Supplementary Materials: Tree Compatibility, Incomplete Directed Perfect Phylogeny, and Dynamic Graph Connectivity: An Experimental Study

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Table S1. Running time of FCT on real data sets.

	Legumes	Seabirds	Mammals
FCT(0)	0.0277 s	0.0051 s	0.1327 s
FCT(1)	0.0926 s	0.0192 s	0.3623 s

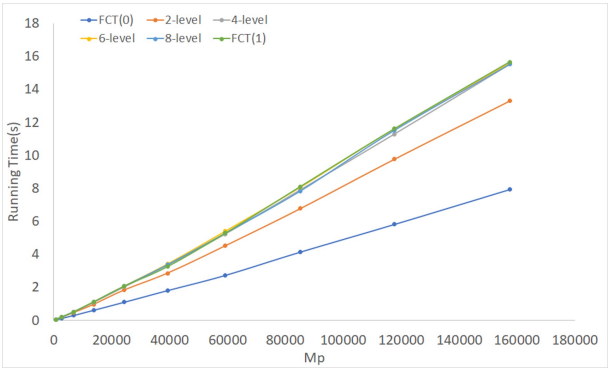

Figure S1. Performance of FCT for varying degrees of level truncation on complete sets of triples.

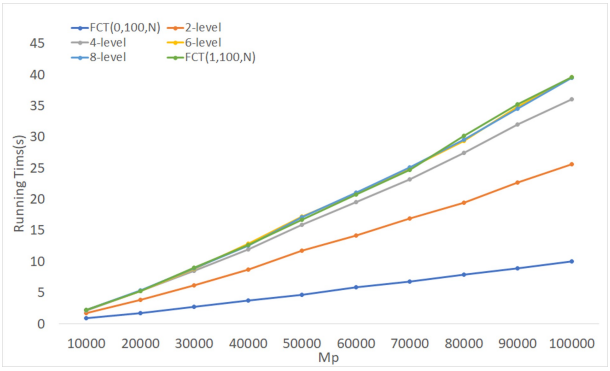

Figure S2. Performance of FCT for varying degrees of level truncation on binary phylogenetic trees with fixed k.

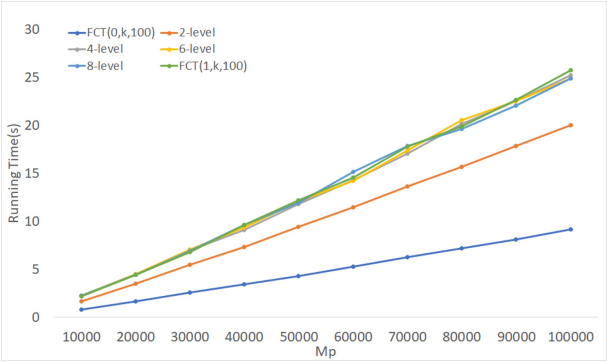

Figure S3. Performance of FCT for varying degrees of level truncation on binary phylogenetic trees with fixed n.

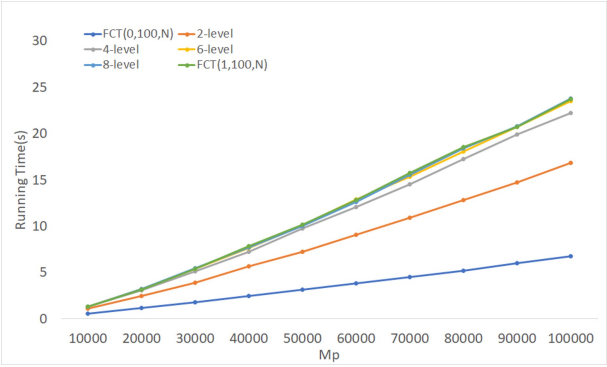

Figure S4. Performance of FCT for varying degrees of level truncation on phylogenetic trees with degree of 4; fixed k.

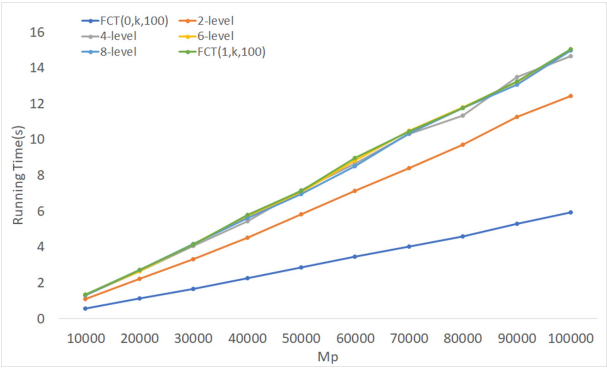

Figure S5. Performance of FCT for varying degrees of level truncation on phylogenetic trees with degree of 4; fixed n.

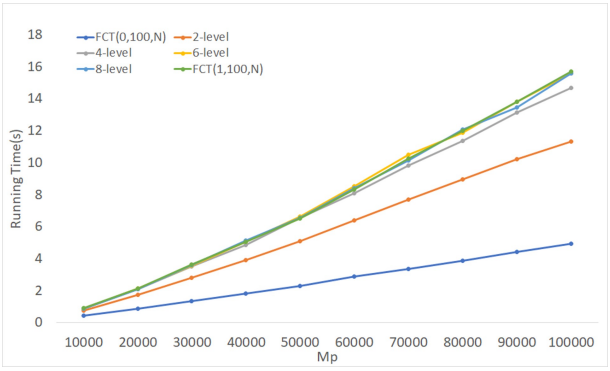

Figure S6. Performance of FCT for varying degrees of level truncation on phylogenetic trees with degree of 7; fixed k.

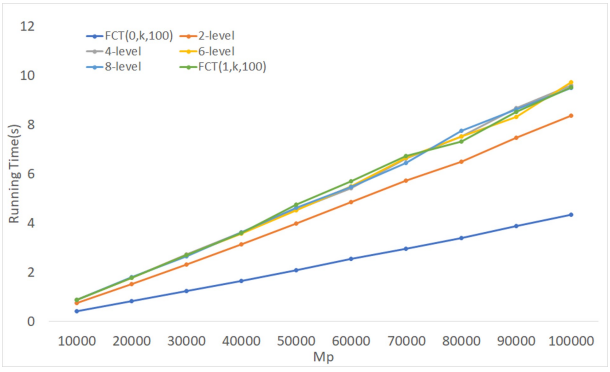

Figure S7. Performance of FCT for varying degrees of level truncation on phylogenetic trees with degree of 7; fixed n.

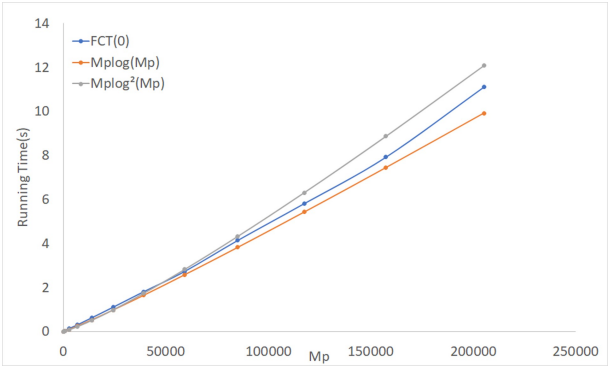

Figure S8. Running time of FCT(0) on complete sets of rooted triples.

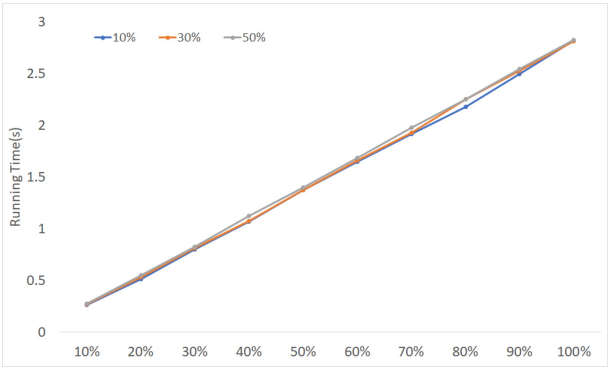

Figure S9. FCT(0) on profiles of rooted triples on 40 labels for different percentages of the maximum number of triples.

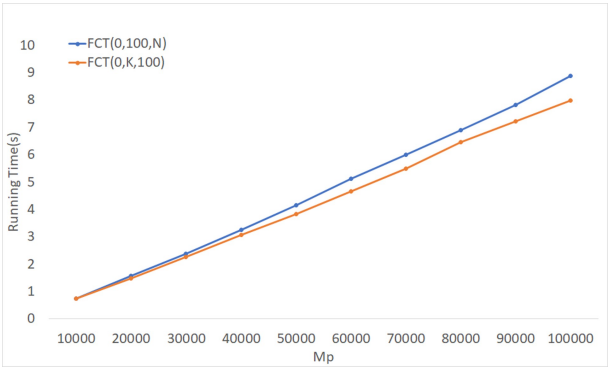

Figure S10. FCT(0) on profiles of binary phylogenetic trees. The orange curve corresponds to varying k, while keeping n fixed at 100. The blue curve corresponds to varying n while keeping k fixed at 100.

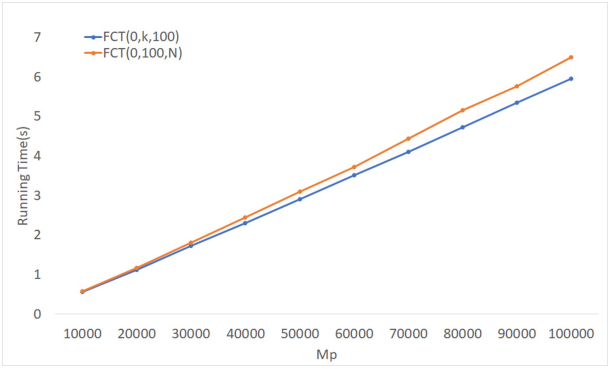

Figure S11. FCT(0) on profiles of phylogenetic trees of degree 4. The blue curve corresponds to varying k, while keeping n fixed at 100. The orange curve corresponds to varying n while keeping k fixed at 100.

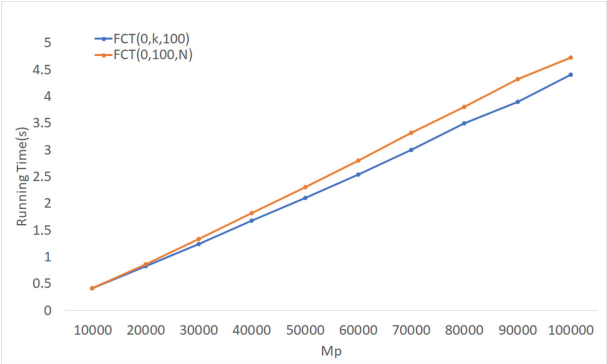

Figure S12. FCT(0) on profiles of phylogenetic trees of degree 7. The blue curve corresponds to varying k, while keeping n fixed at 100. The orange curve corresponds to varying n while keeping k fixed at 100.

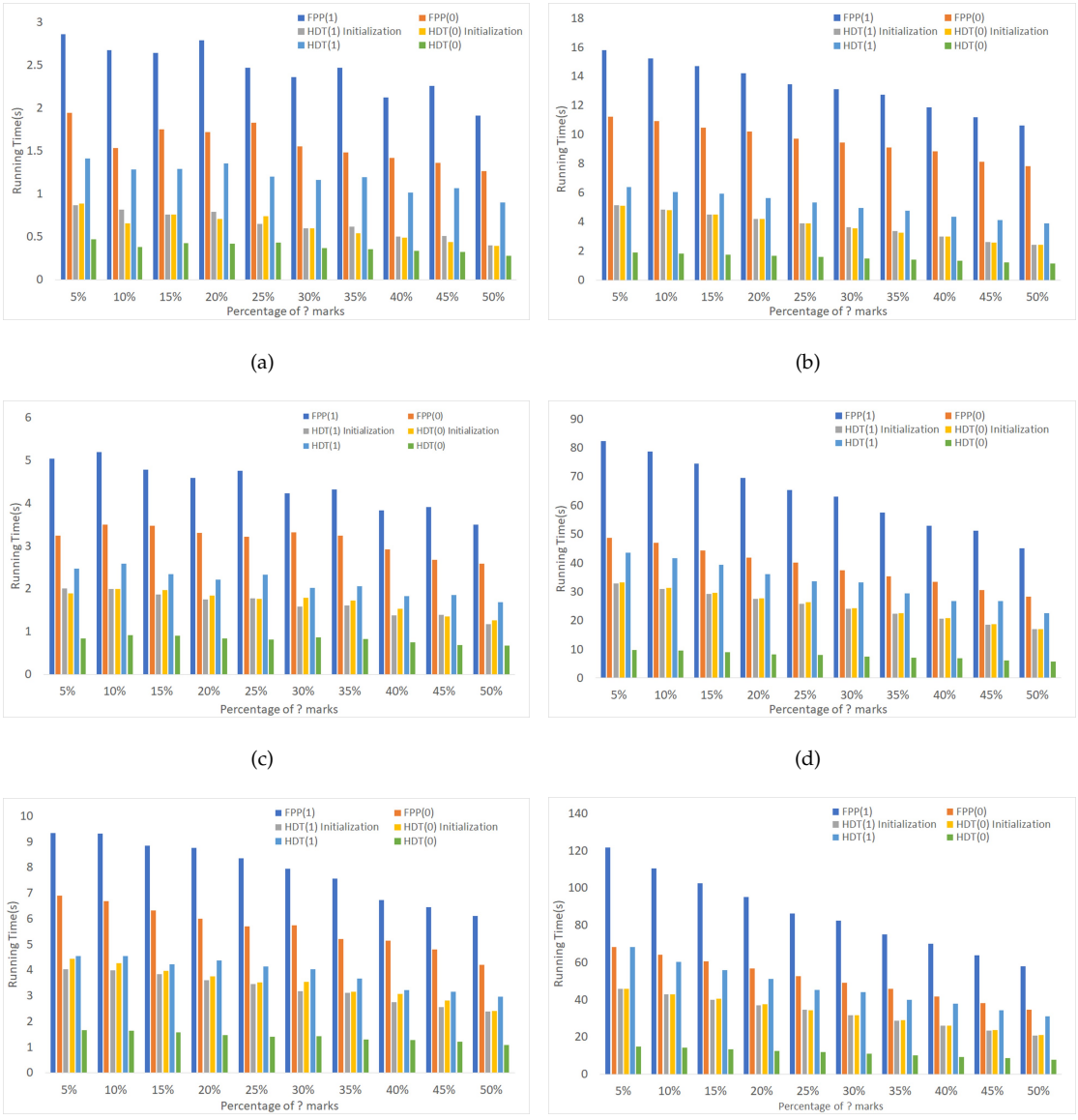

Figure S13. Running time of FPP(0) on matrices of order 100×2000 (left column) and order 300×6000 (right column) with different density levels: low ((a) and (b)), medium ((c) and (d)), and high ((e) and (f)).

Table S2. Comparison between execution time(in seconds) of tree compatibility algorithm on transformed IDPP and original IDPP on low density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
FCT(0)	10.23	9.58	8.96	8.67	7.54	7.11	6.65	6.09	5.79	5.53
FPP(0)	2.33	2.26	2.23	2.38	2.24	2.26	2.21	2.17	2.29	2.17
Connectivity (FCT)	3.53	3.25	3.02	2.96	2.74	2.62	2.52	2.26	2.17	2.18
Connectivity (FPP)	0.49	0.49	0.47	0.50	0.48	0.48	0.47	0.46	0.47	0.46

Table S3. Comparison between execution time (in seconds) of tree compatibility algorithm on transformed IDPP and original IDPP on high-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
FCT(0)	10.07	10.00	9.87	9.61	9.52	9.41	9.36	9.07	9.05	8.80
FPP(0)	6.43	6.48	6.38	6.32	6.43	5.68	6.30	6.55	6.44	6.12
Connectivity (FCT)	3.37	3.37	3.32	3.24	3.28	3.22	3.30	3.16	3.15	3.07
Connectivity (FPP)	1.38	1.36	1.33	1.28	1.36	1.26	1.35	1.42	1.38	1.30

Table S4. FCT(0) on complete sets of triples: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

$M_{\mathcal{P}}$	**65**	**730**	**2745**	**6860**	**13825**	**24390**	**39305**	**59320**	**85185**	**117650**
$\frac{Num(NTE)}{Num(E)}$	9.25%	19.51%	21.88%	21.08%	22.49%	23.00%	23.20%	22.49%	23.05%	23.13%
$\frac{Time(NTE)}{Time(HDT)}$	7.48%	9.49%	10.04%	9.86%	10.09%	10.28%	10.22%	9.99%	9.99%	9.83%

Table S5. FCT(0) on profiles of binary phylogenetic trees for $k = 100$ and varying n: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
$\frac{Num(NTE)}{Num(E)}$	17.13%	17.32%	17.25%	17.27%	17.25%	17.27%	17.35%	17.35%	17.32%	17.42%
$\frac{Time(NTE)}{Time(HDT)}$	8.03%	7.97%	7.64%	7.82%	7.81%	7.58%	7.69%	7.58%	7.62%	7.63%

Table S6. FCT(0) on profiles of binary phylogenetic trees for $n = 100$ and varying k: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
$\frac{Num(NTE)}{Num(E)}$	17.87%	18.76%	18.76%	18.79%	18.76%	19.17%	19.33%	19.13%	19.11%	19.35%
$\frac{Time(NTE)}{Time(HDT)}$	8.21%	8.30%	8.29%	8.35%	8.47%	8.22%	8.30%	8.38%	8.29%	8.39%

Table S7. FCT(0) on profiles of phylogenetic trees of degree 4 with $k = 100$ and varying n: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
$\frac{Num(NTE)}{Num(E)}$	30.79%	30.17%	30.37%	30.14%	29.95%	30.11%	29.88%	30.01%	29.94%	30.08%
$\frac{Time(NTE)}{Time(HDT)}$	10.75%	10.35%	10.10%	10.06%	9.92%	9.93%	9.81%	9.99%	9.68%	9.57%

Table S8. FCT(0) on profiles of phylogenetic trees of degree 4 with $n = 100$ and varying k: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
$\frac{Num(NTE)}{Num(E)}$	30.48%	33.16%	32.79%	33.83%	33.87%	34%	33.91%	33.96%	33.83%	33.76%
$\frac{Time(NTE)}{Time(HDT)}$	10.72%	11.40%	11.17%	11.49%	11.36%	11.62%	11.50%	11.53%	11.32%	11.22%

Table S9. FCT(0) on profiles of phylogenetic trees of degree 7 with $k = 100$ and varying n: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
$\frac{Num(NTE)}{Num(E)}$	42.23%	42.72%	42.69%	42.16%	42.27%	42.13%	42.11%	42.47%	42.14%	42.23%
$\frac{Time(NTE)}{Time(HDT)}$	14.07%	13.85%	13.39%	13.20%	13.32%	12.88%	12.65%	13.02%	12.92%	12.99%

Table S10. FCT(0) on profiles of phylogenetic trees of degree 7 with $n = 100$ and varying k: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
$\frac{Num(NTE)}{Num(E)}$	42.20%	42.59%	44.90%	45.04%	44.88%	44.92%	46.22%	46.44%	44.79%	44.97%
$\frac{Time(NTE)}{Time(HDT)}$	14.00%	13.94%	15.00%	14.93%	15.15%	15.13%	15.34%	15.40%	14.96%	14.94%

Table S11. FPP(0) on low-density matrices of order 300×6000: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
$\frac{Num(NTE)}{Num(E)}$	88.53%	87.95%	87.42%	87.12%	86.54%	85.94%	85.36%	84.62%	83.73%	82.70%
$\frac{Time(NTE)}{Time(HDT)}$	41.77%	41.01%	40.18%	39.86%	38.96%	38.07%	37.83%	36.89%	35.07%	34.10%

Table S12. FPP(0) on medium-density matrices of order 300×6000: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
$\frac{Num(NTE)}{Num(E)}$	96.59%	95.40%	96.17%	96.09%	95.73%	95.40%	95.20%	94.73%	94.35%	93.85%
$\frac{Time(NTE)}{Time(HDT)}$	56.36%	54.65%	54.61%	55.24%	53.33%	52.79%	52.84%	51.24%	50.24%	49.78%

Table S13. FPP(0) on high-density matrices of order 300×6000: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
$\frac{Num(NTE)}{Num(E)}$	97.80%	97.69%	97.58%	97.37%	97.24%	97.11%	96.88%	96.62%	96.39%	96.07%
$\frac{Time(NTE)}{Time(HDT)}$	68.38%	67.74%	67.17%	66.15%	65.99%	65.05%	64.53%	63.25%	62.74%	61.73%

Table S14. FPP(0) on low-density matrices of order 100×2000: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
$\frac{Num(NTE)}{Num(E)}$	77.26%	76.33%	75.50%	74.62%	74.02%	73.10%	72.00%	70.31%	69.02%	67.46%
$\frac{Time(NTE)}{Time(HDT)}$	28.55%	26.49%	26.56%	25.10%	25.47%	24.26%	23.25%	22.06%	20.97%	20.06%

Table S15. FPP(0) on medium-density matrices of order 100×2000: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
$\frac{Num(NTE)}{Num(E)}$	88.78%	88.28%	87.57%	87.03%	86.32%	85.00%	83.87%	82.61%	82.12%	79.89%
$\frac{Time(NTE)}{Time(HDT)}$	40.22%	39.76%	38.45%	38.06%	36.94%	35.61%	34.14%	32.68%	31.89%	30.48%

Table S16. FPP(0) on high-density matrices of order 100×2000: Ratio of number of deleted non-tree edges to total number of edges and ratio of time spent on deleting non-tree edges to total HDT execution time.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
$\frac{Num(NTE)}{Num(E)}$	94.07%	93.64%	92.98%	93.05%	92.47%	92.04%	91.60%	91.13%	90.26%	89.15%
$\frac{Time(NTE)}{Time(HDT)}$	54.88%	54.00%	51.90%	52.16%	50.92%	49.99%	48.68%	47.83%	46.01%	43.80%

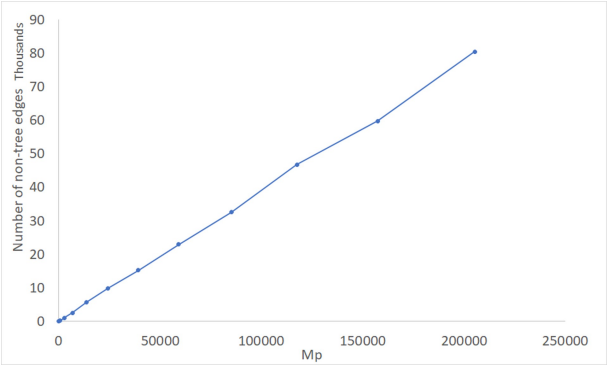

Figure S14. Number of non-tree edges scanned by FCT(0) for complete sets of rooted triples.

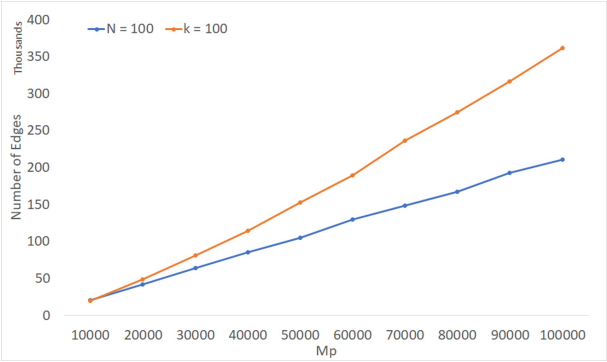

Figure S15. Number of non-tree edges scanned by FCT(0) for profiles of binary phylogenetic trees. The blue curve corresponds to varying k, while keeping n fixed at 100. The orange curve corresponds to varying n while keeping k fixed at 100.

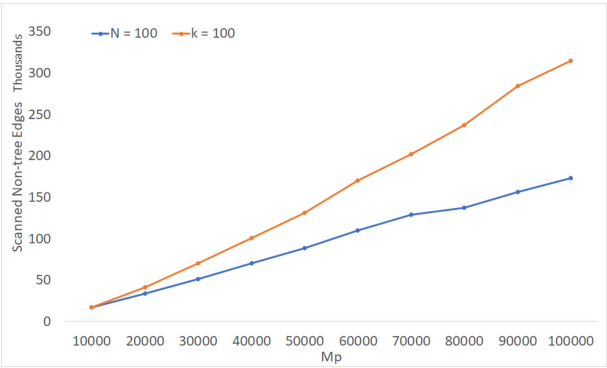

Figure S16. Number of non-tree edges scanned by FCT(0) for profiles of phylogenetic trees of degree of 4. The blue curve corresponds to varying k, while keeping n fixed at 100. The orange curve corresponds to varying n while keeping k fixed at 100.

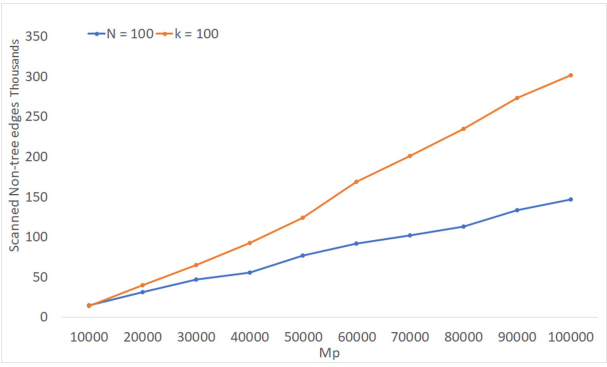

Figure S17. Number of non-tree edges scanned by FCT(0) for profiles of phylogenetic trees of degree 7. The blue curve corresponds to varying k, while keeping n fixed at 100. The orange curve corresponds to varying n while keeping k fixed at 100.

Table S17. Number of non-tree edges scanned by FPP(0) on matrices of order of 100 × 2000 with different density levels.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Low	19410	18650	16894	16499	15155	12714	11611	10107	9299.1	8400
Medium	34855	35560	27803	27185	27032	26305	24370	22992	21301	19245
High	8533	8623	7596	7376	7937	8547	7084	8891	6820	6345

Table S18. Number of non-tree edges scanned by FPP(0) on matrices of order of 300×6000 with different density levels.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Low	69133	65415	61975	57305	53638	47722	44590	40668	35545	31668
Medium	461875	420106	372561	400387	387020	353904	334324	292983	250507	251315
High	42210	32766	41489	36797	40709	29770	29068	29206	26317	26811

Table S19. Comparison of execution times (in seconds) of tree compatibility algorithm on transformed IDPP and original IDPP on low-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
FCT(0)	30.06	28.62	27.42	26.22	23.98	20.85	19.50	18.24	17.28	16.21
FPP(0)	2.83	2.76	2.75	2.93	2.78	2.82	2.78	2.74	2.87	2.75

Table S20. Comparison between entire execution time (in seconds) of tree compatibility algorithm on transformed IDPP and original IDPP on high-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
FCT(0)	27.00	26.71	26.36	25.83	25.53	25.34	25.12	24.63	24.46	23.93
FPP(0)	6.93	7.00	6.88	6.83	7.17	6.15	6.83	7.1	6.96	6.64

Table S21. FCT(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for complete set of triples.

$M_{\mathcal{P}}$	**65**	**730**	**2745**	**6860**	**13825**	**24390**	**39305**	**59320**	**85185**	**117650**
Num. ≤ 2	29	355	1385	3498	7130	12460	20110	30433	43911	61348
Num. > 2	2	23	50	81	113	151	191	233	277	318

Table S22. FCT(0) Number of subtrees of size at most two versus number of subtrees of size greater than two for profiles of binary trees with $n = 100$ and varying k.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
Num. ≤ 2	7453	14985	22566	29957	37532	44959	52500	59630	66984	74780
Num. > 2	667	1156	1648	2104	2555	3201	3460	3966	4643	4975

Table S23. FCT(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for profiles of binary trees with $k = 100$ and varying n.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
Num. ≤ 2	7473	15098	22711	30403	37903	45636	53083	60638	68336	75946
Num. > 2	683	1340	2037	2722	3387	4047	4759	5394	6017	6743

Table S24. FCT(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for profiles of binary trees of degree of 4 with $n = 100$ and varying k.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
Num. ≤ 2	4906	9770	14558	19519	24556	29378	34258	38899	43788	48769
Num. > 2	421	602	760	956	1081	1377	1478	1543	1727	1890

Table S25. FCT(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for profiles of trees of degree 4 with $k = 100$ and varying n.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
Num. ≤ 2	4897	9847	14949	19767	24854	29863	34844	39771	44824	49917
Num. > 2	397	864	1313	1747	2183	2618	3055	3477	3889	4330

Table S26. FCT(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for profiles of trees of degree 7 with $n = 100$ and varying k.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
Num. ≤ 2	3533	7078	10444	14000	17662	20975	24460	27944	31372	35200
Num. > 2	266	388	411	507	563	608	670	778	771	726

Table S27. FCT(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for profiles of trees of degree of with $k = 100$ and varying n.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
Num. ≤ 2	3543	6989	10505	14118	17589	21066	24665	28281	31638	35283
Num. > 2	273	587	883	1162	1507	1821	2096	2394	2719	3027

Table S28. FPP(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for low-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Num. ≤ 2	3384	3306	3238	3048	3001	2786	2673	2540	2377	2237
Num. > 2	375	369	374	380	387	375	376	374	368	355

Table S29. FPP(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for medium-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Num. ≤ 2	6238	6107	6251	6274	6124	6063	5964	5973	5854	5828
Num. > 2	304	317	317	329	323	337	345	345	359	355

Table S30. FPP(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for high-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Num. ≤ 2	8438	8438	8598	9138	7941	8016	7753	7557	7684	7260
Num. > 2	279	290	300	301	309	310	310	318	315	323

Table S31. FPP(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for low-density matrices of order 300×6000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Num. ≤ 2	12465	12129	11972	11522	11304	10913	10528	10076	9762	9188
Num. > 2	1130	1175	1201	1199	1218	1209	1204	1191	1199	1185

Table S32. FPP(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for medium-density matrices of order 300×6000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Num. ≤ 2	27592	27793	26951	27358	27358	27126	26711	26839	26590	26003
Num. > 2	924	926	940	972	969	983	983	976	976	979

Table S33. FPP(0): Number of subtrees of size at most two versus number of subtrees of size greater than two for high-density matrices of order 300×6000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
Num. ≤ 2	31130	30709	31054	30443	30194	29359	29393	28839	27603	26696
Num. > 2	882	908	905	907	901	908	892	912	900	879

Table S34. FCT(0): Number of subtrees of sizes $1, 2, \ldots, 8$ and greater than 8 for complete sets of triples.

$M_{\mathcal{P}}$	**65**	**730**	**2745**	**6860**	**13825**	**24390**	**39305**	**59320**	**85185**	**117650**
1	26	331	1286	3280	6757	11659	18876	28474	41200	57738
2	3	24	99	218	373	801	1234	1959	2711	3610
3	1	2	3	5	5	7	8	8	11	11
4	0	1	2	3	4	5	6	7	8	9
5	1	1	2	3	3	4	5	5	6	7
6	0	1	2	2	4	5	4	5	6	6
7	0	1	1	2	3	3	3	5	5	6
8	0	1	1	2	2	3	3	4	5	5
> 8	0	16	39	64	92	124	162	199	236	274

Table S35. FCT(0) Number of subtrees of sizes $1, 2, \ldots, 8$ and greater than 8 for profiles of binary trees with $n = 100$ and varying k.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
1	6983	14130	21315	28336	35531	42544	49581	56490	63285	70719
2	470	855	1251	1621	2001	2415	2919	3140	3699	4061
3	154	305	464	635	777	985	1140	1237	1529	1671
4	86	176	256	373	420	562	618	712	850	915
5	55	112	166	249	271	365	388	474	543	546
6	35	76	114	172	188	249	269	337	378	369
7	27	55	85	19	138	183	195	234	267	263
8	19	40	64	86	105	132	140	168	203	188
> 8	291	392	499	570	656	725	710	804	873	1023

Table S36. FCT(0): Number of subtrees of sizes $1, 2, \ldots, 8$ and greater than 8 for profiles of binary trees with $k = 100$ and varying n.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
1	7012	14188	21324	28550	35603	42883	49848	56924	64181	71334
2	461	910	1387	1853	2300	2753	3235	3714	4155	4612
3	158	287	434	586	721	862	1016	1154	1288	1448
4	88	157	239	315	393	472	560	615	689	794
5	56	101	149	197	242	290	350	392	431	495
6	38	70	102	137	166	197	238	266	294	348
7	27	48	71	101	125	143	174	190	218	245
8	20	37	54	78	95	113	132	151	167	182
> 8	296	640	988	1308	1645	1970	2289	2626	2930	3231

Table S37. FCT(0): Number of subtrees of sizes $1, 2, \ldots, 8$ and greater than 8 for profiles of trees of degree of 4 with $n = 100$ and varying k.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
1	4622	9261	13879	18581	23463	28039	32663	37129	41787	46578
2	284	509	679	938	1093	1339	1595	1770	2001	2191
3	106	169	232	309	364	475	546	607	702	755
4	55	86	109	143	179	244	248	302	309	337
5	32	48	59	84	100	145	148	155	144	174
6	22	31	36	52	56	79	88	88	80	100
7	14	19	25	34	35	49	48	51	49	63
8	11	14	18	21	19	33	30	31	32	40
> 8	181	235	281	313	328	352	370	309	411	421

Table S38. FCT(0): Number of subtrees of sizes $1, 2, \ldots, 8$ and greater than 8 for profiles of trees of degree 4 with $k = 100$ and varying n.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
1	4628	9311	14129	18704	23534	28243	32977	37643	42429	47249
2	269	536	820	1063	1320	1620	1867	2128	2395	2668
3	98	203	307	403	498	603	688	795	882	978
4	47	104	159	214	266	322	369	414	469	521
5	30	65	99	133	167	196	234	254	284	319
6	20	45	69	92	115	140	162	182	198	226
7	13	34	52	68	87	102	118	134	152	167
8	11	26	40	52	67	81	91	107	116	129
> 8	178	387	587	785	983	1174	1393	1591	1788	1990

Table S39. FCT(0): Number of subtrees of sizes $1, 2, \ldots, 8$ and greater than 8 for profiles of trees of degree 7 with $n = 100$ and varying k.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
1	3359	6774	10061	13505	17062	20261	23654	26977	30350	34174
2	174	304	383	495	600	714	806	967	1022	1026
3	61	100	99	136	163	182	210	256	248	236
4	31	49	44	59	70	77	89	111	110	87
5	20	26	26	35	35	40	46	53	54	43
6	13	17	19	22	22	22	25	28	29	24
7	9	14	13	15	15	15	18	18	18	17
8	9	9	11	12	12	12	12	14	13	13
Num. > 8	123	173	199	228	246	260	270	298	299	306

Table S40. FCT(0): Number of subtrees of sizes $1, 2, \ldots, 8$ and greater than 8 for profiles of trees of degree 7 with $k = 100$ and varying n.

$M_{\mathcal{P}}$	**10000**	**20000**	**30000**	**40000**	**50000**	**60000**	**70000**	**80000**	**90000**	**100000**
1	3363	6641	9977	13426	16738	20040	23454	26924	30102	33569
2	180	348	528	692	851	1026	1211	1357	1536	1714
3	66	130	200	255	330	409	454	513	595	654
4	33	69	107	140	177	218	250	290	335	363
5	23	45	67	91	119	144	162	187	213	242
6	13	32	48	62	83	99	112	129	146	171
7	10	24	35	45	62	74	89	97	112	125
8	8	17	28	38	49	59	68	78	88	100
> 8	120	270	398	531	687	818	961	1100	1230	1372

Table S41. FPP(0): Number of subtrees of sizes $1, 2, 3, 4$ and greater than 4 for low-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
1	3323	3244	3173	2982	2937	2720	2610	2477	2314	2178
2	61	62	66	66	65	66	64	63	63	59
3	23	26	27	29	27	28	30	30	28	29
4	16	17	19	20	20	19	18	19	19	20
> 4	335	326	328	331	340	329	328	325	321	306

Table S42. FPP(0): Number of subtrees of sizes $1, 2, 3, 4$ and greater than 4 for medium-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
1	6141	6008	6142	6167	6020	5956	5856	5864	5750	5714
2	97	99	108	107	104	107	109	109	104	114
3	50	53	49	51	49	50	51	47	48	51
4	26	25	27	27	27	29	28	27	31	28
> 4	229	239	241	250	247	258	266	271	279	276

Table S43. FPP(0): Number of subtrees of sizes $1, 2, 3, 4$ and greater than 4 for high-density matrices of order 100×2000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
1	8341	8340	8497	8030	7838	7913	7644	7443	7574	7153
2	97	98	101	108	103	103	109	113	110	108
3	51	54	53	52	55	54	49	55	52	52
4	28	27	30	30	30	31	32	31	30	33
> 4	200	209	217	219	224	226	229	232	233	238

Table S44. FPP(0): Number of subtrees ofsizes $1, 2, 3, 4$ and greater than 4 for low-density matrices of order 300×6000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
1	12251	11899	11751	11300	11076	10682	10301	9857	9552	8981
2	214	229	221	222	228	231	228	220	211	207
3	68	75	81	82	86	87	87	84	85	85
4	40	49	50	56	55	55	55	56	57	55
> 4	1021	1052	1070	1061	1077	1067	1062	1050	1057	1045

Table S45. FPP(0): Number of subtrees of sizes $1, 2, 3, 4$ and greater than 4 for medium-density matrices of order 300×6000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
1	27244	27442	26589	26991	26981	26749	26321	26445	26185	25606
2	348	351	362	367	377	377	391	394	405	398
3	152	146	153	156	160	163	167	175	161	167
4	108	107	103	104	107	108	103	98	98	96
> 4	664	673	685	712	702	713	713	703	716	716

Table S46. FPP(0): Number of subtrees of sizes $1, 2, 3, 4$ and greater than 4 for high-density matrices of order 300×6000.

	5%	**10%**	**15%**	**20%**	**25%**	**30%**	**35%**	**40%**	**45%**	**50%**
1	30822	30388	30730	30119	29865	29023	29064	28497	27258	26353
2	308	321	323	325	329	335	330	343	345	343
3	149	153	156	152	149	156	151	164	153	151
4	98	99	91	99	97	98	97	93	96	93
> 5	634	657	658	656	655	654	643	655	651	634