

Article

Energy Conservation in a Charged Retarded Field Engine

Prachi Sharma ¹ and Asher Yahalom ^{1,2,3,*}¹ Faculty of Engineering, Department of Electrical & Electronic Engineering, Ariel University, Ariel 40700, Israel; prachis@ariel.ac.il² Center for Astrophysics, Geophysics, and Space Sciences (AGASS), Ariel University, Ariel 40700, Israel³ FEL User Facility, Ariel University, Ariel 40700, Israel

* Correspondence: asya@ariel.ac.il; Tel.: +972-54-7740294

Abstract

Energy conservation, rooted in the time invariance of physical laws and formalized by Noether's theorem, requires that systems with space-time translational symmetry conserve momentum and energy. This work examines how this principle applies to a charged retarded field engine, where the rate of change of total energy—mechanical plus field energy—is balanced by the energy flux through the system's boundary. Using electric and magnetic field expressions from a Taylor expansion to incorporate retardation effects, we analyze the energy equation order by order for two arbitrary charged bodies. Our results show that total energy is conserved up to the fourth order, with mechanical and field energy changes exactly offset by boundary energy flux. Consequently, the work done by the internal electromagnetic field precisely equals the engine's gained mechanical kinetic energy, addressing the central focus of this study.

Keywords: electromagnetism; retardation; relativity

1. Introduction

Charged retarded systems represent a ground-breaking approach to space propulsion, aimed at addressing the major limitations of traditional rocket fuel. Robertson et al. [1] reviewed the principal barriers slowing advancements in space propulsion technologies and highlighted several theoretical ideas that, within the last twenty years, have evolved from speculative fiction into scientific theory. Thus, it is essential to explore various aspects of these systems, such as linear momentum and energy conservation, as all engines operate within the constraints of the laws of mechanics and the first law of thermodynamics: energy cannot be created or destroyed—only transformed.

To articulate clearly the research gap, we must show explicitly what is the source of energy of the charged retarded field engine and how it converts stored electromagnetic energy into kinetic energy. The present study specifically advances prior work, in which we calculated engine characteristics such as force and gained linear momentum, but no attention was given to the source of the engine's kinetic energy. Thus, we aim to explicitly connect the problem statement, which is the source of energy, to a methodology of analytical calculations, which will allow us to pinpoint this source. This illuminating calculation is the main result of the current paper.

This work discusses the retardation effects caused by the relativistic finite speed of signal propagation in a charged engine. As signals must propagate at a finite speed, it follows that an action and its corresponding reaction cannot occur simultaneously. Although retardation already exists in Maxwell's equations, relativity went even further to postulate that



Academic Editor: Paweł Ligeza

Received: 10 July 2025

Revised: 22 August 2025

Accepted: 29 August 2025

Published: 2 September 2025

Citation: Sharma, P.; Yahalom, A. Energy Conservation in a Charged Retarded Field Engine. *Energies* **2025**, *18*, 4661. <https://doi.org/10.3390/en18174661>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

no object, message, signal (of any kind, including non-electromagnetic), or field can travel faster than the speed of light in a vacuum [2–4]. In their paper, Griffiths and Heald [5] highlighted in that Coulomb’s law and the Biot–Savart law are strictly valid only for determining electric and magnetic fields from static sources. To extend their applicability beyond the static domain, Jefimenko’s time-dependent generalizations [6] of these laws were formulated, taking into account the retardation effect. In this case, momentum and energy are exchanged between the field and the material, enabling the relativistic engine effect. This result had already been observed and highlighted by several authors [7–11]. This should not be confused with photonic sails [12], in which external photons need to transfer their momentum to the device. Basically, photons can only be used to gain linear momentum (p , which is the concept of photonic sales) at amounts of $p = \frac{E}{c}$, in which E is the photons’ energy and c is the speed of light in vacuum; thus, it seems much better to store the energy (E) and use it as a source for a relativistic kinetic engine.

This process differs significantly from the transfer of momentum and energy between two material bodies, which is the basis of standard transportation (a rocket ejects material to propagate and a car moves forward by pushing against the road). Relativistic effects indicate a novel type of motor, consisting not of two material components but of a combination of matter and an electromagnetic field. In [13], we provided a detailed analysis of the retardation effect and presented the conceptual framework and design principles underlying a charged relativistic motor. In particular, we have shown that if retardation is not neglected in a two-body charged system, then the system will be affected by a total force (despite the lack of an external force) and will gain linear momentum. The system’s center of mass will move with a velocity equal to the total linear momentum divided by the system’s mass. Mathematical expressions for total force, total linear momentum, and system velocity were derived and shown to be of order $\frac{1}{c^2}$. Recently, we also investigated the interaction between charged bodies and magnetic currents generated by permanent magnetic materials, analyzing the potential implications for a charged relativistic engine utilizing a permanent magnet [14]. Energy conservation has already been discussed in the case of an uncharged relativistic engine [15], in which the energy transfers between the mechanical component of the relativistic engine and the electromagnetic field for a non charged engine. In this case, the total utilized electromagnetic energy is six times the kinetic energy gained by the relativistic motor. Furthermore, previous research [16] has provided a mathematical proof that demonstrates that linear momentum is conserved in a retarded electric motor that is uncharged. More explicitly, we calculated the electromagnetic linear momentum in a retarded electric motor and found it to be equal in quantity and opposite in sign to the gained mechanical linear momentum in the same retarded electric motor, as calculated in [17].

In the current work, we investigate a relativistic charged engine consisting of two subsystems of arbitrary geometry as in [13], with a clear intention to show that the kinetic energy acquired by the system is deducted from its stored field energy. We begin by analyzing the energy exchange between mechanical and electromagnetic components in the relativistic regime. Subsequently, we carry out a detailed examination of the engine’s energy dynamics by expanding the relevant field quantities as a power series in $1/c$, where c denotes the speed of light in vacuum. This expansion systematically incorporates the effects of retardation. The total energy, comprising mechanical energy, electromagnetic field energy, and energy flux via the Poynting vector, is analyzed to verify the energy balance equation at each order. Specifically, we evaluate the contributions to the energy equation for $n = 0, 1, 2, 3$, and 4, demonstrating how retardation influences energy exchange and confirming that the energy balance is satisfied at each order within the accuracy of the

expansion. Finally, we demonstrate that the kinetic energy acquired by the engine exactly equals the amount of electromagnetic energy lost.

2. Energy Conservation

The conservation of energy in the general electromagnetic system is expressed by the following equation:

$$\frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = - \oint_S \vec{S}_P \cdot \hat{n} da. \quad (1)$$

This equation reflects the fact that any decrease (or increase) in the total energy within the system is accounted for by the energy carried away (or supplied) by the electromagnetic field through the system's boundary. Here, S denotes a closed surface enclosing the volume of interest, $\hat{n}$ is a unit vector normal to the surface, and da represents a surface element.

Poynting's vector is defined as follows:

$$\vec{S}_P = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{\epsilon_0}{\mu_0 \epsilon_0} \vec{E} \times \vec{B} = \epsilon_0 c^2 \vec{E} \times \vec{B}, \quad (2)$$

where $\epsilon_0 = 8.85 \times 10^{-12} \text{ Fm}^{-1}$ is the vacuum permittivity, $\mu_0 = 4\pi \times 10^{-7} \text{ (H/m)}$, and $c \simeq 3 \times 10^8 \text{ m/s}$ is the velocity of light in vacuum. The term $\oint_S \vec{S}_P \cdot \hat{n} da$ represents the total electromagnetic power flux through the closed surface (S). The field energy density is defined as follows:

$$E_{field} = \frac{\epsilon_0}{2} \int (\vec{E}^2 + c^2 \vec{B}^2) d^3x. \quad (3)$$

The derivative of the mechanical energy is the mechanical power given by the following expression:

$$Power \equiv \frac{dE_{mech}}{dt} = \int d^3x \vec{j} \cdot \vec{E}, \quad (4)$$

where $\vec{j}$ is the current density. Hence,

$$\int d^3x \vec{j} \cdot \vec{E} + \frac{dE_{field}}{dt} = - \oint_S \vec{S}_P \cdot \hat{n} da \quad (5)$$

or

$$\int d^3x \vec{j} \cdot \vec{E} = - \frac{dE_{field}}{dt} - \oint_S \vec{S}_P \cdot \hat{n} da. \quad (6)$$

In what follows, we interpret Equation (6) in terms of its role in governing energy conservation for a charged relativistic engine system.

Kinetic Energy

According to [13] (see also Appendix K), the mechanical momentum associated with the time-dependent charged retarded field engine is expressed as follows:

$$\vec{P}(t) = \frac{\mu_0}{4\pi} \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} (\rho_2 \partial_t \rho_1 - \rho_1 \partial_t \rho_2) \hat{R} - (\rho_1 \vec{j}_2 + \rho_2 \vec{j}_1) R^{-1} \right], \quad (7)$$

or taking into account the fact that $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ and, as is customary, defining $k = \frac{1}{4\pi \epsilon_0}$,

$$\vec{P}(t) = \frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} (\rho_2 \partial_t \rho_1 - \rho_1 \partial_t \rho_2) \hat{R} - (\rho_1 \vec{j}_2 + \rho_2 \vec{j}_1) R^{-1} \right]. \quad (8)$$

It is clear from the expression that the mechanical momentum is of the order of $\frac{1}{c^2}$. The kinetic mechanical energy associated with this momentum is given by the following expression:

$$E_k = \frac{\vec{P}^2}{2M} = \frac{1}{2} \vec{P} \cdot \vec{v}_s, \quad (9)$$

where M is the mass of the relativistic engine and the engine velocity ($\vec{v}_s$) is defined as follows:

$$\vec{v}_s \equiv \frac{\vec{P}}{M} \propto \frac{1}{c^2} \quad (10)$$

We notice that the expression for kinetic energy is of order $\frac{1}{c^4}$; lower order corrections do not exist, and higher order corrections are neglected.

3. Field Energy for Two Charged Sub-Systems

Consider two arbitrary charged sub-systems denoted system 1 and system 2 (see Figure 1).

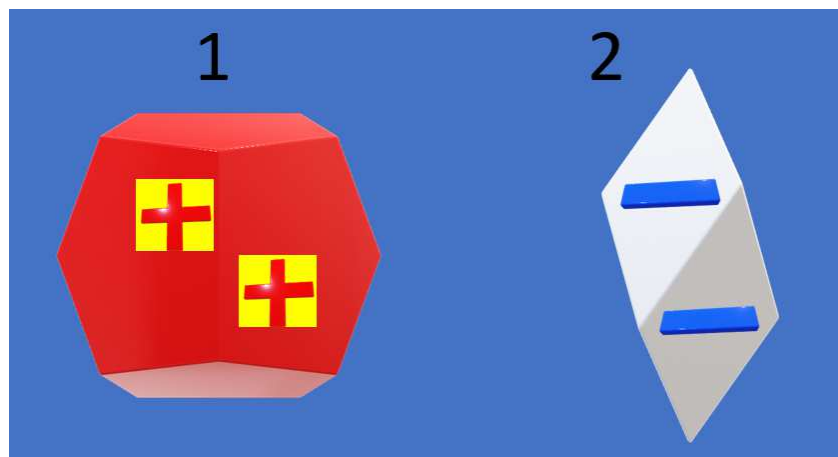


Figure 1. Two charged bodies.

The charged subsystems are far apart such that their interaction is negligible. In this case, Equation (6) is correct for each sub-system separately, that is,

$$\int d^3x \vec{j}_1 \cdot \vec{E}_1 = -\frac{dE_{field1}}{dt} - \oint_S \vec{S}_{P1} \cdot \hat{n} da. \quad (11)$$

$$\int d^3x \vec{j}_2 \cdot \vec{E}_2 = -\frac{dE_{field2}}{dt} - \oint_S \vec{S}_{P2} \cdot \hat{n} da. \quad (12)$$

Next, we put them closer together such that they may interact but without modifying the charge or the current densities of each of the subsystems. The total fields of the combined system are expressed as follows:

$$\vec{E} = \vec{E}_1 + \vec{E}_2, \quad \vec{B} = \vec{B}_1 + \vec{B}_2. \quad (13)$$

Since both the field energy (Equation (3)) and Poynting's vector Equation (2) are quadratic in the fields, the following result is obtained:

$$E_{field} \equiv \frac{\epsilon_0}{2} \int (\vec{E}^2 + c^2 \vec{B}^2) d^3x = E_{field1} + E_{field2} + E_{field12}. \quad (14)$$

$$E_{field1} \equiv E_{Efield1} + E_{Mfield1} \equiv \frac{\epsilon_0}{2} \int (\vec{E}_1^2 + c^2 \vec{B}_1^2) d^3x, \quad (15)$$

$$E_{field2} \equiv E_{Efield2} + E_{Mfield2} \equiv \frac{\epsilon_0}{2} \int (\vec{E}_2^2 + c^2 \vec{B}_2^2) d^3x, \quad (16)$$

$$E_{field12} \equiv E_{Efield12} + E_{Mfield12} \equiv \epsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2 + c^2 \vec{B}_1 \cdot \vec{B}_2) d^3x. \quad (17)$$

$$\vec{S}_P \equiv \epsilon_0 c^2 \vec{E} \times \vec{B} = \vec{S}_{P1} + \vec{S}_{P2} + \vec{S}_{P12}. \quad (18)$$

$$\vec{S}_{P1} \equiv \epsilon_0 c^2 \vec{E}_1 \times \vec{B}_1, \quad (19)$$

$$\vec{S}_{P2} \equiv \epsilon_0 c^2 \vec{E}_2 \times \vec{B}_2, \quad (20)$$

$$\vec{S}_{P12} \equiv \epsilon_0 c^2 (\vec{E}_1 \times \vec{B}_2 + \vec{E}_2 \times \vec{B}_1). \quad (21)$$

The power invested in the combined system is bilinear for the current density and electric field according to Equation (4). This will lead to the following expression:

$$Power = \int d^3x \vec{j} \cdot \vec{E} = Power_1 + Power_2 + Power_{12}. \quad (22)$$

$$Power_1 = \int d^3x \vec{j}_1 \cdot \vec{E}_1, \quad (23)$$

$$Power_2 = \int d^3x \vec{j}_2 \cdot \vec{E}_2, \quad (24)$$

$$Power_{12} = \int d^3x (\vec{j}_1 \cdot \vec{E}_2 + \vec{j}_2 \cdot \vec{E}_1). \quad (25)$$

Subtracting the expressions given in Equations (11) and (12) from Equation (6) yields the following:

$$Power - Power_1 - Power_2 = -\frac{d(E_{field} - E_{field1} - E_{field2})}{dt} - \oint_S (\vec{S}_P - \vec{S}_{P1} - \vec{S}_{P2}) \cdot \hat{n} da. \quad (26)$$

Taking into account Equations (14)–(25), we arrive at the following:

$$Power_{12} = -\frac{dE_{field12}}{dt} - \oint_S \vec{S}_{P12} \cdot \hat{n} da. \quad (27)$$

The significance of the above equation is best illustrated by the flow chart presented in Figure 2, which illustrates the possible exchange between stored electromagnetic energy, mechanical work and radiated energy.

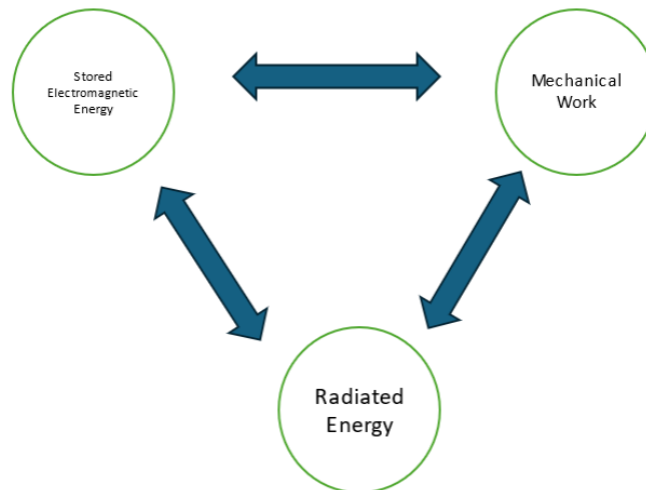


Figure 2. An illustration of the possible exchange between stored electromagnetic energy, mechanical work, and radiated energy.

4. Power Expansion in $\frac{1}{c}$

Let us now expand Equation (27) in power of $\frac{1}{c}$; to this end, we introduce the notation of $G^{[n]}$ to designate a term of order $\frac{1}{c}$ and quantity G . Thus, any quantity (G) may be written as the following sum:

$$G = \sum_{n=0}^{\infty} G^{[n]}. \quad (28)$$

Specifically,

$$Power_{12}^{[n]} = -\frac{dE_{field12}^{[n]}}{dt} - \oint_S \vec{S}_{P12}^{[n]} \cdot \hat{n} da. \quad (29)$$

It follows from Equation (25) that

$$Power_{12}^{[n]} = \int d^3x \left(\vec{J}_1 \cdot \vec{E}_2^{[n]} + \vec{J}_2 \cdot \vec{E}_1^{[n]} \right). \quad (30)$$

It also follows from Equation (17) that

$$E_{field12}^{[n]} = \epsilon_0 \int \left(\sum_{k=0}^n \vec{E}_1^{[n-k]} \cdot \vec{E}_2^{[k]} + c^2 \sum_{k=0}^{n+2} \vec{B}_1^{[n+2-k]} \cdot \vec{B}_2^{[k]} \right) d^3x. \quad (31)$$

Finally, from Equation (21), it follows that

$$\vec{S}_{P12}^{[n]} = \epsilon_0 c^2 \sum_{k=0}^{n+2} \left(\vec{E}_1^{[n+2-k]} \times \vec{B}_2^{[k]} + \vec{E}_2^{[n+2-k]} \times \vec{B}_1^{[k]} \right). \quad (32)$$

In the subsections below, we discuss Equation (29) for different values of n —from 0 up to 4. An analysis of the field contributions can now be carried out, considering each order in turn.

4.1. $n = 0$

Let us look at Equation (27) and study it for the zeroth order in $\frac{1}{c}$:

$$Power_{12}^{[0]} = -\frac{dE_{field12}^{[0]}}{dt} - \oint_S \vec{S}_{P12}^{[0]} \cdot \hat{n} da. \quad (33)$$

4.1.1. Power

We start by calculating $Power_{12}^{[0]}$ according to Equation (30):

$$Power_{12}^{[0]} = \int d^3x \left(\vec{j}_1 \cdot \vec{E}_2^{[0]} + \vec{j}_2 \cdot \vec{E}_1^{[0]} \right). \quad (34)$$

After some calculations elaborated upon in Appendix A, it follows from Equation (A18) that

$$Power_{12}^{[0]} = -\frac{d}{dt} \left[k \int \int d^3x_1 d^3x_2 \frac{\rho_1 \rho_2}{R} \right]. \quad (35)$$

4.1.2. Field Energy

Next, we calculate field energy according to Equation (31).

$$E_{field12}^{[0]} = \epsilon_0 \int \left(\vec{E}_1^{[0]} \cdot \vec{E}_2^{[0]} + c^2 \left[\vec{B}_1^{[2]} \cdot \vec{B}_2^{[0]} + \vec{B}_1^{[1]} \cdot \vec{B}_2^{[1]} + \vec{B}_1^{[0]} \cdot \vec{B}_2^{[2]} \right] \right) d^3x \quad (36)$$

The magnetic field does not have terms that are lower than second order. We use the null magnetic field expressions from Equation (58) of [13]:

$$\vec{B}_1^{[0]} = \vec{B}_2^{[0]} = \vec{B}_1^{[1]} = \vec{B}_2^{[1]} = 0. \quad (37)$$

Accordingly, the field energy takes the following form:

$$E_{field12}^{[0]} = \epsilon_0 \int \left(\vec{E}_1^{[0]} \cdot \vec{E}_2^{[0]} \right) d^3x. \quad (38)$$

After calculations elaborated upon in Appendix B and taking into account the identity expressed as $k = \frac{1}{4\pi\epsilon_0}$, it follows from Equation (A28) that

$$E_{field12}^{[0]} = k \left[\int \int d^3x_1 d^3x_2 \frac{\rho_1 \rho_2}{R} \right]. \quad (39)$$

Based on Equations (35) and (39), it is now clear that

$$Power_{12}^{[0]} = -\frac{dE_{field12}^{[0]}}{dt}. \quad (40)$$

It follows that the power related to mechanical work equals less than the temporal rate of change of the stored field energy.

4.1.3. Poynting Vector

Finally, we study the Poynting vector according to Equation (32):

$$\vec{S}_{P12}^{[0]} = \epsilon_0 c^2 \left(\vec{E}_1^{[2]} \times \vec{B}_2^{[0]} + \vec{E}_1^{[1]} \times \vec{B}_2^{[1]} + \vec{E}_1^{[0]} \times \vec{B}_2^{[2]} + \vec{E}_2^{[2]} \times \vec{B}_1^{[0]} + \vec{E}_2^{[1]} \times \vec{B}_1^{[1]} + \vec{E}_2^{[0]} \times \vec{B}_1^{[2]} \right). \quad (41)$$

It is clear from Equation (37) that

$$\vec{S}_{P12}^{[0]} = \epsilon_0 c^2 \left(\vec{E}_1^{[0]} \times \vec{B}_2^{[2]} + \vec{E}_2^{[0]} \times \vec{B}_1^{[2]} \right), \quad (42)$$

As both $\vec{E}^{[0]}$ and $\vec{B}^{[2]}$ decrease as $\frac{1}{r^2}$ at infinity, it follows that their cross product decreases as $\frac{1}{r^4}$; thus, a surface integral over an infinite sphere will also vanish, which is what should be expected from Equation (40), balancing the changes in the electromagnetic energy and mechanical power without any Poynting contribution. For $n = 0$, we conclude that the energy equation of order zero is, indeed, balanced, as is expected.

4.1.4. Summary

The key results of the $n = 0$ calculation is that in this order, stored field energy, which is basically electric (Coulomb type), can be transferred to mechanical energy and vice-versa, but no energy can be radiated. This is illustrated in Figure 3.

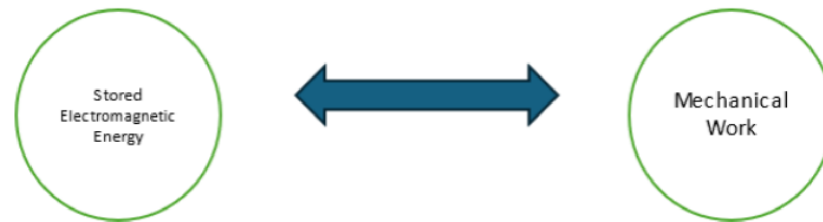


Figure 3. Energy exchange for $n \leq 2$. Stored electromagnetic energy can be converted to kinetic energy and vice-versa, but energy cannot be radiated.

4.2. $n = 1$

Let us look at Equation (27) and study it for the first order in $\frac{1}{c}$:

$$Power_{12}^{[1]} = -\frac{dE_{field12}^{[1]}}{dt} - \oint_S \vec{S}_{P12}^{[1]} \cdot \hat{n} da. \quad (43)$$

4.2.1. Power

The first-order expression for the power invested in mechanical work is expressed as follows:

$$Power_{12}^{[1]} = \int d^3x \left(\vec{j}_1 \cdot \vec{E}_2^{[1]} + \vec{j}_2 \cdot \vec{E}_1^{[1]} \right). \quad (44)$$

According to Equations (51) and (54) of [13], the first-order term in the electric field expansion is expressed as follows:

$$\vec{E}_1^{[1]} = \vec{E}_2^{[1]} = 0. \quad (45)$$

Therefore,

$$Power_{12}^{[1]} = 0. \quad (46)$$

4.2.2. Field Energy

Now, we calculate the field energy according to Equation (31) for $n = 1$.

$$E_{field12}^{[1]} = \epsilon_0 \int \left(\vec{E}_1^{[0]} \cdot \vec{E}_2^{[1]} + \vec{E}_1^{[1]} \cdot \vec{E}_2^{[0]} + c^2 \sum_{k=0}^3 \vec{B}_1^{[3-k]} \cdot \vec{B}_2^{[k]} \right) d^3x. \quad (47)$$

From Equations (37) and (45), we obtain the following:

$$E_{field12}^{[1]} = 0. \quad (48)$$

4.2.3. Poynting Vector

Taking into account Equations (37) and (45), the first-order expression for the Poynting vector is given by the following expression:

$$\vec{S}_{P12}^{[1]} = \epsilon_0 c^2 \left(\vec{E}_1^{[0]} \times \vec{B}_2^{[3]} + \vec{E}_2^{[0]} \times \vec{B}_1^{[3]} \right), \quad (49)$$

According to Equation (61) of [13], it follows that

$$\vec{B}^{[3]} = 0. \quad (50)$$

Hence,

$$\vec{S}_{P12}^{[1]} = 0. \quad (51)$$

Thus, for $n = 1$, we conclude that the energy equation of order one is, indeed, balanced in a trivial way. Equation (27) is satisfied in the sense that $0 = 0$.

4.2.4. Summary

The key result of the $n = 1$ calculation is that in this order, both electric and magnetic-field components vanish. Thus, no mechanical work can be done, no energy can be stored, and no energy can be radiated.

4.3. $n = 2$

Let us look at Equation (27) and study it for the second order in $\frac{1}{c}$:

$$Power_{12}^{[2]} = -\frac{dE_{field12}^{[2]}}{dt} - \oint_S \vec{S}_{P12}^{[2]} \cdot \hat{n} da. \quad (52)$$

4.3.1. Power

The second-order expression for the power invested in mechanical work is given below:

$$Power_{12}^{[2]} = \int d^3x \left(\vec{J}_1 \cdot \vec{E}_2^{[2]} + \vec{J}_2 \cdot \vec{E}_1^{[2]} \right). \quad (53)$$

After calculations elaborated upon in Appendix C and following Equation (A40), we obtain the following:

$$Power_{12}^{[2]} = -\frac{k}{c^2} \frac{d}{dt} \left[\int \int d^3x_1 d^3x_2 \left(\frac{R(\partial_t \rho_1(\vec{x}_1) \partial_t \rho_2(\vec{x}_2))}{2} + \frac{\vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2)}{R} \right) \right]. \quad (54)$$

Thus, we obtain a power expression accurate to the second order.

4.3.2. Field Energy

We now proceed to calculate the field energy using Equation (31) for $n = 2$:

$$E_{field12}^{[2]} = \epsilon_0 \int \left(\sum_{k=0}^2 \vec{E}_1^{[2-k]} \cdot \vec{E}_2^{[k]} + c^2 \sum_{k=0}^4 \vec{B}_1^{[4-k]} \cdot \vec{B}_2^{[k]} \right) d^3x \quad (55)$$

As we already know that

$$\vec{E}_1^{[1]} = \vec{E}_2^{[1]} = 0, \quad \vec{B}_1^{[0]} = \vec{B}_2^{[0]} = 0, \quad \vec{B}_1^{[1]} = \vec{B}_2^{[1]} = 0, \quad (56)$$

it follows that the field energy takes the following simplified form:

$$E_{field12}^{[2]} = \epsilon_0 \int \left(\vec{E}_1^{[0]} \cdot \vec{E}_2^{[2]} + \vec{E}_1^{[2]} \cdot \vec{E}_2^{[0]} + c^2 \vec{B}_1^{[2]} \cdot \vec{B}_2^{[2]} \right) d^3x. \quad (57)$$

After calculations elaborated upon in Appendix D and combining Equations (A73), (A64) and (A65), we obtain the following expression for the second-order field energy:

$$E_{field12}^{[2]} = \frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left[\frac{\vec{J}_1 \cdot \vec{J}_2}{R} + \frac{R}{2} \partial_t \rho_1 \partial_t \rho_2 \right]. \quad (58)$$

4.3.3. Poynting Vector

Next, we study the Poynting vector according to Equation (32) for $n = 2$:

$$\vec{S}_{P12}^{[2]} = \epsilon_0 c^2 \sum_{k=0}^4 \left(\vec{E}_1^{[4-k]} \times \vec{B}_2^{[k]} + \vec{E}_2^{[4-k]} \times \vec{B}_1^{[k]} \right), \quad (59)$$

According to Equation (56), we obtain the following resulting form of the Poynting vector:

$$\vec{S}_{P12}^{[2]} = \epsilon_0 c^2 \left(\vec{E}_1^{[2]} \times \vec{B}_2^{[2]} + \vec{E}_1^{[0]} \times \vec{B}_2^{[4]} + \vec{E}_2^{[2]} \times \vec{B}_1^{[2]} + \vec{E}_2^{[0]} \times \vec{B}_1^{[4]} \right). \quad (60)$$

The Poynting vector term, typically associated with radiation, contributes to the energy balance only through the surface integral over a closed surface enclosing the system. When this surface is taken to be spherical and located at a far distance (R_{max}), only the asymptotic forms of the electric and magnetic fields are relevant for evaluating the Poynting vector contribution. Accordingly, the expression for the Poynting flux on a far-away sphere of radius R_{max} is given by the following expression:

$$PF^{[2]} = \oint \vec{S}_{P12}^{[2]} \cdot \hat{n} da = R_{max}^2 \oint_{R_{max}} \vec{S}_{P12}^{[2]} \cdot \hat{r} d\Omega. \quad (61)$$

After calculations elaborated upon in Appendix E and plugging Equations (A85), (A86), (A90) and (A91) into Equation (A74), we obtain the following:

$$PF^{[2]} = 0, \quad (62)$$

in which the equality sign is only correct for the current approximation. Hence, there is no Poynting vector contribution to the energy balance. This is expected, as the mechanical work (Equation (54)) is balanced by the field energy loss Equation (58) in the second-order terms of $\frac{1}{c}$:

$$Power_{12}^{[2]} = -\frac{dE_{field12}^{[2]}}{dt}. \quad (63)$$

It is remarkable that no radiation losses need to be considered up to the second-order approximation. This will have implications for the retarded field motor that we discuss in later sections of this paper.

4.3.4. Summary

The key results of the $n = 2$ calculation are outlined as follows. First, we notice that this is the first order in which retardation effects are present in a non-trivial way. However, as we have shown, no energy is lost to radiation; thus, the only exchange that is possible is between stored energy and mechanical work and vice-versa. This is very significant for the development of a retarded field engine, as we emphasized above and will elaborate upon later in this paper. This is illustrated in Figure 3.

4.4. $n = 3$

Let us look at Equation (27) and study it for the third order in $\frac{1}{c}$:

$$Power_{12}^{[3]} = -\frac{dE_{field12}^{[3]}}{dt} - \oint_S \vec{S}_{P12}^{[3]} \cdot \hat{n} da. \quad (64)$$

4.4.1. Power

The third-order expression for the power invested in mechanical work is given below:

$$Power_{12}^{[3]} = \int d^3x \left(\vec{J}_1 \cdot \vec{E}_2^{[3]} + \vec{J}_2 \cdot \vec{E}_1^{[3]} \right) \quad (65)$$

After calculations elaborated upon in Appendix F and combining Equations (A104) and (A105), we obtain the following:

$$Power_{12}^{[3]} = \left(\frac{2k}{3c^3} \right) \int \int d^3x_1 d^3x_2 \left(\vec{J}_1(\vec{x}_1) \cdot \partial_t^2 \vec{J}_2(\vec{x}_2) + \vec{J}_2(\vec{x}_2) \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) \right). \quad (66)$$

The obtained expression represents the third-order mechanical power.

4.4.2. Field Energy

We now proceed to calculate the field energy using Equation (31) for $n = 3$:

$$E_{field12}^{[3]} = \epsilon_0 \int \left(\sum_{k=0}^3 \vec{E}_1^{[3-k]} \cdot \vec{E}_2^{[k]} + c^2 \sum_{k=0}^5 \vec{B}_1^{[5-k]} \cdot \vec{B}_2^{[k]} \right) d^3x. \quad (67)$$

It is known from [13] (see also Equation (56)) that

$$\vec{E}_1^{[1]} = \vec{E}_2^{[1]} = 0, \quad \vec{B}_1^{[0]} = \vec{B}_2^{[0]} = 0, \quad \vec{B}_1^{[1]} = \vec{B}_2^{[1]} = 0, \quad \vec{B}_1^{[3]} = \vec{B}_2^{[3]} = 0. \quad (68)$$

Therefore, the field energy can be written as follows:

$$E_{field12}^{[3]} = \epsilon_0 \int \left(\vec{E}_1^{[3]} \cdot \vec{E}_2^{[0]} + \vec{E}_1^{[0]} \cdot \vec{E}_2^{[3]} \right) d^3x. \quad (69)$$

After calculations elaborated upon in Appendix G and by inserting the results of Equations (A117) and (A118) into Equation (69), we obtain a null result:

$$E_{field12}^{[3]} = 0. \quad (70)$$

Remarkably, it follows that no electromagnetic energy can be stored at the third order of $\frac{1}{c}$.

4.4.3. Poynting Vector

Next, we study the Poynting vector according to Equation (32) for $n = 3$.

$$\vec{S}_{P12}^{[3]} = \epsilon_0 c^2 \sum_{k=0}^5 \left(\vec{E}_1^{[5-k]} \times \vec{B}_2^{[k]} + \vec{E}_2^{[5-k]} \times \vec{B}_1^{[k]} \right) \quad (71)$$

According to Equation (68), the Poynting vector takes the following form:

$$\vec{S}_{P12}^{[3]} = \epsilon_0 c^2 \left(\vec{E}_1^{[3]} \times \vec{B}_2^{[2]} + \vec{E}_1^{[0]} \times \vec{B}_2^{[5]} + \vec{E}_2^{[3]} \times \vec{B}_1^{[2]} + \vec{E}_2^{[0]} \times \vec{B}_1^{[5]} \right). \quad (72)$$

The Poynting flux on a far-away sphere of radius (R_{max}) is given by the following expression:

$$PF^{[3]} = \oint \vec{S}_{P12}^{[3]} \cdot \hat{n} da = R_{max}^2 \oint \vec{S}_{P12}^{[3]} \cdot \hat{r} d\Omega. \quad (73)$$

After calculations elaborated upon in Appendix H and by inserting Equations (A131), (A132), (A136) and (A137) into Equation (A119), we obtain the total Poynting flux contribution of the third order in $\frac{1}{c}$:

$$PF^{[3]} = -\frac{2k}{3c^3} \int \int d^3x_1 d^3x_2 \left(\vec{J}_1 \cdot \partial_t^2 \vec{J}_2 + \partial_t^2 \vec{J}_1 \cdot \vec{J}_2 \right). \quad (74)$$

This term is minus Equation (66), that is, minus the power invested by the electromagnetic field to do mechanical work to the third order in $\frac{1}{c}$. It follows that in the third order, no energy can be stored in the electromagnetic field and the energy equation is balanced between radiation and work. That incoming radiation can cause charges to do work, or charges can radiate at the expense of their mechanical energy. However, if the currents in the two subsystems are orthogonal, no radiation is radiated and no work is done.

4.4.4. Summary

The key results of the $n = 3$ calculation are outlined as follows. First, we notice that in this order, no energy is stored. This is a surprising result and can only be justified based on calculations. However, energy can be radiated by doing work or vice-versa—work can be done by absorbing radiation. From the point of view of the development of a retarded field engine, one should take special care such that radiation losses are avoided. This can be done by avoiding currents in either subsystem or by designing the engine such that the currents are perpendicular. The energy exchange is illustrated in Figure 4.

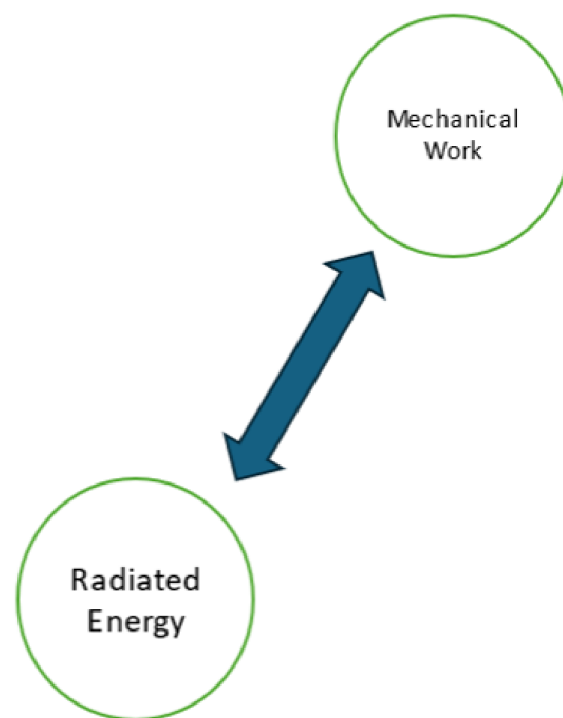


Figure 4. Energy exchange for $n = 3$. Radiation can be converted to kinetic energy and vice-versa, but energy cannot be stored.

4.5. $n = 4$

Let us look at Equation (27) and study it for the fourth order in $\frac{1}{c}$:

$$Power_{12}^{[4]} = -\frac{dE_{field12}^{[4]}}{dt} - \oint_S \vec{S}_{p12}^{[4]} \cdot \hat{n} da. \quad (75)$$

4.5.1. Power

The fourth-order expression for the power invested in mechanical work is given below:

$$Power_{12}^{[4]} = \int d^3x \left(\vec{J}_1 \cdot \vec{E}_2^{[4]} + \vec{J}_2 \cdot \vec{E}_1^{[4]} \right). \quad (76)$$

According to Equations (51) and (54) of [13], the fourth-order term in the electric field expansion is expressed as follows:

$$\vec{E}_1^{[4]}(\vec{x}) = -\frac{k}{2c^4} \int d^3x_1 \left[\frac{1}{4} \partial_t^4 \rho_1 R_1^2 \hat{R}_1 + \partial_t^3 \vec{J}_1 R_1 \right]. \quad (77)$$

Let us calculate both integrals of Equation (76) one by one:

$$I_1 = \int d^3x \vec{J}_2 \cdot \vec{E}_1^{[4]} = -\frac{k}{2c^4} \int d^3x \vec{J}_2(\vec{x}) \cdot \int d^3x_1 \left[\frac{1}{4} \partial_t^4 \rho_1(\vec{x}_1) R_1^2 \hat{R}_1 + \partial_t^3 \vec{J}_1(\vec{x}_1) R_1 \right]. \quad (78)$$

Changing the integration symbol from $\vec{x}$ to $\vec{x}_2$ results in the following form:

$$I_1 = \frac{k}{2c^4} \int d^3x_1 d^3x_2 R \left[\frac{1}{4} \partial_t^4 \rho_1(\vec{x}_1) \left(\vec{J}_2(\vec{x}_2) \cdot \vec{R} \right) - \partial_t^3 \vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2) \right]. \quad (79)$$

Exchanging indices 1 and 2, we obtain the second integral of Equation (76):

$$I_2 = \frac{k}{2c^4} \int d^3x_1 d^3x_2 R \left[-\frac{1}{4} \partial_t^4 \rho_2(\vec{x}_2) \left(\vec{J}_1(\vec{x}_1) \cdot \vec{R} \right) - \partial_t^3 \vec{J}_2(\vec{x}_2) \cdot \vec{J}_1(\vec{x}_1) \right]. \quad (80)$$

Adding the contributions, we finally obtain the following fourth-order power expression:

$$\begin{aligned} Power_{12}^{[4]} &= \frac{k}{2c^4} \int d^3x_1 d^3x_2 R \left[\frac{1}{4} \left(\partial_t^4 \rho_1(\vec{x}_1) \vec{J}_2(\vec{x}_2) - \partial_t^4 \rho_2(\vec{x}_2) \vec{J}_1(\vec{x}_1) \right) \cdot \vec{R} \right. \\ &\quad \left. - \left(\partial_t^3 \vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2) + \partial_t^3 \vec{J}_2(\vec{x}_2) \cdot \vec{J}_1(\vec{x}_1) \right) \right]. \end{aligned} \quad (81)$$

4.5.2. Poynting Vector

Next, we analyze the Poynting vector for $n = 4$ using Equation (32):

$$\vec{S}_{P12}^{[4]} = \epsilon_0 c^2 \sum_{k=0}^6 \left(\vec{E}_1^{[6-k]} \times \vec{B}_2^{[k]} + \vec{E}_2^{[6-k]} \times \vec{B}_1^{[k]} \right) \quad (82)$$

According to Equation (68), the Poynting vector takes the following fourth-order form:

$$\vec{S}_{P12}^{[4]} = \epsilon_0 c^2 \left(\vec{E}_1^{[4]} \times \vec{B}_2^{[2]} + \vec{E}_2^{[4]} \times \vec{B}_1^{[2]} + \vec{E}_1^{[2]} \times \vec{B}_2^{[4]} + \vec{E}_2^{[2]} \times \vec{B}_1^{[4]} + \vec{E}_1^{[0]} \times \vec{B}_2^{[6]} + \vec{E}_2^{[0]} \times \vec{B}_1^{[6]} \right). \quad (83)$$

The Poynting flux on a far-away sphere of radius R_{max} is given by the following expression:

$$PF^{[4]} = \oint \vec{S}_{P12}^{[4]} \cdot \hat{n} da = R_{max}^2 \oint_{R_{max}} \vec{S}_{P12}^{[4]} \cdot \hat{r} d\Omega. \quad (84)$$

After calculations elaborated upon in Appendix I and by inserting Equations (A145), (A146), (A150), (A151) and (A153) into Equation (A138), we obtain the total Poynting flux contribution of the fourth order in $\frac{1}{c}$:

$$PF^{[4]} = \frac{k R_{max}}{2c^4} \int \int d^3x_1 d^3x_2 \left(\frac{1}{2} \left(\partial_t^3 \vec{J}_1 \cdot \vec{J}_2 + \partial_t^3 \vec{J}_2 \cdot \vec{J}_1 \right) - \frac{1}{3} \left(\partial_t \vec{J}_1 \cdot \partial_t^2 \vec{J}_2 + \partial_t \vec{J}_2 \cdot \partial_t^2 \vec{J}_1 \right) \right). \quad (85)$$

4.5.3. Field Energy

Now, the contribution of the field energy to the energy balance at the fourth order is evaluated as follows:

$$E_{field12}^{[4]} = \epsilon_0 \int \left(\sum_{k=0}^4 \vec{E}_1^{[4-k]} \cdot \vec{E}_2^{[k]} + c^2 \sum_{k=0}^6 \vec{B}_1^{[6-k]} \cdot \vec{B}_2^{[k]} \right) d^3x. \quad (86)$$

Taking into account Equation (68), the field energy simplifies as follows:

$$E_{field12}^{[4]} = \epsilon_0 \int \left[\vec{E}_1^{[4]} \cdot \vec{E}_2^{[0]} + \vec{E}_1^{[0]} \cdot \vec{E}_2^{[4]} + \vec{E}_1^{[2]} \cdot \vec{E}_2^{[2]} + c^2 \left(\vec{B}_1^{[4]} \cdot \vec{B}_2^{[2]} + \vec{B}_1^{[2]} \cdot \vec{B}_2^{[4]} \right) \right] d^3x. \quad (87)$$

After calculations elaborated upon in Appendix J and by adding the fourth-order electric field energy given in Equation (A186) with the fourth-order magnetic field energy contribution given in Equation (A199), we arrive at the total fourth-order field energy:

$$\begin{aligned} E_{field12}^{[4]} &= E_{field12}^{[4] \text{ electric}} + E_{field12}^{[4] \text{ magnetic}} = \\ &= \frac{k}{8c^4} \int \int d^3x_1 d^3x_2 \left[-\frac{1}{3} \partial_t^2 \rho_1 \partial_t^2 \rho_2 \left(R^3 + 2R^2 R_{max} \right) + \frac{1}{3} R^3 \left(\partial_t^3 \rho_2 \partial_t \rho_1 + \partial_t^3 \rho_1 \partial_t \rho_2 \right) + \right. \\ &\quad \left. 4\partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 (2R_{max} - R) + 4(R - R_{max}) (\partial_t^2 \vec{J}_2 \cdot \vec{J}_1 + \partial_t^2 \vec{J}_1 \cdot \vec{J}_2) \right]. \end{aligned} \quad (88)$$

This expression can be partitioned into a volume contribution and surface contribution. The latter type of contribution involves R_{max} designating the maximal distance to which our expansion is applicable and, thus, also the radius of the sphere in which our energy calculations are performed. Thus,

$$\begin{aligned} E_{field12}^{[4]} &= E_{field12}^{[4] \text{ volume}} + E_{field12}^{[4] \text{ surface}} \\ E_{field12}^{[4] \text{ volume}} &\equiv \frac{k}{8c^4} \int \int d^3x_1 d^3x_2 \left[\frac{1}{3} R^3 \left(\partial_t^3 \rho_2 \partial_t \rho_1 + \partial_t^3 \rho_1 \partial_t \rho_2 - \partial_t^2 \rho_1 \partial_t^2 \rho_2 \right) + \right. \\ &\quad \left. 4R \left(\partial_t^2 \vec{J}_2 \cdot \vec{J}_1 + \partial_t^2 \vec{J}_1 \cdot \vec{J}_2 - \partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 \right) \right]. \\ E_{field12}^{[4] \text{ surface}} &\equiv \frac{kR_{max}}{4c^4} \int \int d^3x_1 d^3x_2 \left[-\frac{1}{3} R^2 \partial_t^2 \rho_1 \partial_t^2 \rho_2 - 2 \left(\partial_t^2 \vec{J}_2 \cdot \vec{J}_1 + \partial_t^2 \vec{J}_1 \cdot \vec{J}_2 - \right. \right. \\ &\quad \left. \left. 2\partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 \right) \right]. \end{aligned} \quad (89)$$

After a few lines of calculations, it can now be shown that

$$\frac{dE_{field12}^{[4] \text{ volume}}}{dt} = -Power_{12}^{[4]}, \quad (90)$$

in which $Power_{12}^{[4]}$ is the mechanical work done by the system and is given in Equation (81). For the surface term, we may only write the following:

$$\frac{dE_{field12}^{[4] \text{ surface}}}{dt} \approx -PF^{[4]}, \quad (91)$$

in which $PF^{[4]}$ is the Poynting flux given in Equation (85). Although both sides of Equation (91) contain the same type of expressions, the numerical factors are only of the same order of magnitude—not identical. This is probably due to the crude approximations used in Section 4.5.2 and in the [appendices]. Notice, however, that this will not change the final results of the current paper.

4.5.4. Summary

The key results of the $n = 4$ calculation are outlined as follows. First, we notice that in this order, we have, for the first time, all energy components: energy is stored and radiated and does mechanical work. This, however, is not relevant to the retarded field engine, as it involves higher derivatives of the field sources that are not needed for the functioning of the retarded field engine. The radiation terms do not balance exactly in the above calculation for the following reasons. First, in the Poynting flux, we keep only terms that are proportional to R_{max} , neglecting lower order terms. Second, in the integral calculation of the energy, we neglect corrections associated with a small change of boundary terms due to the change of variable (from $\vec{x}$ to $\vec{R}_1$), which considerably simplifies the calculation. A more accurate and detailed calculation should balance those terms as well; however, this has no bearing on retarded field engine theory and, from this point of view, is redundant. The energy exchange is illustrated in Figure 2.

5. Discussion

In order to verify the validity of the energy balance equation at various orders of approximation, we systematically evaluate the mechanical work, field energy, and Poynting vector contributions corresponding to successive terms in the Taylor expansion of the electromagnetic fields. Each term, labeled by index n , represents a distinct order in the $1/c$ expansion and contributes differently to the total energy dynamics of the system. Below, we summarize the outcome of this analysis for $n = 0$ through $n = 4$, highlighting the nature of energy exchange and the role of radiation at each order. The results are also given in Table 1.

For $n = 0$, the zeroth-order energy equation is found to be satisfied. Mechanical work performed on or by the system leads to a corresponding increase or decrease in the field energy, as expected in the absence of any contribution from the Poynting vector. For $n = 1$, we find that the first-order energy equation is trivially satisfied, that is, it reduces to an identity of $0 = 0$ under the condition in which the mechanical work, field energy, and Poynting contribution all vanish individually. Vanishing mechanical work implies that, to the first order in $1/c$, no mechanical energy is either supplied to or extracted from the relativistic engine. Furthermore, the first-order terms in the $1/c$ expansion do not contribute to either the field energy or the Poynting vector associated with the engine. In other words, in the first order, the electromagnetic fields neither stores energy nor transports it through radiation or absorption, indicating the absence of any real energy exchange. For $n = 2$, we conclude that the second-order energy equation is, indeed, balanced: the power associated with mechanical work equals the negative time derivative of the field energy, as expected in the absence of any Poynting contribution. In the case of $n = 3$, the absence of field energy implies that the electromagnetic field cannot store energy, leading to an energy balance strictly between radiation and mechanical work. Thus, incident radiation has the potential to drive charged particles to perform work. However, when the currents in the two subsystems are orthogonal, radiation is not emitted, and consequently, no work is done. It is also important to note that, if not properly configured, a relativistic motor may emit radiation. This behavior has been demonstrated in earlier studies involving uncharged relativistic motors [18]. For $n = 4$, we conclude that the fourth-order energy balance holds. At this order, both the electric and magnetic-field energies are influenced by the mechanical work. At the order of $1/c^4$, the structure of our series expansion limits our ability to evaluate the radiation flux at infinity. Instead, we must restrict our analysis to the radiation flux through a spherical surface of radius R_{max} , beyond which the validity of our approximation breaks down. We also notice that for terms dependent on R_{max} , the approximation we use here is not sufficient to exactly balance the terms of the energy change and the Poynting flux, for which a better approximation is required.

Table 1. Table of energy balance results.

n	Field Energy Derivative	Mechanical Power	Radiated Power
0	$\frac{d}{dt} [k \iint d^3x_1 d^3x_2 \frac{\rho_1 \rho_2}{R}]$	$-\frac{d}{dt} [k \iint d^3x_1 d^3x_2 \frac{\rho_1 \rho_2}{R^2}]$	0
1	0	0	0
2	$\frac{k}{c^2} \frac{d}{dt} \left[\iint d^3x_1 d^3x_2 \left(\frac{R(\partial_t \rho_1 \partial_t \rho_2)}{2} + \frac{\vec{J}_1 \cdot \vec{J}_2}{R} \right) \right]$	$-\frac{k}{c^2} \frac{d}{dt} \left[\iint d^3x_1 d^3x_2 \left(\frac{R(\partial_t \rho_1 \partial_t \rho_2)}{2} + \frac{\vec{J}_1 \cdot \vec{J}_2}{R} \right) \right]$	0
3	0	$\left(\frac{2k}{3c^3} \right) \iint d^3x_1 d^3x_2 \left(\vec{J}_1(\vec{x}_1) \cdot \partial_t^2 \vec{J}_2(\vec{x}_2) + \vec{J}_2(\vec{x}_2) \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) \right)$	$-\left(\frac{2k}{3c^3} \right) \iint d^3x_1 d^3x_2 \left(\vec{J}_1(\vec{x}_1) \cdot \partial_t^2 \vec{J}_2(\vec{x}_2) + \vec{J}_2(\vec{x}_2) \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) \right)$
4	$\frac{k}{8c^4} \iint d^3x_1 d^3x_2 \left[-\frac{1}{3} \partial_t^2 \rho_1 \partial_t^2 \rho_2 (R^3 + 2R^2 R_{max}) + \frac{1}{3} R^3 (\partial_t^3 \rho_2 \partial_t \rho_1 + \partial_t^3 \rho_1 \partial_t \rho_2) + 4 \partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 (2R_{max} - R) + 4(R - R_{max}) (\partial_t^2 \vec{J}_2 \cdot \vec{J}_1 + \partial_t^2 \vec{J}_1 \cdot \vec{J}_2) \right]$	$\frac{k}{2c^4} \iint d^3x_1 d^3x_2 R \left[\frac{1}{4} \left(\partial_t^4 \rho_1(\vec{x}_1) \vec{J}_2(\vec{x}_2) - \partial_t^4 \rho_2(\vec{x}_2) \vec{J}_1(\vec{x}_1) \right) \cdot \vec{R} - \left(\partial_t^3 \vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2) + \partial_t^3 \vec{J}_2(\vec{x}_2) \cdot \vec{J}_1(\vec{x}_1) \right) \right]$	$\frac{k R_{max}}{2c^4} \iint d^3x_1 d^3x_2 \left(\frac{1}{2} \left(\partial_t^3 \vec{J}_1 \cdot \vec{J}_2 + \partial_t^3 \vec{J}_2 \cdot \vec{J}_1 \right) - \frac{1}{3} \left(\partial_t \vec{J}_1 \cdot \partial_t^2 \vec{J}_2 + \partial_t \vec{J}_2 \cdot \partial_t^2 \vec{J}_1 \right) \right)$

Our motivation for this study is to understand the energy exchange in a relativistic motor; however, for this motor, one needs to consider only the first-order derivative of the charge density. On the other hand, $n \geq 3$ terms in the energy equation depend on higher derivatives. This problem will be resolved in the next section.

We conclude this section by listing the limiting assumptions of our analysis:

1. Our analysis is within the framework of classical electromagnetism; this means that we exclude extremely high electric and magnetic fields, which are beyond the Schwinger limit and for which a proper physical treatment will require quantum electrodynamics.
2. We assume that each of the interacting subsystems (subsystems 1 and 2) is small enough such that the retardation effects can be neglected within each system. The precise meaning of “small enough” for slow, non-relativistic systems was developed in [19] and has to do with the size (R) of each subsystem satisfying $R < \frac{2c^2}{a}$, in which a is the acceleration of any microscopic constituent of those systems. The abovementioned paper concentrates on gravitational systems, but the analysis should be approximately relevant to electrodynamic systems as well.
3. We assume that the subsystems have the same charge and current densities with respect to the engine frame but not necessarily with respect to the laboratory frame. The engine may move and even accelerate with respect to the laboratory frame.

6. The Source of the Charged Retarded Field Engine Energy

So far, our discussion has been quite general. We have calculated the energy balance terms for various orders of $1/c^n$ up to $n = 4$. We have not paid specific attention to the characteristics of the relativistic engine, which we consider now. Suppose we have two static charge distributions consisting of a retarded field engine and suppose one induces a time-dependent change in one of the charge distributions generating momentum according to Equation (8), after which the charge distribution becomes static again, as shown in Figure 5.

Notice that at this point, we have two static charge distributions in the engine’s frame but not in the laboratory frame. In the laboratory frame, the two subsystems connected to the engine are now moving with a velocity of $\vec{v}_s$ (see Equation (10)), which is the velocity of the engine. The engine has gained kinetic energy, as represented by Equation (9). We now inquire regarding the source of this energy.

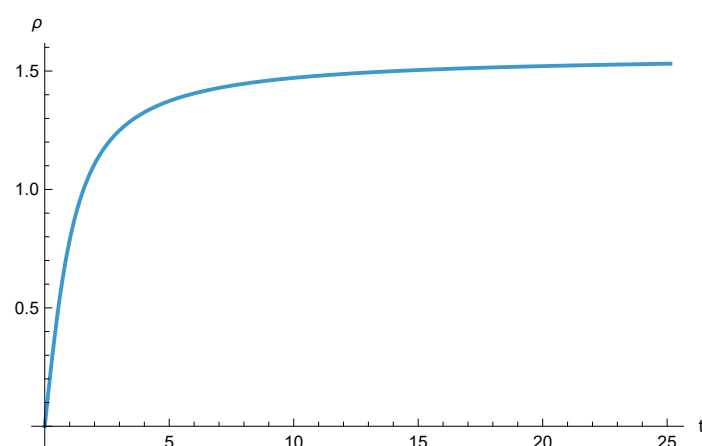


Figure 5. A change is introduced momentarily to ρ , after which it becomes static.

Let $\vec{x}$ be the location vector of a point in the laboratory (inertial) frame and let $\vec{x}_c(t)$ be the location of the engine in that frame, as shown in Figure 6.

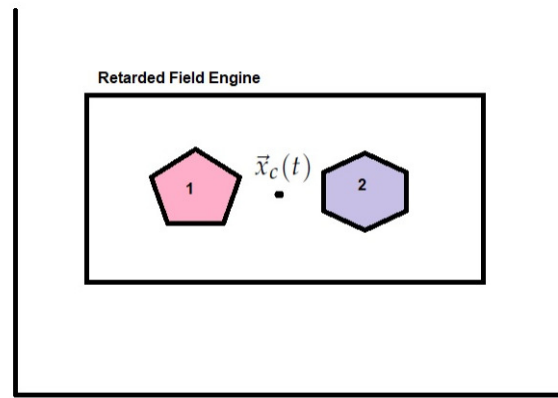


Figure 6. The retarded field engine comprised of charged subsystems 1 and 2. The engine is located at $\vec{x}_c(t)$ with respect to the inertial system.

It follows that the location of the same point with respect to the engine is expressed as follows:

$$\vec{x}' = \vec{x} - \vec{x}_c(t), \Rightarrow \vec{x} = \vec{x}' + \vec{x}_c(t). \quad (92)$$

It also follows that

$$\vec{v}_s = \frac{d\vec{x}_c(t)}{dt}. \quad (93)$$

The charge density can be expressed in either the laboratory (inertial) frame or in the engine frame. In the first case, it is designated by ρ , and in the second, it is designated by ρ_m ; these two functions are related as follows:

$$\rho(\vec{x}, t) = \rho_m(\vec{x}', t) = \rho_m(\vec{x} - \vec{x}_c(t), t). \quad (94)$$

In the case in which the charge distribution is “static” in the engine frame, this reduces to the following:

$$\rho(\vec{x}, t) = \rho_m(\vec{x} - \vec{x}_c(t)). \quad (95)$$

Similarly, the partial temporal derivative can be calculated as follows:

$$\partial_t \rho(\vec{x}, t)|_{\vec{x}} = \partial_t \rho_m(\vec{x} - \vec{x}_c(t), t)|_{\vec{x}} = -\vec{v}_s \cdot \vec{\nabla} \rho_m + \partial_t \rho_m|_{\vec{x}'}. \quad (96)$$

For the case in which the charge density is “static” in the engine frame, this reduces to the following:

$$\partial_t \rho(\vec{x}, t)|_{\vec{x}} = -\vec{v}_s \cdot \vec{\nabla} \rho_m = -\vec{\nabla} \cdot (\rho \vec{v}_s). \quad (97)$$

Taking into account the continuity (Equation (A9)), we notice that although no current densities were initially assumed in the system, it now follows that, from the laboratory point of view, the engine now carries current density of the following form:

$$\vec{J} = \rho \vec{v}_s \quad (98)$$

We are now at a position to calculate the work done by the internal electric field in such a system. This is done by using Equation (54), which describes the time derivative of this work. We remind the reader that we have shown this work to be exactly equal to the amount of energy change in the electromagnetic field (see Equation (58)). Thus,

$$E_{mech} = -\frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left(\frac{R(\partial_t \rho_1(\vec{x}_1) \partial_t \rho_2(\vec{x}_2))}{2} + \frac{\vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2)}{R} \right). \quad (99)$$

This can be written in the following form:

$$E_{mech} = -\frac{k}{2c^2} \int \int d^3x_1 d^3x_2 \left(\frac{R}{2} (\partial_t \rho_1 \partial_t \rho_2 + \partial_t \rho_1 \partial_t \rho_2) + \frac{\vec{J}_1 \cdot \vec{J}_2 + \vec{J}_1 \cdot \vec{J}_2}{R} \right). \quad (100)$$

Taking into account Equations (97) and (98), it follows that

$$E_{mech} = \frac{k}{2c^2} \int \int d^3x_1 d^3x_2 \left(\frac{R}{2} (\partial_t \rho_1 \vec{v}_s \cdot \vec{\nabla} \rho_2 + \vec{v}_s \cdot \vec{\nabla} \rho_1 \partial_t \rho_2) - \frac{\vec{J}_1 \cdot \vec{v}_s \rho_2 + \rho_1 \vec{v}_s \cdot \vec{J}_2}{R} \right). \quad (101)$$

Notice that

$$\int d^3x_2 R \vec{v}_s \cdot \vec{\nabla} \rho_2 = \int d^3x_2 [\vec{\nabla}_2 \cdot (R \rho_2 \vec{v}_s) - \rho_2 \vec{v}_s \cdot \vec{\nabla}_2 R]. \quad (102)$$

Using the Gaussian theorem and Equation (A34), this can be written in the following form:

$$\int d^3x_2 R \vec{v}_s \cdot \vec{\nabla} \rho_2 = \oint d\vec{S}_2 \cdot (R \rho_2 \vec{v}_s) + \int d^3x_2 \rho_2 \vec{v}_s \cdot \hat{R} = \int d^3x_2 \rho_2 \vec{v}_s \cdot \hat{R}, \quad (103)$$

in which we assume, as usual, that there is no charge on a far encapsulating surface. Similarly,

$$\int d^3x_1 R \vec{v}_s \cdot \vec{\nabla} \rho_1 = - \int d^3x_1 \rho_1 \vec{v}_s \cdot \hat{R}. \quad (104)$$

Plugging Equations (103) and (104) into Equation (101) it follows that

$$E_{mech} = \frac{1}{2} \vec{v}_s \cdot \frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left(\frac{\hat{R}}{2} (\partial_t \rho_1 \rho_2 - \rho_1 \partial_t \rho_2) - \frac{\vec{J}_1 \rho_2 + \rho_1 \vec{J}_2}{R} \right) = \frac{1}{2} \vec{v}_s \cdot \vec{P} = E_k. \quad (105)$$

The linear momentum is given by Equation (8), and the kinetic energy was defined in Equation (9). Hence, the work done by the retarded electromagnetic field is converted into the kinetic energy of the retarded field engine. We use an $n = 2$ expression for the work, but since $\vec{v}_s$ is proportional to $\frac{1}{c^2}$, it turns out to be proportional to, $\frac{1}{c^4}$, as previously discussed, thereby resolving the dilemma stated in the end of the previous section.

7. Conclusions

A relativistic engine is not a self-sustaining device, nor is it in perpetuum mobile; it requires an input of energy to function. This energy is drawn from the electromagnetic field, reducing its energy in the process. In this study, we explore the validity of energy conservation in relation to a charged retarded engine. This research focuses on how energy is conserved in charged retarded systems, proposing an exciting new way to power spacecraft. By using electromagnetic fields instead of traditional rocket fuel, this technology could help spaceships travel faster and farther while using much less energy.

The practical applications of the proposed “retarded field engine” are described in several papers. In [13], we describe a relativistic engine based on a charged capacitor/electret (system 1) and a coil carrying current (system 2) (see Figure 7). In this case, the capacitor is charged for a brief moment, after which both currents and charges are stationary, the force and momentum are in the direction of the current in the upper part of the figure.

In [20], we describe a relativistic engine based on two wires carrying periodic currents (see Figure 8). In this case, the upper wire is system 1, and the lower wire is system 2. The force is generated in the y direction, which is perpendicular to both wires. The current in each wire propagates along the wire.

In [14], we describe a relativistic engine composed of a permanent magnet (system 1) and a capacitor (system 2) (see Figure 9). In this case, the capacitor is charged for a brief moment, after which both (magnetic) currents and charges are stationary. The force and momentum are in the direction of the magnetic current in the upper part of the figure.

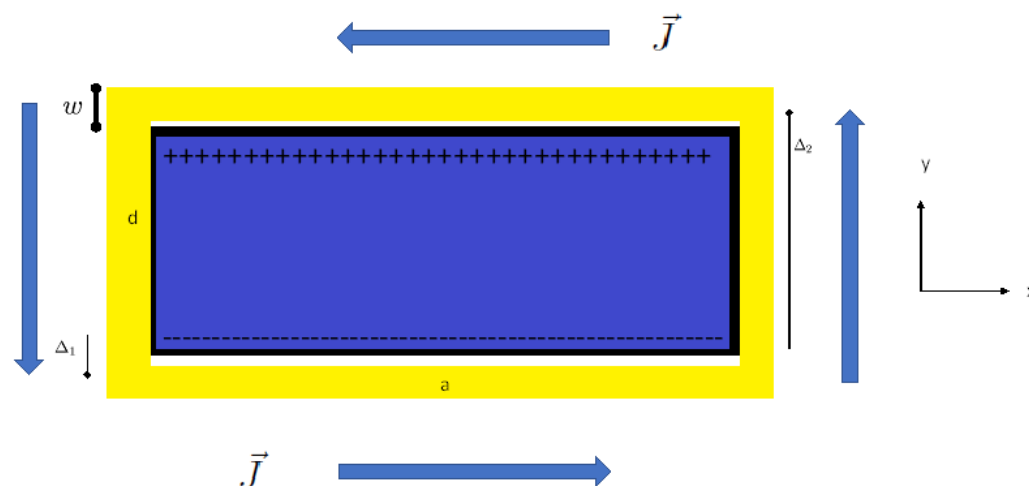


Figure 7. A cross section of the relativistic engine composed of a charged capacitor/electret (system 1) and a coil carrying current (system 2). In this implementation, the capacitor is charged for a brief moment, after which both currents and charges are stationary, the force and momentum are in the direction of the current in the upper part of the figure. The distance between the capacitor plates is d , and the width of the capacitor is a . The coil winding width is w , and it carries a current density of $\vec{J}$. The negative charged plate is located a distance of Δ_1 from the middle of the lower part of the winding and a distance of Δ_2 from the middle of the upper part of the winding.

The major obstacles to obtaining a useful retarded field engine have to do with the phenomenon of dielectric breakdown, which limits the amount of charge density that can be accumulated before a discharge occurs and the amount of current density that a wire can carry, even if it is superconducting. Modest figures that are relevant to the configuration described in Figure 7 are presented in Table 2.

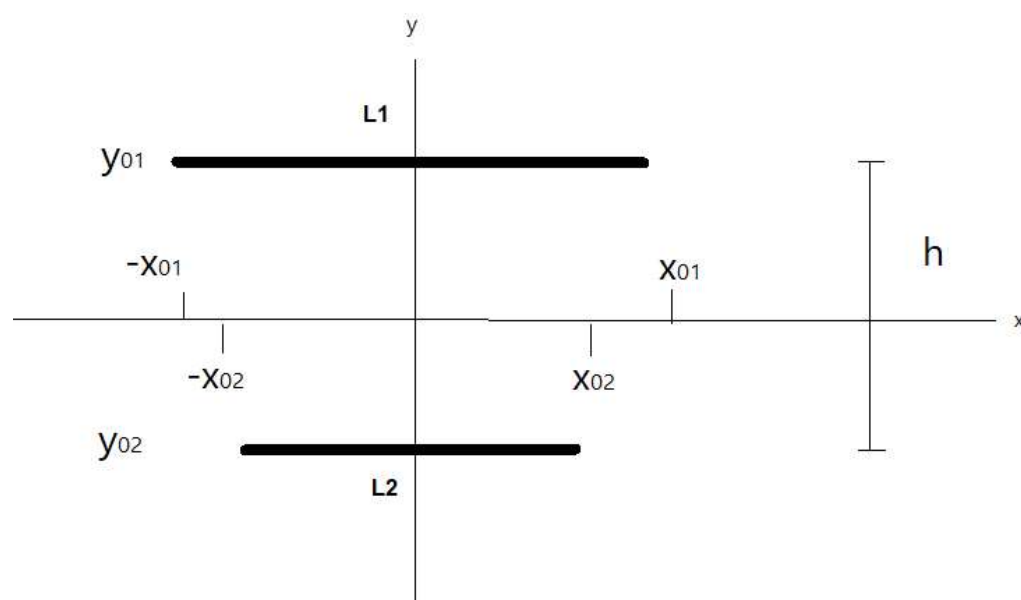


Figure 8. Two current strips. The upper wire is system 1, and the lower wire is system 2. The force is generated in the y direction, which is perpendicular to both wires. The current in each wire propagates along the wire. One strip is of length L_1 ; its ends have coordinates $(-x_{01}, y_{01})$ (left end) and $(+x_{01}, y_{01})$ (right end). The other strip is of length L_2 ; its ends have coordinates $(-x_{02}, y_{02})$ (left end) and $(+x_{02}, y_{02})$ (right end). The distance between the wires is h .

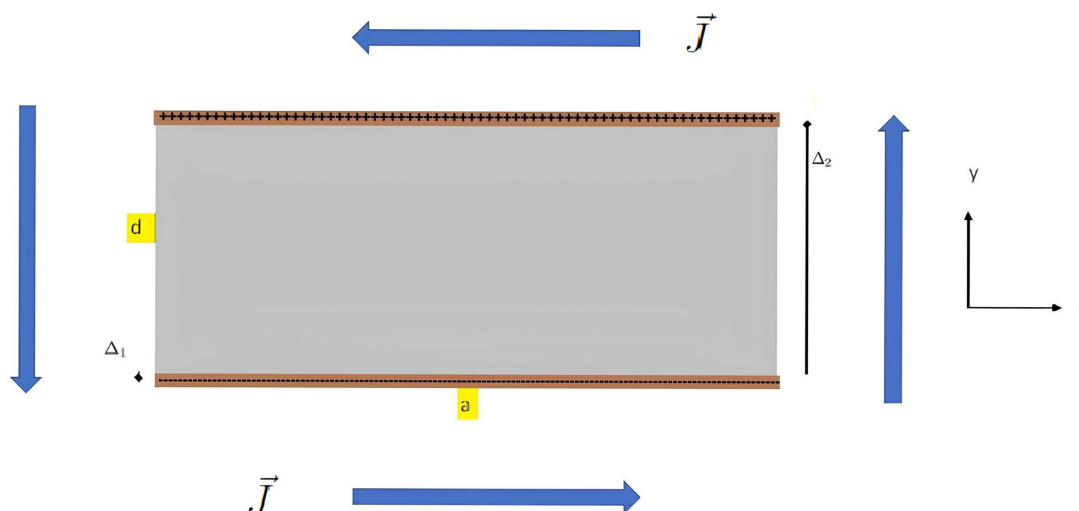


Figure 9. A cross section of the relativistic engine based on a permanent magnet (gray) (system 2) and a capacitor (system 1). $\vec{J}$ denotes the magnetic current. The force and momentum are in the direction of the current in the upper part of the figure. The distance between the capacitor plates is d , and the width of the capacitor is a . The permanent magnet has a magnetization of $\vec{M}$, and it carries a magnetic current density of $\vec{J} = \vec{\nabla} \times \vec{M}$. The negative charged plate is located a distance of Δ_1 from the lower magnet surface and a distance of Δ_2 from the upper magnet surface.

Table 2. Maximal momentum gained by a relativistic motor for three cases of parameters. We assume an extreme case of charge density of $\sigma = 3.7 \times 10^{-3}$ Coulomb/m² and current density pf $J_0 = 5 \times 10^7$ Ampere/m².

	Car	Rocket Engine Size	Giant Cube	Units
a	6	200	1000	m
b	2	10	1000	m
d	1	10	1000	m
w	0.2	0.4	0.4	m
P_T	0.3	868	3.1×10^7	kg m/s

Obviously, one needs a very sensitive and precise experimental setup to measure such effects. On the atomic scale, the figures are quite different; for example, one can reach a volume charge density of

$$\bar{\rho} = \frac{e}{\frac{4}{3}\pi a_0^3} \simeq 2.6 \times 10^{11} \text{ Coulomb/m}^3, \quad (106)$$

in which $e \simeq 1.6 \times 10^{-19}$ Coulomb is the electron charge and $a_0 \simeq 0.53 \times 10^{-10}$ m is Bohr's radius. The surface charge density of an electron in a hydrogen atom has an order of magnitude of

$$\bar{\sigma} = \frac{e}{4\pi a_0^2} \simeq 4.5 \text{ Coulomb/m}^2. \quad (107)$$

This is a thousand times larger than what is available for macroscopic charge densities. The order of magnitude of the current density is

$$\bar{J} = \frac{e\hbar}{m_e a_0^4} \simeq 2.3 \times 10^{18} \text{ Ampere/m}^2 = 2.3 \times 10^{14} \text{ Ampere/cm}^2 \quad (108)$$

This current density is ten orders of magnitude larger than what can be achieved at a macroscopic scale. This motivates suggestions for implementations involving material design [21,22] (see, for example, Figures 10 and 11).

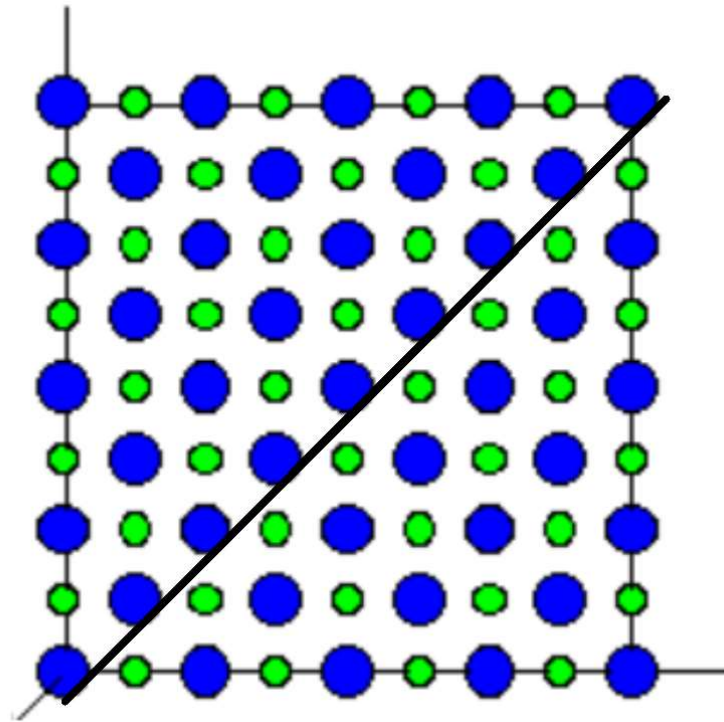


Figure 10. The 100 plane of a lattice of Na^+Cl^- (table salt). Blue circles depict sodium-positive ions, and green circles depict chlorine-negative ions. The trajectory of relativistic electrons is represented by a thick black line. If the electron is close enough to the ion, a significant net force should appear in the direction of the electron trajectory.

The above examples explicitly highlight the practical implications of this paper's findings. The authors welcome potential experimental demonstrations of any of the above configurations. They also direct readers to possible future work in the fields of material science and solid-state physics, in which it is hoped that materials that are relevant to the retarded field engine could be produced.

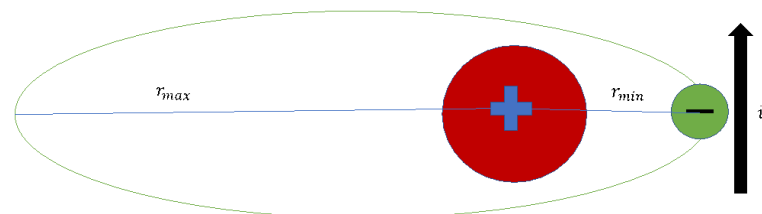


Figure 11. A schematic of an elliptical orbit of an electron around a proton. If one is able to introduce asymmetry to the electron orbital such that when the electron is going upwards, the electron is closer to the nucleus than in the case in which it is going downward, a retarded engine effect is expected.

Author Contributions: Conceptualization, A.Y.; methodology, A.Y. and P.S.; formal analysis, A.Y. and P.S.; investigation, A.Y. and P.S.; writing—original draft preparation, P.S.; writing—review and editing, A.Y. and P.S.; supervision, A.Y.; project administration, A.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Equations (51) and (54) of [13] provide detailed expressions for the zeroth-order term in the electric field expansion, which is just the Coulomb field:

$$\vec{E}_1^{[0]} = k \int d^3x_1 \frac{\rho_1(\vec{x}_1)}{R_1^2} \hat{R}_1, \quad (\text{A1})$$

Similarly,

$$\vec{E}_2^{[0]} = k \int d^3x_2 \frac{\rho_2(\vec{x}_2)}{R_2^2} \hat{R}_2, \quad (\text{A2})$$

where $\vec{R}_1$ and $\vec{R}_2$ define the relative position vectors:

$$\vec{R}_1 \equiv \vec{x} - \vec{x}_1, \quad \vec{R}_2 \equiv \vec{x} - \vec{x}_2, \quad R_1 \equiv |\vec{R}_1|, \quad R_2 \equiv |\vec{R}_2|, \quad \hat{R}_1 \equiv \frac{\vec{R}_1}{R_1}, \quad \hat{R}_2 \equiv \frac{\vec{R}_2}{R_2}. \quad (\text{A3})$$

By substituting Equations (A1) and (A2) into Equation (34), we obtain the following:

$$Power_{12}^{[0]} = k \int d^3x \left[\vec{J}_1(\vec{x}) \cdot \left(\int d^3x_2 \frac{\rho_2(\vec{x}_2)}{R_2^2} \hat{R}_2 \right) + \vec{J}_2(\vec{x}) \cdot \left(\int d^3x_1 \frac{\rho_1(\vec{x}_1)}{R_1^2} \hat{R}_1 \right) \right]. \quad (\text{A4})$$

Applying the identities expressed as

$$\frac{\hat{R}_1}{R_1^2} = -\vec{\nabla} \left(\frac{1}{R_1} \right), \quad \frac{\hat{R}_2}{R_2^2} = -\vec{\nabla} \left(\frac{1}{R_2} \right), \quad (\text{A5})$$

the expression in Equation (A4) can be simplified as follows:

$$Power_{12}^{[0]} = -k \int d^3x \left[\vec{J}_1(\vec{x}) \cdot \int d^3x_2 \left(\rho_2(\vec{x}_2) (\vec{\nabla} R_2^{-1}) \right) + \vec{J}_2(\vec{x}) \cdot \int d^3x_1 \left(\rho_1(\vec{x}_1) (\vec{\nabla} R_1^{-1}) \right) \right]. \quad (\text{A6})$$

The same can be achieved by changing the order of integration in the following form:

$$\begin{aligned} Power_{12}^{[0]} &= -k \left[\int d^3x_2 \rho_2(\vec{x}_2) \int d^3x \vec{J}_1(\vec{x}) \cdot \vec{\nabla} \left(\frac{1}{R_2} \right) \right. \\ &\quad \left. + \int d^3x_1 \rho_1(\vec{x}_1) \int d^3x \vec{J}_2(\vec{x}) \cdot \vec{\nabla} \left(\frac{1}{R_1} \right) \right]. \end{aligned} \quad (\text{A7})$$

Next, we compute the individual integral terms to simplify the above equation:

$$\int d^3x \vec{J}_1(\vec{x}) \cdot \vec{\nabla} \left(\frac{1}{R_2} \right) = \int d^3x \left[\vec{\nabla} \cdot \left(\frac{\vec{J}_1(\vec{x})}{R_2} \right) - \frac{1}{R_2} \vec{\nabla} \cdot \vec{J}_1(\vec{x}) \right] \quad (\text{A8})$$

Using the Gaussian theorem and continuity equation, i.e.,

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0, \quad (\text{A9})$$

we obtain the following result:

$$\int d^3x \vec{J}_1(\vec{x}) \cdot \vec{\nabla} \left(\frac{1}{R_2} \right) = \oint dS \cdot \left(\frac{\vec{J}_1(\vec{x})}{R_2} \right) + \int d^3x \frac{1}{R_2} \partial_t \rho_1(\vec{x}). \quad (\text{A10})$$

The surface integral is applied over a surface encapsulating the volume of the volume integral. If the volume integral is applied over all space, the surface is at infinity. Provided

that there are no currents at infinity. i.e., $\vec{J}_1(\infty) = 0$,

$$\int d^3x \vec{J}_1(\vec{x}) \cdot \vec{\nabla} \left(\frac{1}{R_2} \right) = \int d^3x \frac{1}{R_2} \partial_t \rho_1(\vec{x}). \quad (\text{A11})$$

Similarly,

$$\int d^3x \vec{J}_2(\vec{x}) \cdot \vec{\nabla} \left(\frac{1}{R_1} \right) = \int d^3x \frac{1}{R_1} \partial_t \rho_2(\vec{x}). \quad (\text{A12})$$

Substituting these identities into Equation (A7), we obtain the following:

$$\begin{aligned} \text{Power}_{12}^{[0]} &= -k \left[\int d^3x_2 \rho_2(\vec{x}_2) \int d^3x \frac{1}{R_2} \partial_t \rho_1(\vec{x}) + \int d^3x_1 \rho_1(\vec{x}_1) \int d^3x \frac{1}{R_1} \partial_t \rho_2(\vec{x}) \right] \\ &= -k \int d^3x_2 \rho_2(\vec{x}_2) \int d^3x \frac{1}{R_2} \partial_t \rho_1(\vec{x}) + k \int d^3x_1 \rho_1(\vec{x}_1) \int d^3x \frac{1}{R_1} \partial_t \rho_2(\vec{x}) \end{aligned} \quad (\text{A13})$$

Thus, the power contains two contributions:

$$\text{Power}_{12_1}^{[0]} = -k \int d^3x_2 \rho_2(\vec{x}_2) \int d^3x \frac{1}{R_2} \partial_t \rho_1(\vec{x}) \quad (\text{A14})$$

and

$$\text{Power}_{12_2}^{[0]} = -k \int d^3x_1 \rho_1(\vec{x}_1) \int d^3x \frac{1}{R_1} \partial_t \rho_2(\vec{x}) \quad (\text{A15})$$

We now make a change to the integration variable ($\vec{x} \rightarrow \vec{x}_1$) in $\text{Power}_{12_1}^{[0]}$; thus, $\vec{R}_2 = \vec{x} - \vec{x}_2 \rightarrow \vec{x}_1 - \vec{x}_2 = \vec{R}$. We obtain the following:

$$\begin{aligned} \text{Power}_{12_1}^{[0]} &= -k \int d^3x_2 \rho_2(\vec{x}_2) \int d^3x_1 \frac{1}{R} \partial_t \rho_1(\vec{x}_1) \\ &= -k \int \int d^3x_1 d^3x_2 \frac{\rho_2(\vec{x}_2) \partial_t \rho_1(\vec{x}_1)}{R}. \end{aligned} \quad (\text{A16})$$

Similarly,

$$\text{Power}_{12_2}^{[0]} = -k \int \int d^3x_1 d^3x_2 \frac{\rho_1(\vec{x}_1) \partial_t \rho_2(\vec{x}_2)}{R}. \quad (\text{A17})$$

We now present the complete expression for the power:

$$\begin{aligned} \text{Power}_{12}^{[0]} &= \text{Power}_{12_1}^{[0]} + \text{Power}_{12_2}^{[0]} \\ &= -k \int \int d^3x_1 d^3x_2 \frac{1}{R} [\rho_1(\vec{x}_1) \partial_t \rho_2(\vec{x}_2) + \rho_2(\vec{x}_2) \partial_t \rho_1(\vec{x}_1)] \\ &= -k \int \int d^3x_1 d^3x_2 \frac{\partial_t (\rho_1 \rho_2)}{R}. \end{aligned} \quad (\text{A18})$$

Appendix B

According to Equation (38), the field energy takes the following form:

$$E_{field12}^{[0]} = \epsilon_0 k^2 \int \left[\int \int d^3x_1 d^3x_2 \frac{\rho_1(\vec{x}_1) \rho_2(\vec{x}_2)}{R_1^2 R_2^2} \hat{R}_1 \cdot \hat{R}_2 \right] d^3x. \quad (\text{A19})$$

Now, we separately solve the integral:

$$II = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \frac{\hat{R}_2}{R_2^2} \right) d^3x = \int \left(\vec{\nabla} \frac{1}{R_1} \cdot \vec{\nabla} \frac{1}{R_2} \right) d^3x. \quad (\text{A20})$$

Using the identity expressed as

$$\vec{\nabla} \frac{1}{R_1} \cdot \vec{\nabla} \frac{1}{R_2} = \vec{\nabla} \cdot \left(\frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} \right) - \frac{1}{R_1} \vec{\nabla}^2 \frac{1}{R_2} \quad (\text{A21})$$

and taking into account the fact that [3]

$$\vec{\nabla}^2 \frac{1}{R_2} = -4\pi\delta(\vec{R}_2), \quad (\text{A22})$$

in which $\delta(\vec{R}_2)$ is a three-dimensional delta function, we obtain the following:

$$\vec{\nabla} \frac{1}{R_1} \cdot \vec{\nabla} \frac{1}{R_2} = \vec{\nabla} \cdot \left(\frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} \right) + \frac{4\pi}{R_1} \delta(\vec{R}_2). \quad (\text{A23})$$

The first term on the right is a divergence. Thus, using Gaussian theorem, its volume integral will become a surface integral, and the second term contains a delta function. This means that there is no contribution to the volume integral from the delta term unless $\vec{x} = \vec{x}_2$. Thus, Equation (A20) simplifies to

$$II = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \frac{\hat{R}_2}{R_2^2} \right) d^3x = \int d\vec{S} \cdot \left(\frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} \right) + \frac{4\pi}{R} \quad (\text{A24})$$

Let us look at the surface integral and assume that the system is contained inside an infinite sphere such that the surface integral is taken over a spherical surface of radius $r = |\vec{x}| \rightarrow \infty$:

$$\lim_{r \rightarrow \infty} \int d\vec{S} \cdot \frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} = \lim_{r \rightarrow \infty} \int d\vec{S} \cdot \frac{1}{R_1} \frac{\hat{R}_2}{R_2^2}. \quad (\text{A25})$$

where $d\vec{S} = r^2 d\Omega \hat{r}$, in which Ω is the solid angle. Furthermore, $R_1 \rightarrow r \rightarrow \infty$, $R_2 \rightarrow r \rightarrow \infty$, and $\hat{R}_2 \rightarrow \hat{r}$. It follows that

$$\lim_{r \rightarrow \infty} \int d\vec{S} \cdot \frac{1}{R_1} \vec{\nabla} \frac{1}{R_2} = \lim_{r \rightarrow \infty} \int r^2 d\Omega \frac{1}{r^3} = \lim_{r \rightarrow \infty} \int d\Omega \frac{1}{r} = 0. \quad (\text{A26})$$

(see also Appendix A.1.2. of [15] for a complete explanation of Equation (A25)). We simply conclude the following:

$$II = \frac{4\pi}{R} \quad (\text{A27})$$

Hence, the expression for the field energy becomes the following:

$$E_{field12}^{[0]} = \epsilon_0 k^2 \left[\int \int d^3x_1 d^3x_2 \frac{\rho_1(\vec{x}_1) \rho_2(\vec{x}_2)}{R} 4\pi \right] \quad (\text{A28})$$

Appendix C

We start by addressing the first integral in Equation (53):

$$I_1 = \int d^3x \vec{J}_1(\vec{x}) \cdot \vec{E}_2^{[2]}(\vec{x}). \quad (\text{A29})$$

According to Equations (51) and (54) of [13], the second-order term in the electric field expansion is expressed as follows:

$$\vec{E}_2^{[2]}(\vec{x}) = \frac{k}{c^2} \int d^3x_2 \left[-\frac{1}{2} \partial_t^2 \rho_2(\vec{x}_2) \hat{R}_2 - \partial_t \vec{J}_2(\vec{x}_2) R_2^{-1} \right]. \quad (\text{A30})$$

Inserting this in the I_1 expression will result in the following:

$$I_1 = \frac{k}{c^2} \int d^3x \left[\vec{J}_1(\vec{x}) \cdot \int d^3x_2 \left[-\frac{1}{2} \partial_t^2 \rho_2(\vec{x}_2) \hat{R}_2 - \partial_t \vec{J}_2(\vec{x}_2) R_2^{-1} \right] \right]. \quad (\text{A31})$$

We now change the integration variable ($\vec{x} \rightarrow \vec{x}_1$) as before such that

$$\begin{aligned} I_1 &= k \left(\frac{1}{c^2} \right) \int d^3x_1 \vec{J}_1(\vec{x}_1) \cdot \int d^3x_2 \left[-\frac{1}{2} \partial_t^2 \rho_2(\vec{x}_2) \hat{R} - \partial_t \vec{J}_2(\vec{x}_2) R^{-1} \right] \\ &= k \left(\frac{1}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[-\frac{1}{2} \vec{J}_1(\vec{x}_1) \cdot \hat{R} \partial_t^2 \rho_2(\vec{x}_2) - \vec{J}_1(\vec{x}_1) \cdot \partial_t \vec{J}_2(\vec{x}_2) R^{-1} \right]. \end{aligned} \quad (\text{A32})$$

Let us look at the following expression:

$$\int \int d^3x_1 d^3x_2 \vec{J}_1(\vec{x}_1) \cdot \hat{R} \partial_t^2 \rho_2(\vec{x}_2) = \int d^3x_2 \partial_t^2 \rho_2(\vec{x}_2) \int d^3x_1 \vec{J}_1(\vec{x}_1) \cdot \hat{R}, \quad (\text{A33})$$

Notice that

$$\vec{\nabla}_{\vec{x}_2} R = -\hat{R}, \quad \vec{\nabla}_{\vec{x}_1} R = \hat{R}. \quad (\text{A34})$$

Based on the continuity (Equation (A9)) and taking into account the Gaussian theorem and the vanishing of current densities on a far-away encapsulating surface, it follows that

$$\int d^3x_1 \vec{J}_1(\vec{x}_1) \cdot \vec{\nabla}_{\vec{x}_1} R = \int d^3x_1 R \partial_t \rho_1(\vec{x}_1), \quad \int d^3x_2 \vec{J}_2(\vec{x}_2) \cdot \vec{\nabla}_{\vec{x}_2} R = \int d^3x_2 R \partial_t \rho_2(\vec{x}_2). \quad (\text{A35})$$

Thus, Equation (A33) takes the following form:

$$\int \int d^3x_1 d^3x_2 \vec{J}_1(\vec{x}_1) \cdot \hat{R} \partial_t^2 \rho_2(\vec{x}_2) = \int \int d^3x_1 d^3x_2 R \partial_t \rho_1(\vec{x}_1) \partial_t^2 \rho_2(\vec{x}_2). \quad (\text{A36})$$

Upon substitution of Equation (A36) into Equation (A32), the resulting expression is

$$I_1 = k \left(\frac{1}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[-\frac{1}{2} R \partial_t \rho_1(\vec{x}_1) \partial_t^2 \rho_2(\vec{x}_2) - \vec{J}_1(\vec{x}_1) \cdot \partial_t \vec{J}_2(\vec{x}_2) R^{-1} \right]. \quad (\text{A37})$$

Exchanging indices 1 and 2, we obtain the second integral of Equation (53):

$$I_2 = k \left(\frac{1}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[-\frac{1}{2} R \partial_t \rho_2(\vec{x}_2) \partial_t^2 \rho_1(\vec{x}_1) - \vec{J}_2(\vec{x}_2) \cdot \partial_t \vec{J}_1(\vec{x}_1) R^{-1} \right]. \quad (\text{A38})$$

Combining Equations (A37) and (A38), we obtain the complete expression for the power:

$$\begin{aligned} Power_{12}^{[2]} &= k \left(\frac{1}{c^2} \right) \left[- \int \int d^3x_1 d^3x_2 \frac{R}{2} \left(\partial_t^2 \rho_2(\vec{x}_2) \partial_t \rho_1(\vec{x}_1) + \partial_t^2 \rho_1(\vec{x}_1) \partial_t \rho_2(\vec{x}_2) \right) \right. \\ &\quad \left. - \int \int d^3x_1 d^3x_2 \frac{1}{R} \left(\vec{J}_1(\vec{x}_1) \cdot \partial_t \vec{J}_2(\vec{x}_2) + \vec{J}_2(\vec{x}_2) \cdot \partial_t \vec{J}_1(\vec{x}_1) \right) \right]. \end{aligned} \quad (\text{A39})$$

Applying the product rule of derivatives, we obtain the following:

$$\begin{aligned} Power_{12}^{[2]} &= k \left(\frac{1}{c^2} \right) \left[- \int \int d^3x_1 d^3x_2 \frac{R}{2} \partial_t (\partial_t \rho_1(\vec{x}_1) \partial_t \rho_2(\vec{x}_2)) \right. \\ &\quad \left. - \int \int d^3x_1 d^3x_2 \frac{1}{R} \partial_t (\vec{J}_1(\vec{x}_1) \cdot \vec{J}_2(\vec{x}_2)) \right]. \end{aligned} \quad (\text{A40})$$

Appendix D

Equation (57) can be expressed as the sum of the following terms:

$$E_{field12}^{[2]} = i_1 + i_2 + i_3, \quad (\text{A41})$$

where

$$i_1 = \epsilon_0 \left(\int \vec{E}_1^{[0]} \cdot \vec{E}_2^{[2]} d^3x \right), \quad (\text{A42})$$

$$i_2 = \epsilon_0 \left(\int \vec{E}_1^{[2]} \cdot \vec{E}_2^{[0]} d^3x \right), \quad (\text{A43})$$

and

$$i_3 = \epsilon_0 c^2 \left(\int \vec{B}_1^{[2]} \cdot \vec{B}_2^{[2]} d^3x \right). \quad (\text{A44})$$

We now compute the integral of each term separately:

$$i_1 = \epsilon_0 \left(\int \vec{E}_1^{[0]} \cdot \vec{E}_2^{[2]} d^3x \right) \quad (\text{A45})$$

The relevant field expressions are provided in [13] (as well as being previously mentioned in the current paper):

$$\vec{E}_1^{[0]}(\vec{x}) = k \int d^3x_1 \frac{\rho_1(\vec{x}_1)}{R_1^2} \hat{R}_1, \quad (\text{A46})$$

and

$$\vec{E}_2^{[2]}(\vec{x}) = k \left(\frac{1}{c^2} \right) \int d^3x_2 \left[\frac{1}{2} \partial_t^2 \rho_2(\vec{x}_2) (-\hat{R}_2) - \partial_t \vec{J}_2(\vec{x}_2) R_2^{-1} \right]. \quad (\text{A47})$$

Substituting these field expressions into Equation (A45), we obtain the following:

$$i_1 = \epsilon_0 k \frac{k}{c^2} \int d^3x \left[\left(\int d^3x_1 \frac{\rho_1(\vec{x}_1)}{R_1^2} \hat{R}_1 \right) \cdot \left(\int d^3x_2 \left[\frac{1}{2} \partial_t^2 \rho_2(\vec{x}_2) (-\hat{R}_2) - \partial_t \vec{J}_2(\vec{x}_2) R_2^{-1} \right] \right) \right], \quad (\text{A48})$$

which can also be expressed in the following form:

$$\begin{aligned} i_1 &= -\frac{k}{4\pi c^2} \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} \partial_t^2 \rho_2(\vec{x}_2) \rho_1(\vec{x}_1) \left(\int d^3x \frac{\hat{R}_1}{R_1^2} \cdot \hat{R}_2 \right) \right. \\ &\quad \left. + \rho_1(\vec{x}_1) \partial_t \vec{J}_2(\vec{x}_2) \cdot \left(\int d^3x \frac{\hat{R}_1}{R_1^2} \frac{1}{R_2} \right) \right]. \end{aligned} \quad (\text{A49})$$

Let us look at the following expression:

$$i_{11} = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \hat{R}_2 \right) d^3x. \quad (\text{A50})$$

Taking into account Equation (A229),

$$i_{11} = 2\pi R \int_0^{y_{max}} dy \int_{-1}^1 ds \frac{y+s}{\sqrt{1+y^2+2ys}}. \quad (\text{A51})$$

Notice that

$$\int_{-1}^1 ds \frac{s}{\sqrt{1+y^2+2ys}} = -\frac{2}{3} \begin{cases} \frac{1}{y^2} & y \geq 1 \\ y & y < 1 \end{cases} \quad (\text{A52})$$

and

$$y \int_{-1}^1 ds \frac{1}{\sqrt{1+y^2+2ys}} = \begin{cases} 2 & y \geq 1 \\ 2y & y < 1. \end{cases} \quad (\text{A53})$$

Substituting these integral forms into Equation (A51), we obtain the following:

$$i_{11} = 4\pi R \left[y_{\max} - 1 + \frac{1}{3y_{\max}} \right] \simeq 4\pi(R_{\max} - R) \quad (\text{A54})$$

Next, we simplify the second integral appearing in Equation (A49):

$$\vec{i}_{12} = \int d^3x \frac{\hat{R}_1}{R_1^2} \frac{1}{R_2} = - \int d^3x \left(\vec{\nabla} \frac{1}{R_1} \right) \frac{1}{R_2}, \quad (\text{A55})$$

as is known from [16]:

$$\int d^3x \frac{1}{R_1} \left(\vec{\nabla} \frac{1}{R_2} \right) = -2\pi \hat{R}. \quad (\text{A56})$$

Subsequently, the required integral follows from a straightforward exchange of indices.

$$\int d^3x \frac{\hat{R}_1}{R_1^2} \frac{1}{R_2} = 2\pi \hat{R}. \quad (\text{A57})$$

Thus,

$$\vec{i}_{12} = -2\pi \hat{R}. \quad (\text{A58})$$

Inserting the evaluated integrals into Equation (A49) yields the following:

$$i_1 = -\frac{k}{4\pi c^2} \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} \partial_t^2 \rho_2(\vec{x}_2) \rho_1(\vec{x}_1) (4\pi(R_{\max} - R)) \right. \\ \left. + \rho_1(\vec{x}_1) \partial_t \vec{j}_2(\vec{x}_2) \cdot (-2\pi \hat{R}) \right]. \quad (\text{A59})$$

Alternatively,

$$i_1 = -\frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} \partial_t^2 \rho_2 \rho_1 (R_{\max} - R) - \frac{1}{2} \rho_1 \partial_t \vec{j}_2 \cdot \hat{R} \right]. \quad (\text{A60})$$

We now focus on the second term to enable further simplification:

$$\int d^3x_2 \partial_t \vec{j}_2(\vec{x}_2) \cdot \hat{R} = -\partial_t \left(\int d^3x_2 \vec{j}_2(\vec{x}_2) \cdot \vec{\nabla}_{\vec{x}_2} R \right) = -\partial_t \left(\int d^3x_2 R \partial_t \rho_2(\vec{x}_2) \right) \\ = - \int d^3x_2 R \partial_t^2 \rho_2(\vec{x}_2), \quad (\text{A61})$$

In the above, we use Equations (A34) and (A35). By substituting this result into Equation (A60), we arrive at the following:

$$i_1 = -\frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} \rho_1 \partial_t^2 \rho_2 (R_{\max} - R) + \frac{1}{2} \rho_1 \partial_t^2 \rho_2 R \right] \\ = -\frac{k R_{\max}}{2c^2} \int \int d^3x_1 d^3x_2 \left[\rho_1 \partial_t^2 \rho_2 \right]. \quad (\text{A62})$$

Notice that the following terms would be zero considering Equation (A9), the Gaussian theorem, and the null current densities at the convergence radius (R_{max}):

$$\int (\partial_t^n \rho) d^3x = -\partial_t^{n-1} \left(\int \vec{\nabla} \cdot \vec{J} d^3x \right) = -\partial_t^{n-1} \left(\oint \vec{J} \cdot d\vec{S} \right) = 0, \quad n \geq 1. \quad (\text{A63})$$

Hence, as the double integral in Equation (A62) decouples, it follows that

$$i_1 = 0. \quad (\text{A64})$$

Analogously, the second integral in the field energy expression also nullifies:

$$i_2 = 0. \quad (\text{A65})$$

Next, we calculate the third integral to finalize the field energy equation, employing Equation (61) of [13]:

$$\vec{B}_1^{[2]}(\vec{x}) = \frac{k}{c^2} \int d^3x_1 \frac{-\hat{R}_1}{R_1^2} \times \vec{J}_1(\vec{x}_1) = -\frac{k}{c^2} \int d^3x_1 \frac{\hat{R}_1}{R_1^2} \times \vec{J}_1(\vec{x}_1). \quad (\text{A66})$$

Similarly,

$$\vec{B}_2^{[2]}(\vec{x}) = -\frac{k}{c^2} \int d^3x_2 \frac{\hat{R}_2}{R_2^2} \times \vec{J}_2(\vec{x}_2). \quad (\text{A67})$$

Hence,

$$\begin{aligned} i_3 &= \epsilon_0 c^2 \left(\int \vec{B}_1^{[2]} \cdot \vec{B}_2^{[2]} d^3x \right) \\ &= \frac{k}{4\pi c^2} \int \left[\int \int d^3x_1 d^3x_2 \frac{(\hat{R}_1 \times \vec{J}_1) \cdot (\hat{R}_2 \times \vec{J}_2)}{R_1^2 R_2^2} \right] d^3x \end{aligned} \quad (\text{A68})$$

Applying the following vector identity expressed as

$$(\hat{R}_1 \times \vec{J}_1) \cdot (\hat{R}_2 \times \vec{J}_2) = (\hat{R}_1 \cdot \hat{R}_2)(\vec{J}_1 \cdot \vec{J}_2) - (\hat{R}_1 \cdot \vec{J}_2)(\hat{R}_2 \cdot \vec{J}_1), \quad (\text{A69})$$

Equation (A68) can be rewritten as

$$\begin{aligned} i_3 &= \frac{k}{4\pi c^2} \left[\int \int d^3x_1 d^3x_2 (\vec{J}_1 \cdot \vec{J}_2) \left(\int \frac{\hat{R}_1 \cdot \hat{R}_2}{R_1^2 R_2^2} d^3x \right) \right. \\ &\quad \left. - \int d^3x \left(\int d^3x_1 \frac{\vec{J}_1 \cdot \hat{R}_1}{R_1^2} \right) \left(\int d^3x_2 \frac{\vec{J}_2 \cdot \hat{R}_2}{R_2^2} \right) \right]. \end{aligned} \quad (\text{A70})$$

Similar to Equation (A11), it can be shown that

$$\int d^3x_1 \frac{\vec{J}_1 \cdot \hat{R}_1}{R_1^2} = \int d^3x_1 \frac{1}{R_1} \partial_t \rho_1. \quad (\text{A71})$$

Taking into account Equations (A27) and (A71), it follows that

$$i_3 = \frac{k}{4\pi c^2} \left[\int \int d^3x_1 d^3x_2 \frac{4\pi}{R} \vec{J}_1 \cdot \vec{J}_2 - \int \int d^3x_1 d^3x_2 \partial_t \rho_1 \partial_t \rho_2 \left(\int d^3x \frac{1}{R_1 R_2} \right) \right]. \quad (\text{A72})$$

Taking into account Equation (A235), we obtain the following:

$$\begin{aligned}
i_3 &= \frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left[\frac{\vec{J}_1 \cdot \vec{J}_2}{R} + \frac{1}{2} (1 - 2y_{max}) \partial_t \rho_1 \partial_t \rho_2 R \right] \\
&= \frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left[\frac{\vec{J}_1 \cdot \vec{J}_2}{R} + \frac{R}{2} \partial_t \rho_1 \partial_t \rho_2 - R_{max} \partial_t \rho_1 \partial_t \rho_2 \right] \\
&= \frac{k}{c^2} \int \int d^3x_1 d^3x_2 \left[\frac{\vec{J}_1 \cdot \vec{J}_2}{R} + \frac{R}{2} \partial_t \rho_1 \partial_t \rho_2 \right], \tag{A73}
\end{aligned}$$

The last equation sign is due to the fact that in the term containing R_{max} , the volume integrals decouple; thus, according to Equation (A63), this term is null.

Appendix E

Following the approach used in the field energy analysis, the Poynting flux may, likewise, be represented as a sum of distinct terms, enabling the successive calculation of the corresponding integrals, i.e.,

$$PF^{[2]} = PF_1^{[2]} + PF_2^{[2]} + PF_3^{[2]} + PF_4^{[2]}, \tag{A74}$$

such that

$$PF_1^{[2]} \equiv R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[2]} \times \vec{B}_2^{[2]} \right) \cdot \hat{r} d\Omega, \tag{A75}$$

$$PF_2^{[2]} \equiv R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_2^{[2]} \times \vec{B}_1^{[2]} \right) \cdot \hat{r} d\Omega, \tag{A76}$$

$$PF_3^{[2]} \equiv R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[0]} \times \vec{B}_2^{[4]} \right) \cdot \hat{r} d\Omega, \tag{A77}$$

and

$$PF_4^{[2]} \equiv R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_2^{[0]} \times \vec{B}_1^{[4]} \right) \cdot \hat{r} d\Omega. \tag{A78}$$

According to Equation (57) of [13],

$$\vec{E}_1^{[2]}(\vec{x}) = -\frac{k}{c^2} \int d^3x_1 \left[\frac{1}{2} \partial_t^2 \rho_1(\vec{x}_1) \hat{R}_1 + \partial_t \vec{J}_1(\vec{x}_1) R_1^{-1} \right]. \tag{A79}$$

Let us now write an asymptotic expression for the second-order electric field:

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[2]}(\vec{x}) = -\frac{k}{c^2} \left[\frac{1}{2} \int d^3x_1 \partial_t^2 \rho_1 \left(\hat{r} - \frac{\vec{x}_1}{R_{max}} + \hat{r} \left(\frac{\vec{x}_1 \cdot \hat{r}}{R_{max}} \right) \right) + \int d^3x_1 \partial_t \vec{J}_1 \frac{1}{R_{max}} \right]. \tag{A80}$$

According to Equation (A63), we have

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[2]}(\vec{x}) = -\frac{k}{c^2 R_{max}} \left[\frac{1}{2} \int d^3x_1 \partial_t^2 \rho_1 (-\vec{x}_1 + \hat{r}(\vec{x}_1 \cdot \hat{r})) + \int d^3x_1 \partial_t \vec{J}_1 \right]. \tag{A81}$$

Taking into account Equation (A243), we obtain the following:

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[2]}(\vec{x}) = -\frac{k}{2c^2 R_{max}} \int d^3x_1 \left[\partial_t \vec{J}_1 + \hat{r}(\partial_t \vec{J}_1 \cdot \hat{r}) \right], \tag{A82}$$

The above expression decreases as $\frac{1}{R_{max}}$. According to Equation (62) of [13],

$$\vec{B}_2^{[2]}(\vec{x}) = -\frac{k}{c^2} \int d^3x_2 \frac{\vec{R}_2}{R_2^2} \times \vec{J}_2(\vec{x}_2). \quad (\text{A83})$$

Let us now write an asymptotic expression for the second-order magnetic field:

$$\lim_{r \rightarrow R_{max}} \vec{B}_2^{[2]}(\vec{x}) = -\frac{k}{c^2} \hat{r} \times \frac{\int d^3x_2 \vec{J}_2(\vec{x}_2)}{R_{max}^2}. \quad (\text{A84})$$

The above expression decreases as $\frac{1}{R_{max}^2}$. From Equations (A82) and (A84), it follows that the Poynting vector decays as $1/R_{max}^3$, and the associated surface integral over a large sphere with a radius of R_{max} , whose surface area grows as R_{max}^2 , will decrease as $1/R_{max}$. Thus, provided R_{max} is large enough, we consider this contribution to be negligible, that is,

$$PF_1^{[2]} \simeq 0. \quad (\text{A85})$$

Similarly,

$$PF_2^{[2]} \simeq 0. \quad (\text{A86})$$

Next, we compute the third integral:

$$PF_3^{[2]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[0]} \times \vec{B}_2^{[4]} \right) \cdot \hat{r} d\Omega. \quad (\text{A87})$$

According to Equation (55) of [13]:

$$\vec{E}_1^{[0]}(\vec{x}) = k \int d^3x_1 \frac{\rho_1(\vec{x}_1)}{R_1^2} \hat{R}_1 \quad (\text{A88})$$

Let us now write an asymptotic expression for the zeroth-order electric field:

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[0]}(\vec{x}) = k \frac{\int d^3x_1 \rho_1(\vec{x}_1)}{R_{max}^2} \hat{r}. \quad (\text{A89})$$

As this vector is radially oriented, the cross product of the above with any other vector must be perpendicular to the radial direction. It follows that said cross product will have a zero scalar product with a radially oriented surface vector; thus,

$$PF_3^{[2]} \simeq 0. \quad (\text{A90})$$

Similarly,

$$PF_4^{[2]} \simeq 0. \quad (\text{A91})$$

Appendix F

Equations (51) and (54) of [13] combine to yield the third-order terms in the electric field expansions:

$$\vec{E}_1^{[3]}(\vec{x}) = \frac{k}{c^3} \int d^3x_1 \left[\frac{1}{3} \partial_i^3 \rho_1(\vec{x}_1) R_1 \hat{R}_1 + \partial_i^2 \vec{J}_1(\vec{x}_1) \right] \quad (\text{A92})$$

We proceed to evaluate each integral individually.

$$\begin{aligned} I_1 &\equiv \int d^3x \left(\vec{J}_2 \cdot \vec{E}_1^{[3]} \right) \\ &= \frac{k}{c^3} \int d^3x \left[\vec{J}_2(\vec{x}) \cdot \left(\int d^3x_1 \left(\frac{1}{3} \partial_i^3 \rho_1(\vec{x}_1) R_1 \hat{R}_1 + \partial_i^2 \vec{J}_1(\vec{x}_1) \right) \right) \right]. \end{aligned} \quad (\text{A93})$$

Changing the order of integration, we obtain the following:

$$I_1 = \frac{k}{c^3} \left[\int d^3x_1 \frac{1}{3} \partial_t^3 \rho_1(\vec{x}_1) \int d^3x \vec{J}_2(\vec{x}) \cdot \vec{R}_1 + \int d^3x_1 \int d^3x \vec{J}_2(\vec{x}) \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) \right]. \quad (\text{A94})$$

Let us look at the following expression:

$$I_{1f} \equiv \int d^3x_1 \partial_t^3 \rho_1(\vec{x}_1) \int d^3x \vec{J}_2(\vec{x}) \cdot \vec{R}_1 = \int d^3x J_2^k(\vec{x}) \int d^3x_1 \partial_t^3 \rho_1(\vec{x}_1) R_1^k, \quad (\text{A95})$$

in which we have changed the order of integrals and use the Einstein summation convention. Now, by continuity,

$$\partial_t \rho_1 = -\vec{\nabla}_1 \cdot \vec{J}_1; \quad (\text{A96})$$

hence,

$$I_{1f} = - \int d^3x J_2^k(\vec{x}) \int d^3x_1 \partial_t^2 \vec{\nabla}_1 \cdot \vec{J}_1 R_1^k = - \int d^3x J_2^k(\vec{x}) \int d^3x_1 \vec{\nabla}_1 \cdot (\partial_t^2 \vec{J}_1) R_1^k. \quad (\text{A97})$$

Now,

$$\vec{\nabla}_1 \cdot (\partial_t^2 \vec{J}_1) R_1^k = \vec{\nabla}_1 \cdot (\partial_t^2 \vec{J}_1 R_1^k) - \partial_t^2 \vec{J}_1 \cdot \vec{\nabla}_1 R_1^k. \quad (\text{A98})$$

However,

$$\vec{\nabla}_1 R_1^k = \vec{\nabla}_1 (x^k - x_1^k) = -\vec{\nabla}_1 x_1^k = -\hat{x}^k. \quad (\text{A99})$$

Thus,

$$\vec{\nabla}_1 \cdot (\partial_t^2 \vec{J}_1) R_1^k = \vec{\nabla}_1 \cdot (\partial_t^2 \vec{J}_1 R_1^k) + \partial_t^2 \vec{J}_1 \cdot \hat{x}^k = \vec{\nabla}_1 \cdot (\partial_t^2 \vec{J}_1 R_1^k) + \partial_t^2 J_1^k. \quad (\text{A100})$$

Inserting Equation (A100) into Equation (A97) and using the Gaussian theorem, we obtain the following:

$$\begin{aligned} I_{1f} &= - \int d^3x J_2^k(\vec{x}) \int d^3x_1 \left[\vec{\nabla}_1 \cdot (\partial_t^2 \vec{J}_1 R_1^k) + \partial_t^2 J_1^k \right] \\ &= - \int d^3x \vec{J}_2(\vec{x}) \cdot \int d^3x_1 \partial_t^2 \vec{J}_1 - \int d^3x J_2^k(\vec{x}) \oint d\vec{S}_1 \cdot (\partial_t^2 \vec{J}_1 R_1^k). \end{aligned} \quad (\text{A101})$$

Assuming a lack of currents on a far-away encapsulating surface, the above expression simplifies as follows:

$$I_{1f} = - \int d^3x \vec{J}_2(\vec{x}) \cdot \int d^3x_1 \partial_t^2 \vec{J}_1. \quad (\text{A102})$$

Inserting Equation (A102) into Equation (A94), we obtain the following form:

$$I_1 = \frac{k}{c^3} \int d^3x_1 \int d^3x \left(1 - \frac{1}{3}\right) \vec{J}_2(\vec{x}) \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) = \frac{2k}{3c^3} \int d^3x_1 \int d^3x \vec{J}_2(\vec{x}) \cdot \partial_t^2 \vec{J}_1(\vec{x}_1). \quad (\text{A103})$$

We now change the integration variable ($\vec{x} \rightarrow \vec{x}_2$). Thus, $\vec{J}_2(\vec{x}) \rightarrow \vec{J}_2(\vec{x}_2)$, and also:

$$I_1 = \left(\frac{2k}{3c^3} \right) \int d^3x_1 d^3x_2 \left[\vec{J}_2(\vec{x}_2) \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) \right]. \quad (\text{A104})$$

By switching the indices, we obtain the second integral:

$$I_2 = \left(\frac{2k}{3c^3} \right) \int d^3x_1 d^3x_2 \left[\vec{J}_1(\vec{x}_1) \cdot \partial_t^2 \vec{J}_2(\vec{x}_2) \right]. \quad (\text{A105})$$

Appendix G

As the first- and third-order electric field expression is given in Equation (A92), we now focus on the first integral in Equation (69):

$$i_1 \equiv \epsilon_0 \int \left[\left(k \left(\frac{1}{c^3} \right) \int d^3 x_1 \left[\frac{1}{3} \partial_t^3 \rho_1(\vec{x}_1) R_1 \hat{R}_1 + \partial_t^2 \vec{J}_1(\vec{x}_1) \right] \right) \cdot \left(k \int d^3 x_2 \frac{\rho_2(\vec{x}_2)}{R_2^2} \hat{R}_2 \right) \right] d^3 x. \quad (\text{A106})$$

By separating the terms, we obtain the following:

$$\begin{aligned} i_1 &= \frac{\epsilon_0 k^2}{c^3} \int \left[\int d^3 x_1 d^3 x_2 \frac{1}{3} \partial_t^3 \rho_1(\vec{x}_1) \rho_2(\vec{x}_2) \left(R_1 \hat{R}_1 \cdot \frac{\hat{R}_2}{R_2^2} \right) \right. \\ &\quad \left. + \int d^3 x_1 d^3 x_2 \rho_2(\vec{x}_2) \left(\partial_t^2 \vec{J}_1(\vec{x}_1) \cdot \frac{\hat{R}_2}{R_2^2} \right) \right] d^3 x. \end{aligned} \quad (\text{A107})$$

Using Equation (A5), we obtain the following:

$$\begin{aligned} \int \left(\partial_t^2 \vec{J}_1(\vec{x}_1) \cdot \frac{\hat{R}_2}{R_2^2} \right) d^3 x &= - \int \left(\partial_t^2 \vec{J}_1(\vec{x}_1) \cdot \vec{\nabla} \frac{1}{R_2} \right) d^3 x \\ &= - \int \left(\vec{\nabla} \cdot \left(\partial_t^2 \vec{J}_1(\vec{x}_1) \frac{1}{R_2} \right) - \vec{\nabla} \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) \frac{1}{R_2} \right) d^3 x, \end{aligned} \quad (\text{A108})$$

since:

$$\vec{\nabla} \cdot \partial_t^2 \vec{J}_1(\vec{x}_1) = \partial_t^2 \vec{\nabla} \cdot \vec{J}_1(\vec{x}_1) = 0. \quad (\text{A109})$$

As $\vec{J}_1$ is a function of $\vec{x}_1$ and does not depend on $\vec{x}$, it follows that

$$\int \left(\partial_t^2 \vec{J}_1(\vec{x}_1) \cdot \frac{\hat{R}_2}{R_2^2} \right) d^3 x = - \oint d\vec{S} \cdot \left(\partial_t^2 \vec{J}_1(\vec{x}_1) \frac{1}{R_2} \right) = 0, \quad (\text{A110})$$

in which we use Gaussian theorem and assume, as usual, that there are no currents on the far-away encapsulating surface. Thus, it follows from Equation (A107) that

$$i_1 = \frac{k}{4\pi c^3} \int \int d^3 x_1 d^3 x_2 \frac{1}{3} \partial_t^3 \rho_1 \rho_2 \left(\int d^3 x \left(R_1 \hat{R}_1 \cdot \frac{\hat{R}_2}{R_2^2} \right) \right). \quad (\text{A111})$$

We now evaluate the following integral independently:

$$i_{1f} \equiv \int d^3 x \left(R_1 \hat{R}_1 \cdot \frac{\hat{R}_2}{R_2^2} \right) = - \int \vec{R}_1 \cdot \vec{\nabla} \frac{1}{R_2} d^3 x. \quad (\text{A112})$$

This can also be written in the following form:

$$i_{1f} = - \int \left[\vec{\nabla} \cdot \left(\frac{\vec{R}_1}{R_2} \right) - \frac{1}{R_2} \vec{\nabla} \cdot \vec{R}_1 \right] d^3 x = - \int \vec{\nabla} \cdot \left(\frac{\vec{R}_1}{R_2} \right) d^3 x + 3 \int \frac{1}{R_2} d^3 x. \quad (\text{A113})$$

Using Gaussian theorem, we can recast i_{1f} in the following form:

$$i_{1f} = - \oint d\vec{S} \cdot \left(\frac{\vec{R}_1}{R_2} \right) + 3 \int \frac{1}{R_2} d^3 x. \quad (\text{A114})$$

Calculating the surface integral on a far-away spherical surface of radius R_{max} and assuming that all currents and charges are close to the origin of the axis with distances much smaller than R_{max} such that $R_2 \simeq R_{max}$ and $\vec{R}_1 \simeq R_{max} \hat{r}$, it follows that

$$\oint d\vec{S} \cdot \left(\frac{\vec{R}_1}{R_2} \right) = \oint R_{max}^2 d\Omega \hat{r} \cdot \left(\frac{\vec{R}_1}{R_2} \right) \simeq \oint R_{max}^2 d\Omega \hat{r} \cdot \hat{r} = 4\pi R_{max}^2. \quad (A115)$$

Thus, Equation (A114) takes the following form:

$$i_{1f} \simeq -4\pi R_{max}^2 + 3 \int \frac{1}{R_2} d^3x. \quad (A116)$$

At this point, we notice that up to rather small terms, i_{1f} depends only on $\vec{x}_2$ and not on $\vec{x}_1$; thus, we may partition the double integral given in Equation (A111) in the form of

$$i_1 = \frac{k}{12\pi c^3} \int d^3x_1 \partial_t^3 \rho_1 \int d^3x_2 \rho_2 i_{1f} = 0 \quad (A117)$$

because $\int d^3x_1 \partial_t^3 \rho_1 = 0$ (see Equation (A63)). Similarly,

$$\epsilon_0 \int \left(\vec{E}_1^{[0]} \cdot \vec{E}_2^{[3]} \right) d^3x = 0. \quad (A118)$$

Appendix H

For ease of computation, we divide the Poynting flux expression given in Equation (73) into four independent integrals:

$$PF^{[3]} = PF_1^{[3]} + PF_2^{[3]} + PF_3^{[3]} + PF_4^{[3]} \quad (A119)$$

We begin by evaluating the first integral:

$$PF_1^{[3]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[3]} \times \vec{B}_2^{[2]} \right) \cdot \hat{r} d\Omega \quad (A120)$$

Let us now write an asymptotic expression for the third-order electric field given in Equation (A92):

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[3]} = k \left(\frac{1}{c^3} \right) \left[\frac{R_{max}}{3} \left(\int d^3x_1 \partial_t^3 \rho_1 \right) \hat{r} + \int d^3x_1 \partial_t^2 \vec{J}_1 \right] \quad (A121)$$

Using Equation (A63), we obtain

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[3]} = k \left(\frac{1}{c^3} \right) \left[\int d^3x_1 \partial_t^2 \vec{J}_1 \right] \quad (A122)$$

and an asymptotic expression for the second-order magnetic field:

$$\lim_{r \rightarrow R_{max}} \vec{B}_2^{[2]} = -\frac{k}{c^2} \hat{r} \times \frac{\int d^3x_2 \vec{J}_2}{R_{max}^2}. \quad (A123)$$

Thus,

$$\begin{aligned} PF_1^{[3]} &= -\frac{k}{4\pi c^3} \oint \left[\int \int d^3x_1 d^3x_2 \left(\partial_t^2 \vec{J}_1 \times \left(\hat{r} \times \vec{J}_2 \right) \right) \right] \cdot \hat{r} d\Omega \\ &= -\frac{k}{4\pi c^3} \oint \left[\int \int d^3x_1 d^3x_2 \left(\hat{r} \left(\partial_t^2 \vec{J}_1 \cdot \vec{J}_2 \right) - \left(\partial_t^2 \vec{J}_1 \cdot \hat{r} \right) \vec{J}_2 \right) \right] \cdot \hat{r} d\Omega \\ &= -\frac{k}{4\pi c^3} \oint \left[\int \int d^3x_1 d^3x_2 \left(\left(\partial_t^2 \vec{J}_1 \cdot \vec{J}_2 \right) - \left(\partial_t^2 \vec{J}_1 \cdot \hat{r} \right) \left(\vec{J}_2 \cdot \hat{r} \right) \right) \right] d\Omega \\ &= -\frac{k}{4\pi c^3} \int \int d^3x_1 d^3x_2 \left[4\pi \left(\partial_t^2 \vec{J}_1 \cdot \vec{J}_2 \right) - \left(\partial_t^2 \vec{J}_1^k \left(\oint \hat{r}_k \hat{r}_l d\Omega \right) \vec{J}_2^l \right) \right] \end{aligned} \quad (A124)$$

In the above, we use the Einstein summation convention. Now, the radial unit vector can be written in Cartesian coordinates in the following form:

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \quad (\text{A125})$$

in which ϕ is the azimuthal angle; θ is the angle of latitude; and $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors pointing in the direction of the Cartesian axes. Now,

$$I_{kl} \equiv \oint \hat{r}_k \hat{r}_l d\Omega = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \hat{r}_k \hat{r}_l. \quad (\text{A126})$$

As every non-diagonal element of I_{kl} is calculated by integrating over a complete period, for a term containing either $\sin \phi$ or $\cos \phi$, it follows that the non-diagonal elements of this matrix vanish. The remaining step is to calculate the diagonal elements:

$$I_{11} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \hat{r}_1 \hat{r}_1 = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^3 \theta \cos^2 \phi = \frac{4\pi}{3}, \quad (\text{A127})$$

$$I_{22} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \hat{r}_2 \hat{r}_2 = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^3 \theta \sin^2 \phi = \frac{4\pi}{3}, \quad (\text{A128})$$

$$I_{33} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \hat{r}_3 \hat{r}_3 = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \cos^2 \theta = \frac{4\pi}{3}, \quad (\text{A129})$$

Thus,

$$I_{kl} = \frac{4\pi}{3} \delta_{kl}, \quad (\text{A130})$$

in which δ_{kl} is Kronecker's delta. Inserting Equation (A130) into Equation (A124), it follows that

$$PF_1^{[3]} = -\frac{k}{c^3} \int \int d^3 x_1 d^3 x_2 \left[\left(\partial_t^2 \vec{J}_1 \cdot \vec{J}_2 \right) - \frac{1}{3} \left(\partial_t^2 \vec{J}_1^k \delta_{kl} \vec{J}_2^l \right) \right] = -\frac{2k}{3c^3} \int d^3 x_1 \partial_t^2 \vec{J}_1 \cdot \int d^3 x_2 \vec{J}_2 \quad (\text{A131})$$

Analogously, the second integral of the Poynting flux takes the following form:

$$\begin{aligned} PF_2^{[3]} &= R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_2^{[3]} \times \vec{B}_1^{[2]} \right) \cdot \hat{r} d\Omega \\ &= -\frac{2k}{3c^3} \int d^3 x_2 \partial_t^2 \vec{J}_2 \cdot \int d^3 x_1 \vec{J}_1 \end{aligned} \quad (\text{A132})$$

We now proceed to evaluate the third integral:

$$PF_3^{[3]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[0]} \times \vec{B}_2^{[5]} \right) \cdot \hat{r} d\Omega. \quad (\text{A133})$$

As referenced earlier, we adopt the electric and magnetic field expressions provided by [13]:

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[0]} = k \frac{\int d^3 x_1 \rho_1}{R_{max}^2} \hat{r}. \quad (\text{A134})$$

and

$$\lim_{r \rightarrow R_{max}} \vec{B}_2^{[5]} = -\frac{k}{3c^5} \int d^3 x_2 R_{max} \hat{r} \times \partial_t^3 \vec{J}_2. \quad (\text{A135})$$

Therefore,

$$PF_3^{[3]} = -\frac{k}{3c^3} \left(\int \int d^3 x_1 d^3 x_2 \rho_1 R_{max} \left(\hat{r} \times \left(\hat{r} \times \partial_t^3 \vec{J}_2 \right) \right) \right) \cdot \hat{r} = 0. \quad (\text{A136})$$

In a similar manner,

$$PF_4^{[3]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_2^{[0]} \times \vec{B}_1^{[5]} \right) \cdot \hat{r} d\Omega = 0. \quad (A137)$$

Appendix I

For each of the terms appearing in Equation (83), we may write a Poynting flux; thus, we obtain the following:

$$PF^{[4]} = PF_1^{[4]} + PF_2^{[4]} + PF_3^{[4]} + PF_4^{[4]} + PF_5^{[4]} + PF_6^{[4]}. \quad (A138)$$

Let us now calculate the first term ($PF_1^{[4]}$):

$$PF_1^{[4]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[4]} \times \vec{B}_2^{[2]} \right) \cdot \hat{r} d\Omega. \quad (A139)$$

Calculating the electric field on a far-away sphere, we may write Equation (77) in the following asymptotic form:

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[4]} = -\frac{k}{2c^4} \left[\frac{1}{4} \int d^3 x_1 \left(\partial_t^4 \rho_1 \left((R_{max}^2 \hat{r} - R_{max} ((\vec{x}_1 \cdot \hat{r}) \hat{r} + \vec{x}_1)) \right) + R_{max} \int d^3 x_1 \partial_t^3 \vec{J}_1 \right) \right]. \quad (A140)$$

Taking into account Equation (A63), we obtain the following:

$$\lim_{r \rightarrow R_{max}} \vec{E}_1^{[4]} = \frac{k R_{max}}{2c^4} \left[\frac{1}{4} \int d^3 x_1 \left(\partial_t^4 \rho_1 ((\vec{x}_1 \cdot \hat{r}) \hat{r} + \vec{x}_1) \right) - \int d^3 x_1 \partial_t^3 \vec{J}_1 \right]. \quad (A141)$$

As current density integrals are related to charge density integrals through Equation (A243), it follows that

$$\begin{aligned} \lim_{r \rightarrow R_{max}} \vec{E}_1^{[4]} &= \frac{k R_{max}}{2c^4} \left[\frac{1}{4} \left(\left(\int d^3 x_1 \partial_t^3 \vec{J}_1 \cdot \hat{r} \right) \hat{r} + \int d^3 x_1 \partial_t^3 \vec{J}_1 \right) - \int d^3 x_1 \partial_t^3 \vec{J}_1 \right] \\ &= \frac{k R_{max}}{2c^4} \left[\frac{1}{4} \left(\int d^3 x_1 \partial_t^3 \vec{J}_1 \cdot \hat{r} \right) \hat{r} - \frac{3}{4} \int d^3 x_1 \partial_t^3 \vec{J}_1 \right]. \end{aligned} \quad (A142)$$

The asymptotic expression for $B_2^{[2]}$ is given in Equation (A84). Using the asymptotic expressions, we are now in a position to calculate $PF_1^{[4]}$. Taking into account the fact that radial components of neither the electric nor magnetic field contribute to the Poynting flux, we obtain the following expression:

$$\begin{aligned} PF_1^{[4]} &= \frac{3k R_{max}}{32\pi c^4} \oint \left[\int \int d^3 x_1 d^3 x_2 \left(\partial_t^3 \vec{J}_1 \times (\hat{r} \times \vec{J}_2) \right) \right] \cdot \hat{r} d\Omega \\ &= \frac{3k R_{max}}{32\pi c^4} \oint \left[\int \int d^3 x_1 d^3 x_2 \hat{r} \left(\partial_t^3 \vec{J}_1 \cdot \vec{J}_2 \right) - \int \int d^3 x_1 d^3 x_2 \left(\partial_t^3 \vec{J}_1 \cdot \hat{r} \right) \vec{J}_2 \right] \cdot \hat{r} d\Omega \\ &= \frac{3k R_{max}}{32\pi c^4} \oint \left[\int \int d^3 x_1 d^3 x_2 \left(\partial_t^3 \vec{J}_1 \cdot \vec{J}_2 \right) - \int \int d^3 x_1 d^3 x_2 \left(\partial_t^3 \vec{J}_1 \cdot \hat{r} \right) (\vec{J}_2 \cdot \hat{r}) \right] d\Omega \\ &= \frac{3k R_{max}}{32\pi c^4} \left[\int \int d^3 x_1 d^3 x_2 4\pi \left(\partial_t^3 \vec{J}_1 \cdot \vec{J}_2 \right) - \int \int d^3 x_1 d^3 x_2 \left(\partial_t^3 \vec{J}_1^k \left(\oint \hat{r}_k \hat{r}_l d\Omega \right) \vec{J}_2^l \right) \right]. \end{aligned} \quad (A143)$$

Inserting Equation (A130) into Equation (A143), it follows that

$$PF_1^{[4]} = \frac{3k R_{max}}{8c^4} \int \int d^3 x_1 d^3 x_2 \left[\left(\partial_t^3 \vec{J}_1 \cdot \vec{J}_2 \right) - \frac{1}{3} \left(\partial_t^3 \vec{J}_1^k \delta_{kl} \vec{J}_2^l \right) \right]. \quad (A144)$$

$$PF_1^{[4]} = \frac{kR_{max}}{4c^4} \int \int d^3x_1 d^3x_2 \partial_t^3 \vec{J}_1 \cdot \vec{J}_2. \quad (A145)$$

Similarly, the second integral of the Poynting flux takes the following form:

$$PF_2^{[4]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_2^{[4]} \times \vec{B}_1^{[2]} \right) \cdot \hat{r} d\Omega = \frac{kR_{max}}{4c^4} \int \int d^3x_1 d^3x_2 \partial_t^3 \vec{J}_2 \cdot \vec{J}_1. \quad (A146)$$

We now proceed to calculate the following:

$$PF_3^{[4]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[2]} \times \vec{B}_2^{[4]} \right) \cdot \hat{r} d\Omega. \quad (A147)$$

The fourth-order term of the magnetic field can be calculated using the asymptotic limit of Equation (61) of [13]:

$$\lim_{r \rightarrow R_{max}} \vec{B}_2^{[4]}(\vec{x}) = \frac{k}{2c^4} \hat{r} \times \int d^3x_2 \partial_t^2 \vec{J}_2(\vec{x}_2). \quad (A148)$$

Using the asymptotic form of the second-order electric field given in Equation (A82) and recalling that the radial part of any field does not contribute to the Poynting flux, we now obtain the following:

$$\begin{aligned} PF_3^{[4]} &= -\frac{kR_{max}}{16\pi c^4} \oint \left[\int \int d^3x_1 d^3x_2 \left(\partial_t \vec{J}_1 \times \left(\hat{r} \times \partial_t^2 \vec{J}_2 \right) \right) \right] \cdot \hat{r} d\Omega \\ &= -\frac{kR_{max}}{16\pi c^4} \oint \left[\int \int d^3x_1 d^3x_2 \left(\hat{r} (\partial_t \vec{J}_1 \cdot \partial_t^2 \vec{J}_2) - (\partial_t \vec{J}_1 \cdot \hat{r}) \partial_t^2 \vec{J}_2 \right) \right] \cdot \hat{r} d\Omega \\ &= -\frac{kR_{max}}{16\pi c^4} \oint \left[\int \int d^3x_1 d^3x_2 \left((\partial_t \vec{J}_1 \cdot \partial_t^2 \vec{J}_2) - (\partial_t \vec{J}_1 \cdot \hat{r}) (\partial_t^2 \vec{J}_2 \cdot \hat{r}) \right) \right] d\Omega \\ &= -\frac{kR_{max}}{16\pi c^4} \int \int d^3x_1 d^3x_2 \left[4\pi (\partial_t \vec{J}_1 \cdot \partial_t^2 \vec{J}_2) - \left(\partial_t \vec{J}_1^k \left(\oint \hat{r}_k \hat{r}_l d\Omega \right) \partial_t^2 \vec{J}_2^l \right) \right]. \end{aligned} \quad (A149)$$

Using Equation (A130), we obtain the following:

$$\begin{aligned} PF_3^{[4]} &= -\frac{kR_{max}}{4c^4} \int \int d^3x_1 d^3x_2 \left[\partial_t \vec{J}_1 \cdot \partial_t^2 \vec{J}_2 - \frac{1}{3} \partial_t \vec{J}_1^k \delta_{kl} \partial_t^2 \vec{J}_2^l \right] \\ &= -\frac{kR_{max}}{6c^4} \int \int d^3x_1 d^3x_2 \partial_t \vec{J}_1 \cdot \partial_t^2 \vec{J}_2. \end{aligned} \quad (A150)$$

In a similar manner,

$$PF_4^{[4]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_2^{[2]} \times \vec{B}_1^{[4]} \right) \cdot \hat{r} d\Omega = -\frac{kR_{max}}{6c^4} \int \int d^3x_1 d^3x_2 \partial_t \vec{J}_2 \cdot \partial_t^2 \vec{J}_1. \quad (A151)$$

Subsequently, we calculate the fifth and sixth contributions appearing in Equation (A138):

$$PF_5^{[4]} = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \left(\vec{E}_1^{[0]} \times \vec{B}_2^{[6]} \right) \cdot \hat{r} d\Omega = R_{max}^2 \epsilon_0 c^2 \oint_{R_{max}} \vec{B}_2^{[6]} \cdot \left(\hat{r} \times \vec{E}_1^{[0]} \right) d\Omega. \quad (A152)$$

The asymptotic form of the zeroth-order electric field is given in Equation (A89) and has only a radial component; hence, it follows that

$$PF_5^{[4]} = PF_6^{[4]} = 0. \quad (A153)$$

Appendix J

Equation (87) can be separated into a sum of terms such that for each term, the integral is calculated separately, i.e.,

$$\begin{aligned} i_1 &= \epsilon_0 \int (\vec{E}_1^{[4]} \cdot \vec{E}_2^{[0]}) d^3x \\ &= \epsilon_0 \int \left[\left(-\left(\frac{k}{2c^4} \right) \int d^3x_1 \left[\frac{1}{4} \partial_t^4 \rho_1 R_1^2 \hat{R}_1 + \partial_t^3 \vec{J}_1 R_1 \right] \right) \cdot \left(k \int d^3x_2 \frac{\rho_2}{R_2^2} \hat{R}_2 \right) \right] d^3x, \end{aligned} \quad (\text{A154})$$

or in the following form:

$$\begin{aligned} i_1 &= -\frac{k}{8\pi c^4} \int \left[\int d^3x_1 d^3x_2 \frac{1}{4} \partial_t^4 \rho_1 \rho_2 \left(R_1^2 \hat{R}_1 \cdot \frac{\hat{R}_2}{R_2^2} \right) \right. \\ &\quad \left. + \int d^3x_1 d^3x_2 \rho_2 \partial_t^3 \vec{J}_1 \cdot \left(R_1 \frac{\hat{R}_2}{R_2^2} \right) \right] d^3x \end{aligned} \quad (\text{A155})$$

The following simplifies the expression with the aid of Equation (A238):

$$\int \frac{R_1 \hat{R}_2}{R_2^2} d^3x = \int R_1 \vec{\nabla}_2 \frac{1}{R_2} d^3x = \vec{\nabla}_2 \int \frac{R_1}{R_2} d^3x = \pi R^2 \hat{R}. \quad (\text{A156})$$

Now, let us evaluate the following term:

$$i_{2f} \equiv \int d^3x \left(R_1^2 \hat{R}_1 \cdot \frac{\hat{R}_2}{R_2^2} \right) = - \int R_1 \vec{R}_1 \cdot \vec{\nabla} \frac{1}{R_2} d^3x, \quad (\text{A157})$$

which can also be written in the following form:

$$i_{2f} = - \int \left[\vec{\nabla} \cdot \left(\frac{R_1 \vec{R}_1}{R_2} \right) - \frac{1}{R_2} \vec{\nabla} \cdot (R_1 \vec{R}_1) \right] d^3x = - \int \vec{\nabla} \cdot \left(\frac{R_1 \vec{R}_1}{R_2} \right) d^3x + 4 \int \frac{R_1}{R_2} d^3x. \quad (\text{A158})$$

Using Gaussian theorem, we can recast i_{2f} in the following form:

$$i_{2f} = - \oint d\vec{S} \cdot \left(\frac{R_1 \vec{R}_1}{R_2} \right) + 4 \int \frac{R_1}{R_2} d^3x. \quad (\text{A159})$$

Calculating the surface integral on a far-away spherical surface of radius R_{max} and assuming that all currents and charges are close to the origin of the axis with distances much smaller than R_{max} such that $R_1 \simeq R_2 \simeq R_{max}$ and $\vec{R}_1 \simeq R_{max} \hat{r}$, we obtain the following:

$$\oint d\vec{S} \cdot \left(\frac{R_1 \vec{R}_1}{R_2} \right) = \oint R_{max}^2 d\Omega \hat{r} \cdot \left(\frac{R_1 \vec{R}_1}{R_2} \right) \simeq \oint R_{max}^2 d\Omega R_{max} \hat{r} \cdot \hat{r} = 4\pi R_{max}^3. \quad (\text{A160})$$

Hence, using Equation (A238), it follows that

$$i_{2f} = -4\pi R_{max}^3 + 4 \int \frac{R_1}{R_2} d^3x = -4\pi R_{max}^3 + \frac{4\pi}{3} (4R_{max}^3 - R^3) = \frac{4\pi}{3} (R_{max}^3 - R^3). \quad (\text{A161})$$

We now can write the triple volume integral given in Equation (A155) as a double volume integral:

$$\begin{aligned} i_1 &= -\frac{k}{8\pi c^4} \left[\int \int d^3x_1 d^3x_2 \frac{1}{4} \partial_t^4 \rho_1 \rho_2 \left(\frac{4\pi}{3} (R_{max}^3 - R^3) \right) \right. \\ &\quad \left. + \int \int d^3x_1 d^3x_2 \rho_2 \partial_t^3 \vec{J}_1 \cdot (\pi R^2 \hat{R}) \right]. \end{aligned} \quad (\text{A162})$$

Alternatively, it can be expressed more simply as

$$i_1 = -\frac{k}{8c^4} \left[\int \int d^3x_1 d^3x_2 \partial_t^4 \rho_1 \rho_2 \left(\frac{1}{3} (R_{max}^3 - R^3) \right) + \int \int d^3x_1 d^3x_2 \rho_2 \partial_t^3 \vec{J}_1 \cdot (R^2 \hat{R}) \right]. \quad (A163)$$

Now, for the R_{max}^3 term, the double volume integrals decouple, and according to Equation (A63), we obtain the following:

$$i_1 = \frac{k}{8c^4} \left[\int \int d^3x_1 d^3x_2 \partial_t^4 \rho_1 \rho_2 \left(\frac{1}{3} R^3 \right) - \int \int d^3x_1 d^3x_2 \rho_2 \partial_t^3 \vec{J}_1 \cdot (R^2 \hat{R}) \right]. \quad (A164)$$

Let us look at the following term:

$$i_{1a} = \int d^3x_1 \partial_t^3 \vec{J}_1 \cdot (R^2 \hat{R}) = \frac{1}{3} \int d^3x_1 \partial_t^3 \vec{J}_1 \cdot \vec{\nabla}_1 R^3. \quad (A165)$$

This can be written in the following form:

$$i_{1a} = \frac{1}{3} \int d^3x_1 \left[\vec{\nabla}_1 \cdot (\partial_t^3 \vec{J}_1 R^3) - R^3 \partial_t^3 \vec{\nabla}_1 \cdot \vec{J}_1 \right]. \quad (A166)$$

Using Gaussian theorem and the continuity (Equation (A9)), this can be written as follows:

$$i_{1a} = \frac{1}{3} \left[\oint d\vec{S}_1 \cdot (\partial_t^3 \vec{J}_1 R^3) + \int d^3x_1 R^3 \partial_t^4 \rho_1 \right]. \quad (A167)$$

Assuming no currents at far-away surfaces, this reduces to the following:

$$i_{1a} = \frac{1}{3} \int d^3x_1 R^3 \partial_t^4 \rho_1. \quad (A168)$$

Plugging i_{1a} back into Equation (A164), we obtain the following:

$$\epsilon_0 \int (\vec{E}_1^{[4]} \cdot \vec{E}_2^{[0]}) d^3x = i_1 = 0. \quad (A169)$$

In a similar fashion, the second integral of Equation (87) is obtained by interchanging the indices:

$$i_2 = \epsilon_0 \int (\vec{E}_2^{[4]} \cdot \vec{E}_1^{[0]}) d^3x = 0. \quad (A170)$$

Next, using Equation (A30), we calculate the following:

$$\begin{aligned} i_3 &= \epsilon_0 \int (\vec{E}_1^{[2]} \cdot \vec{E}_2^{[2]}) d^3x = \epsilon_0 \int \left(-\frac{k}{c^2} \int d^3x_1 \left[\frac{1}{2} \partial_t^2 \rho_1 \hat{R}_1 + \partial_t \vec{J}_1 R_1^{-1} \right] \right. \\ &\quad \cdot \left. \left(-\frac{k}{c^2} \int d^3x_2 \left[\frac{1}{2} \partial_t^2 \rho_2 \hat{R}_2 + \partial_t \vec{J}_2 R_2^{-1} \right] \right) d^3x, \end{aligned} \quad (A171)$$

which can be cast in the following form:

$$\begin{aligned} i_3 &= \frac{k}{4\pi c^4} \int \int d^3x_1 d^3x_2 \left[\frac{1}{4} \partial_t^2 \rho_1 \partial_t^2 \rho_2 \left(\int d^3x \hat{R}_1 \cdot \hat{R}_2 \right) + \frac{1}{2} \partial_t^2 \rho_1 \partial_t \vec{J}_2 \cdot \left(\int \frac{\hat{R}_1}{R_2} d^3x \right) \right. \\ &\quad \left. + \frac{1}{2} \partial_t^2 \rho_2 \partial_t \vec{J}_1 \cdot \left(\int \frac{\hat{R}_2}{R_1} d^3x \right) + \partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 \left(\int \frac{1}{R_1 R_2} d^3x \right) \right]. \end{aligned} \quad (A172)$$

To reduce the above triple volume integral into a double volume integral, we notice that by using Equation (A238), it follows that

$$\int \frac{\hat{R}_1}{R_2} d^3x = - \int \frac{\vec{\nabla}_1 R_1}{R_2} d^3x = - \vec{\nabla}_1 \int \frac{R_1}{R_2} d^3x = - \vec{\nabla}_1 \left(\frac{\pi}{3} (4R_{max}^3 - R^3) \right) = \pi R^2 \hat{R}. \quad (A173)$$

Exchanging the 1 and 2 indices, we obtain the following:

$$\int \frac{\hat{R}_2}{R_1} d^3x = -\pi R^2 \hat{R}. \quad (\text{A174})$$

Taking into account Equation (A235), it follows that

$$\int \left(\frac{1}{R_1 R_2} \right) d^3x = 2\pi(2R_{\max} - R). \quad (\text{A175})$$

The following integral can be computed using methods similar to those used in Appendix L such that

$$\int d^3x \hat{R}_1 \cdot \hat{R}_2 = 2\pi R^3 \int_0^{y_{\max}} y^2 dy \int_{-1}^1 \frac{y+s}{\sqrt{1+y^2+2ys}} ds. \quad (\text{A176})$$

Using Equations (A52) and (A53), we obtain the following:

$$\int d^3x \hat{R}_1 \cdot \hat{R}_2 = 2\pi R^3 \left[\frac{1}{3} + \frac{2}{3} (y_{\max}^3 - y_{\max}) \right] = \frac{2}{3} \pi (2R_{\max}^3 - 2R^2 R_{\max} + R^3). \quad (\text{A177})$$

Thus,

$$\begin{aligned} i_3 &= \frac{k}{4\pi c^4} \int \int d^3x_1 d^3x_2 \left[\frac{1}{4} \partial_t^2 \rho_1 \partial_t^2 \rho_2 \left(\frac{2}{3} \pi (2R_{\max}^3 - 2R^2 R_{\max} + R^3) \right) \right. \\ &+ \left. \frac{1}{2} \partial_t^2 \rho_1 \partial_t \vec{J}_2 \cdot (\pi R^2 \hat{R}) + \frac{1}{2} \partial_t^2 \rho_2 \partial_t \vec{J}_1 \cdot (-\pi R^2 \hat{R}) + \partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 (2\pi(2R_{\max} - R)) \right], \end{aligned} \quad (\text{A178})$$

which can be slightly simplified as follows:

$$\begin{aligned} i_3 &= \frac{k}{4c^4} \int \int d^3x_1 d^3x_2 \left[\frac{1}{4} \partial_t^2 \rho_1 \partial_t^2 \rho_2 \left(\frac{2}{3} (2R_{\max}^3 - 2R^2 R_{\max} + R^3) \right) \right. \\ &+ \left. \frac{1}{2} \partial_t^2 \rho_1 \partial_t \vec{J}_2 \cdot (R^2 \hat{R}) - \frac{1}{2} \partial_t^2 \rho_2 \partial_t \vec{J}_1 \cdot (R^2 \hat{R}) + \partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 (2(2R_{\max} - R)) \right]. \end{aligned} \quad (\text{A179})$$

The $R_{\max}^3$ term decouples into a multiplication of two volume integrals and, thus, according to Equation (A63), vanishes; thus, we obtain the following:

$$\begin{aligned} i_3 &= \frac{k}{4c^4} \int \int d^3x_1 d^3x_2 \left[\frac{1}{4} \partial_t^2 \rho_1 \partial_t^2 \rho_2 \left(\frac{2}{3} (R^3 - 2R^2 R_{\max}) \right) \right. \\ &+ \left. \frac{1}{2} \partial_t^2 \rho_1 \partial_t \vec{J}_2 \cdot (R^2 \hat{R}) - \frac{1}{2} \partial_t^2 \rho_2 \partial_t \vec{J}_1 \cdot (R^2 \hat{R}) + \partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 (2(2R_{\max} - R)) \right]. \end{aligned} \quad (\text{A180})$$

The above can also be written in the following form:

$$\begin{aligned} i_3 &= \frac{k}{4c^4} \int \int d^3x_1 d^3x_2 \left[\frac{1}{6} \partial_t^2 \rho_1 \partial_t^2 \rho_2 (R^3 - 2R^2 R_{\max}) - \frac{1}{2} \partial_t^2 \rho_1 \partial_t \vec{J}_2 \cdot \left(\frac{1}{3} \vec{\nabla}_2 R^3 \right) \right. \\ &- \left. \frac{1}{2} \partial_t^2 \rho_2 \partial_t \vec{J}_1 \cdot \left(\frac{1}{3} \vec{\nabla}_1 R^3 \right) + 2\partial_t \vec{J}_1 \cdot \partial_t \vec{J}_2 (2R_{\max} - R) \right]. \end{aligned} \quad (\text{A181})$$

Now:

$$\int d^3x_1 \partial_t \vec{J}_1 \cdot \vec{\nabla}_1 R^3 = \int d^3x_1 \left[\vec{\nabla}_1 \cdot (\partial_t \vec{J}_1 R^3) - R^3 \partial_t \vec{\nabla}_1 \cdot \vec{J}_1 \right] = \oint d\vec{S}_1 \cdot \vec{J}_1 R^3 + \int d^3x_1 R^3 \partial_t^2 \rho_1, \quad (\text{A182})$$

in which we use Gaussian theorem and the continuity (Equation (A9)). Taking a far-away encapsulating surface in which no currents are to be found, it follows that

$$\int d^3x_1 \partial_t \vec{J}_1 \cdot \vec{\nabla}_1 R^3 = \int d^3x_1 R^3 \partial_t^2 \rho_1. \quad (\text{A183})$$

Similarly,

$$\int d^3x_2 \partial_t \vec{j}_2 \cdot \vec{\nabla}_2 R^3 = \int d^3x_2 R^3 \partial_t^2 \rho_2. \quad (\text{A184})$$

Plugging Equations (A183) and (A184) into Equation (A182), we obtain the following simplified expression:

$$i_3 = \frac{k}{4c^4} \int \int d^3x_1 d^3x_2 \left[-\frac{1}{6} \partial_t^2 \rho_1 \partial_t^2 \rho_2 (R^3 + 2R^2 R_{max}) + 2\partial_t \vec{j}_1 \cdot \partial_t \vec{j}_2 (2R_{max} - R) \right]. \quad (\text{A185})$$

In summary, the electric part of the fourth-order electromagnetic energy can be written in terms of the temporal derivatives of charge and current densities in the following form:

$$\begin{aligned} E_{field12}^{[4]} &= \epsilon_0 \int \left[\vec{E}_1^{[4]} \cdot \vec{E}_2^{[0]} + \vec{E}_1^{[0]} \cdot \vec{E}_2^{[4]} + \vec{E}_1^{[2]} \cdot \vec{E}_2^{[2]} \right] d^3x \\ &= \frac{k}{4c^4} \int \int d^3x_1 d^3x_2 \left[-\frac{1}{6} \partial_t^2 \rho_1 \partial_t^2 \rho_2 (R^3 + 2R^2 R_{max}) + 2\partial_t \vec{j}_1 \cdot \partial_t \vec{j}_2 (2R_{max} - R) \right]. \end{aligned} \quad (\text{A186})$$

We remind the reader that R_{max} is the maximal radius for which Taylor-series expansion is applicable. The remaining step is to calculate the magnetic terms' contributions, a task which we now attend to. First, we set

$$i_4 = \epsilon_0 c^2 \int \left(\vec{B}_1^{[4]} \cdot \vec{B}_2^{[2]} \right) d^3x. \quad (\text{A187})$$

Now, according to Equation (61) of [13],

$$\vec{B}_1^{[4]}(\vec{x}) = \frac{k}{2c^4} \int d^3x_1 \hat{R}_1 \times \partial_t^2 \vec{j}_1(\vec{x}_1), \quad (\text{A188})$$

$$\vec{B}_2^{[2]}(\vec{x}) = -\frac{k}{c^2} \int d^3x_2 \frac{\hat{R}_2}{R_2^2} \times \vec{j}_2(\vec{x}_2). \quad (\text{A189})$$

Plugging Equations (A188) and (A189) into Equation (A187), it follows that

$$i_4 = -\frac{k}{8\pi c^4} \int \left[\int \int d^3x_1 d^3x_2 \frac{(\hat{R}_1 \times \partial_t^2 \vec{j}_1) \cdot (\hat{R}_2 \times \vec{j}_2)}{R_2^2} \right] d^3x. \quad (\text{A190})$$

According to a well known vector identity, i.e.,

$$(\hat{R}_1 \times \partial_t^2 \vec{j}_1) \cdot (\hat{R}_2 \times \vec{j}_2) = (\hat{R}_1 \cdot \hat{R}_2) (\partial_t^2 \vec{j}_1 \cdot \vec{j}_2) - (\hat{R}_1 \cdot \vec{j}_2) (\hat{R}_2 \cdot \partial_t^2 \vec{j}_1), \quad (\text{A191})$$

we may write Equation (A190) in the following form:

$$\begin{aligned} i_4 &= -\frac{k}{8\pi c^4} \left[\int \int d^3x_1 d^3x_2 (\partial_t^2 \vec{j}_1 \cdot \vec{j}_2) \left(\int \frac{\hat{R}_1 \cdot \hat{R}_2}{R_2^2} d^3x \right) \right. \\ &\quad \left. - \int d^3x \left(\int d^3x_1 \partial_t^2 \vec{j}_1 \cdot \hat{R}_1 \right) \left(\int d^3x_2 \frac{\vec{j}_2 \cdot \hat{R}_2}{R_2^2} \right) \right]. \end{aligned} \quad (\text{A192})$$

Notice that

$$\begin{aligned} \int d^3x_1 \partial_t^2 \vec{j}_1 \cdot \hat{R}_1 &= -\int d^3x_1 \partial_t^2 \vec{j}_1 \cdot \vec{\nabla}_1 R_1 = -\int d^3x_1 \left[\vec{\nabla}_1 \cdot (\partial_t^2 \vec{j}_1 R_1) - R_1 \partial_t^2 \nabla_1 \cdot \vec{j}_1 \right] \\ &= -\oint d\vec{S}_1 \cdot (\partial_t^2 \vec{j}_1 R_1) - \int d^3x_1 R_1 \partial_t^3 \rho_1 = -\int d^3x_1 R_1 \partial_t^3 \rho_1. \end{aligned} \quad (\text{A193})$$

In the above, we have use the Gaussian theorem, assuming that there is no current density on far-away surfaces, and the continuity Equation (A9). Similarly,

$$\begin{aligned} \int d^3x_2 \vec{j}_2 \cdot \frac{\hat{R}_2}{R_2^2} &= \int d^3x_2 \vec{j}_2 \cdot \vec{\nabla}_2 \frac{1}{R_2} = \int d^3x_2 \left[\vec{\nabla}_2 \cdot \left(\frac{\vec{j}_2}{R_2} \right) - \frac{1}{R_2} \vec{\nabla}_2 \cdot \vec{j}_2 \right] \\ &= \oint d\vec{S}_2 \cdot \frac{\vec{j}_2}{R_2} + \int d^3x_2 \frac{\partial_t \rho_2}{R_2} = \int d^3x_2 \frac{\partial_t \rho_2}{R_2}. \end{aligned} \quad (A194)$$

Plugging Equations (A54), (A193) and (A194) into Equation (A192), we arrive at the following expression:

$$\begin{aligned} i_4 &= -\frac{k}{8\pi c^4} \int \int d^3x_1 d^3x_2 \left[(\partial_t^2 \vec{j}_1 \cdot \vec{j}_2) (4\pi(R_{max} - R)) - \int d^3x \left(-R_1 \partial_t^3 \rho_1 \right) \left(\frac{1}{R_2} \partial_t \rho_2 \right) \right] \\ &= -\frac{k}{8\pi c^4} \int \int d^3x_1 d^3x_2 \left[(\partial_t^2 \vec{j}_1 \cdot \vec{j}_2) (4\pi(R_{max} - R)) + (\partial_t^3 \rho_1 \partial_t \rho_2) \int d^3x \frac{R_1}{R_2} \right]. \end{aligned} \quad (A195)$$

Taking into account Equation (A238), the above expression reduces as follows:

$$\begin{aligned} i_4 &= -\frac{k}{8\pi c^4} \int \int d^3x_1 d^3x_2 \left[(\partial_t^2 \vec{j}_1 \cdot \vec{j}_2) (4\pi(R_{max} - R)) + (\partial_t^3 \rho_1 \partial_t \rho_2) \frac{\pi}{3} (4R_{max}^3 - R^3) \right] \\ &= -\frac{k}{8c^4} \int \int d^3x_1 d^3x_2 \left[4(\partial_t^2 \vec{j}_1 \cdot \vec{j}_2) (R_{max} - R) + \frac{1}{3} (\partial_t^3 \rho_1 \partial_t \rho_2) (4R_{max}^3 - R^3) \right]. \end{aligned} \quad (A196)$$

The term containing R_{max}^3 decouples into a multiplication of two volume integrals. However, according to Equation (A63) and due to the fact that we assume no current density on the far-away encapsulating surface, it follows that $\int d^3x_1 \partial_t^3 \rho_1 = 0$ and $\int d^3x_2 \partial_t \rho_2 = 0$. Thus, we obtain a simplified form of Equation (A196):

$$i_4 = \frac{k}{8c^4} \int \int d^3x_1 d^3x_2 \left[4(R - R_{max}) (\partial_t^2 \vec{j}_1 \cdot \vec{j}_2) + \frac{1}{3} R^3 (\partial_t^3 \rho_1 \partial_t \rho_2) \right]. \quad (A197)$$

By analogy,

$$\epsilon_0 c^2 \int (\vec{B}_2^{[4]} \cdot \vec{B}_1^{[2]}) d^3x = \frac{k}{8c^4} \int \int d^3x_1 d^3x_2 \left[4(R - R_{max}) (\partial_t^2 \vec{j}_2 \cdot \vec{j}_1) + \frac{1}{3} R^3 (\partial_t^3 \rho_2 \partial_t \rho_1) \right]. \quad (A198)$$

Summing up Equations (A197) and (A198), we arrive at the fourth-order contribution to the magnetic field energy in terms of a double volume integral:

$$\begin{aligned} E_{field12}^{[4]} &= \epsilon_0 c^2 \int [\vec{B}_1^{[4]} \cdot \vec{B}_2^{[2]} + \vec{B}_1^{[2]} \cdot \vec{B}_2^{[4]}] d^3x = \\ &= \frac{k}{8c^4} \int \int d^3x_1 d^3x_2 \left[4(R - R_{max}) (\partial_t^2 \vec{j}_2 \cdot \vec{j}_1 + \partial_t^2 \vec{j}_1 \cdot \vec{j}_2) + \frac{1}{3} R^3 (\partial_t^3 \rho_2 \partial_t \rho_1 + \partial_t^3 \rho_1 \partial_t \rho_2) \right]. \end{aligned} \quad (A199)$$

Appendix K

What follows is aimed at deriving the momentum (Equation (7)) as outline in [13]. Using the expressions of the electric and magnetic fields given in [13], we may now calculate the force that system 1 exerts on system 2 in the n^{th} order through the Lorentz force given in Equation (10) of [13]:

$$\vec{F}_{21}^{[n]} = \int d^3x_2 \left(\rho_2(\vec{x}_2) \vec{E}_1^{[n]}(\vec{x}_2) + \vec{j}_2(\vec{x}_2) \times \vec{B}_1^{[n]}(\vec{x}_2) \right). \quad (A200)$$

It follows that for the zeroth order, we obtain the following:

$$\vec{F}_{21}^{[0]} = \int d^3x_2 \rho_2(\vec{x}_2) \vec{E}_1^{[0]}(\vec{x}_2). \quad (A201)$$

Inserting Equation (55) of [13], we obtain Coulomb's force:

$$\vec{F}_{21}^{[0]} = -k \int \int d^3x_1 d^3x_2 \rho_1(\vec{x}_1) \rho_2(\vec{x}_2) \frac{1}{R^2} \hat{R} = -\vec{F}_{12}^{[0]}. \quad (\text{A202})$$

This type of force, which is just the quasi-static force, satisfies Newton's third law; hence, the total force on the system is null:

$$\vec{F}_T^{[0]} = \vec{F}_{21}^{[0]} + \vec{F}_{12}^{[0]} = 0. \quad (\text{A203})$$

The first-order force in $\frac{1}{c}$ is null, since the first-order electric and magnetic fields are null; thus,

$$\vec{F}_T^{[1]} = \vec{F}_{21}^{[1]} = \vec{F}_{12}^{[1]} = 0. \quad (\text{A204})$$

We now proceed with the calculation of the second-order force term; this will suffice, as $\frac{1}{c}$ is a rather small number. To do this, we first divide the force given in Equation (A200) into electric and magnetic terms:

$$\begin{aligned} \vec{F}_{21}^{[2]} &= \vec{F}_{21e}^{[2]} + \vec{F}_{21m}^{[2]} \\ \vec{F}_{21e}^{[2]} &\equiv \int d^3x_2 \rho_2(\vec{x}_2) \vec{E}_1^{[2]}(\vec{x}_2) \\ \vec{F}_{21m}^{[2]} &\equiv \int d^3x_2 \vec{j}_2(\vec{x}_2) \times \vec{B}_1^{[2]}(\vec{x}_2) \end{aligned} \quad (\text{A205})$$

Inserting Equation (57) of [13] into Equation (A205), we readily obtain the electric force as follows:

$$\vec{F}_{21e}^{[2]} = \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} \rho_2 \partial_t^2 \rho_1 \hat{R} - \rho_2 \partial_t \vec{j}_1 R^{-1} \right] \quad (\text{A206})$$

The magnetic force can be obtained by inserting Equation (62) of [13] into Equation (A205):

$$\begin{aligned} \vec{F}_{21m}^{[2]} &= \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \vec{j}_2 \times \left(R^{-2} \hat{R} \times \vec{j}_1 \right) \\ &= \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[\hat{R} \frac{\vec{j}_1 \cdot \vec{j}_2}{R^2} - \vec{j}_1 R^{-2} (\hat{R} \cdot \vec{j}_2) \right] \end{aligned} \quad (\text{A207})$$

However, given

$$R^{-2} \hat{R} = \vec{\nabla}_{\vec{x}_2} R^{-1}, \quad (\text{A208})$$

it follows that

$$\vec{F}_{21m}^{[2]} = \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[\hat{R} \frac{\vec{j}_1 \cdot \vec{j}_2}{R^2} - \vec{j}_1 \left((\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{j}_2 \right) \right]. \quad (\text{A209})$$

Let us look at the following integral:

$$\int d^3x_2 (\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{j}_2 = \int d^3x_2 \left[\vec{\nabla}_{\vec{x}_2} \cdot \left(\frac{\vec{j}_2}{R} \right) - \frac{1}{R} \vec{\nabla}_{\vec{x}_2} \cdot \vec{j}_2 \right] \quad (\text{A210})$$

Using Gaussian theorem and the continuity (Equation (A9)), we arrive at the following expression:

$$\int d^3x_2 (\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{j}_2 = \oint d\vec{S}_2 \cdot \left(\frac{\vec{j}_2}{R} \right) + \int d^3x_2 \frac{1}{R} \partial_t \rho_2. \quad (\text{A211})$$

The surface integral is applied over a surface encapsulating the volume of the volume integral. If the volume integral is applied over all space, the surface is at infinity, provided there are no currents at infinity:

$$\int d^3x_2 (\vec{\nabla}_{\vec{x}_2} R^{-1}) \cdot \vec{J}_2 = \int d^3x_2 \frac{1}{R} \partial_t \rho_2. \quad (\text{A212})$$

Inserting the result of Equation (A212) into Equation (A209), we arrive at the following expression for the second-order magnetic force:

$$\vec{F}_{21m}^{[2]} = \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[\hat{R} \frac{\vec{J}_1 \cdot \vec{J}_2}{R^2} - \frac{\vec{J}_1}{R} \partial_t \rho_2 \right]. \quad (\text{A213})$$

We remark that for the magnetic field, the second order is the lowest order for the force, as the zeroth and first orders are null. Moreover, we observe that the force is the sum of two parts; the first one satisfies Newton's third law, and the second does not. We can now calculate the total electromagnetic force by adding Equation (A206) to Equation (A213):

$$\begin{aligned} \vec{F}_{21}^{[2]} &= \vec{F}_{21e}^{[2]} + \vec{F}_{21m}^{[2]} \\ &= \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} \rho_2 \partial_t^2 \rho_1 \hat{x}_{12} - \partial_t (\rho_2 \vec{J}_1) R^{-1} + \hat{x}_{12} \frac{\vec{J}_1 \cdot \vec{J}_2}{R^2} \right] \end{aligned} \quad (\text{A214})$$

We now use the notation of $\hat{R} = \hat{x}_{12}$ for clarity. Based on the above expressions, it is easy to calculate $F_{12}^{[2]}$ by exchanging indices 1 and 2:

$$\begin{aligned} \vec{F}_{12}^{[2]} &= \\ &= \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} \rho_1 \partial_t^2 \rho_2 \hat{x}_{21} - \partial_t (\rho_1 \vec{J}_2) R^{-1} + \hat{x}_{21} \frac{\vec{J}_1 \cdot \vec{J}_2}{R^2} \right] \end{aligned} \quad (\text{A215})$$

Combining Equations (A214) and (A215) and taking into account the fact that $\hat{x}_{12} = -\hat{x}_{21}$, it follows that

$$\begin{aligned} \vec{F}_T^{[2]} &= \vec{F}_{12}^{[2]} + \vec{F}_{21}^{[2]} = \\ &= \left(\frac{k}{c^2} \right) \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} (\rho_2 \partial_t^2 \rho_1 - \rho_1 \partial_t^2 \rho_2) \hat{R} - \partial_t (\rho_1 \vec{J}_2 + \rho_2 \vec{J}_1) R^{-1} \right] \end{aligned} \quad (\text{A216})$$

Now, as $\frac{k}{c^2} = \frac{\mu_0}{4\pi}$ and since

$$\rho_2 \partial_t^2 \rho_1 - \rho_1 \partial_t^2 \rho_2 = \partial_t (\rho_2 \partial_t \rho_1 - \rho_1 \partial_t \rho_2), \quad (\text{A217})$$

it follows that

$$\vec{F}_T^{[2]} = \frac{\mu_0}{4\pi} \partial_t \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} (\rho_2 \partial_t \rho_1 - \rho_1 \partial_t \rho_2) \hat{R} - (\rho_1 \vec{J}_2 + \rho_2 \vec{J}_1) R^{-1} \right] \quad (\text{A218})$$

We remark that in some fast-changing systems, the second-order correction will not suffice and higher order terms will be needed. According to Newton's second law, a system with a non-zero total force in its center of mass must have a change in its total linear momentum ($\vec{P}(t)$):

$$\vec{F}_T^{[2]} = \frac{d\vec{P}}{dt} \quad (\text{A219})$$

Assuming that $\vec{P}(-\infty) = 0$ and that there are no current or charge densities at $t = -\infty$, it follows from Equation (A218) that

$$\vec{P}(t) = \frac{\mu_0}{4\pi} \int \int d^3x_1 d^3x_2 \left[\frac{1}{2} (\rho_2 \partial_t \rho_1 - \rho_1 \partial_t \rho_2) \hat{R} - (\rho_1 \vec{J}_2 + \rho_2 \vec{J}_1) R^{-1} \right], \quad (\text{A220})$$

which is the same as Equation (7).

Appendix L

We know that $\vec{R}_1 = \vec{x} - \vec{x}_1$, $\vec{R}_2 = \vec{x} - \vec{x}_2$, and as $\vec{x}_1$ is constant with respect to the $\vec{x}$ integration, it follows that $d\vec{R}_1 = d\vec{x}$, so

$$i_{11} = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \hat{R}_2 \right) d^3x = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \hat{R}_2 \right) d^3\vec{R}_1 \quad (\text{A221})$$

Using spherical coordinates (R_1, θ_1, ϕ_1) , i.e.,

$$d^3R_1 = R_1^2 dR_1 d\Omega_1, \quad d\Omega_1 = -ds d\phi_1, \quad s = \cos \theta_1, \quad (\text{A222})$$

we may write Equation (A221) in the following form:

$$i_{11} = \int (\hat{R}_1 \cdot \hat{R}_2) dR_1 d\Omega_1. \quad (\text{A223})$$

We now recall the following definitions:

$$\vec{R}_2 \equiv \vec{x} - \vec{x}_2 = \vec{R}_1 + \vec{x}_1 - \vec{x}_2 = \vec{R}_1 + \vec{R}, \quad \vec{R} \equiv \vec{x}_{12} \equiv \vec{x}_1 - \vec{x}_2. \quad (\text{A224})$$

Using the above equation for R_2 , we obtain the following:

$$i_{11} = \int \frac{\hat{R}_1 \cdot \vec{R}_2}{|\vec{R}_2|} dR_1 d\Omega_1 = \int \frac{\hat{R}_1 \cdot (\vec{R}_1 + \vec{R})}{|\vec{R}_1 + \vec{R}|} dR_1 d\Omega_1 = \int \frac{R_1 + \hat{R}_1 \cdot \vec{R}}{|\vec{R}_1 + \vec{R}|} dR_1 d\Omega_1 \quad (\text{A225})$$

Let us choose the z axis in our spherical coordinates to be in the $\vec{R}$ direction.

$$\hat{R}_1 = \sin \theta_1 \cos \phi_1 \hat{x}_1 + \sin \theta_1 \sin \phi_1 \hat{y}_1 + \cos \theta_1 \hat{z}_1, \quad \hat{R} = \hat{z}_1 \quad (\text{A226})$$

Furthermore, let us expand as follows:

$$|\vec{R}_1 + \vec{R}| = \sqrt{R_1^2 + R^2 + 2\vec{R}_1 \cdot \vec{R}} = \sqrt{R_1^2 + R^2 + 2R_1 R \hat{R}_1 \cdot \hat{R}} = \sqrt{R_1^2 + R^2 + 2R_1 R s}, \quad (\text{A227})$$

since $\hat{R}_1 \cdot \hat{R} = \hat{R}_1 \cdot \hat{z}_1 = \cos \theta_1 = s$. We now introduce the dimensionless variable: $y = \frac{R_1}{R}$. Thus, Equation (A225) takes the following form:

$$i_{11} = \int \frac{y + s}{\sqrt{1 + y^2 + 2ys}} (-R dy d\Omega_1). \quad (\text{A228})$$

Taking into account the solid angle expression given in Equation (A222), we finally obtain the following:

$$i_{11} = \int \left(\frac{\hat{R}_1}{R_1^2} \cdot \hat{R}_2 \right) d^3x = 2\pi R \int_0^{y_{max}} dy \int_{-1}^1 \frac{y + s}{\sqrt{1 + y^2 + 2ys}} ds, \quad y_{max} \equiv \frac{R_{max}}{R}. \quad (\text{A229})$$

where R_{max} is the radius of a sphere whose surface is presumably far away from the charged systems and defines the limit when Taylor-series expansion is applicable.

Appendix M

$$\int \left(\frac{1}{R_1 R_2} \right) d^3 x = \int d^3 R_1 \frac{1}{R_1} \frac{1}{|\vec{R}_1 + \vec{R}|} = \int \frac{R_1 dR_1 d\Omega_1}{\sqrt{R_1^2 + R^2 + 2R_1 R \hat{R}_1 \cdot \hat{R}}} \quad (\text{A230})$$

where

$$d^3 R_1 = R_1^2 dR_1 d\Omega_1 \quad d\Omega_1 = -ds d\phi_1 \quad (\text{A231})$$

Let $y = \frac{R_1}{R}$, $s = \cos \theta_1$

$$\begin{aligned} \int \left(\frac{1}{R_1 R_2} \right) d^3 x &= R \int \frac{y dy d\Omega_1}{\sqrt{1 + y^2 + 2ys}}, \\ &= R \int_0^{y_{\max}} \int_1^{-1} \int_0^{2\pi} \frac{-y dy ds d\phi_1}{\sqrt{1 + y^2 + 2ys}} \\ &= 2\pi R \int_0^{y_{\max}} y dy \int_{-1}^1 \frac{ds}{\sqrt{1 + y^2 + 2ys}} \end{aligned} \quad (\text{A232})$$

$$\int_{-1}^1 ds \frac{1}{\sqrt{1 + y^2 + 2ys}} = \begin{cases} \frac{2}{y} & y \geq 1. \\ 2 & y < 1. \end{cases} \quad (\text{A233})$$

$$\int \left(\frac{1}{R_1 R_2} \right) d^3 x = 2\pi R \left[\int_0^1 2y dy + \int_1^{y_{\max}} y \frac{2}{y} dy \right] = 2\pi R (1 + 2(y_{\max} - 1)) \quad (\text{A234})$$

Thus,

$$\int \left(\frac{1}{R_1 R_2} \right) d^3 x = 2\pi R (-1 + 2y_{\max}) = 2\pi (2R_{\max} - R). \quad (\text{A235})$$

Appendix N

As in Appendix L,

$$\int \frac{R_1}{R_2} d^3 x = \int d^3 R_1 \frac{R_1}{|\vec{R}_1 + \vec{R}|} = \int \frac{R_1^3 dR_1 d\Omega_1}{\sqrt{R_1^2 + R^2 + 2R_1 R \hat{R}_1 \cdot \hat{R}}}, \quad (\text{A236})$$

in which we replace the $\vec{x}$ integration variable with the $\vec{R}_1$ integration variable and we use spherical coordinates as in Equation (A222). Again, we use the dimensionless variable ($y = \frac{R_1}{R}$) and obtain the following:

$$\begin{aligned} \int \frac{R_1}{R_2} d^3 x &= R^3 \int \frac{y^3 dy d\Omega_1}{\sqrt{1 + y^2 + 2ys}}, = R^3 \int_0^{y_{\max}} \int_1^{-1} \int_0^{2\pi} \frac{-y^3 dy ds d\phi_1}{\sqrt{1 + y^2 + 2ys}} \\ &= 2\pi R^3 \int_0^{y_{\max}} y^3 dy \int_{-1}^1 \frac{ds}{\sqrt{1 + y^2 + 2ys}}, \quad y_{\max} \equiv \frac{R_{\max}}{R}. \end{aligned} \quad (\text{A237})$$

Now, taking into account Equation (A233), it follows that

$$\int \frac{R_1}{R_2} d^3 x = 4\pi R^3 \left[\int_0^1 y^3 dy + \int_1^{y_{\max}} y^2 dy \right] = \frac{\pi R^3}{3} (4y_{\max}^3 - 1) = \frac{\pi}{3} (4R_{\max}^3 - R^3). \quad (\text{A238})$$

Appendix O

Let us calculate a volume integral of a current density component:

$$\int d^3x J_i = \int d^3x \delta_i^k J_k = \int d^3x \partial_k x_i J_k, \quad (\text{A239})$$

in which δ_i^k is Kronecker's delta and the Einstein summation convention is assumed regarding repeated indices. Thus,

$$\int d^3x J_i = \int d^3x [\partial_k (x_i J_k) - x_i \partial_k J_k] = \int d^3x [\vec{\nabla} \cdot (x_i \vec{J}) - x_i \vec{\nabla} \cdot \vec{J}], \quad (\text{A240})$$

where in the last equation sign, we have switch to vector notation. Taking into account Gaussian theorem and the continuity (Equation (A9)), it follows that

$$\int d^3x J_i = \oint d\vec{S} \cdot (x_i \vec{J}) + \int d^3x x_i \partial_t \rho. \quad (\text{A241})$$

Assuming no currents on the (far-away) encapsulating surface, the surface integral vanishes. Adopting vector notation, we have the following:

$$\int d^3x \vec{J} = \int d^3x \vec{x} \partial_t \rho = \partial_t \vec{d}, \quad \vec{d} \equiv \int d^3x \vec{x} \rho, \quad (\text{A242})$$

In the above, $\vec{d}$ is the system (or subsystem) electric dipole moment. It follows that

$$\int d^3x \partial_t^n \vec{J} = \int d^3x \vec{x} \partial_t^{n+1} \rho = \partial_t^{n+1} \vec{d}. \quad (\text{A243})$$

References

- Robertson, G.A.; Murad, P.; Davis, E. New frontiers in space propulsion sciences. *Energy Convers. Manag.* **2008**, *49*, 436–452. [\[CrossRef\]](#)
- Einstein, A. *On the Electrodynamics of Moving Bodies*; John Walker: Edinburgh, UK, 1905.
- Jackson, J.D. *Classical Electrodynamics*, 3rd ed.; Wiley: New York, NY, USA, 1999.
- Feynman, R.P.; Leighton, R.B.; Sands, M. *The Feynman Lectures on Physics, Vol. I: The New Millennium Edition: Mainly Mechanics, Radiation, and Heat*; Basic Books: New York, NY, USA, 2011; Volume 1.
- Griffiths, D.J.; Heald, M.A. Time-dependent generalizations of the Biot–Savart and Coulomb laws. *Am. J. Phys.* **1991**, *59*, 111–117. [\[CrossRef\]](#)
- Jefimenko, O. *Electricity and Magnetism*, 2nd ed.; Electret Scientific: Star City, WV, USA, 1989.
- Breitenberger, E. Magnetic interaction between charged particles. *Am. J. Phys.* **1968**, *36*, 505–515. [\[CrossRef\]](#)
- Scanio, J. Conservation of momentum in electrodynamics—An example. *Am. J. Phys.* **1975**, *43*, 258–260. [\[CrossRef\]](#)
- Portis, A. *Electromagnetic Fields, Source and Media*; John Wiley & Sons Inc.: New York, NY, USA, 1978.
- Jefimenko, O. A Relativistic Paradox Seemingly Violating Conservation of Momentum in Electromagnetic Systems. *Eur. J. Phys.* **1999**, *20*, 39. [\[CrossRef\]](#)
- Jefimenko, O. *Causality, Electromagnetic Induction and Gravitation*, 2nd ed.; Electret Scientific: Star City, WV, USA, 2000.
- Lin, J.Y.; de Sterke, C.M.; Ilic, O.; Kuhlmeier, B.T. Photonic Lightsails: Fast and Stable Propulsion for Interstellar Travel. *arXiv* **2025**, arXiv:2502.17828.
- Rajput, S.; Yahalom, A. Newton's Third Law in the Framework of Special Relativity for Charged Bodies. *Symmetry* **2021**, *13*, 1250. [\[CrossRef\]](#)
- Sharma, P.; Yahalom, A. A Charged Relativistic Engine Based on a Permanent Magnet. *Appl. Sci.* **2024**, *14*, 11764. [\[CrossRef\]](#)
- Rajput, S.; Yahalom, A.; Qin, H. Lorentz symmetry group, retardation and energy transformations in a relativistic engine. *Symmetry* **2021**, *13*, 420. [\[CrossRef\]](#)
- Tuval, M.; Yahalom, A. Momentum conservation in a relativistic engine. *Eur. Phys. J. Plus* **2016**, *131*, 374. [\[CrossRef\]](#)
- Tuval, M.; Yahalom, A. Newton's Third Law in the Framework of Special Relativity. *Eur. Phys. J. Plus* **2014**, *129*, 240. [\[CrossRef\]](#)
- Rajput, S.; Yahalom, A. Material Engineering and Design of a Relativistic Engine: How to Avoid Radiation Losses. In *Proceedings of the Advanced Engineering Forum*; Trans Tech Publications: Bäch, Switzerland, 2020; Volume 36, pp. 126–131.
- Yahalom, A. Gravitational Hamiltonian Systems and the Retarded Gravity Inequality. *Entropy* **2024**, *26*, 986. [\[CrossRef\]](#) [\[PubMed\]](#)

20. Yahalom, A. Time-Dependent Retarded Microwave Electromagnetic Motors. *Appl. Sci.* **2024**, *14*, 10975. [[CrossRef](#)]
21. Yahalom, A. Newton's Third Law in the Framework of Special Relativity for Charged Bodies Part 2: Preliminary Analysis of a Nano Relativistic Motor. *Symmetry* **2022**, *14*, 94. [[CrossRef](#)]
22. Yahalom, A. Implementing a Relativistic Motor over Atomic Scales. *Symmetry* **2023**, *15*, 1613. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.