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Multi-Period Portfolio Optimization Model with Cone Constraints and Discrete Decisions

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Abstract: This work develops a practical multi-period optimization approach that incorporates real-world constraints, including discrete decisions and conic risk constraints. Expanding upon earlier single-period models, our framework employs a binary scenario tree derived from monthly returns of randomly selected S&P 500 stocks to represent market evolution across multiple periods. The formulation captures essential portfolio constraints, such as transaction fees, sector diversification, and minimum investment thresholds, resulting in a robust and comprehensive optimization approach. To efficiently solve the resulting mixed-integer second-order cone programming (MISOCP) problem, we employ an outer approximation algorithm with a warmstart strategy, which significantly improves solution runtimes and computational efficiency. Numerical experiments demonstrate the model's effectiveness, showing an average improvement of 10.71% in iteration count and 15.24% in computational time when using the warmstart approach.

Keywords: mixed-integer linear and nonlinear optimizer (MILANO); mixed-integer second-order cone programming (MISOCP); multi-period portfolio optimization (MPO); outer approximation

JEL Classification: C61; C63; C88; G11

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1. Introduction

Significant progress has been made in the field of portfolio optimization since the foundational work of Markowitz (1952). Many of these advancements have been made possible by improvements in solver technology, increased computational power, and the widespread use of parallel processing. These developments now allow researchers and practitioners to solve not only larger-scale models but also more realistic ones that incorporate second-order cone constraints and discrete decision variables.

At the same time, financial markets have evolved in complexity, requiring optimization models to reflect a broader array of real-world considerations. This increased realism naturally leads to more sophisticated formulations. Despite these advances, there remains a need for further integration of practical investment constraints and for developing efficient solution techniques to handle such complexity.

In this context, we present a multi-period portfolio optimization (MPO) model rooted in the classical mean-variance approach of Markowitz (1952). The investor's goal is to

determine a trading strategy that maximizes expected return over the investment horizon while adhering to predefined limits on acceptable risk levels.

$$\begin{array}{ll} \max\limits_{x} & r^T x \\ \text{s.t.} & x^T \Sigma x \leq \sigma^2 \\ & \sum_{i=1}^{n} x_i = 1 \\ & x \geq 0 \end{array} \tag{1}$$

We examine a portfolio consisting of n assets, each with an estimated return vector $r \in \mathbb{R}^n$ and an associated variance–covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$, where each entry Σ_{ij} represents the covariance between assets i and j. The portfolio allocation is determined by decision variables $x \in \mathbb{R}^n$, representing asset weights. To prohibit short selling, we impose non-negativity constraints on the weights.

The investor's objective is to maximize expected return, subject to a risk constraint that limits portfolio variance to a predefined level σ^2, which reflects the investor's risk appetite. While this classical structure originates from single-period models, we extend it to a multi-period context in this study.

The formulation incorporates several real-world investment features, including transaction costs, Conditional Value-at-Risk (CVaR) constraints to manage tail risk, sector-based diversification rules, and minimum investment requirements. The inclusion of these elements is guided by both practical relevance and computational feasibility, particularly with respect to advances in mixed-integer second-order cone programming (MISOCP) solvers. These components result in a robust and detailed formulation that ranks among the most comprehensive in the current literature.

Earlier works by Benson and Sağlam (2013); Liechty and Sağlam (2017) explored a single-period context, and this paper expands upon that foundation by addressing the multi-period case. The modeling choices and techniques presented here draw on established contributions in the literature, including those of Adcock and Meade (1994), Bonami and Lejeune (2009), Garleanu and Pedersen (2009), Gulpinar et al. (2003), and Lobo et al. (2007).

In this study, we develop an MPO model that addresses these real-world complexities through an MISOCP formulation. Our approach incorporates discrete investment decisions, sector-based diversification, transaction costs, and CVaR as a risk measure. To address the computational challenges posed by a scenario tree structure and integer variables, we employ an outer approximation algorithm with warmstarting. We evaluate our model using historical data from the S&P 500, demonstrating its practicality and performance in optimizing terminal wealth over multiple planning horizons.

Our study makes several key contributions to the portfolio optimization literature:

- Modeling Advancement: We develop a novel MPO model formulated as an MISOCP, incorporating real-world investment features, including transaction costs, sector-based diversification constraints, minimum investment requirements, and CVaR.
- Computational Strategy: To address the computational challenges posed by the scenario tree and discrete decisions, we implement an outer approximation algorithm enhanced with a warmstart strategy. This approach improves solution efficiency and scalability.
- Empirical Implementation: We apply the model to historical return data from randomly selected S&P 500 stocks and evaluate the performance under various asset and time-period configurations. We report not only expected returns but also computational benchmarks.
- Practical Relevance: The framework supports real-world decision-making in institutional finance, offering tools for dynamic asset allocation under realistic risk, diversification, and trading constraints.

The remainder of the paper is structured as follows. Section 2 introduces the MPO model and details its mathematical formulation. It also describes the solution strategy, which leverages a novel algorithm built upon modern MISOCP techniques and is implemented using the MATLAB-based solver MILANO. Section 3 presents numerical experiments that evaluate the performance and scalability of the proposed approach. Concluding remarks and suggestions for future research directions are provided in Section 4.

2. Methods

2.1. Multi-Period Portfolio Optimization (MPO) Model

In this paper, we consider the MPO model, which has n periods in the investor's investment horizon, and their objective function is to choose the optimal trading strategies by making a series of decisions to rebalance their portfolio in each time period that maximize the-end-of-period expected return. In this study, we obtain the multi-period model by constructing a scenario tree. In this model, we follow Gulpinar et al. (2003) for formulating balance constraints between the time periods when the investor is penalized by transaction costs when rebalancing the portfolio. Table 1 summarizes the key notation used throughout the paper.

Table 1. Notation: The following notational scheme is adopted in the formulation of our model.

Notation	Description
r	is the vector of expected returns on each asset,
w	is the vector of initial holdings in each asset,
x^+	is the vector of amounts bought in each asset,
x^-	is the vector of amounts sold in each asset,
Σ	is the variance-covariance matrix of n assets,
Φ	is the cumulative standard normal distribution function,
W	is at the end of period wealth,
W^{low}	is at the end of period threshold wealth level,
s_{min}	is the threshold investment level for each economic sector,
L_{min}	is the minimum number of economic sectors in the portfolio,
ζ	is a binary variable that is associated with each economic sector,
w_{min}	is the minimum amount transacted in each asset,
δ	is a binary variable that is associated with each asset,
s	is the short position threshold,
ξ_t	vector of stochastic data observed at time t,
ξ^t	history of stochastic data up to t,
μ_e	stochastic realization of r_t in event e: $r_e \sim N(r_t(\xi^t), \lambda)$,
$\widehat{\mu}_e$	expectation of $r_t(\xi^t)$ for event e, conditional on ξ^{t-1},
$e \equiv (s,t)$	index denoting an event and time period,
$a(e)$	ancestor of event $e \in \mathcal{N}$,
p_e	branching probability of event e : $p_e = Prob[e\|a(e)]$,
P_e	probability of event e: if e = (s,t), then $P_e = \prod_{i=1..T} p_(s,i)$,
$\mathcal{N}$	set of all nodes.

2.1.1. Scenario Tree

Figure 1 depicts a binary scenario tree, which we use to illustrate key modeling concepts and notation. Investment decisions are made at discrete time steps $t = 1, \ldots, T$, where the investor reallocates holdings across n risky assets and one risk-free asset. The set of all nodes in the scenario tree is denoted by $\mathcal{N}$, and each node $e \in \mathcal{N}$ corresponds to a specific event, defined as an ordered pair (s,t), where s is the scenario and t is the time period.

For each event e, its immediate predecessor or parent node in the tree is denoted by $a(e)$. The conditional branching probability at node e, given its parent, is represented as $p_e = \text{Prob}[e \mid a(e)]$. The total probability of reaching event e in the tree, denoted $\mathcal{P}_e$, is calculated as the product of conditional probabilities along the path from the root to node e, i.e.,

$$\mathcal{P}_e = \prod_{i=1}^{t} p_{(s,i)}.$$

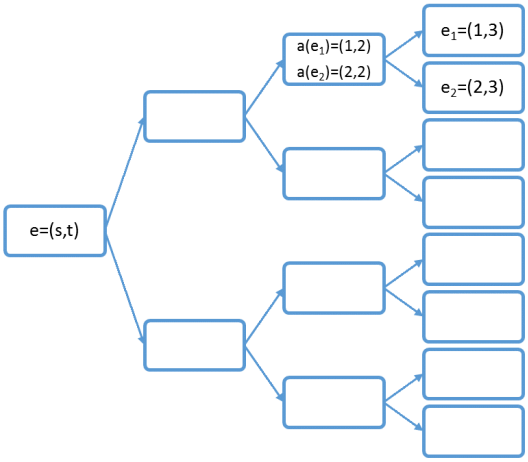

Figure 1. A scenario tree with 3 time periods.

2.1.2. Objective Function

The investor aims to determine an optimal trading strategy, represented as $x_e = x_e^+ - x_e^-$, that maximizes the expected return at the end of the investment horizon. For each event e, the expected return vector is denoted by $r_e \in \mathbb{R}^{n+1}$, and the portfolio composition at that point is given by $w_e \in \mathbb{R}^{n+1}$. The objective function capturing the expected terminal return is expressed as follows:

$$\begin{aligned}
\mathcal{W}_T &= \mathbb{E}[r_T(\xi^T) w_{T-1}] \\
&= \mathbb{E}[r_T(\xi_T | \xi^{T-1}) w_{T-1}] \\
&= \mathbb{E}\left[\sum_{e \in \mathcal{N}_T} \mathcal{P}_e r_T^T w_{a(e)}\right] \\
&= \sum_{e \in \mathcal{N}_T} \mathcal{P}_e \hat{r}_e w_{a(e)}
\end{aligned}$$

where ξ_t denotes the random variables observed at time t, while ξ^t captures the history of all realizations up to and including period t. The term $\hat{r}_e$ refers to the realized value of the return at terminal time T, conditioned on the historical data ξ^{T-1}, i.e., $r_T(\xi_T \mid \xi^{T-1})$.

2.1.3. Transaction Costs Constraints

Incorporating transaction costs into portfolio optimization models allows for more realistic representations by accounting for practical considerations such as bid–ask spreads, brokerage fees, taxes, and market impact. In this multi-period context, we adopt a linear transaction cost model, where costs are incurred at each rebalancing stage. Accordingly, these costs are explicitly embedded within the flow balance constraints, which are formulated as equalities.

We define transaction costs separately for asset purchases (c_b) and sales (c_s). This leads to the following balance equation that governs asset flows at each intermediate node $e \in \mathcal{N}_I$:

$$w_e = \hat{r}_e \circ w_{a(e)} + x_e^+ \circ (1 - c_b) - x_e^- \circ (1 + c_s), \quad \forall e \in \mathcal{N}_I$$

Additionally, we enforce that the total volume of purchases equals total sales at each interior node, ensuring portfolio rebalancing is self-financing:

$$1^T x_e^+ = 1^T x_e^- \quad \forall e \in \mathcal{N}_I$$

Here, $\mathcal{N}_I$ denotes the set of all non-terminal (interior) nodes within the scenario tree.

2.1.4. Shortfall Risk Constraints

As previously noted, this model incorporates both return objectives and risk management. One commonly used metric for assessing potential losses over a specified time horizon and confidence level is Value-at-Risk (VaR). However, VaR is criticized for being non-convex, non-sub-additive, not controlling scenarios exceeding VaR threshold, and being computationally challenging to optimize for non-normal distributions. To overcome these problems, Rockafellar and Uryasev (2000) developed CVaR which is convex, a coherent risk measure, and also computationally tractable. To model downside risk, we incorporate a CVaR constraint based on the approach proposed by Lobo et al. (2007), who examined a single-period portfolio optimization (SPO) problem involving both linear and fixed transaction costs. Their framework includes a shortfall risk condition that ensures terminal wealth exceeds a specified minimum threshold.

We should note that in Benson and Sağlam (2013), we had considered the impact of our investments on the asset prices in the long run as a part of our transaction costs. In the multi-period case, we can model direct transaction costs and build their indirect impact directly into the scenario tree.

To control downside risk, we impose CVaR constraints indexed by $k = 1, \ldots, M$, each of which ensures that the terminal wealth remains above a specified threshold W_k^{low} with a confidence level of at least η_k. In the single-period context, this constraint is formulated as follows, as presented in Benson and Sağlam (2013):

$$\Phi^{-1}(\eta_k) \left\| \Sigma^{\frac{1}{2}} (w + x^+ - x^-) \right\| \leq r^T (w + x^+ - x^-) - W_k^{\text{low}}, \qquad k = 1, \ldots, M.$$

For the multi-period formulation, the corresponding CVaR constraint is adapted to the following:

$$\Phi^{-1}(\eta_k) \left\| \Sigma^{\frac{1}{2}} w_e \right\| \leq \hat{r}_e w_{a(e)} - W_k^{\text{low}}, \qquad k = 1, \ldots, M. \tag{2}$$

Additionally, to maintain a minimum proportion of wealth at each non-terminal node in the scenario tree, we incorporate the following constraint:

$$1^T w_e \geq 0.90 \left(1^T w_{a(e)} \right), \qquad e(s,t) \in \mathcal{N}_I, \quad t = 1, \ldots, T-1.$$

This intermediate constraint allows for controlled deviations from the previous period's wealth, enabling temporary drawdowns where appropriate while supporting potential long-term gains in future decision periods.

2.1.5. Diversification by Sectors

An essential component of robust portfolio construction is diversification, particularly across economic sectors. In alignment with the broader risk management objectives of this study, we incorporate a diversification mechanism into our multi-period model. This extension builds upon the formulation proposed by Bonami and Lejeune (2009), who introduced diversification constraints in a single-period framework with stochastic and integer decision variables.

To prevent concentration in a small number of sectors, the investor is required to allocate capital across at least $L_{\min}$ out of L available sectors. To implement this condition, we define binary variables $\zeta_{ke} \in \{0,1\}^{L\times\mathcal{N}}$, where $k = 1,\ldots,L$ indexes economic sectors and $e \in \mathcal{N}$ refers to scenario tree nodes.

The sector-level diversification condition is formulated as follows:

$$-\sum_{e\in\mathcal{N}} s_e + (\sum_{e\in\mathcal{N}} s_e + s_{min})\zeta_{ke} \leq \sum_{e\in S_k} w_e \leq \sum_{e\in\mathcal{N}} s_e + s_{min} + (1 - (\sum_{e\in\mathcal{N}} s_e + s_{min}))\zeta_{ke}$$

where S_k represents the set of assets categorized under sector k, for $k = 1,\ldots,L$ and $e \in \mathcal{N}$.

To ensure the investor holds positions in a sufficient number of sectors, we enforce the following cardinality constraint:

$$\sum_{k=1}^{L} \zeta_{ke} \geq L_{\min}, \quad \text{where} \quad k = 1,\ldots,L, \quad \zeta \in \{0,1\}^{L\times\mathcal{N}}, \quad \text{and} \quad \forall e \in \mathcal{N}.$$

This constraint guarantees that portfolio allocations are distributed across a minimum number of sectors throughout the scenario tree, thereby enhancing diversification and reducing the risk of sector-specific concentration effects.

2.1.6. Buy-in-Threshold Constraints

In line with our inclusion of transaction costs, we also control the number and significance of trades made across decision periods. Specifically, to avoid unnecessarily small active positions—often impractical in real-world trading environments—we impose minimum investment requirements on each asset position, as discussed in Bonami and Lejeune (2009).

To implement this, we introduce binary variables $\delta_e \in \{0,1\}^{n\times\mathcal{N}}$ for each node $e \in \mathcal{N}$, which indicate whether a trade occurs in a given asset. These binary indicators are used to define a minimum buy-in threshold $w_{\min}$, representing the smallest allowable position size as a fraction of total wealth. The following constraints ensure that any nonzero investment in an asset is of meaningful size:

$$w_{\min}\delta_e^+ \leq x_e^+ \leq \delta_e^+, \quad \text{and} \quad w_{\min}\delta_e^- \leq x_e^- \leq \delta_e^-, \quad \forall e \in \mathcal{N},$$

where $w_{\min}$ is a user-defined threshold that prevents insignificant allocations. Given that the positive and negative trade variables satisfy $x_e^+ \cdot x_e^- = 0$, we can equivalently express the constraint as follows:

$$w_{\min}\delta_e \leq x_e^+ + x_e^- \leq \delta_e, \qquad \forall e \in \mathcal{N}, \quad \text{and} \quad \delta \in \{0,1\}^{n\times\mathcal{N}}.$$

These constraints ensure that any transaction leads to a position that exceeds the minimum investment size, reducing model sensitivity to trivial trading actions and promoting computational tractability.

2.1.7. Portfolio Constraints

To enhance model flexibility while still reflecting realistic trading limits, we permit restricted short selling of non-liquid assets. This is achieved by introducing lower bounds on portfolio positions, ensuring that no short position exceeds a specified limit. The short-sale constraint is defined as follows:

$$w_e \geq -s_e, \qquad \forall e \in \mathcal{N},$$

where s_e denotes the maximum allowable short position for each asset at event $e \in \mathcal{N}$. These limits prevent unbounded exposure to assets with limited liquidity.

In addition, we enforce non-negativity constraints on the trade variables associated with asset purchases and sales. Specifically, both x_e^+ and x_e^-, which represent the quantities bought and sold at each node, must satisfy the following:

$$x_e^+ \geq 0, \qquad x_e^- \geq 0, \qquad \forall e \in \mathcal{N}.$$

These conditions ensure the feasibility of transaction flows and maintain consistency with the directional interpretation of buy and sell actions.

2.1.8. The Model

The complete MPO model, incorporating all relevant constraints and decision variables, can be formulated as the following MISOCP:

$$\begin{aligned}
\max_{w_e, x_e^+, x_e^-, \zeta_e, \delta_e} \quad & \sum_{e \in \mathcal{N}_T} \mathcal{P}_e \widehat{r}_e w_{a(e)} \\
\text{s.t.} \quad & w_e = \widehat{r}_e \circ w_{a(e)} + x_e^+ \circ (1 - c_b) - x_e^- \circ (1 + c_s), \quad \forall e \in N_I \\
& 1^T x_e^+ = 1^T x_e^- \quad \forall e \in N_I \\
& \Phi^{-1}(\eta_k) \|\Sigma^{\frac{1}{2}} w_e\| \leq \widehat{r}_e w_{a(e)} - W_k^{low}, \quad k = 1, \ldots, m, \quad \forall e \in N_T \\
& 1^T w_e \geq 0.90(1^T w_{a(e)}) \qquad \forall e \in N_I \\
& -\sum_{e \in \mathcal{N}} s_e + (\sum_{e \in \mathcal{N}} s_e + s_{min})\zeta_{ke} \leq \sum_{e \in S_k} w_e \leq \sum_{e \in \mathcal{N}} s_e + s_{min} + (1 - (\sum_{e \in \mathcal{N}} s_e + s_{min}))\zeta_{ke} \quad \forall e \in \mathcal{N} \\
& \sum_{k=1}^{L} \zeta_{ke} \geq L_{min}, \quad k = 1, \ldots, L, \quad \forall e \in \mathcal{N} \\
& \zeta \in \{0,1\}^{L \times \mathcal{N}} \\
& w_{min}\delta_e \leq x_e^+ + x_e^- \leq \delta_e, \quad \forall e \in \mathcal{N} \\
& \delta \in \{0,1\}^{n \times \mathcal{N}} \\
& x_e^+ \geq 0, \qquad \forall e \in \mathcal{N} \\
& x_e^- \geq 0, \qquad \forall e \in \mathcal{N} \\
& w_e \geq -s_e, \quad \forall e \in \mathcal{N}
\end{aligned}$$

This formulation brings together all structural components of the MPO problem, including rebalancing logic, risk management through CVaR, diversification by economic sector, transaction cost modeling, minimum investment thresholds, and restricted short selling. It defines a robust and flexible optimization framework that can be efficiently solved using MISOCP solution techniques described in previous sections.

2.2. MISOCP Formulation and Solution Approach

The MPO problem presented in this work can be categorized as an MISOCP. This section provides a formal definition of the problem class and outlines the computational strategy employed for its solution.

The general form of an MISOCP can be written as follows:

$$\begin{array}{lll} \min\limits_{x\in\mathcal{X}} & c^T x & \\ \text{s.t.} & \|A_i x + b_i\| & \leq \quad a_{0i}^T x + b_{0i}, \qquad i = 1, \ldots, m \end{array} \tag{3}$$

In this formulation, the decision variable $x \in \mathbb{R}^n$ is partitioned into a vector $y \in \mathbb{Z}^p$ of integer variables and a vector $z \in \mathbb{R}^k$ of continuous variables, where $p + k = n$, and the feasible region is defined by $\mathcal{X} = \{(y, z) : y \in \mathbb{Z}^p, z \in \mathbb{R}^k\}$. The problem parameters include a cost vector $c \in \mathbb{R}^n$, matrices $A_i \in \mathbb{R}^{m_i \times n}$, vectors $b_i \in \mathbb{R}^{m_i}$, and scalars $a_{0i} \in \mathbb{R}^n$ and $b_{0i} \in \mathbb{R}$, for $i = 1, \ldots, m$. The norm $\|\cdot\|$ denotes the standard Euclidean norm, and each constraint forms a second-order (Lorentz) cone.

When all variables are continuous (i.e., $p = 0$), the problem reduces to a second-order cone program (SOCP), a convex optimization class that has been extensively explored in the literature; see Alizadeh and Goldfarb (2003); Lobo et al. (1998) for foundational theory and applications. While MISOCPs are less mature in comparison, their relevance is growing rapidly—particularly in finance—where advanced formulations involving discrete decisions and conic risk measures are increasingly required Benson and Sağlam (2013).

In our previous work, Benson and Sağlam (2013), we evaluated two prominent solution techniques for single-period MISOCPs: a branch-and-bound framework by Land and Doig (1960) and an outer approximation method by Duran and Grossmann (1986). The former constructs a tree of SOCP subproblems by iteratively tightening bounds on the integer variables. Although effective, it is often computationally demanding.

The outer approximation approach, by contrast, is well suited for cases where the continuous relaxation remains convex—as is the case with SOCPs. This method alternates between solving a SOCP and a mixed-integer linear program (MIP), where the nonlinear constraints are approximated via gradient-based linearizations. In this study, we employ the outer approximation strategy to solve the proposed multi-period portfolio model.

Each iteration of the algorithm performs the following steps:

1. Solve an SOCP to generate a candidate solution for the continuous variables.
2. Solve an MIP to update integer variables and refine the approximation.
3. Apply warmstart techniques to accelerate convergence by reusing information from previous iterations.

Although interior-point methods are efficient for solving individual SOCPs, they are generally not designed to support warmstarts or detect infeasibility efficiently (Alizadeh and Goldfarb (2003)). Moreover, the presence of the Euclidean norm introduces nondifferentiability, which creates additional challenges:

1. Many optimization solvers rely on gradients, which are undefined at nondifferentiable points.
2. Outer approximation relies explicitly on gradient information to construct valid linear cuts.

We previously described the design of this algorithm for single-period cases in Benson and Sağlam (2013). Here, we extend the same methodology to handle multi-period scenarios. This generalization allows for modeling more complex investment horizons and provides a foundation for tackling large-scale problems in portfolio optimization.

2.2.1. Reformulation of the SOCP for Differentiability

One potential numerical issue in solving second-order cone programs (SOCPs) is the nondifferentiability that occurs when the argument inside the Euclidean norm becomes zero. This situation was addressed by Benson and Vanderbei (2003), who proposed alternative formulations to mitigate difficulties stemming from such nonsmooth points. While the probability of encountering nondifferentiability at the solution is minimal—especially when using interior-point methods with randomized starting points—it is still advisable to account for it in the model structure.

In our framework, second-order cone constraints arise in the context of both transaction cost modeling and shortfall risk control. It is plausible that the optimal policy might recommend maintaining the current portfolio allocation or investing solely in the risk-free asset, leading to cases where the norm expression becomes zero at optimality.

To address this, we adopt a reformulated version of the shortfall constraint introduced in Benson and Sağlam (2013), which avoids the nondifferentiability by expressing the condition as follows:

$$\frac{(\Phi^{-1}(\eta_{ke}))^2 (w_e)^T \Sigma (w_e)}{\widehat{r}_e w_{a(e)} - W_k^{\text{low}}} \quad \leq \quad \widehat{r}_e w_{a(e)} - W_k^{\text{low}}, \quad k = 1, \ldots, M, \quad \forall e \in \mathcal{N}.$$

To ensure this reformulation remains mathematically valid, the denominator must be strictly greater than zero. We can guarantee this by relying on the structure of the initial portfolio and the model's design objectives:

The model begins with a normalized portfolio such that $\sum_{j=0}^{n} w_j = 1$. Given that the shortfall threshold W_k^{low} is strictly less than 1, and assuming the initial portfolio satisfies sector diversification constraints, a feasible solution can be constructed without initiating any trades. Since the objective is to maximize expected returns, the optimal solution is expected to yield at least the same performance as this baseline. Hence, we have

$$\sum_{j=0}^{n} \widehat{r}_e w_{a(e)} \geq 1 > W_k^{\text{low}}, \qquad k = 1, \ldots, M,$$

ensuring that the denominator in the reformulated constraint remains strictly positive.

2.2.2. The Primal and Dual Penalty Problems

To solve the SOCPs that arise during the execution of both the branch-and-bound and outer approximation strategies, we utilize a primal–dual penalty interior-point algorithm. This method was originally developed for linear programming, later extended to nonlinear programming Benson and Shanno (2008), and further adapted for SOCPs in Benson and Sağlam (2013).

This approach augments both the primal and dual formulations with penalty terms. These modifications preserve the original SOCP structure while providing several important computational benefits. First, the inclusion of penalty and relaxation terms enables effective warmstarting and facilitates the detection of infeasible instances during the optimization process. Importantly, these enhancements do not alter the fundamental problem class—solutions remain within the SOCP framework and can be efficiently computed using interior-point techniques.

Moreover, this regularization strategy ensures that both the primal and dual feasible regions possess strict interiors, even in situations where the standard assumptions required for conventional interior-point methods may not hold. This added robustness increases solver reliability in edge cases where feasible solutions lie on or near the boundary of the feasible set.

Finally, the algorithm supports infeasibility detection by allowing the penalty parameters to grow unbounded or, equivalently, by transitioning into a feasibility detection phase. This capability is particularly valuable in MISOCP contexts, where problem infeasibility may otherwise be difficult to diagnose using traditional solvers.

2.3. Model Rationale and Practical Relevance

The proposed model is designed to reflect realistic, multi-period investment decision-making processes faced by institutional portfolio managers. The inclusion of discrete decision variables—such as minimum investment thresholds and sector-based diversification constraints—mirrors practical considerations commonly imposed by regulatory policies, internal investment mandates, or operational trading rules. These features ensure that the portfolio allocations adhere to minimum investment sizes and maintain exposure across distinct economic sectors.

Risk is modeled using CVaR, a widely adopted measure in financial practice for capturing tail risk. By formulating CVaR as a second-order cone constraint, the model achieves a computationally tractable representation of downside risk that aligns with industry-standard risk management objectives.

Return uncertainty over time is captured through a binary scenario tree constructed from historical asset return data. Each node in the scenario tree corresponds to a decision point where portfolio rebalancing occurs in response to evolving market conditions. This structure offers a transparent and flexible approach to modeling future market states, eliminating the need for strong parametric assumptions.

Despite the exponential growth of the scenario tree with the number of assets and periods, the model remains computationally viable using an outer approximation algorithm with warmstarting. This approach significantly reduces solution times and supports the model's scalability for practical applications.

The framework suits various real-world contexts, including long-term strategic asset allocation, retirement fund management, and automated portfolio rebalancing systems. It provides a robust tool for investors seeking to optimize performance over time while satisfying risk, diversification, and transaction-related constraints.

3. Results

In our numerical experiments, we use datasets consisting of one riskless asset and four, six, eight, or ten risky assets, selected from different economic sectors of the S&P 500. Table 2 presents the descriptive statistics of these assets' monthly returns from September 2005 to December 2010, including the number of observations, mean returns, standard deviations, and quantile measures. The variability in average returns and volatilities underscores the importance of modeling portfolio risk and diversification accurately.

Figure 2 shows a binary scenario tree with five time periods. Each node corresponds to a point in time when stock prices are observed and serves as a decision point within the investment horizon. As the number of periods increases, the number of possible scenarios (i.e., paths from the root to terminal nodes) grows exponentially, reflecting increasing uncertainty over time. At each node, the investor makes buy/sell decisions subject to constraints such as transaction costs, sector diversification, shortfall risk (CVaR), and minimum investment thresholds. The model is designed to enhance the end-of-horizon portfolio value by optimizing asset allocations under these constraints. The binary branching structure offers a flexible and realistic framework for capturing the dynamics of multi-period investment decisions.

Table 2. The descriptive statistics of the datasets for 3, 4, and 5 periods.

3-Period	N	Mean	SE Mean	SD	Min	Q1	Median	Q3	Max
IBM	14	0.0167	0.0116	0.0435	−0.0650	−0.0167	0.0223	0.0458	0.0894
APD	14	0.0166	0.0199	0.0746	−0.1005	−0.0568	0.0229	0.0778	0.1253
AET	14	0.0160	0.0265	0.0990	−0.1585	−0.0548	−0.0029	0.0965	0.1830
BAC	14	−0.0017	0.0264	0.0990	−0.1258	−0.0927	−0.0122	0.0760	0.2186
CL	14	0.0042	0.0109	0.0409	−0.0715	−0.0213	0.0090	0.0375	0.0708
EXC	14	−0.0029	0.0142	0.0532	−0.1034	−0.0399	−0.0096	0.0396	0.1016
T	14	0.0154	0.0133	0.0499	−0.0816	−0.0225	0.0327	0.0515	0.0912
AMZN	14	0.0349	0.0278	0.1040	−0.1291	−0.0589	0.0391	0.0952	0.2582
HES	14	0.0283	0.0230	0.0859	−0.1629	−0.0467	0.0523	0.0731	0.1785
UPS	14	0.0264	0.0173	0.0646	−0.0935	−0.0041	0.0301	0.0749	0.1425
4-Period	**N**	**Mean**	**SE Mean**	**SD**	**Min**	**Q1**	**Median**	**Q3**	**Max**
IBM	30	0.0110	0.0120	0.0657	−0.2052	−0.0191	0.0114	0.0524	0.1293
APD	30	0.0048	0.0189	0.1037	−0.2498	−0.0568	0.0049	0.0656	0.2255
AET	30	−0.0007	0.0240	0.1314	−0.3111	−0.0723	−0.0029	0.0793	0.3061
BAC	30	0.0125	0.0449	0.2457	−0.5325	−0.1144	−0.0122	0.1529	0.7320
CL	30	0.0087	0.0101	0.0555	−0.1618	−0.0201	0.0098	0.0431	0.1178
EXC	30	−0.0197	0.0117	0.0643	−0.1756	−0.0444	−0.0124	0.0216	0.1016
T	30	0.0021	0.0104	0.0570	−0.1273	−0.0331	0.0068	0.0439	0.0912
AMZN	30	0.0379	0.0226	0.1240	−0.2540	−0.0539	0.0553	0.1361	0.2726
HES	30	−0.0094	0.0206	0.1127	−0.2665	−0.0676	0.0168	0.0603	0.2154
UPS	30	0.0118	0.0154	0.0844	−0.2298	−0.0210	0.0167	0.0659	0.1954
5-Period	**N**	**Mean**	**SE Mean**	**SD**	**Min**	**Q1**	**Median**	**Q3**	**Max**
IBM	62	0.0126	0.0074	0.0582	−0.2052	−0.0143	0.0116	0.0515	0.1293
APD	62	0.0124	0.0097	0.0764	−0.2498	−0.0110	0.0166	0.0442	0.2255
AET	62	0.0003	0.0140	0.1104	−0.3111	−0.0644	0.0154	0.0574	0.3061
BAC	62	−0.0001	0.0224	0.1767	−0.5325	−0.0503	−0.0043	0.0690	0.7320
CL	62	0.0097	0.0059	0.0462	−0.1618	−0.0097	0.0093	0.0383	0.1178
EXC	62	0.0011	0.0075	0.0594	−0.1756	−0.0282	0.0054	0.0431	0.1068
T	62	0.0093	0.0074	0.0586	−0.1556	−0.0229	0.0198	0.0588	0.0998
AMZN	62	0.0333	0.0172	0.1356	−0.3048	−0.0435	0.0321	0.1200	0.5413
HES	62	0.0165	0.0143	0.1122	−0.2665	−0.0265	0.0239	0.0607	0.4179
UPS	62	0.0045	0.0086	0.0680	−0.2298	−0.0210	0.0040	0.0392	0.1954

N: number of observations; SE: standard error; SD: standard deviation; Q1: first quartile; Q3: third quartile. The table presents the descriptive statistics for ten stocks. The *4S* dataset includes the first 4 stocks: IBM, APD, AET, and BAC; *6S* includes IBM, APD, AET, BAC, CL, and EXC; and *8S* includes IBM, APD, AET, BAC, CL, EXC, T, and AMZN.

This section reports the computational results for the proposed MPO model. A *period*, denoted by P, represents a discrete rebalancing point in the investor's planning horizon. The uncertainty in asset returns across these P periods is modeled using the binary scenario tree constructed from historical S&P 500 return data.

At each node of the tree, the model determines optimal trading decisions under realistic investment constraints. These include transaction costs, sector-based diversification, CVaR risk limits, minimum allocation limits, and short-sale restrictions. The objective is to maximize expected terminal wealth while maintaining feasibility across all scenarios.

To solve the model, we implement an MISOCP approach using an outer approximation algorithm. We evaluate performance with and without warmstarting by tracking key metrics such as node counts, iteration numbers, and total runtime. The computational results are summarized in Table 3.

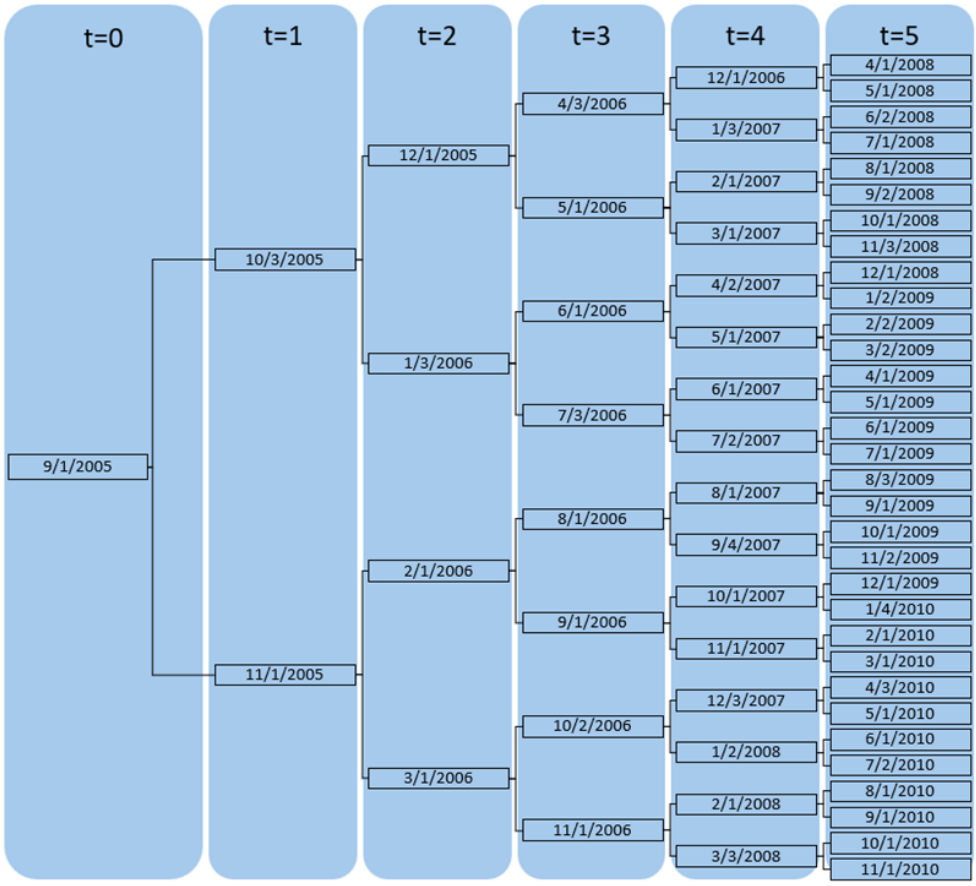

Figure 2. A scenario tree with 5 time periods.

Table 3. The results of the outer approximation algorithm.

	Coldstart Algorithm			Warmstart Algorithm			% Improvement	
Problem	**Nodes**	**Iters.**	**Time (s)**	**Nodes**	**Iters.**	**Time (s)**	**Iteration**	**Time**
3*P*4*S*	2	45	31.71	2	41	28.99	8.89%	8.59%
3*P*6*S*	2	51	77.87	2	46	72.20	9.80%	7.29%
3*P*8*S*	2	59	171.64	2	53	166.48	10.17%	3.00%
3*P*10*S*	2	60	300.07	2	56	292.99	6.67%	2.36%
4*P*4*S*	2	70	313.37	2	66	306.13	5.71%	2.31%
4*P*6*S*	2	78	1432.26	2	73	910.27	6.41%	36.45%
4*P*8*S*	2	82	2669.89	2	69	1752.51	15.85%	34.36%
4*P*10*S*	2	79	4422.48	2	69	3598.61	12.66%	18.63%
5*P*4*S*	2	72	4491.07	2	64	3746.58	11.11%	16.58%
5*P*6*S*	2	84	8424.76	2	72	6883.42	14.29%	18.30%
5*P*8*S*	2	97	17,625.32	2	79	14,595.14	18.56%	17.19%
5*P*10*S*	5	243	98,438.76	2	89	32,363.75	8.44%	17.81%

On average, warmstarting provides 10.71% and 15.24% improvements in the number of iterations and elapsed time per node, respectively.

The computational experiments were conducted using MATLAB (Version 24.1), where all problem instances are modeled and solved through the MILANO solver (Version 1.4) Benson (2007). MILANO is a MATLAB-based optimization tool that implements both branch-and-bound and outer approximation strategies. It incorporates the primal–dual penalty interior-point method described in Benson and Sağlam (2013), which facilitates warmstarting and improves infeasibility detection. The mixed-integer linear programming (MIP) subproblems that arise during the execution of the outer approximation algorithm are solved using Gurobi Optimization (2013).

Table 4 summarizes the structural characteristics of the problem instances, while Table 3 reports the computational results obtained using the outer approximation method applied to the multi-period portfolio optimization (MPO) model. In Table 4, each instance is labeled using the notation *TPRS*, where *T* indicates the number of time periods, and *R* denotes the number of risky assets included. The second column specifies the total number of assets, including a single risk-free asset. The third column lists the number of distinct economic sectors represented in the portfolio. The fourth column shows the number of CVaR constraints incorporated into the formulation. Subsequent columns present additional modeling statistics, including the number of discrete variables (DVs), continuous variables (CVs), and the total number of second-order cone constraint blocks used in the problem.

Table 4. The problem features' MISOCP models.

	Features			Variables			Results	
Problem	**N**	**L**	**M**	**DV**	**CV**	**SOCC**	**Obj. Value**	**Returns**
3P4S	5	4	2	112	168	16	1.0700	7.00%
3P6S	7	6	2	168	252	16	1.4789	47.89%
3P8S	9	8	2	224	336	16	1.9360	93.60%
3P10S	11	10	2	280	420	16	2.3782	137.82%
4P4S	5	4	2	240	360	32	1.1980	19.80%
4P6S	7	6	2	360	540	32	1.6038	60.38%
4P8S	9	8	2	480	720	32	2.0648	106.48%
4P10S	11	10	2	600	900	32	2.4533	145.33%
5P4S	5	4	2	496	744	64	1.1121	11.21%
5P6S	7	6	2	744	1116	64	1.5482	54.82%
5P8S	9	8	2	992	1488	64	2.0652	106.52%
5P10S	11	10	2	1240	1860	64	2.5307	153.07%

P: periods; S: risky stocks; N: assets, L: Global Industry Classification Standard (GICS) sectors; DV: discrete variable; CV: continuous variable; SOCC: second-order cone constraint blocks in the MPO model.

As the table shows, the number of discrete variables increases proportionally with the number of assets and periods, reflecting the growing complexity of decision-making over time. Similarly, continuous variables scale up due to the need to manage evolving asset allocations and risk measures. The number of second-order cone constraints (SOCCs) also increases with additional periods, which makes the problem more challenging to solve computationally. The increase in problem size indicates that solving larger instances will require substantial computational resources, making efficient algorithms (such as warmstarting) crucial for practical implementation. The last column presents the objective function value, which represents the end-of-period expected return. This value increases with the addition of more stocks, suggesting that larger portfolios provide more significant opportunities for optimizing returns, though they also introduce greater complexity. While the marginal contribution of adding two additional stocks diminishes for each portfolio, the objective function values, on average, increase by approximately 30%. However, longer investment horizons do not always guarantee positive profit growth (e.g., the negative growth observed between the $4P4S$ and $5P4S$ portfolios and the $4P6S$ and $5P6S$ portfolios). This highlights that other factors, such as transaction costs or market conditions, may offset the benefits of extended investment periods.

Table 3 presents the results of the outer approximation algorithm for the MPO model. As mentioned above, the first column lists the different portfolios with varying investment horizons. The subsequent six columns detail the number of nodes, iterations, and elapsed time required to solve the problem using either a coldstart or a warmstart approach. The last two columns show the percentage improvements in the average number of iterations per node and the elapsed time per node achieved through warmstarting. The

numerical results clearly demonstrate the efficiency gains from employing warmstarting in the multi-period model with the outer approximation algorithm. Warmstarting reduces both the number of iterations and the elapsed time per node. On average, it provides 10.71% and 15.24% improvements in the number of iterations and elapsed time per node, respectively, underscoring its effectiveness in solving computationally complex portfolio optimization problems.

Table 3 summarizes the performance of various portfolio configurations across different time horizons. The increase in expected terminal wealth with additional assets reflects improved diversification potential and flexibility in satisfying risk and sector constraints. Although adding more assets improves objective values, it also increases computational complexity, highlighting the importance of efficient solution methods such as warmstarting. These results provide practical insights into how portfolio structure and investment horizon impact long-term outcomes.

The results of the proposed model demonstrate its potential applicability in real-world portfolio management settings. For example, institutional investors with long-term horizons, including pension funds, endowments, and insurance firms can benefit from the multi-period structure by periodically rebalancing portfolios while adhering to realistic constraints such as sector allocation rules and risk thresholds. The ability to incorporate transaction costs, diversification criteria, and CVaR limits into the optimization ensures that investment strategies remain aligned with operational and regulatory requirements. Moreover, the warmstarting approach enables faster reoptimization, which is particularly valuable in dynamic environments where periodic decision-making is required. The model may also be a decision support tool for robo-advisors and asset managers seeking to implement algorithmic rebalancing under uncertainty.

4. Discussion and Conclusions

This study builds upon the SPO framework introduced in Benson and Sağlam (2013) by extending it to a multi-period setting. We considered the MPO problem, which was formulated as an MISOCP. We included considerations of diversification by sectors, buy-in-threshold transaction costs, CVaR requirements, and the usual mean-variance framework.

We extended the reformulation/warmstart method described in Benson and Sağlam (2013) to solve MPO problems. Despite the inherent complexity of the problem structure, the proposed solution approach demonstrates both robustness and computational efficiency. The numerical experiments yield favorable outcomes, even for large-scale instances involving thousands of discrete variables and numerous second-order cone constraints. As highlighted in Benson and Sağlam (2013), the use of the primal–dual penalty interior-point method enhances reliability by enabling the solver to either find a solution or conclusively detect infeasibility at each search node. Furthermore, the ability to warmstart successive subproblems contributes significantly to the overall performance of the algorithm.

While the proposed model captures essential features of real-world investing, notable limitations exist. First, the binary scenario tree used to model return uncertainty is constructed using historical monthly returns. It does not account for broader market complexities such as macroeconomic shocks, policy changes, or nonlinear market reactions. As such, the model provides a stylized—though data-driven—representation of uncertainty. Second, the scenario tree grows exponentially in size with the number of assets and periods, increasing computational demands significantly. Although using warmstarting improves efficiency, scalability remains a challenge for large problem instances.

Future research will focus on enhancing portfolio optimization models by introducing round-lot constraints into both single-period and multi-period frameworks. These constraints are inherently nonlinear and will require a mixed-integer nonlinear programming

(MINLP) approach. Handling second-order cone constraints and ensuring efficient warm-starts will remain essential within this framework. In this context, we anticipate significant advantages from using solvers such as MILANO, which are designed for general MINLP problems and can explicitly handle cone constraints.

Additionally, we recognize the value of exploring parallel computing or approximation algorithms to address scalability limitations and reduce solution times for large-scale scenario trees. Another promising direction involves using reinforcement learning (RL) techniques for dynamic asset allocation. Unlike traditional scenario-based optimization, RL algorithms can learn adaptive allocation policies through direct interaction with the environment, making them well suited to navigating evolving market dynamics.

Overall, we believe the proposed framework offers a robust foundation for practical MPO and provides a starting point for future work at the intersection of optimization and learning in finance.

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