

Lemma 5. *Let G be a graph with a CSRF R. If $R(C) = \lceil(|C|+1)/2\rceil$ for every $C \in C(G)$, then $\tau_C^s(G) = 2\tau_R(G) - n$.*

Proof. Assume that D is a minimum R-CTS of G. Then, $|D| = \tau_R(G)$. Let $x \in V(G)$ and let f be a function of G whose domain is $V(G)$ and range is $\{-1, 1\}$, and $f(x) = 1$ if $x \in D$; otherwise, $f(x) = -1$. Since $|D \cap C| \geq \lceil(|C|+1)/2\rceil$ for every $C \in C(G)$, there are at least $\lceil(|C|+1)/2\rceil$ vertices in C with the function value 1. Therefore, $f(C) \geq 1$ for every $C \in C(G)$, and f is an SCTF of G. Then, $\tau_C^s(G) \leq 2\tau_R(G) - n$.

Assume that h is a minimum-weight SCTF of G. Clearly, $h(V(G)) = \tau_C^s(G)$. Since $h(C) \geq 1$ for every $C \in C(G)$, C contains at least $\lceil(|C|+1)\rceil/2$ vertices with the function value 1. Let $D = \{x \mid h(x) = 1, x \in V(G)\}$. The set D is an R-CTS of G. Therefore, $2\tau_R(G) - n \leq 2|D| - n = \tau_C^s(G)$. Hence, we have $\tau_C^s(G) = 2\tau_R(G) - n$. □

Theorem 3. *The SCTP on balanced graphs can be solved in polynomial time.*

Proof. Suppose that a graph G has n vertices $v_1, v_2, \ldots, v_n$ and ℓ maximal cliques $C_1, C_2, \ldots, C_\ell$. Let $i \in \{1, 2, \ldots, \ell\}$ and $j \in \{1, 2, \ldots, n\}$. Let M be an $\ell \times n$ matrix such that an element $M(i, j)$ of M is one if the maximal clique C_i contains the vertex v_j, and $M(i, j) = 0$ otherwise. We call M the *clique matrix* of G. If the clique matrix M of G does not contain a square submatrix of odd order with exactly two ones per row and column, then M is a *balanced* matrix and G is a *balanced* graph. We formulae the GCTP on a balanced graph G with a CSRF R as the following integer programming problem:

$$\left.\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} x_i \\ \\ \text{subject to} & MX \geq \mathcal{C} \end{array}\right\}$$

where $\mathcal{C} = (R(C_1), R(C_2), \ldots, R(C_\ell))$ is a column vector and $X = (x_1, x_2, \ldots, x_n)$ is a column vector such that x_i is either 0 or 1. Since the matrix M is balanced, an optimal 0–1 solution of the integer programming problem above can be found in polynomial time by the results in [15]. By Lemma 5, we know that the SCTP on balanced graphs can be solved in polynomial time. □

4. Split Graphs

Let G be such a graph that $V(G) = I \cup C$ and $I \cap C = \emptyset$. If I is an independent set and C is a clique, G is a *split* graph. Then, every maximal of G is either C itself, or the closed neighborhood $N_G[x]$ of a vertex $x \in I$. We use $G = (I, C, E)$ to represent a split graph. The $\{k\}$-CTP, the k-FCTP, the SCTP, and the MCTP for split graphs are considered in this section. We also consider the $\{k\}$-DP, the k-TDP, the SDP, and the MDP for split graphs.

For split graphs, the $\{k\}$-DP, the k-TDP, and the MDP are NP-complete [16–18], but the complexity of the SDP is still unknown. In the following, we examine the relationships between the $\{k\}$-CTP and the $\{k\}$-DP, the k-FCTP and the k-TDP, the SCTP and the SDP, and the MCTP and the MDP. Then, by the relationships, we prove the NP-completeness of the SDP, the $\{k\}$-CTP, the k-FCTP, the SCTP, and the MCTP for split graphs. We first consider the $\{k\}$-CTP and the k-FCTP and show in Theorems 4 and 5 that $\tau_C^k(G) = \gamma_{\times k}(G)$ and $\tau_C^{\{k\}}(G) = \gamma_{\{k\}}(G)$ for any split graph G. Chordal graphs form a superclass of split graphs [19]. The cardinality of $C(G)$ is at most n for any chordal graph G [20]. The following lemma therefore holds trivially.

Lemma 6. *The k-FCTP, the $\{k\}$-CTP, the SCTP, and the MCTP for chordal graphs are in NP.*

Theorem 4. *Suppose that $k \in \mathbb{N}$ and $G = (I, C, E)$ is a split graph. Then, $\tau_C^k(G) = \gamma_{\times k}(G)$.*

Proof. Let S be a minimum k-FCTS of G. Consider a vertex $y \in I$. By the structure of G, $N_G[y]$ is a maximal clique of G. Then, $|S \cap N_G[y]| \geq k$. We now consider a vertex $x \in C$. If $C \notin \mathcal{C}(G)$, then there exists a vertex $y \in I$ such that $N_G[y] = C \cup \{y\}$. Clearly, $N_G[y] \subseteq N_G[x]$ and $|S \cap N_G[x]| \geq |S \cap N_G[y]| \geq k$. If $C \in \mathcal{C}(G)$, then $|S \cap N_G[x]| \geq |S \cap C| \geq k$. Hence, S is a k-TDS of G. We have $\gamma_{\times k}(G) \leq \tau_C^k(G)$.

Let D be a minimum k-TDS of G. Recall that the closed neighborhood of every vertex in I is a maximal clique. Then, D contains at least k vertices in the maximal clique $N_G[y]$ for every vertex $y \in I$. If $C \notin \mathcal{C}(G)$, D is clearly a k-FCTS of G. Suppose that $C \in \mathcal{C}(G)$. We consider three cases as follows.

Case 1: $y \in I \setminus D$. Then, $|D \cap C| \geq |D \cap N_G(y)| \geq k$. The set D is a k-FCTS of G.

Case 2: $y \in I \cap D$ and $x \in N_G(y) \setminus D$. Then, the set $D' = (D \setminus \{y\}) \cup \{x\}$ is still a minimum k-TDS and $|D' \cap C| \geq |D' \cap N_G(y)| \geq k$. The set D' is a k-FCTS of G.

Case 3: $I \subseteq D$ and $N_G[y] \subseteq D$ for every $y \in I$. Then, $|D \cap C| \geq |D \cap N_G(y)| \geq k - 1$. Since $C \in \mathcal{C}(G)$, there exists $x \in C$ such that $y \notin N_G(x)$. If $N_G(x) \cap I = \emptyset$, then $N_G[x] = C$ and $|D \cap C| = |D \cap N_G[x]| \geq k$. Otherwise, let $y' \in N_G(x) \cap I$. Then, $x \in D$ and $|D \cap C| \geq |D \cap N_G(y)| + 1 \geq k$. The set D is a k-FCTS of G.

By the discussion of the three cases, we have $\tau_C^k(G) \leq \gamma_{\times k}(G)$. Hence, we obtain that $\gamma_{\times k}(G) \leq \tau_C^k(G)$ and $\tau_C^k(G) \leq \gamma_{\times k}(G)$. The theorem holds for split graphs. □

Theorem 5. *Suppose that $k \in \mathbb{N}$ and $G = (I, C, E)$ is a split graph. Then, $\tau_C^{\{k\}}(G) = \gamma_{\{k\}}(G)$.*

Proof. We can verify by contradiction that G has a minimum-weight $\{k\}$-CTF f and a minimum-weight $\{k\}$-DF g of G such that $f(y) = 0$ and $g(y) = 0$ for every $y \in I$. By the structure of G, $N_G[y] \in \mathcal{C}(G)$ for every $y \in I$. Then, $f(N_G[y]) \geq k$ and $g(N_G[y]) \geq k$. Since $f(y) = 0$ and $g(y) = 0$, $f(N_G(y)) \geq k$ and $g(N_G(y)) \geq k$.

For every $y \in I$, $N_G(y) \subseteq C$ and $f(C) \geq f(N_G(y)) \geq k$. For every $x \in C$, $f(N_G[x]) \geq f(C) \geq k$. Therefore, the function f is also a $\{k\}$-DF of G. We have $\gamma_{\{k\}}(G) \leq \tau_C^{\{k\}}(G)$. We now consider $g(C)$ for the clique C. If $C \notin \mathcal{C}(G)$, the function g is clearly a $\{k\}$-CTF of G. Suppose that $C \in \mathcal{C}(G)$. Notice that g is a $\{k\}$-DF and $g(y) = 0$ for every $y \in I$. Then, $g(C) = g(N_G[x]) \geq k$ for any vertex $x \in C$. Therefore, g is also a $\{k\}$-CTF of G. We have $\tau_C^{\{k\}}(G) \leq \gamma_{\{k\}}(G)$. Following what we have discussed above, we know that $\tau_C^{\{k\}}(G) = \gamma_{\{k\}}(G)$. □

Corollary 3. *The $\{k\}$-CTP and the k-FCTP are NP-complete for split graphs.*

Proof. The corollary holds by Theorems 4 and 5 and the NP-completeness of the $\{k\}$-DP and the k-TDP for split graphs [16,18]. □

A graph G is a *complete* if $\mathcal{C}(G) = \{V(G)\}$. Let G be a complete graph and let $x \in V(G)$. The vertex set $V(G)$ is the union of the sets $\{x\}$ and $V(G) \setminus \{x\}$. Clearly, $\{x\}$ is an independent set and $V(G) \setminus \{x\}$ is a clique of G. Therefore, complete graphs are split graphs. It is easy to verify the Lemma 7.

Lemma 7. *If G is a complete graph and $k \in \mathbb{N}$, then*

(1) $\tau_C^k(G) = \gamma_{\times k}(G) = k$ *for* $k \leq n$;

(2) $\tau_C^{\{k\}}(G) = \gamma_{\{k\}}(G) = k$;

(3) $\tau_C^-(G) = \gamma^-(G) = 1$;

(4) $\tau_C^s(G) = \gamma_s(G) = \begin{cases} 1 & \textit{if } n \textit{ is odd;} \\ 2 & \textit{otherwise.} \end{cases}$

For split graphs, however, the signed and minus domination numbers are not necessarily equal to the signed and minus clique transversal numbers, respectively. Figure 2 shows a split graph G with $\tau_C^s(G) = \tau_C^-(G) = -3$. However, $\gamma_s(G) = \gamma^-(G) = 1$. We therefore introduce H_1-*split* graphs and show in Theorem 6 that their signed and minus domination numbers are equal to the signed and minus clique transversal numbers, respectively. H_1-split graphs are motivated by the graphs in [17] for proving the NP-completeness of the MDP on split graphs. Figure 3 shows an H_1-split graph.

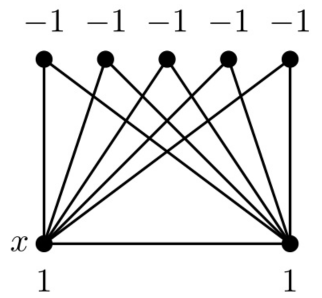

Figure 2. A split graph G with $\tau_C^s(G) = \tau_C^-(G) = -3$.

Definition 9. *Suppose that $G = (I, C, E)$ is a split graph with $3p + 3\ell + 2$ vertices. Let U, S, X, and Y be pairwise disjoint subsets of $V(G)$ such that $U = \{u_i \mid 1 \le i \le p\}$, $S = \{s_i \mid 1 \le i \le \ell\}$, $X = \{x_i \mid 1 \le i \le p + \ell + 1\}$, and $Y = \{y_i \mid 1 \le i \le p + \ell + 1\}$. The graph G is an H_1-split graph if $V(G) = U \cup S \cup X \cup Y$ and G entirely satisfies the following three conditions.*

(1) $I = S \cup Y$ *and* $C = U \cup X$.
(2) $N_G(y_i) = \{x_i\}$ *for* $1 \le i \le p + \ell + 1$.
(3) $|N_G(s_i) \cap U| = 3$ *and* $N_G(s_i) \cap X = \{x_i\}$ *for* $1 \le i \le \ell$.

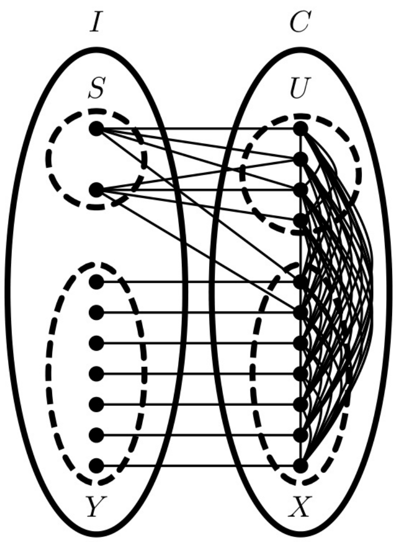

Figure 3. A split graph G with one of its partitions indicated.

Theorem 6. *For any H_1-split graph $G = (I, C, E)$, $\tau_C^s(G) = \gamma_s(G)$ and $\tau_C^-(G) = \gamma^-(G)$.*

Proof. We first prove $\tau_C^s(G) = \gamma_s(G)$. Let $G = (I, C, E)$ be an H_1-split graph. As stated in Definition 9, I can be partitioned into $S = \{s_i \mid 1 \le i \le \ell\}$ and $Y = \{y_i \mid 1 \le i \le p + \ell + 1\}$, and C can be partitioned into $U = \{u_i \mid 1 \le i \le p\}$ and $X = \{x_i \mid 1 \le i \le p + \ell + 1\}$. Assume that f is a minimum-weight SDF of G. For each $y_i \in Y$, $|N_G[y_i]| = 2$ and y_i is adjacent to only the vertex $x_i \in X$. Then, $f(x_i) = f(y_i) = 1$ for $1 \le i \le p + \ell + 1$. Since $C = U \cup X$ and $|U| = p$, we know that $f(C) = f(U) + f(X) \ge (-p) + (p + \ell + 1) \ge \ell + 1$.

Notice that $f(N_G[y]) \geq 1$ and $N_G[y] \in C(G)$ for every $y \in I$. Therefore, f is also an SCTF of G. We have $\tau_C^s(G) \leq \gamma_s(G)$.

Assume that h is a minimum-weight SCTF of G. For each $y_i \in Y$, $|N_G[y_i]| = 2$ and y_i is adjacent to only the vertex $x_i \in X$. Then, $h(x_i) = h(y_i) = 1$ for $1 \leq i \leq p + \ell + 1$. Consider the vertices in I. Since $N_G[y] \in C(G)$ for every $y \in I$, $h(N_G[y]) \geq 1$. We now consider the vertices in C. Recall that $C = U \cup X$. Let $u_i \in U$. Since $|U| = p$ and $|S| = \ell$, we know that $h(N_G[u_i]) = h(U) + h(X) + h(N_G[u_i] \cap S) \geq (-p) + (p + \ell + 1) + (-\ell) \geq 1$. Let $x_i \in X$. Then, $h(N_G[x_i]) = h(U) + h(X) + h(y_i) + h(s_i) \geq (-p) + (p + \ell + 1) + 1 - 1 \geq \ell + 1$. Therefore, h is also an SDF of G. We have $\gamma_s(G) \leq \tau_C^s(G)$.

Following what we have discussed above, we have $\tau_C^s(G) = \gamma_s(G)$. The proof for $\tau_C^-(G) = \gamma^-(G)$ is analogous to that for $\tau_C^s(G) = \gamma_s(G)$. Hence, the theorem holds for any H_1-split graphs. □

Theorem 7. *The SDP on H_1-split graphs is NP-complete.*

Proof. We reduce the (3,2)-*hitting set problem* to the SDP on H_1-split graphs. Let $U = \{u_i \mid 1 \leq i \leq p\}$ and let $\mathcal{C} = \{C_1, C_2, \ldots, C_\ell\}$ such that $C_i \subseteq U$ and $|C_i| = 3$ for $1 \leq i \leq \ell$. A (3,2)-hitting set for the instance $(U, \mathcal{C})$ is a subset U' of U such that $|C_i \cap U'| \geq 2$ for $1 \leq i \leq \ell$. The (3,2)-hitting set problem is to find a minimum (3,2)-hitting set for any instance $(U, \mathcal{C})$. The (3,2)-hitting set problem is NP-complete [17].

Consider an instance $(U, \mathcal{C})$ of the (3,2)-hitting set problem. Let $S = \{s_i \mid 1 \leq i \leq \ell\}$, $X = \{x_i \mid 1 \leq i \leq p + \ell + 1\}$, and $Y = \{y_i \mid 1 \leq i \leq p + \ell + 1\}$. We construct an H_1-split graph $G = (I, C, E)$ by the following steps.

(1) Let $I = S \cup Y$ be an independent set and let $C = U \cup X$ be a clique.
(2) For each vertex $s_i \in S$, a vertex $u \in U$ is connected to s_i if $u \in C_i$.
(3) For $1 \leq i \leq p + \ell + 1$, the vertex y_i is connected to the vertex x_i.
(4) For $1 \leq i \leq \ell$, the vertex s_i is connected to the vertex x_i.

Let $\tau_h(3,2)$ be the minimum cardinality of a (3,2)-hitting set for the instance $(U, \mathcal{C})$. Assume that U' is a minimum (3,2)-hitting set for the instance $(U, \mathcal{C})$. Then, $|U'| = \tau_h(3,2)$. Let f be a function whose domain is $V(G)$ and range is $\{-1, 1\}$, and $f(v) = 1$ if $v \in X \cup Y \cup U'$ and $f(v) = -1$ if $v \in S \cup (U \setminus U')$. Clearly, f is an SDF of G. We have $\gamma_s(G) \leq 2(p + \ell + 1) + |U'| - \ell - (p - |U'|) = p + \ell + 2\tau_h(3,2) + 2$.

Assume that f is minimum-weight SDF of G. For each $y_i \in Y$, $|N_G[y_i]| = 2$ and y_i is adjacent to only the vertex $x_i \in X$. Then, $f(x_i) = f(y_i) = 1$ for $1 \leq i \leq p + \ell + 1$. For any vertex $v \in X \cup Y \cup U$, $f(N_G[v]) \geq 1$ no matter what values the function f assigns to the vertices in U or in S. Consider the vertices in S. By the construction of G, $deg_G(s_i) = 4$ and $|N_G[s_i]| = 5$ for $1 \leq i \leq \ell$. There are at least three vertices in $N_G[s_i]$ with the function value 1. If $f(N_G[s_i]) = 5$, then there exists an SDF g of G such that $g(s_i) = -1$ and $g(v) = f(v)$ for every $v \in V(G) \setminus \{s_i\}$. Then, $g(V(G)) < f(V(G))$. It contradicts the assumption that the weight of f is minimum. Therefore, there exists a minimum-weight SDF h of G such that $h(s_i) = -1$ for $1 \leq i \leq \ell$. Notice that $N_G(s_i) = C_i \cup \{x_i\}$ for $1 \leq i \leq \ell$. There are at least two vertices in C_i with the function value 1. Then, the set $U' = \{u \in U \mid h(u) = 1\}$ is a (3,2)-hitting set for the instance $(U, \mathcal{C})$. We have $p + \ell + 2\tau_h(3,2) + 2 \leq p + \ell + 2|U'| + 2 = \gamma_s(G)$.

Following what we have discussed above, we know that $\gamma_s(G) = p + \ell + 2\tau_h(3,2) + 2$. Hence, the SDP on H_1-split graphs is NP-complete. □

Corollary 4. *The SCTP and the MCTP on split graphs are NP-complete.*

Proof. The corollary holds by Theorems 6 and 7 and the NP-completeness of the MDP on split graphs [17]. □

5. Doubly Chordal and Dually Chordal Graphs

Assume that G is a graph with n vertices $x_1, x_2, \ldots, x_n$. Let $i \in \{1, 2, \ldots, n\}$ and let G_i be the subgraph $G[V(G) \setminus \{x_1, x_2, \ldots x_{i-1}\}]$. For every $x \in V(G_i)$, let $N_i[x] = \{y \mid y \in (N_G[x] \setminus \{x_1, x_2, \ldots, x_{i-1}\})\}$. In G_i, if there exists a vertex $x_j \in N_i[x_i]$ such that $N_i[x_k] \subseteq N_i[x_j]$ for every $x_k \in N_i[x_i]$, then the ordering $(x_1, x_2, \ldots, x_n)$ is a *maximum neighborhood ordering* (abbreviated as MNO) of G. A graph G is *dually chordal* [21] if and only if G has an MNO. It takes linear time to compute an MNO for any dually chordal graph [22]. A graph G is a *doubly chordal* graph if G is both chordal and dually chordal [23]. Lemma 8 shows that a dually chordal graph is not necessarily a chordal graph or a clique perfect graph. Notice that the number of maximal cliques in a chordal graph is at most n [20], but the number of maximal cliques in a dually chordal graph can be exponential [24].

Lemma 8. *For any dually graph G, $\tau_C(G) = \alpha_C(G)$, but G is not necessarily clique perfect or chordal.*

Proof. Brandstädt et al. [25] showed that the CTP is a particular case of the *clique r-domination problem* and the CIP is a particular case of the *clique r-packing problem*. They also showed that the minimum cardinality of a clique r-dominating set of a dually chordal graph G is equal to the maximum cardinality of a clique r-packing set of G. Therefore, $\tau_C(G) = \alpha_C(G)$.

Assume that H is a graph obtained by connecting every vertex of a cycle C_4 of four vertices x_1, x_2, x_3, x_4 to a vertex x_5. Clearly, the ordering $(x_1, x_2, x_3, x_4, x_5)$ is an MNO and thus H is a dually chordal graph. The cycle C_4 is an induced subgraph of H and does not have a chord. Moreover, $\tau_C(H) = \alpha_C(H) = 1$, but $\tau_C(C_4) = 2$ and $\alpha_C(C_4) = 1$. Hence, a dually chordal graph is not necessarily clique perfect or chordal. □

Theorem 8. *Suppose that $k \in \mathbb{N}$ and $k > 1$. The k-FCTP on doubly chordal graphs is NP-complete.*

Proof. Suppose that G is a chordal graph. Let H be a graph such that $V(H) = V(G) \cup \{x\}$ and $E(H) = E(G) \cup \{(x, y) \mid y \in V(G)\}$. Clearly, H is a doubly chordal graph and we can construct H from G in linear time.

Assume that S is a minimum $(k-1)$-FCTS of G. By the construction of H, each maximal clique of H contains the vertex x. Therefore, $S \cup \{x\}$ is a k-FCTS of H. Then $\tau_C^k(H) \leq \tau_C^{k-1}(G) + 1$.

By contradiction, we can verify that there exists a minimum k-FCTS D of H such that $x \in D$. Let $S = D \setminus \{x\}$. Clearly, S is a $(k-1)$-FCTS of G. Then $\tau_C^{k-1}(G) \leq \tau_C^k(H) - 1$. Following what we have discussed above, we have $\tau_C^k(H) = \tau_C^{k-1}(G) + 1$. Notice that $\tau_C(G) = \tau_C^1(G)$ and the CTP on chordal graphs is NP-complete [14]. Hence, the k-FCTP on doubly chordal graphs is NP-complete for doubly chordal graphs. □

Theorem 9. *For any doubly chordal graph G, $\tau_C^{\{k\}}(G)$ can be computed in linear time.*

Proof. The clique r-dominating problem on doubly chordal graphs can be solved in linear time [25]. The CTP is a particular case of the clique r-domination problem. Therefore, the CTP on doubly chordal graphs can also be solved in linear time. By Lemmas 4 and 8, the theorem holds. □

6. k-Trees

Assume that G is a graph with n vertices $x_1, x_2, \ldots, x_n$. Let $i \in \{1, 2, \ldots, n\}$ and let G_i be the subgraph $G[V(G) \setminus \{x_1, x_2, \ldots x_{i-1}\}]$. For every $x \in V(G_i)$, let $N_i[x] = \{y \mid y \in (N_G[x] \setminus \{x_1, x_2, \ldots, x_{i-1}\})\}$. If $N_i[x_i]$ is a clique for $1 \leq i \leq n$, then the ordering $(x_1, x_2, \ldots, x_n)$ is a *perfect elimination ordering* (abbreviated as PEO) of G. A graph G is chordal if and only if G has a PEO [26]. A chordal graph G is a k-tree if and only if either G is a complete graph of k vertices or G has more than k vertices and there exists a PEO

$(x_1, x_2, \ldots, x_n)$ such that $N_i[x_i]$ is a clique of k vertices if $i = n - k + 1$; otherwise, $N_i[x_i]$ is a clique of $k+1$ vertices for $1 \le i \le n-k$. Figure 4 shows a 2-tree with the PEO $(v_1, v_2, \ldots, v_{13})$.

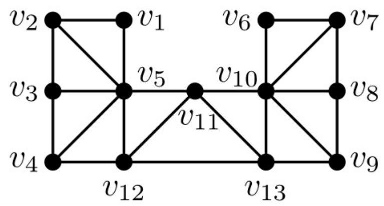

Figure 4. A 2-tree H.

In [3], Chang et al. showed that the MCTP is NP-complete for k-trees with unbounded k by proving $\gamma(G) = \tau_M(G)$ for any k-tree G. However, Figure 4 shows a counterexample that disproves $\gamma(G) = \tau_M(G)$ for any k-tree G. The graph H in Figure 4 is a 2-tree with the perfect elimination ordering $(v_1, v_2, \ldots, v_{13})$. The set $\{v_5, v_{10}\}$ is the minimum dominating set of H and the set $\{v_5, v_{10}, v_{11}\}$ is a minimum MCTS of H. A modified NP-completeness proof is therefore desired for the MCTP on k-tree with unbounded k.

Theorem 10. *The MCTP and the MCIP are NP-complete for k-trees with unbounded k.*

Proof. The CTP and the CIP are NP-complete for k-trees with unbounded k [8]. Since every maximal clique in a k-tree is also a maximum clique [27], an MCTS is a CTS and an MCIS is a CIS. Hence, the MCTP and the MCIP are NP-complete for k-trees with unbounded k. □

Theorem 11. *The SCTP is NP-complete for k-trees with unbounded k.*

Proof. Suppose that $k_1 \in \mathbb{N}$ and G is a k_1-tree with $|V(G)| > k_1$. Let $C(G) = \{C_1, C_2, \ldots, C_\ell\}$. Since G is a k_1-tree, $|C_i| = k_1 + 1$ for $1 \le i \le \ell$.

Let Q be a clique with $k_1 + 1$ vertices. Let H be a graph such that $V(H) = V(G) \cup Q$ and $E(H) = E(G) \cup \{(x, y) \mid x, y \in Q\} \cup \{(x, y) \mid x \in Q, y \in V(G)\}$. Let $X_i = C_i \cup Q$ be a clique for $1 \le i \le \ell$. Clearly, $C(H) = \{X_i \mid 1 \le i \le \ell\}$. Let $k_2 = 2k_1 + 1$. Then, H is a k_2-tree and $|X_i| = k_2 + 1 = 2k_1 + 2$ for $1 \le i \le \ell$. Clearly, we can verify that there exists a minimum-weight SCTF h of H of such that $h(x) = 1$ for every $x \in Q$. Then, $C_i = X_i \setminus Q$ contains at least one vertex x with $h(x) = 1$ for $1 \le i \le \ell$. Let $S = \{x \mid x \in V(H) \setminus Q$ and $h(x) = 1\}$. Then, S is a CTS of G. Since $\tau_C^s(H) = |Q| + 2|S| - |V(G)|$, we have $|Q| + 2\tau_C(G) - |V(G)| \le \tau_C^s(H)$.

Assume that D is a minimum CTS of G. Let f be a function of H whose domain is $V(H)$ and range is $\{-1, 1\}$, and (1) $f(x) = 1$ for every $x \in Q$, (2) $f(x) = 1$ for every $x \in D$, and (3) $f(x) = -1$ for every $x \in V(G) \setminus D$. Each maximal clique of H has at least $k_1 + 2$ vertices with the function value 1. Therefore, f is an SCTF. We have $\tau_C^s(H) \le |Q| + 2\tau_C(G) - |V(G)|$. Following what we have discussed above, we know that $\tau_C^s(H) = |Q| + 2\tau_C(G) - |V(G)|$. The theorem therefore holds by the NP-completeness of the CTP for k-trees [8]. □

Theorem 12. *Suppose that $\kappa \in \mathbb{N}$ the κ-FCTP is NP-complete on k-trees with unbounded k.*

Proof. Assume that $k_1 \in \mathbb{N}$ and G is a k_1-tree with $|V(G)| > k_1$. Let H be a graph such that $V(H) = V(G) \cup \{x\}$ and $E(H) = E(G) \cup \{(x, y) \mid y \in V(G)\}$. Clearly, H is a $(k_1 + 1)$-tree and we can construct H in linear time. Following the argument analogous to the proof of Theorem 8, we have $\tau_C^\kappa(H) = \tau_C^{\kappa-1}(G) + 1$. The theorem therefore holds by the NP-completeness of the CTP for k-trees [8]. □

Theorem 13. *The SCTP and κ-FCTP problems can be solved in linear-time for k-trees with fixed k.*

Proof. Assume that $\kappa \in \mathbb{N}$ and G is a graph. The κ-FCTP is the GCTP with the CSRF R whose domain is $C(G)$ and range is $\{\kappa\}$. By Lemma 5, $\tau_C^s(G)$ can be obtained from the solution to the GCTP on a graph G with a particular CSRF R. Since the GCTP is linear-time solvable for k-trees with fixed k [8], the SCTP and κ-FCTP are also linear-time solvable for k-trees with fixed k. $\square$

7. Planar, Total, and Line Graphs

In a graph, a vertex x and an edge e are *incident* to each other if e connects x to another vertex. Two edges are *adjacent* if they share a vertex in common. Let G and H be graphs such that each vertex $x \in V(H)$ corresponds to an edge $e_x \in E(G)$ and two vertices $x, y \in V(H)$ are adjacent in H if and only if their corresponding edges e_x and e_y are adjacent in G. Then, H is the *line graph* of G and denoted by $L(G)$. Let H' be a graph such that $V(H') = V(G) \cup E(G)$ and two vertices $x, y \in V(H')$ are adjacent in H if and only if x and y are adjacent or incident to each other in G. Then, H' is the *total graph* of G and denoted by $T(G)$.

Lemma 9 ([28]). *The following statements hold for any triangle-free graph G.*

(1) *Every maximal clique of $L(G)$ is the set of edges of G incident to some vertex of G.*
(2) *Two maximal cliques in $L(G)$ intersect if and only if their corresponding vertices (in G) are adjacent in G.*

Theorem 14. *The MCIP is NP-complete for any 4-regular planar graph G with the clique number 3.*

Proof. Since $|C(G)| = O(n)$ for any planar graph G [29], the MCIP on planar graphs is in NP. Let $\mathcal{G}$ be the class of triangle-free, 3-connected, cubic planar graphs. The independent set problem remains NP-complete even when restricted to the graph class $\mathcal{G}$ [30]. We reduce this NP-complete problem to the MCIP for 4-regular planar graphs with the clique number 3 as follows.

Let $G \in \mathcal{G}$ and $H = L(G)$. Clearly, we can construct H in polynomial time. By Lemma 9, we know that H is a 4-regular planar graph with $\omega(H) = 3$ and each maximal clique is a triangle in H.

Assume that $D = \{x_1, x_2, \ldots, x_\ell\}$ is an independent set of G of maximum cardinality. Since $G \in \mathcal{G}$, $deg_G(x) = 3$ for every $x \in V(G)$. Let $e_{i_1}, e_{i_2}, e_{i_3} \in E(G)$ have the vertex x_i in common for $1 \le i \le \ell$. Then, $e_{i_1}, e_{i_2}, e_{i_3}$ form a triangle in H. Let C_i be the triangle formed by $e_{i_1}, e_{i_2}, e_{i_3}$ in H for $1 \le i \le \ell$. For each pair of vertices $x_j, x_k \in D$, x_j is not adjacent to x_k in G. Therefore, C_j and C_k in H do not intersect. The set $\{C_1, C_2, \ldots, C_\ell\}$ is an MCIS of H. We have $\alpha(G) \le \alpha_M(H)$.

Assume that $S = \{C_1, C_2, \ldots, C_\ell\}$ is a maximum MCIS of H. Then, each $C_i \in S$ is a triangle in H. Let C_i be formed by $e_{i_1}, e_{i_2}, e_{i_3}$ in H for $1 \le i \le \ell$. Then, $e_{i_1}, e_{i_2}, e_{i_3}$ are incident to the same vertex in G. For $1 \le i \le \ell$, let $e_{i_1}, e_{i_2}, e_{i_3} \in E(G)$ have the vertex x_i in common. For each pair of $C_j, C_k \in S$, C_j and C_k do not intersect. Therefore, x_j is not adjacent to x_k in G. The set $\{x_1, x_2, \ldots, x_\ell\}$ is an independent set of G. We have $\alpha_M(H) \le \alpha(G)$.

Hence, $\alpha(G) = \alpha_M(H)$. For $k \in \mathbb{N}$, we know that $\alpha(G) \ge k$ if and only if $\alpha_M(G) \ge k$. $\square$

Corollary 5. *The MCIP is NP-complete for line graphs of triangle-free, 3-connected, cubic planar graphs.*

Proof. The corollary holds by the reduction of Theorem 14. $\square$

Theorem 15. *The MCIP problem is NP-complete for total graphs of triangle-free, 3-connected, cubic planar graphs.*

Proof. Since $|C(G)| = O(n)$ for a planar graph G, the MCIP on planar graphs is in NP. Let $\mathcal{G}$ be the classes of traingle-free, 3-connected, cubic planar graphs. The independent set problem remains NP-complete even when restricted to the graph class $\mathcal{G}$ [30]. We reduce this NP-complete problem to MCIP for for total graphs of triangle-free, 3-connected, cubic planar graphs. as follows

Let $G \in \mathcal{G}$ and $H = T(G)$. Clearly, we can construct H in polynomial time. By Lemma 9, we can verify that H is a 6-regular graph with $\omega(H) = 4$.

Assume that $D = \{x_1, x_2, \ldots, x_\ell\}$ is an independent set of G of maximum cardinality. Since $G \in \mathcal{G}$, $deg_G(x) = 3$ for every $x \in V(G)$. Let $e_{i_1}, e_{i_2}, e_{i_3} \in E(G)$ have the vertex x_i in common. Then, $x_i, e_{i_1}, e_{i_2}, e_{i_3}$ form a maximum clique in H. Let C_i be the maximum clique formed by $x_i, e_{i_1}, e_{i_2}, e_{i_3}$ in H for $1 \le i \le \ell$. For each pair of vertices $x_j, x_k \in D$, x_j is not adjacent to x_k in G. Therefore, C_j and C_k in H do not intersect. The set $\{C_1, C_2, \ldots, C_\ell\}$ is an MCIS of H. We have $\alpha(G) \le \alpha_M(H)$.

Assume that $S = \{C_1, C_2, \ldots, C_\ell\}$ is a maximum MCIS of H. By the construction of H, each $C_i \in S$ is formed by three edge-vertices in $E(G)$ and their common end vertex in $V(G)$. Let $x_i \in V$ and $e_{i_1}, e_{i_2}, e_{i_3} \in E(G)$ in H such that C_i is formed by $v_i, e_{i_1}, e_{i_2}, e_{i_3}$ for $1 \le i \le \ell$. For each pair of $C_j, C_k \in C$, C_j and C_k do not intersect. Therefore, x_j is not adjacent to x_k in G. The set $\{x_1, x_2, \ldots, x_\ell\}$ is an independent set of G. We have $\alpha_M(H) \le \alpha(G)$.

Hence, $\alpha(G) = \alpha_M(H)$. For $k \in \mathbb{N}$, we know that $\alpha(G) \ge k$ if and only if $\alpha_M(H) \ge k$. $\square$

Funding: This research is supported by a Taiwanese grant under Grant No. NSC-97-2218-E-130-002-MY2.

Acknowledgments: We are grateful to the anonymous referees for their valuable comments and suggestions to improve the presentation of this paper.

Conflicts of Interest: The author declares no conflict of interest.

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