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Computational Optimization of Free-Piston Stirling Engine by Variable-Step Simplified Conjugate Gradient Method with Compatible Strategies

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Abstract: This study aimed at the development of an algorithm for the computational optimization of free-piston Stirling engines. The design algorithm includes an optimization method and two compatible strategies. The optimization method is an improved version of traditional conjugate gradient method and is named the variable-step simplified conjugate gradient method (VSCGM). The free-piston Stirling engine is operable only in narrow-bounded parameter regions. Using the present approach, the operable variable combinations can be found efficiently. Two compatible strategies, the wake-up and backward-comparison strategies, are integrated with the VSCGM. The present design algorithm can handle multiple-parameter optimization with more flexible objective function definitions. Meanwhile, it features faster convergence as compared with the traditional conjugate gradient methods. Moreover, the feasibility of the VSCGM and the two compatible strategies is demonstrated in two test cases. It was found that the present approach can optimize the ten designed variables simultaneously, and the optimal designs can be yielded in a finite number of iterations. The results show that the inoperable initial designs were successfully optimized to reach a high power output.

Keywords: variable-step simplified conjugate gradient method; free-piston Stirling engines; wake-up strategy; backward-comparison strategy; computational optimization



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1. Introduction

Stirling engines are energy conversion devices suitable for a variety of heat sources. Unlike internal combustion engines that have explosions and combustions in the cylinder, the power piston of a Stirling engine is mainly driven by the expansion and contraction of the isolated working gas under different temperatures. There are typically two moving parts of a regular Stirling engine: the displacer and the piston. The displacer drives the working gas moving back and forth between the high- and low-temperature ends, and the piston converts the expansion and contraction energy into reciprocating movement and power output. The two moving parts do not synchronize in movement but have a specific phase angle between them. In general, the movement of the displacer must lead the movement of the piston, and the phase angle can be implemented through the design of links and flywheel connections [1–3].

In a free-piston Stirling engine (FPSE), the links and the flywheel are replaced by springs connected to the two respective moving parts. As shown in Figure 1, the displacer and the piston are constrained by springs with stiffnesses K_D and K_P , respectively, which means that the strokes and equilibrium positions of the piston and the displacer are determined by the dynamic behavior of the springs. This special feature makes a free-piston Stirling engine work differently from Stirling engines with mechanisms. The cylinder of the engine is divided into six chambers, namely, the expansion chamber, heater, regenerator, cooler, compression chamber and buffer space. Heating and cooling are provided on

the walls of the expansion and compression chambers, respectively. All the geometrical parameters of the engine are illustrated in Figure 1.

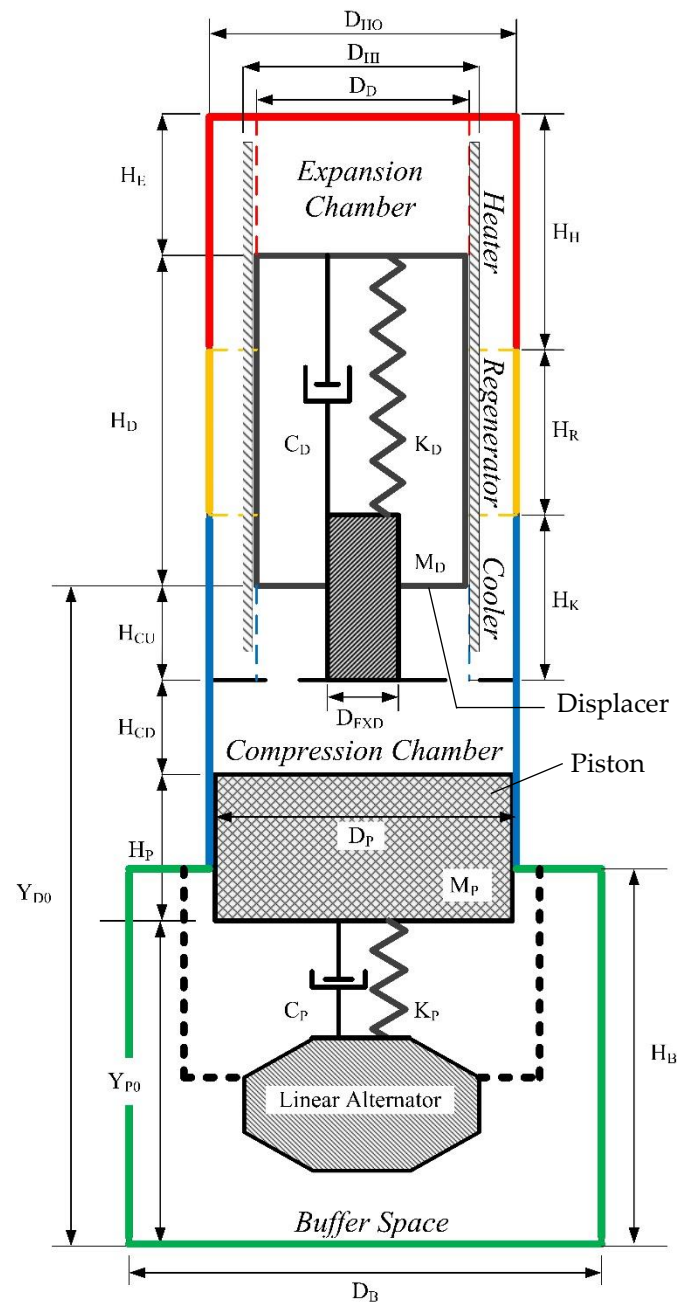


Figure 1. Diagram of a free-piston Stirling engine.

The free-piston Stirling engine has advantages such as a simple mechanism and the ability to self-adjust its operation mode corresponding to different operating conditions [4]. However, its major disadvantage is that it is operable only in narrow-bounded parameter regions in nature. In other words, if it is improperly designed, the engine cannot be started or instantly stopped while the operating conditions are not located in the right parameter regions. This explains why, in the majority of studies on FPSEs, the existing FPSE model RE-1000 is taken as a reference [5]. Although, in some studies, self-developed FPSEs are provided, the cases are operated only under a normal atmosphere [6]. The aim of this study was to provide a complete design tool for a self-developed pressurized FPSE. Therefore, making sure that the free-piston Stirling engine is designed and operated in the operable region is an extremely important issue. To help find the operable parameters and reduce

the time and cost of finding them, this study provides a systemic algorithm for these purposes, which includes an optimization method and two compatible strategies. Instead of using a genetic algorithm (GA) as the optimization method, which consumes a lot of computation time in regular studies [6], the present optimization method is a modified version of the traditional conjugate gradient method (CGM) [7,8]. In the CGM, the objective function must be defined in the form of a sum of squares of the differences between two prescribed physical quantities, and the step sizes and search directions in iterations should be determined by solving a set of simultaneous partial differential equations (PDEs). This is a rather time-consuming process. To remove the restrictions on the definition of the objective functions and reduce the time required, Cheng and Chang [9] presented a simplified conjugate gradient method (SCGM) which suggests a fixed value for the step sizes and uses an efficient sensitivity analysis to determine the search directions. The SCGM has been widely applied in the optimal design of a great number of energy devices. However, it suffers from the convergence problem when the search is close to the optimal points, particularly when the fixed step size is too large. Recently, a variable-step simplified conjugate gradient method (VSCGM) was proposed by Cheng and Lin [10]. In the VSCGM, the step size is automatically adjusted based on the variation in the magnitude of the objective function during iteration. In this manner, the VSCGM allows adjusting the step size such that the convergence can be accelerated when the search is close to the optimal points.

Furthermore, it is important to mention that Cheng and Chang [9] discussed several optimization methods such as genetic algorithms (GAs), the traditional conjugate gradient method (CGM), the Newton–Raphson method and the adaptive weighting input estimation method. In addition, Cheng and Lin [10] presented the VSCGM which was coupled with the neural network method in machine learning. Therefore, for those who are interested in the comparison between the VSCGM and other optimization methods, sufficient information is available in [9,10]. To save space in this manuscript, the discussion is not provided here.

As the baseline design of this engine is readily available, local optimization is carried out to improve the engine performance while keeping the original engine configuration. Thus, the VSCGM is chosen for its deterministic approach and fast convergence. However, the VSCGM is not without its disadvantages. When the search in iteration is trapped in an inoperable region, the objective function cannot provide meaningful gradient variations, and therefore the optimization process will be terminated without the expected results. The present study integrated the wake-up strategy with the VSCGM. By using the strategy, a new point is randomly generated continuously until the design is relocated to an operable region. In addition, although the VSCGM updates multiple variables at the same time, some design parameters are sensitive to the engine operation, which may lead to instability in the optimization process. Thus, a “backward-comparison strategy” is proposed and also integrated with the VSCGM. With this strategy, the results of the sensitivity analysis for an individual design parameter are provided as a conservative updating solution, and the original VSCGM result is then used as an aggressive updating solution, which updates all the design parameters after the same iteration. The feasible updating solution is automatically selected in each iteration, and the calculated results are properly used without significantly increasing the computation time. The VSCGM and the two compatible strategies are described in detail in the subsequent sections.

2. Design Algorithm

The design algorithm used in this study is composed of an optimization method, the VSCGM [10], and two compatible strategies, namely, the “wake-up strategy” and the “backward-comparison strategy”. The idea of the approach is briefly illustrated in Figure 2, and the schemes are introduced in detail as follows.

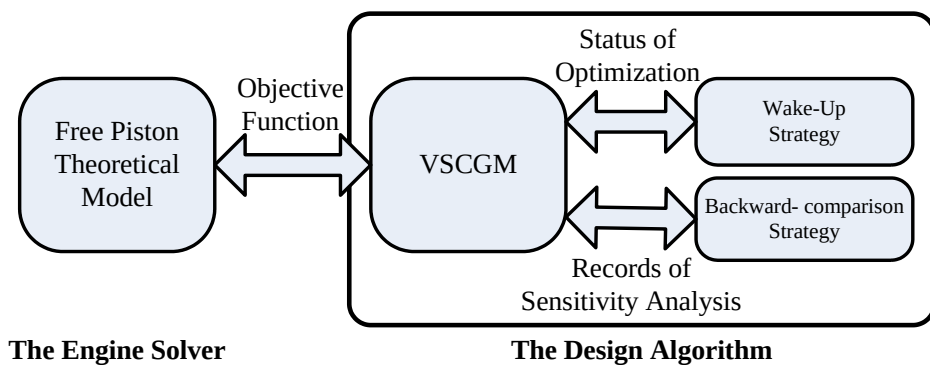


Figure 2. Design algorithm combining theoretical model and design algorithm for free-piston Stirling engine optimization.

2.1. Variable-Step Simplified Conjugate Gradient Method, VSCGM

The variable-step simplified conjugate gradient method (VSCGM) is an updated version of the CGM, and hence it is regarded as a gradient-series optimization method. In a typical gradient-series method, the conjugate direction is determined by the steepest descent method [11] and the Newton method [12] in search of the optimal point. However, the CGM has an important limitation that the objective function must be defined in the form of a sum of squares of the differences between two prescribed physical quantities [13] as

$$F = \sum_{i=1}^I (v_i - \bar{v}_i)^2 \quad (1)$$

where v_i and $\bar{v}_i$ represent the iterative and the desired values of the physical quantity of interest, respectively, and I is the total data number. The restriction of the objective function is due to the step size calculation in the iteration process. The gradient of the objective function is expressed as

$$\frac{\partial F}{\partial x_i} = \sum_{i=0}^I 2(v_i - \bar{v}_i) \frac{\partial v_i}{\partial x_i}, \quad i = 1, 2, \dots, k \quad (2)$$

As described in [9], in the CGM, the step size and the gradient of the objective function in iterations are determined by solving a set of simultaneous PDEs for $\frac{\partial v_i}{\partial x_i}$. The computation process could be greatly simplified by the SCGM. The SCGM is suitable for any objective function, and it is not necessary to solve PDEs for the step sizes because the step sizes are fixed at constant values as

$$S_i = \text{constant}, \quad i = 1, 2, \dots, k \quad (3)$$

Moreover, in the SCGM, the objective function gradient is determined by direct numerical differentiation with small perturbations of each designed variable:

$$\frac{\partial F}{\partial x_i} = \frac{\Delta F}{\Delta x_i}, \quad i = 1, 2, \dots, k \quad (4)$$

In the SCGM, Equation (4) is used to evaluate the objective function gradient, instead of Equation (2). Thus, the applicability of the optimization method is greatly extended to a number of different energy devices [14–17]. However, as stated earlier, the SCGM suffers from the convergence problem when the search is close to the optimal points, particularly when the fixed step size is too large. The VSCGM proposed by Cheng and Lin [10] is used to automatically adjust the step size based on the variation in the magnitude of the objective function during iteration. In this manner, the convergence can be accelerated toward the optimal point. The main concept of the VSCGM is that when the iteration is far from the

optimal point with $R_i^{(n)} > 1$, the step size will be increased to facilitate the optimization process; when it is close to the optimal point with $R_i^{(n)} \leq 1$, the step size can be gradually reduced. The concept is illustrated in Figure 3.

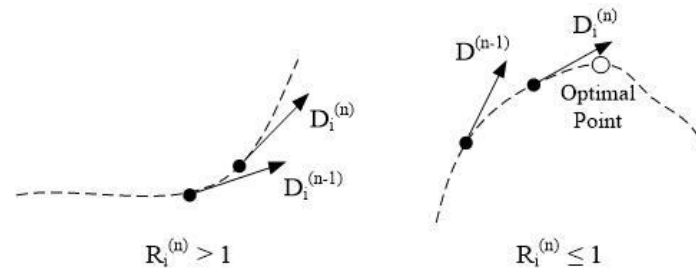


Figure 3. The adjustment of the step size is related to the ratio of the searching directions.

The step size $S_i^{(n)}$ is adjusted based on the ratio of the searching direction (R_i) and the gain value (G_i) as

$$\begin{cases} S_i^{(n)} = S_i^{(n-1)} \times (R_{i,min})^{G_{j,min}}, & R_i^{(n)} \leq R_{i,min} \\ S_i^{(n)} = S_i^{(n-1)} \times (R_i^{(n)})^{G_i^{(n)}}, & R_{i,min} < R_i^{(n)} < R_{i,Max} \\ S_i^{(n)} = S_i^{(n-1)} \times (R_{i,Max})^{G_{j,Max}}, & R_{i,Max} \leq R_i^{(n)} \end{cases} \quad (5)$$

where the searching direction ratio (R_i) is defined as

$$R_i^{(n)} = \frac{D_i^{(n)}}{D_i^{(n-1)}} \quad (6)$$

where $\{D_i^{(n)}, i = 1, 2, \dots, k\}$ is the vector of the search direction. The searching direction ratio (R_i) is confined within two limiting values, $R_{i,min} = 0.9$ and $R_{i,Max} = 1.1$. In addition, the gain value (G_i) is expressed with a linear interpolation between the lower and upper bounds ($G_{j,min}$ and $G_{j,Max}$).

$$G_i^{(n)} = \left(\frac{R_i^{(n)} - R_{i,min}}{R_{i,Max} - R_{i,min}} \right) \times (G_{j,min} - G_{j,Max}) + G_{j,Max} \quad (7)$$

In this study, the lower and upper bounds of the gain value are both assigned the value of 1.0, and hence the gain value (G_i) is 1.0. When the value of G_i is higher than 1.0, the search for the optimal point will be accelerated; when G_i is lower than 1.0, the search for the optimal point will be decelerated. One can adjust the values of the lower and upper bounds in order to yield the gain value using Equation (7). In order to improve the accuracy of the sensitivity analysis, the perturbation of the designed variable ΔX_i appearing in Equation (4) is assumed to be proportional to the step size.

In summary, the iteration process of the VSCGM consists of four steps:

1. Use Equation (4) for the objective function gradient $\frac{\partial F}{\partial x_i}$ with the perturbation of each designed variable. Meanwhile, the step size is determined by Equation (5).
2. Evaluate the conjugate gradient coefficient by the ratio of the objective function gradients.

$$\gamma_i^{(n)} = \left[\frac{\left(\frac{\partial F}{\partial x_i} \right)^{(n)}}{\left(\frac{\partial F}{\partial x_i} \right)^{(n-1)}} \right]^2, \quad i = 1, 2, \dots, k \quad (8)$$

3. Calculate the searching direction $D_i^{(n)}$ with a linear combination of the objective function gradients $(\frac{\partial F}{\partial x_i})^{(n)}$ and conjugate gradients $\gamma_i^{(n)} D_i^{(n-1)}$.

$$D_i^{(n)} = (\frac{\partial F}{\partial x_i})^{(n)} + \gamma_i^{(n)} D_i^{(n-1)}, i = 1, 2, \dots, k \quad (9)$$

4. Update the designed variables in terms of the variable step size $S_i^{(n)}$ and the searching direction $D_i^{(n)}$.

$$x_i^{(n+1)} = x_i^{(n)} - S_i^{(n)} D_i^{(n)}, i = 1, 2, \dots, k \quad (10)$$

The flow chart of the iteration process of the VSCGM is displayed in Figure 4. It is noted that the solver for the objective function may be a theoretical model of the free-piston Stirling engine or a database from experiments. In the present study, a theoretical model is developed and incorporated with the design algorithm for optimization of the engine. The theoretical model will be described later.

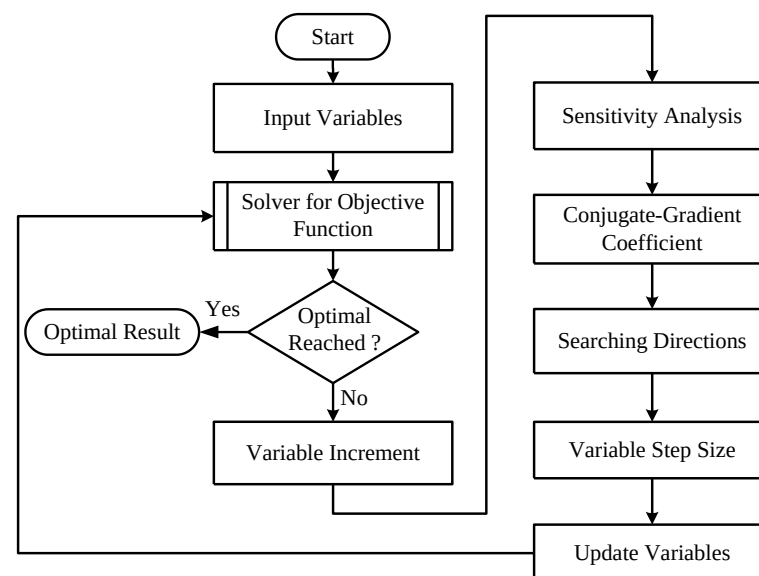


Figure 4. Flow chart of the variable-step size simplified conjugate gradient method.

2.2. Wake-Up and Backward-Comparison Strategies

In theory, the free-piston Stirling engine is operable only in narrow-bounded parameter regions. Thus, it is very possible that the iterative designs will become trapped in the inoperable regions, which means the optimization terminates before the optimal result is reached. In general, it is difficult for engineers to properly design the geometrical and operating parameters beforehand. Herein, the wake-up strategy is developed to resolve this problem. Using the scheme, one is able to design the parameters of free-piston Stirling engines in the operable regions.

The wake-up strategy is very straightforward. When the optimization search is trapped in an inoperable region, one of the designed variables may be randomly selected and altered by adding a random value to it. The variable and the added values are selected within the maximum and minimum bounds based on a random number generator. The process repeats by randomly selecting another variable until the optimization jumps from the traps in the inoperable region, and then the optimization search resumes its search for the optimal point. Figure 5 shows the variable updated by the wake-up strategy. In this figure, as an example, it is Variable 2 that is updated.

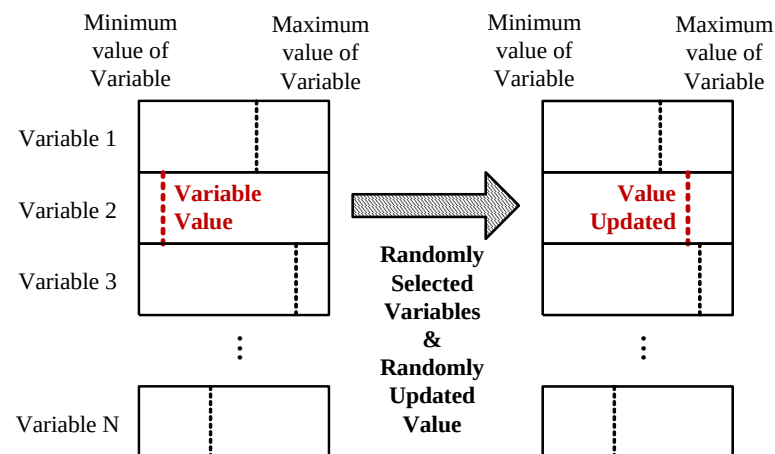


Figure 5. Variables updated by the wake-up strategy.

Furthermore, since the VSCGM is a multiple-variable optimization method, all the designed variables are updated at the same time by Equation (10) after all the searching directions and all step sizes are obtained from sensitivity analyses. Actually, updating all the designed variables at the same time as suggested originally by the VSCGM may not always lead to the best solution. In this sense, it may be helpful to look back into the objective function gradient for individual designed variables. A backward-comparison strategy is proposed. That is, the original VSCGM result is regarded as an aggressive updating solution, and the results from the sensitivity analyses for individual design parameters are treated as conservative updating solutions. The aggressive updating solution is compared to all the conservative updating solutions, and then the best updated solution is selected. The backward-comparison strategy provides an efficient method to accelerate the search for the optimal point in the operable region and to help the trapped search out of the inoperable region. It is compatible with the VSCGM with no extra computation, as conveyed in Figure 6.

2.3. Theoretical Model of Free-Piston Stirling Engine

Next, a theoretical model of a free-piston Stirling engine is developed and used as the engine solver already indicated in Figure 2. The theoretical model should be capable of predicting the temperature and pressure of the working gas in the six chambers, the pressure force acting on the moving parts and the oscillations of the two moving parts. The effects of the stiffnesses of the springs, the dampers and the masses of the piston and displacer are included. In the theoretical model, the engine is divided into six chambers, namely, the expansion chamber, heater, regenerator, cooler, compression chamber and buffer space. The volumes of the heater, cooler and regenerator (V_H , V_K , V_R) remain unchanged in the computation, while the volumes of the expansion and compression chambers and buffer space (V_E , V_C , V_B) are varied with time due to the periodic movements of the piston and displacer as follows:

$$V_E = V_{E_0} - X_D \cdot A_P \quad (11)$$

$$V_C = V_{C_0} + (X_D - X_P)(A_P - A_R) \quad (12)$$

$$V_B = V_{B_0} + X_P \cdot A_P \quad (13)$$

where V_{E_0} , V_{C_0} and V_{B_0} are the volumes of the expansion and compression chambers and the buffer space when the displacer and piston are at their equilibrium positions, and X_D and X_P are the positions of the displacer and piston, respectively. As a function of the working gas temperatures in all the chambers, the pressure in the cylinder can be calculated with

$$P_W = \frac{M_W \cdot R}{\frac{V_H}{T_H} + \frac{V_E}{T_E} + \frac{2 \cdot V_R}{T_H + T_E} + \frac{V_C}{T_C} + \frac{V_K}{T_K}} \quad (14)$$

The above equation is derived from the ideal-gas equation of state and the mass conservation equation. The instantaneous pressure is assumed to be equal in each engine space. The equation also contains the assumption that the gas volume in the regenerator is at the arithmetic mean temperature of the gas temperatures in the heater and cooler. In addition, the pressure in the buffer space is determined as

$$P_B = P_{B_0} \cdot \left(\frac{V_{B_0}}{V_B} \right)^{1.67} \quad (15)$$

Herein, it is assumed that the working gas is assumed to be an ideal gas with constant specific heats in the buffer space which experiences an adiabatic process in expansion or contraction.

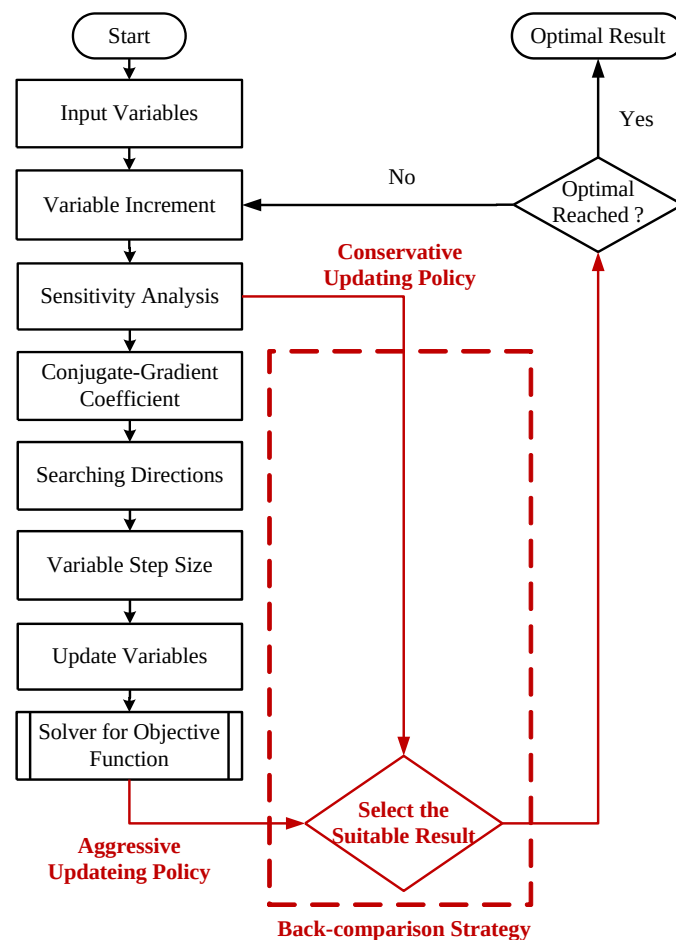


Figure 6. Integration of the backward-comparison strategy with the VSCGM.

The oscillations of the displacer and the piston are governed by the springs (K_D and K_P), the dampers (C_D and C_P), the masses (M_D and M_P) and the pressure forces. The pressure forces are obtained from Equations (14) and (15). The function of the piston, while it is moving, is to drive the linear alternator for power output. On the other hand, one of the driving forces for the displacer comes from the difference in pressure forces exerted by the working gas pressure on the top and bottom surfaces of the displacer. Therefore, the dynamic equations of the displacer and the piston can be expressed as follows:

$$M_D \ddot{X}_D + C_D \dot{X}_D + K_D X_D = P_W (A_{DB} - A_{DU}) - M_D g \quad (16)$$

$$M_P \ddot{X}_P + C_P \dot{X}_P + K_P X_P = (P_B - P_W) A_{DB} - M_P g \quad (17)$$

These two equations can be solved by numerical integration with respect to the time required to obtain the instantaneous positions of the moving parts (X_D and X_P), which are then used to determine the volumes of the expansion and compression chambers and the buffer space. Based on the obtained volumes, the pressure and the temperatures of the working gas are calculated, and the computation can be marched to the next time step. As the computation reaches the prescribed maximum time, it is terminated, and the objective function can be determined and transferred to the optimization algorithm for further analysis. This simulation model is suitable for designing an FPSE in the initial stage, which demonstrates the specified operable and inoperable behaviors with different major geometric parameters. It is important to mention that, pending more advanced models that allow the assessment of mechanical friction losses, the optimal design is considered to be identified with the maximum stroke.

The present optimization of the free-piston Stirling engine can be stated mathematically as follows:

Find: multiple designed variables, $\{x_i, i = 1, 2, \dots, k\}$;

Minimize: objective function, F ;

Subject to: free-piston Stirling engine solver (theoretical model).

The optimal design of the engine is performed by determining the optimal combination of the designed variables $\{x_i, i = 1, 2, \dots, k\}$ leading to a maximum stroke of the oscillation of the displacer. The engine is operable if the oscillation of the displacer can be successfully initiated and kept stable. Therefore, in this study, the objective function F is defined with the reciprocal of the displacer's stroke.

$$F = \frac{1}{S_D} \quad (18)$$

By doing so, when the value of the objective function is minimized, a maximum stroke is achieved. Since initial velocity is applied on the piston, and the power generation unit (linear alternator) is connected to the piston, instead of identifying the amplitude of the piston, the displacer's amplitude is much more suitable.

3. Test Cases

In order to demonstrate the feasibility of the present approach, two initial designs were considered, labeled Cases 1 and 2. Case 1 has an initial design that is operable, but the amplitude of the displacer is very small. Using the VSCGM for Case 1, an optimal design with a larger amplitude of the displacer can be yielded.

As for Case 2, initially, it is inoperable with zero amplitude. Nonetheless, using the VSCGM plus the two compatible strategies, one can again obtain an optimal design with a much larger amplitude of the displacer and, hence, achieve a high power output.

The designed variables of the initial design of the two test cases are provided in Table 1. The designed variables are adjusted in the optimization process until the optimal design is reached, while some other variables are fixed during optimization. These fixed variables are displayed in Table 2.

Table 1. Initial designs of Cases 1 and 2.

Designed Variables	Initial Design of Case 1	Initial Design of Case 2
Displacer Spring Stiffness (N/m)	68,700	67,691.38
Displacer Mass (kg)	0.303	2.58
Piston Spring Stiffness (N/m)	27,631	28,303.86
Piston Mass (kg)	2.750	13.75
Displacer Diameter (m)	0.07	0.07
Piston Diameter (m)	0.07	0.07

Table 1. *Cont.*

Designed Variables	Initial Design of Case 1	Initial Design of Case 2
Displacer Height (m)	0.148	0.148
Piston Height (m)	0.060	0.060
Displacer Install Position (m)	0.197	0.19766
Piston Install Position (m)	0.106	0.106

Table 2. Fixed variables.

Fixed Variables	Value
Charged pressure (bar)	10
Heating temperature (K)	900
Cooling temperature (K)	300
Porosity of regenerator	0.612
Heater diameter (m)	0.084
Heater inner fin number	10
Heater inner fin angle (degree)	18
Heater height (m)	0.062
Regenerator diameter (m)	0.084
Regenerator height (m)	0.4156
Cooler diameter (m)	0.084
Cooler inner fin number	10
Cooler inner fin angle (degree)	18
Cooler height (m)	0.065
Lower compression chamber height (m)	0.2316
Back pressure chamber diameter (m)	0.13
Back pressure chamber height (m)	0.106

4. Results and Discussion

4.1. Case 1 by VSCGM

Figure 7 shows the growth of the amplitudes of the displacer with the initial and the optimal designs for Case 1. The stable oscillation amplitude of the initial design is merely 0.533 cm. Through the VSCGM, after 26 optimization steps, the displacer amplitude can be enlarged to its maximum, 1.649 cm, as shown in Figure 7b. Note that in Figure 7b, the curve of displacement of the displacer has a zig-zag pattern, which means the oscillation of the displacer is so strong that the displacer even collides with its neighboring parts. In this particular case, the operating frequency changes from 28.03 Hz to 15.89 Hz. The VSCGM can deal with the ten designed variables well.

By comparing the PV plots before and after optimization as in Figure 8, one can observe the variation in the area of the PV diagrams to view the change in power output. In this figure, it is seen that after optimization by the VSCGM, the area of the PV diagram associated with the optimal design is larger than that associated with the initial design. Moreover, the PV plot moves to right side of the figure, and this means that the total volume of the cylinder tends to become larger after optimization. By calculating the area of the PV diagram for the cycles in the stable operation regime, one can obtain the work output per cycle. Based on the obtained work output per cycle and frequency of oscillation, the output power can be further determined. The resulting optimal designed variables for Case 1 are provided in Table 3.

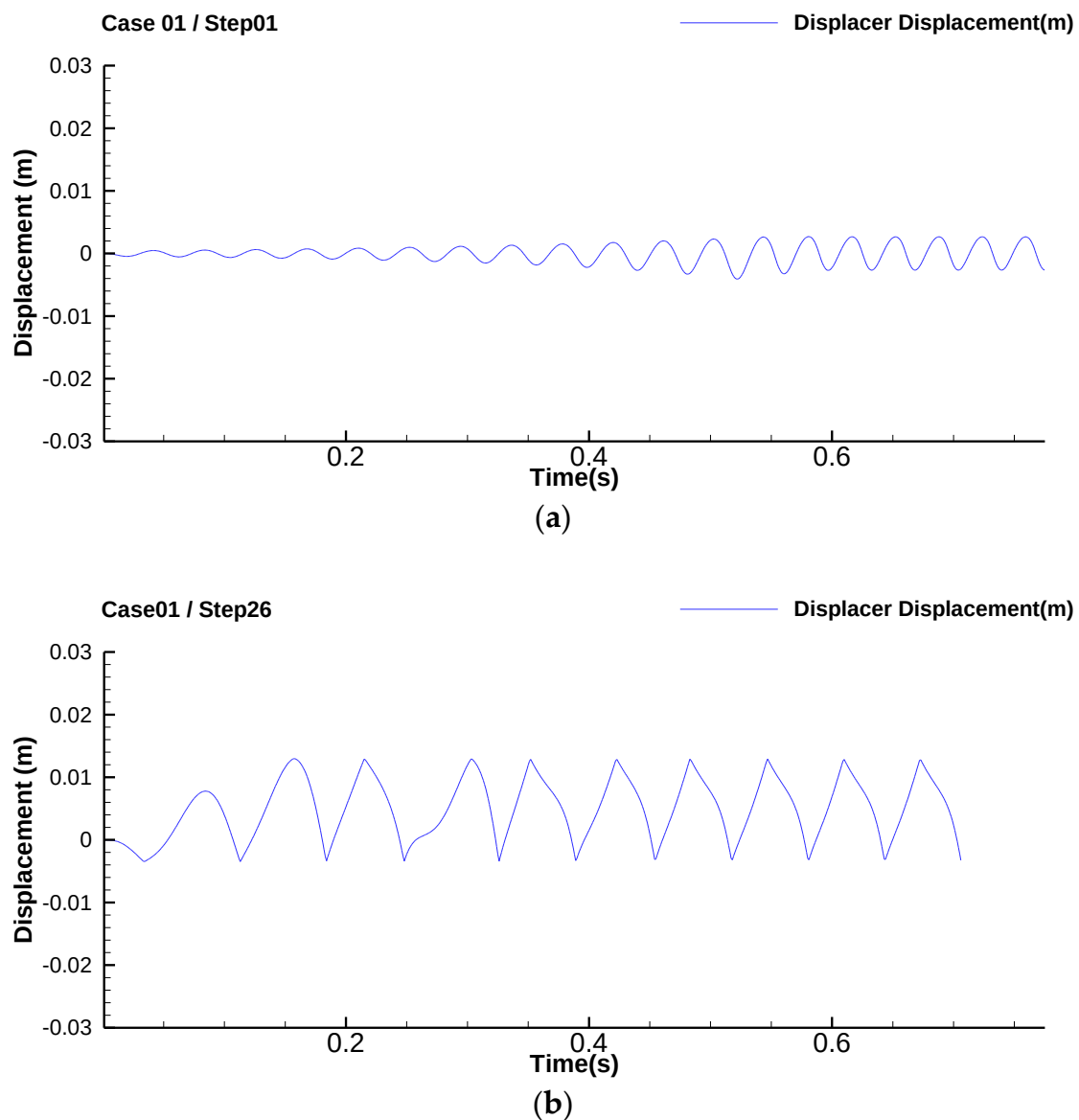


Figure 7. Optimization with the VSCGM for the maximum stroke of the displacer in Case 1: (a) initial design; (b) optimal design reached at the 26th step.

Table 3. Optimal design of Case 1.

Variable	Optimal Value
Displacer Spring Stiffness (N/m)	67,783.38
Displacer Mass (kg)	22.6
Piston Spring Stiffness (N/m)	29,070.53
Piston Mass (kg)	34.375
Displacer Diameter (m)	0.0669
Piston Diameter (m)	0.0660
Displacer Height (m)	0.108
Piston Height (m)	0.0504
Displacer Installation Position (m)	0.1141
Piston Installation Position (m)	0.038

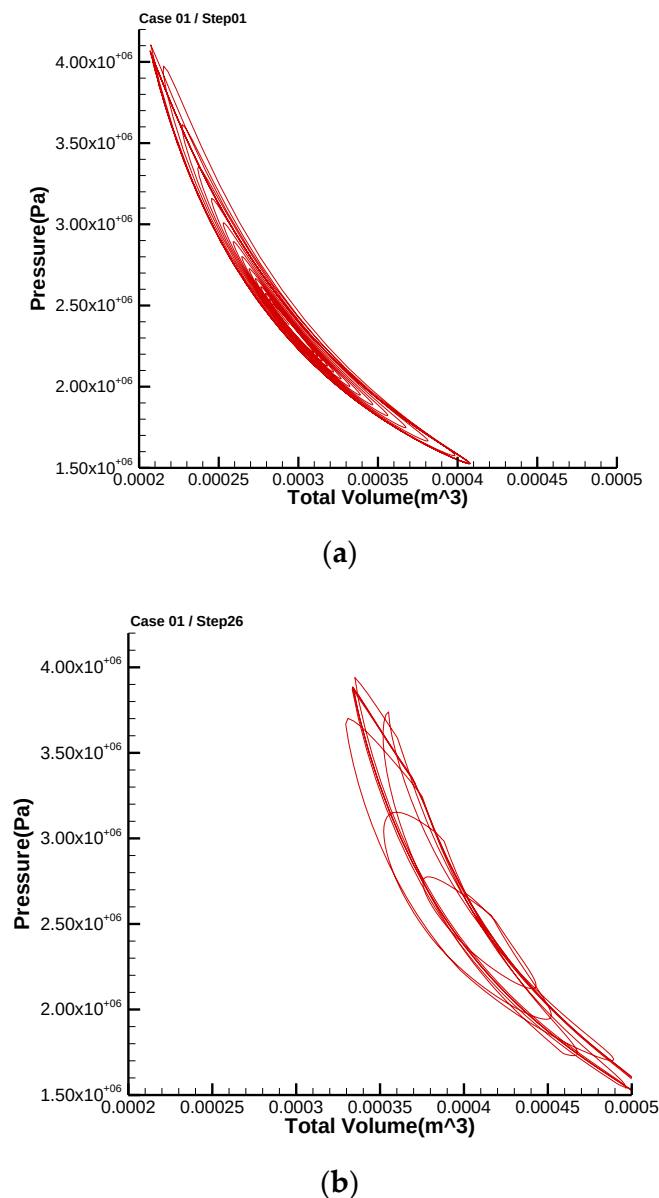


Figure 8. Comparison of the pressure–volume plots for Case 1: (a) initial design; (b) optimal design by the VSCGM.

4.2. Case 2 by VSCGM + Wake-Up and Backward-Comparison Strategies

In the above case, a small amplitude can be remarkably increased to a larger amplitude in a finite number of iterations simply by using the VSCGM. However, in most of the cases, the VSCGM may not lead to a satisfactory oscillation. The search direction of the VSCGM could be trapped in the inoperable region. This is because the search direction is evaluated based on the objective function gradients. When the objective function gradients approach infinity, the optimization method will fail to figure out a direction toward the optimal point. The initial design with Case 2 presented in Table 1 is another inoperable design, of which the amplitude of the displacer is very small, as shown in Figure 9a. For this initial design, the VSCGM cannot help the iteration jump from the inoperable region. In order to improve this initial design, the wake-up and the backward-comparison strategies are incorporated with the VSCGM, and the result of the optimal design is shown in Figure 9b. It is interesting to see that the amplitude of the displacer with the initial design is only 0.356 mm. Through this design algorithm, the amplitude of the displacer is successfully increased to 23.998 mm.

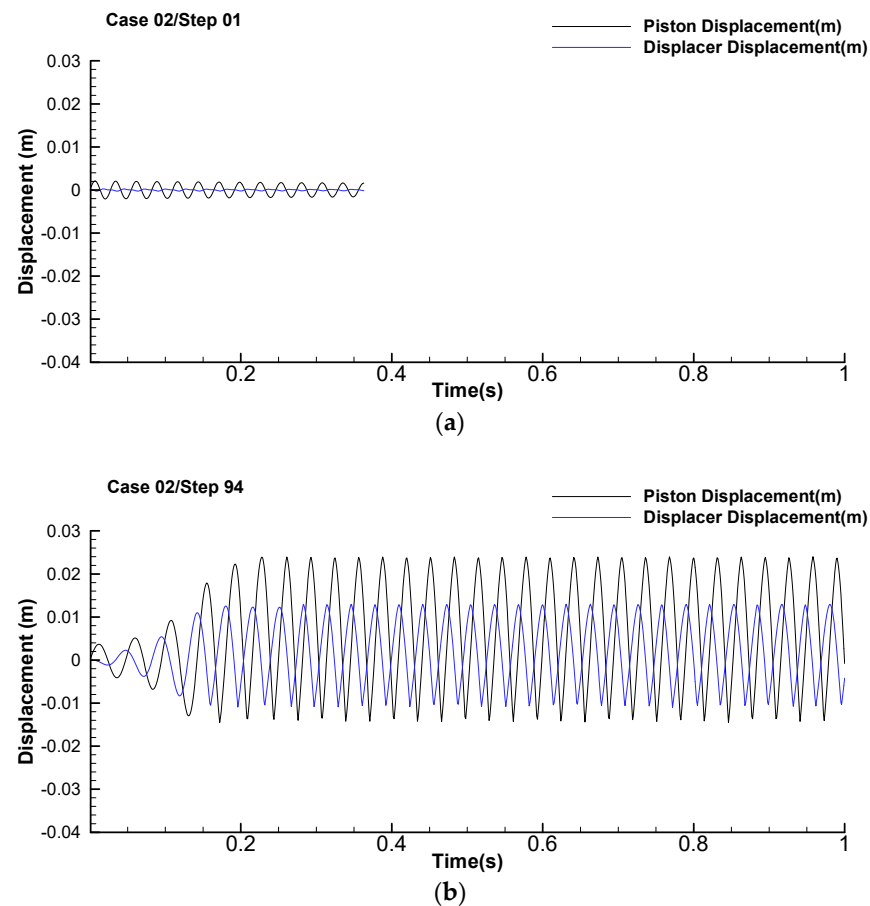


Figure 9. Optimization with the VSCGM + the wake-up and backward-comparison strategies for the maximum stroke of the displacer in Case 2: (a) initial design; (b) design generated by the VSCGM and the backward-comparison strategy.

Figure 10 conveys the variation in the objective function (F) by the VSCGM and the two strategies for Case 2. In this figure, the steps intervened by the wake-up and backward-comparison strategies are marked with the inverted triangle and circle symbols, respectively. It is found that, in this case, the wake-up strategy only intervenes in the optimization process twice (at the 1st and 35th steps), whereas the backward-comparison strategy continuously intervenes around 80 times. With the help of the two strategies, the optimization task can be successfully completed in 94 steps. The resulting variables of the optimal design of Case 2 are presented in Table 4.

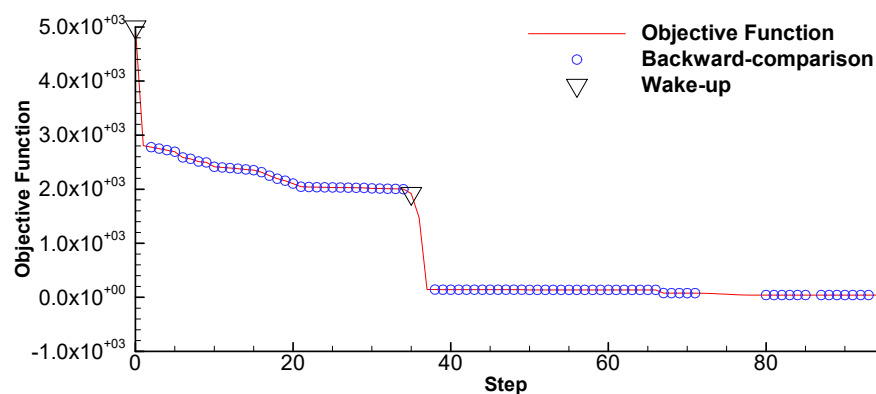


Figure 10. Variation in the objective function by the VSCGM + the wake-up and backward-comparison strategies, for Case 2.

Table 4. Optimal design of Case 2.

Variable	Value
Displacer Spring Stiffness (N/m)	64,969.5
Displacer Mass (kg)	3.03
Piston Spring Stiffness (N/m)	27,592.4
Piston Mass (kg)	5.13
Displacer Diameter (m)	0.0790
Piston Diameter (m)	0.0563
Displacer Height (m)	0.142
Piston Height (m)	0.069
Displacer Installation Position (m)	0.200
Piston Installation Position (m)	0.085

The power outputs and oscillation parameters of the optimal designs obtained in the two test cases are displayed in Table 5. The results show that the present approach can improve the inoperable initial designs to reach a high power output. According to this table, the optimal designs of Cases 1 and 2 can deliver approximately 668 and 890 W in power output.

Table 5. Power output and oscillation parameters with the optimal designs for Cases 1 and 2.

Case	Amplitude (m)	Frequency (Hz)	Power Output (W)
Case 1 (VSCGM)	0.0164	15.849	667.775
Case 2 (VSCGM + wake-up and backward-comparison)	0.0239	31.110	889.430

5. Conclusions

In the present study, a computational optimization method for the design of free-piston Stirling engines was developed. The method was built by incorporating two compatible strategies, namely, the wake-up strategy and the backward-comparison strategy, with the variable-step simplified conjugate gradient method (VSCGM). One major disadvantage of the free-piston Stirling engine is that it is operable only in narrow-bounded parameter regions. The present design algorithm can handle multiple-parameter optimization with more flexible objective function definitions. Moreover, the wake-up and backward-comparison strategies can deal with situations where the designed variables are trapped in the inoperable region during the optimization process.

The feasibility of the VSCGM and the two compatible strategies was demonstrated in two test cases, in which a group of designed variables were optimized simultaneously. The results show that the optimal designs can be yielded in a finite number of iterations. In the two test cases, the inoperable initial designs were successfully optimized to reach high power outputs.

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Nomenclature

A	area (m ²)
C	damping (Ns/m)
D	searching direction
F	objective function
g	gravitational constant (m/s ²)
G	gain value
I	total data number
K	stiffness of spring (N/m)
M	mass (kg)
P	pressure (Pa)
R	the ratio of searching direction
s	stroke (m)
S	step size for the optimization
T	temperature (K)
v	iterative values
V	volume of chamber (m ³)
x	designed variable
ΔX	perturbation of designed variable

Greek symbols

γ	conjugate gradient coefficient
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Superscripts

n	number of iteration
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Subscripts

B	back pressure chamber
B_0	initial condition of back pressure chamber
C	compression chamber
C_0	initial condition of compression chamber
D	displacer
DB	bottom surface of displacer
DU	upper surface of displacer
E	expansion chamber
E_0	initial condition of expansion chamber
H	heater chamber
i	index of iterative value
K	cooler chamber
Max	maximum value
min	minimum value
P	piston
R	regenerator chamber
W	working gas

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