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Inflation Targeting with an Optimal Nonlinear Monetary Rule—The Case Study of Colombia

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Abstract: This article examines whether Banco de la República (Banrep), Colombia’s central bank, has operated under a dual-regime policy framework—one for recessionary periods and another for periods of economic overheating—since adopting inflation targeting (IT) from Q4 2000 to Q4 2019. We modify the canonical New Keynesian inflation model to accommodate an optimal nonlinear monetary rule aligned with a two-regime policy framework. Using a LSTAR model estimated over the study period, with the output gap lagged by three periods as the transition variable, we identify two distinct monetary regimes. Our findings reveal that the smooth transitions between regimes were driven by shifts in Banrep’s preferences related to its loss function, alongside adjustments in the parameters of the aggregate demand and supply curves within the Colombian economy. Notably, we observe that a modified Taylor principle is not met in either identified monetary regime. This suggests that, in this context, IT has been a successful policy framework even without requiring the policy interest rate to respond aggressively to inflation gaps, as the Taylor principle would otherwise dictate.

Keywords: non-linear monetary rules; inflation targeting; taylor principle; non-linear STAR models

JEL Classification: E52; E58



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1. Introduction

Inflation targeting (IT) was first introduced as a monetary policy framework in New Zealand in 1990. It is a rule-based approach that requires three components: (1) a publicly announced numerical inflation target, (2) a monetary policy process with a major focus on inflation forecasts, and (3) a high degree of transparency and accountability Svensson (2007). IT aligns with a central bank’s institutional mandate to control inflation, a semi-independent central bank structure, and accountability to the public. This framework commits central banks to achieving and maintaining low inflation rates. Since the 1990s, over 30 developed and emerging countries have adopted IT (Hammond 2012; Schmidt-Hebbel and Carrasco 2016).

In practice, IT requires a central bank to set and announce a credible long run target for inflation, such as 3%, within a specific range, like 2 to 4%, as Colombia has done since the first quarter of 2003 and has maintained. However, IT should be flexible rather than strict. As Güler (2021) and Carvalho et al. (2023) have noted, IT regimes emphasize a credible commitment to long run inflation, which requires anchoring inflation expectations.

In this view, credibility means that an independent or semi-independent central bank is “believed to be firmly committed to low inflation” (Friedman 2003). As Güler (2021) argues, “credibility is also defined and measured by the degree to which inflation expectations

are anchored to a specific inflation target". This connection between credibility and the anchoring of inflation expectations is central to controlling inflation in an IT regime.

Empirical evidence from Güler (2021) supports this perspective, demonstrating that six countries implementing an IT regime—Turkey, Brazil, the Czech Republic, Chile, Poland, and South Africa—successfully anchored long-term inflation expectations, thereby establishing strong credibility.

Achieving anchored expectations requires public confidence in the central bank's commitment to its IT framework and official statements. When expectations are anchored, the public trusts the bank's adherence to its long-term target. In contrast, when credibility in central bank announcements erodes, expectations become unanchored, potentially leading to policy ineffectiveness in the long run and greater volatility in inflation outcomes (Carvalho et al. 2023).

IT central banks aim to stabilize inflation around the target while also considering the real economy, as they typically have a dual mandate that includes stabilizing measures like the output gap—the difference between actual and potential output—to promote balanced economic growth (Svensson 2010).

The flexibility of IT relies heavily on the central bank's credibility in the eyes of the public. As long as inflation stays around the announced IT range, the central bank's credibility remains strong. This suggests that a central bank can adopt monetary regimes aligned with the target range, yet vary how the policy rule is applied, especially during periods of inflation surges or economic downturns.

In Colombia, the inflation rate remained outside the target range 45% of the time from Q1 2003 to Q4 2019. Significantly, 84% of these instances exceeded the target, suggesting an asymmetric pattern in meeting inflation goals. Such asymmetry could indicate a state-dependent monetary policy regime designed to mitigate potential reductions in Banrep's credibility—much as it would for any central bank that fails to consistently achieve its targets.

This observation aligns with the concept of IT as a state-dependent monetary policy rule, where strong central bank credibility reduces the impact of forward-looking public expectations. Recently, following the pandemic, the issue of exogenous environmental pressures on central banks has regained focus. Afrouzi et al. (2024) argue that central banks face political economy pressures, especially in emerging economies.

Similarly, Hioppa (1994) argued nearly 30 years ago that, in the real world—unlike in the smoothly functioning economies of abstract models—there are shocks and unexpected circumstances that challenge the notion of excessive rigidity in the interpretation of price stability or reliance on mechanical rules for monetary management. This observation motivates us to move away from rigid monetary rules that are state-independent.

From this perspective, we construct a framework within the canonical NKM, as in Walsh (2017). In our model, an optimal nonlinear monetary policy rule emerges, with the central bank's preferences adapting to economic conditions. This adaptability represents the novel aspect of our approach.

We are interested in examining whether, within the nonlinear monetary policy rule we develop, transitions between monetary regimes occur due to changes in central bank preferences or due to structural changes in the economy (Caporale et al. 2018). Distinguishing between these possibilities is important for understanding monetary policy under an IT framework, under the assumption that the long-term expectations are anchored.

As noted by Debortoli and Nunes (2014), in the standard NKM of inflation, simple interest rate rules have been widely used. These rules often rely on reduced-form representations that link monetary regime shifts to changes in central bank preferences. This approach, however, has limitations.

Debortoli and Nunes (2014) argues that such reduced-form rules, while reflecting policymaker behavior, cannot generally identify the structural sources of these changes. Nonetheless, we show that, under certain conditions, partial identification of key structural parameters within an IT framework can reveal whether shifts in the monetary rule stem

from preference changes or structural economic shifts. Unlike our approach, which combines reduced-form and structural econometric methods, [Debortoli and Nunes \(2014\)](#) relies on calibration methodology.

In a NKM of inflation with interest rate smoothing, the optimal monetary rule that arises does not necessarily need to satisfy the so-called Taylor principle. In theoretical models within the standard NKM framework, the Taylor principle is sometimes assumed to ensure determinacy of the macroeconomic system. We take a different approach. Our focus is on whether the Taylor principle is empirically satisfied in our case study, given that, theoretically, it is not required for the optimal monetary rule to meet invertibility requirements under rational expectations.

We demonstrate that the optimal linear monetary rule, which is not state-dependent, operates with the first difference of the policy interest rate rather than its level. This gives rise to what we call a modified Taylor principle, due to the non-contemporaneous relationship between the inflation gap and the expected nominal interest rate level one period ahead. The Taylor principle traditionally implies that a central bank should respond to inflation shocks above the target by raising the nominal interest rate by more than the increase in inflation [Woodford and Walsh \(2005\)](#). Our findings indicate that this modified Taylor principle does not need to be fulfilled for the optimal monetary rule to be consistent with rational expectations.

[Ascari et al. \(2019\)](#) propose a generalization of the rational expectations framework in which the optimal policy rule does not necessarily ensure stability and determinacy of a macroeconomic dynamic system. This implies that an increase in the policy interest rate does not need to match or exceed an unanticipated inflation shock. This insight helps explain why the Taylor principle may not hold in an IT monetary framework when a central bank follows the optimal rule from the canonical NKM. In our approach, we find that the optimal monetary policy across the two regimes does not satisfy the modified Taylor principle, consistent with these recent findings.

Our approach makes several key contributions: (a) we develop a conceptual framework within the canonical NKM with interest rate smoothing. In this framework, IT builds high central bank credibility, which reduces the importance of forward-looking expectations by the public; (b) we show that if structural parameters are state-dependent, the optimal monetary rule becomes nonlinear, allowing the central bank to transition smoothly between two monetary regimes; (c) the structural econometric model derived from this framework can be represented as a nonlinear STAR (Smooth Transition Autoregressive) time series model, enabling us to (partially) identify critical structural coefficients of the NKM.

Our main contribution is linking reduced-form estimates to structural parameters within an IT framework. This is important because, in many cases, reduced-form econometric estimates are seldom connected to structural parameters from theoretical models. By recovering some of these structural preference parameters, we can study the conditions under which the modified Taylor principle holds.

Some articles in the literature assume a nonlinear monetary rule and estimate it using nonlinear estimation procedures, as seen in [Petersen \(2007\)](#). We take a different approach. We derive an optimal monetary rule that is linear for a single monetary regime within a NKM of inflation. The nonlinearity of the structural model arises from the smooth transition between two monetary regimes.

This represents a novel contribution to the literature. Typically, nonlinearities are assumed, and the rules are estimated without a supporting theoretical model. Exceptions include [Martin and Milas \(2004\)](#); [Dolado et al. \(2005\)](#), and [Surico \(2007\)](#). In these cases, the nonlinearity of the monetary rule originates from the adoption of the IT framework, nonlinear weights of the inflation gap and output gap in the loss function, or the interaction between expected inflation and the output gap.

The nonlinear monetary rule we derive incorporates a delay variable. When this variable falls below a certain threshold, the rule behaves in one way, but when it exceeds this threshold, the rule changes, resulting in the observed nonlinearity. This characteristic

of the theoretical model aligns with the main features of time series nonlinear STAR models. Recent literature on IT monetary regimes has adopted this approach, estimating STAR or TAR (Threshold Autoregressive) models with thresholds that generate nonlinear responses by central banks, especially in emerging economies, as shown by [Caporale et al. \(2018\)](#).

These models allow flexibility in coefficients and enable smooth transitions between policy regimes. We contend that interest rate smoothing, as a core objective of central banks, is best modeled through a smooth transition process rather than a discontinuous Markov switching framework. While commonly used in empirical studies ([Bae et al. 2012](#); [Gonzalez-Astudillo 2014](#); [Kuzin 2006](#); [Murray et al. 2015](#)), the Markov approach has been critiqued for limiting the ability to capture gradual regime shifts ([Castro 2011](#)). We hope our smooth transition approach encourages further inflation-targeting case studies across both emerging and advanced economies.

State-contingent preferences for a central bank are more likely to arise in emerging economies. These small open economies can experience significant supply and/or demand shocks. We focus on the performance of IT in an emerging economy like Colombia.

Banrep's IT monetary policy has successfully controlled inflation since its implementation. However, the COVID-19 pandemic limits our study period to Q4 2000 through Q4 2019. Our objective is to assess whether Banrep's behavior aligns with a two-regime IT monetary policy in which state-contingent preferences respond to recessionary or overheating conditions in the economy. We assume that Banrep's commitment to low inflation has anchored long-term inflation expectations across these monetary regimes.

2. Conceptual Framework

2.1. Setup

Consider a standard New Keynesian model (NKM) of inflation, as in [Walsh \(2017\)](#), where a central bank takes as given an aggregated IS curve denoted as

$$x_t = \eta E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - r_t^n) + \varepsilon_{1t} \quad (1)$$

and an aggregate forward-looking Phillips supply curve

$$\pi_t = \xi E_t \pi_{t+1} + kx_t + \varepsilon_{2t} \quad (2)$$

where x denotes the output gap, i the of nominal (short run) intervention interest rate, π the inflation rate, $r_t = i_t - E_t \pi_{t+1}$ represents the real interest rate where $r^n = i^n - \bar{\pi}$ represents the long run level of the real interest rate which is assumed to be determined by exogenous factors. The parameters are assumed to have the following signs: $\sigma > 0$, $k > 0$, $0 \leq \xi < 1$ and $0 \leq \eta < 1$. The variables ε_1 and ε_2 denote respectively the demand and supply disturbances/shocks assumed to be stationary processes. Denote $E_t(\cdot)$ as the operator that reflects expected value at time t of a variable using all observable information up to time t . As [Svensson \(2010\)](#) argues "the objectives under flexible inflation targeting seem well approximated by a standard quadratic loss function consisting of the sum of the squared inflation gap to the target and a weight times the squared output gap, and possibly also a weight times squared policy-rate changes (the last part corresponding to a preference for interest-rate smoothing)". Following [Woodford \(2003\)](#) we include in the loss function a weight on lagged interest rates in order to allow for interest rate smoothing. We follow the canonical framework of the New Keynesian approach and assume accordingly that the central bank has a loss function for period t represented by

$$\frac{L_t}{2} \equiv \lambda_\pi (\pi_t - \bar{\pi})^2 + \lambda_x x_t^2 + \lambda_i (i_t - i_{t-1})^2 \quad (3)$$

where the lambdas are weights assumed to be between zero and one such that $\lambda_\pi + \lambda_x + \lambda_i \equiv 1$. The first two terms are common to all monetary rules where λ_π and λ_x denote the preference weights of the inflation and output gaps in the loss function while the last term λ_i denotes the preference weight on interest rate smoothing. Since the central bank does not

observe directly the output gap it faces a measurement error represented by $x_t = y_t(1 - \zeta)$ where y_t is the true output gap and $\zeta \in \mathbb{R}$. Evidently when $\zeta = 0$, we are back to the standard NKM. Finally assume that the central bank's objective function is

$$E_t \left\{ \sum_{t=0}^{\infty} \beta^t \frac{L_t}{2} \right\} \quad (4)$$

where $\beta \in (0, 1)$.

2.2. Optimal Monetary Rule in a Single Monetary Regime

Solving the NKM of inflation yields the optimal monetary rule for a single monetary regime as reported in the following proposition.

Proposition 1. *Under rational expectations the optimal monetary rule that minimizes (4) subject to (1) and (2) is given by*

$$\begin{aligned} \Delta i_t^e = & -\frac{\sigma k \lambda_{\pi} \Psi(L)(\pi_{t-1} - \bar{\pi}) - \frac{\sigma \lambda_x(1 - \zeta) \Psi(L) y_{t-1} + \frac{\sigma \zeta(1 - \zeta) \lambda_x \Psi(L) y_{t-2}}{\beta^2 \lambda_i}}{\beta \lambda_i} \\ & + \frac{\sigma k + \eta + \zeta}{\beta} \Psi(L) \Delta i_{t-1} - \frac{\sigma k + \eta + \zeta}{\beta^2} \Psi(L) \Delta i_{t-2} + \frac{\zeta \eta}{\beta^3} \Psi(L) \Delta i_{t-3} \end{aligned} \quad (5)$$

where $\Delta i_t^e \equiv E_{t-1}(i_t - i_{t-1})$, $\Psi(L) = \sum_{k=0}^{\infty} \Psi_k L^k$ such that $0 < \Psi_k < 1$ for all k given that $\beta \in (\bar{\beta}, 1)$ for some threshold $\bar{\beta} = g(\zeta, \eta, \sigma k) \in (0, 1)$.

Proof of Proposition 1. See Appendix A. $\square$

At steady state $i_t = i^n = r^n + \bar{\pi}$, $\pi_t = \bar{\pi}$, $y_t = 0$ for all t . Therefore $\Delta i_t = 0$ for all t and the optimal monetary rule involves no change in the nominal interest rate in the absence of any shock.

In an IT framework in which a central bank builds high credibility, transparency, and accountability from the public its monetary policy would be able to anchor more likely forward looking inflation expectations in the public. As Williams (2020) argues that "The most critical element is anchoring inflation expectations at the right level". If forward looking expectations are kept under control through IT then parameters η and ζ in Equations (1) and (2) would be small. This means that one of the benefits of an IT framework is to allow a central bank to anchor forward looking expectations from the public and reduce the persistence of shocks in the monetary rule. The following corollary establishes the result for our empirical case study.

Corollary 1. *If forward looking expectations are anchored in the public i.e., $\zeta = \eta = 0$, then the monetary rule in Equation (5) simplifies to*

$$\begin{aligned} \Delta i_t^e = & -\frac{\sigma k \lambda_{\pi}}{\beta \left(1 - \frac{\sigma k}{\beta}\right) \lambda_i} (\pi_{t-1} - \bar{\pi}) - \frac{\sigma(1 - \zeta) \lambda_x}{\beta \left(1 - \frac{\sigma k}{\beta}\right) \lambda_i} y_{t-1} \\ & + \frac{\sigma k}{\beta \left(1 - \frac{\sigma k}{\beta}\right)} \Delta i_{t-1} - \frac{\sigma k}{\beta^2 \left(1 - \frac{\sigma k}{\beta}\right)} \Delta i_{t-2} \end{aligned} \quad (6)$$

where $0 < \frac{\sigma k}{\beta} < 1$ is the stability condition for $\Psi(L)$.

Proof of Corollary 1. See Appendix A. $\square$

Equation (5) represents the optimal linear monetary rule generated by the NKM of inflation, incorporating constant weights in the central bank's loss function and interest

rate smoothing. However, this rule does not necessarily ensure the credibility of the regime implemented by the central bank. As argued by Caporale et al. (2018) and Güler (2021), the long-term credibility of a monetary regime relies on the successful anchoring of inflation expectations, which serves as a key policy mechanism derived from rational expectations in IT regimes. Therefore, Equation (6) can be understood as the optimal monetary linear rule that maintains long-term credibility through the central bank's commitment to anchoring expectations, even if those expectations are unanchored in the short run. We label this as the optimal monetary linear rule under credibility.

2.2.1. The Modified Taylor Principle

We study here the conditions under which the optimal monetary rule satisfies the so called Taylor principle. From the optimal monetary rule in Equation (6) it is clear that there is something akin to what the Taylor principle proposes. Recall that the Taylor principle establishes a positive proportional or more than proportional change in the policy interest rate *level* due to a change in the inflation gap. Since $E_{t-1}i_t = i_{t-1} + E_{t-1}\Delta i_t$ or equivalently using our notation $i_t^e = i_{t-1} + \Delta i_t^e$ then a *modified* Taylor principle would be $\frac{\partial i_t^e}{\partial(\pi_{t-1} - \bar{\pi})} = \frac{\partial \Delta i_t^e}{\partial(\pi_{t-1} - \bar{\pi})}$ given that i_{t-1} is predetermined at $t - 1$. We label it *modified* since it is *not a contemporaneous* relation as the original Taylor principle establishes. As Walsh (2017) shows the adhoc Taylor principle has been used as an assumption to guarantee determinacy of a macroeconomic equilibrium in a NKM under rational expectations. Here we do not impose the Taylor principle, on the contrary we study whether the NKM with weak forward looking expectations generates a similar principle as the adhoc Taylor one.

From Equation (6) it is not clear how to determine if the modified Taylor principle is satisfied since we have lagged dependent variables as explanatory variables. To study this issue the following proposition shows that the optimal monetary rule of Equation (6) under rational expectations can be written as an infinite accumulation of inflation and output gaps.

Proposition 2. *Under rational expectations ($\Delta i_t^e = \Delta i_t$ for all t) the optimal monetary rule of Equation (6) can be written as*

$$\Delta i_t = \frac{\lambda_\pi}{\lambda_i} \Phi(L)(\pi_{t-1} - \bar{\pi}) + \frac{(1 - \zeta)\lambda_x}{k\lambda_i} \Phi(L)y_{t-1} \quad (7)$$

where $\Phi(L) = \sum_{k=0}^{\infty} \Phi_k L^k$ such that $-1 < \Phi_k < 1$ for all k given that $0 < \sigma k < \frac{\beta^2}{1+\beta}$ is satisfied.

Proof of Proposition 2. See Appendix A. $\square$

Since $\Phi(0) = 1$, according to the proof of Proposition 2, then it is straightforward to determine that

$$\frac{\partial i_t^e}{\partial(\pi_{t-1} - \bar{\pi})} = \frac{\lambda_\pi}{\lambda_i}.$$

Hence, the modified Taylor principle is satisfied only if $\frac{\lambda_\pi}{\lambda_i} \geq 1$, which is a quite intuitive result. It suffices to identify and estimate consistently the parameter $\frac{\lambda_\pi}{\lambda_i}$ in order to determine if the modified Taylor principle is satisfied.

2.2.2. Adhoc Monetary Rules

We can rewrite the optimal monetary rule in Equation (5) where the dependent variable is the interest rate in levels (not as a first difference) while the independent variables are the inflation and output gaps as the original adhoc Taylor rule while the remaining variables could be grouped in an error term. This way of writing the optimal monetary rule could be seen as an empirical mis-specification according to our model if there is a trend term that explains the interest rate, given that first differencing eliminates a linear trend empirically,

while more importantly, omission of relevant variables generate serial correlation in the error term. Therefore empirical specifications used in the literature like

$$i_t = \theta_\pi(\pi_t - \bar{\pi}) + \theta_x y_t + e_t \quad (8)$$

and

$$i_t = \psi_\pi(\pi_t - \bar{\pi}) + \psi_x y_t + \psi_i i_{t-1} + e_t \quad (9)$$

would have serial correlation either in Equation (5) or in (6). This could generate inconsistent estimates of the parameters θ_π and ψ_π that could be close to one making one conclude erroneously that the Taylor principle is satisfied when in reality it might not.

2.2.3. Empirical Linear Reduced Form Equations

The linear reduced form equation that corresponds to Equation (5) is

$$\begin{aligned} \Delta i_t = & \delta_0 + \delta_1 \pi_{t-1} + \delta_2 y_{t-1} + \delta_3 y_{t-2} \\ & + \delta_4 \Delta i_{t-1} + \delta_5 \Delta i_{t-2} + \delta_6 \Delta i_{t-3} + u_t \end{aligned} \quad (10)$$

LS estimation of this reduced form would yield consistent (not unbiased) estimates of the reduced form coefficients. The reduced form coefficients are related to the structural ones according to the following definitions

$$\begin{aligned} \delta_1 & \equiv -\frac{\sigma k \lambda_\pi}{\beta \lambda_i} \Psi(1), \delta_2 \equiv -\frac{\sigma \lambda_x (1 - \zeta)}{\beta \lambda_i} \Psi(1), \\ \delta_3 & \equiv \frac{\sigma \zeta \lambda_x (1 - \zeta)}{\beta^2 \lambda_i} \Psi(1), \delta_4 \equiv \frac{\sigma k + \eta + \zeta}{\beta} \Psi(1), \\ \delta_5 & \equiv -\frac{\sigma k + \eta + \zeta}{\beta^2} \Psi(1), \delta_6 \equiv \frac{\zeta \eta}{\beta^3} \Psi(1) \end{aligned} \quad (11)$$

According to the theoretical model the signs of the reduced form coefficients would be the following since $\Psi(1) > 0$

$$\delta_1 < 0, \delta_2 \begin{matrix} > \\ < \end{matrix} 0, \delta_3 \begin{matrix} > \\ < \end{matrix} 0, \delta_4 > 0, \delta_5 < 0, \delta_6 > 0$$

where the ambiguity of δ_2 and δ_3 comes from ζ which can be greater or less than one. Parameters are not identified without further assumptions. If $\zeta = \eta = 0$ holds empirically then the reduced form empirical model, corresponds to Equation (6) which is written in error form

$$\Delta i_t = \rho_0 + \rho_1 \pi_{t-1} + \rho_2 y_{t-1} + \rho_3 \Delta i_{t-1} + \rho_4 \Delta i_{t-2} + u_t \quad (12)$$

where the reduced form coefficients are functions of the structural parameters in the following way

$$\begin{aligned} \rho_1 & = -\frac{\sigma k \lambda_\pi}{(\beta - \sigma k) \lambda_i}, \quad \rho_2 = -\frac{\sigma (1 - \zeta) \lambda_x}{(\beta - \sigma k) \lambda_i}, \\ \rho_3 & = \frac{\sigma k}{\beta - \sigma k}, \quad \rho_4 = -\frac{\sigma k}{(\beta - \sigma k) \beta} \end{aligned} \quad (13)$$

According to the theoretical model, given that we assume $0 < \frac{\sigma k}{\beta} < 1$ and $0 < \sigma k < \frac{\beta^2}{1 + \beta}$ in accordance to Propositions 1 and 2, the signs of the reduced form coefficient would be

$$\rho_1 < 0, \rho_2 \begin{matrix} > \\ < \end{matrix} 0, \rho_3 > 0 \text{ and } \rho_4 < 0. \quad (14)$$

In this particular case we are able to identify key parameters as the following proposition shows.

Proposition 3. From reduced form equation coefficients in (12) then: (i) coefficients β , σk and $\frac{\lambda_\pi}{\lambda_i}$ are identified.

Proof. (i) From the definitions of ρ_3 and ρ_4 in (13) β is identified since it corresponds to $\beta^* = -\frac{\rho_3}{\rho_4} > 0$. From the definition of ρ_3 we can then identify $(\sigma k)^* = \frac{1}{\frac{1}{\rho_3} + \frac{1}{\beta^*}}$ while from the definition of ρ_1 we identify $\left(\frac{\lambda_\pi}{\lambda_i}\right)^* = -\left(\frac{\beta^*}{(\sigma k)^*} - 1\right)\rho_1$. $\square$

As argued above in an IT monetary framework forward looking expectations might not be present in the aggregate demand and supply equations in (1) and (2). To determine if these forward looking expectations are relevant or not in the optimal monetary rule we can test empirically, at say an α level of statistical significance with appropriate standard errors that take into account serial correlation if detected, whether we have the general optimal monetary rule as in Equation (5) or the simplified version in Equation (6).

We can use the reduced form Equation (10) where we first verify if there is serial correlation. The absence of serial correlation would suggest that we do not omit lags of the dependent variable and gives us some confidence that it is dynamically complete specification. We then test the null $\delta_6 = 0$ where if we fail to reject it, at say α significance level, then we would have either $\zeta = 0$ and/or $\eta = 0$ given that $\Psi(1) > 0$. We can test then the null $\delta_3 = 0$ in Equation (10) where if we fail to reject it we conclude either $\zeta = 0$ or $\zeta = 1$, according to the theory. We then test the null $\delta_2 = 0$ where if we fail to reject it we conclude that $\zeta = 1$. If $\delta_6 = 0$ is not rejected while $\delta_3 = 0$ is rejected then $\eta = 0$ holds as well as $\zeta \neq 1$. If $\delta_2 = 0$, $\delta_3 = 0$ and $\delta_6 = 0$ cannot be rejected individually then we must have $\zeta = 1$ and $\eta = 0$ even if $\zeta > 0$. If $\delta_2 = 0$ is rejected while $\delta_3 = 0$ and $\delta_6 = 0$ cannot be rejected at α then $\zeta \neq 1$ and $\zeta = 0$ even if $\eta > 0$. In conclusion the simplified optimal monetary rule in Equation (12) holds ($\zeta = \eta = 0$) if we reject $\delta_2 = 0$ but fail to reject $\delta_3 = 0$ and $\delta_6 = 0$. Finally, if $\delta_2 > 0$ (resp. $\delta_2 < 0$) and $\delta_3 > 0$ while $\delta_1 < 0$ (resp. $\delta_1 > 0$) then we would conclude that measurement error exists in the output gap such that $\zeta > 1$ (resp. $\zeta < 1$).

2.3. Optimal Non Linear Monetary Rule: Transitioning Between Two Monetary Regimes

As pointed out by Debortoli and Nunes (2014), within the standard NKM of inflation, there has been extensive use of simple interest monetary rules that come from reduced form representations that equate monetary regime change, due to a central bank's preference change, with a monetary rule switch. They argue that this is problematic since simple rules that come from reduced form representations that reflect policymakers' behavior cannot be used in general to identify the structural sources of behavioral changes.

This is true if key structural parameters are not identified. Nonetheless, we have shown above that in the NKM developed, we have partially identified key structural parameters from Equation (6) under long-run commitment. Evidently, identification has its limits, as it is not guaranteed in all scenarios unless stronger assumptions are imposed.

As argued above an IT monetary policy is rule based but flexible enough to allow the monetary policy to depend on the state of the economy. A change in the state of the economy can change the preferences of the central bank due to demand and/or supply shocks that can alter the assessment of the central bank in order to adjust the monetary rule that is followed. It also can change the non behavioral parameters of the central bank that governs aggregate demand and supply in the economy.

IT can be viewed therefore as generating a state dependent monetary policy rule that changes with the central bank's preference and discount rate due to the state of the economy. We take this point of view and construct a generalized NKM in which an optimal non linear monetary policy rule can emerge in which preference parameters of the central bank are state dependent.

To make things precise define a *monetary regime* as a pair $(\lambda, \beta)^g$ where $\lambda = (\lambda_\pi, \lambda_x, \lambda_i)$ is the vector of preference weights that add up to one and β is the discount rate that can vary between two monetary regimes denoted as $g = 0, 1$. The optimal monetary rule

that emerges could be different depending on the monetary regime that the central bank prefers within the IT framework and/or a change in structural coefficients (σ, k) that govern Equations (1) and (2). We focus on the monetary rule under credibility, as represented by Equation (6).

We can write the optimal monetary rule more compactly in the following way

$$\Delta i_t = \phi z_t + u_t \quad (15)$$

where $z_t = [1, \pi_{t-1}, x_{t-1}, \Delta i_{t-1}, \Delta i_{t-2}]$ and $\phi = [\rho_0, \rho_1, \rho_2, \rho_3, \rho_4]$ as in Equation (12). Let G_t denote a continuous random variable between zero and one that indicates if the monetary rule followed by the central bank of an economy is changing towards regime 0 when $0 < G_t < \frac{1}{2}$ while it indicates that it is changing towards regime 1 when $\frac{1}{2} < G_t < 1$. Since the interest rate is not observed in the transition towards these two regimes at the same time then the observed interest rate difference arises depending on variable G_t in the following way

$$\Delta i_t = (1 - G_t)\Delta i_t^0 + G_t\Delta i_t^1 \quad (16)$$

where Δi_t^0 denotes the first difference of the interest rate that arises in the transition towards regime 0 and Δi_t^1 denotes the first difference of the interest rate that arises in transition towards regime 1. Take Equation (15) and multiply it on both side by $1 - G_t$ (resp. G_t) and evaluate the dependent variable when transitioning towards regime 0 (resp. regime 1) to get

$$\Delta i_t^0(1 - G_t) = (\phi^0 z_t + u_t^0)(1 - G_t) \quad (17)$$

and

$$\Delta i_t^1 G_t = (\phi^1 z_t + u_t^1)G_t. \quad (18)$$

We can add Equations (17) and (18), use Equation (16), due to the fact that Δi_t^0 and Δi_t^1 cannot be observed directly independent of G_t , to replace the left hand side of the resulting equation in order to get

$$\Delta i_t = \phi^0 z_t(1 - G_t) + \phi^1 z_t G_t + \eta_t \quad (19)$$

where $\eta_t \equiv u_t^0(1 - G_t) + u_t^1 G_t$. Alternatively this equation can also be written as

$$\Delta i_t = (\phi^0 + \psi G_t)z_t + \eta_t \quad (20)$$

where $\psi + \phi^0 \equiv \phi^1$ when $G_t = 1$. This representation shows that ϕ^0 is the vector of the reduced parameters of the linear part of the optimal monetary rule since it arises when $G_t = 0$, which would be considered regime 0. On the other hand, ψ is the vector of the reduced parameters of the non linear part when $0 < G_t < 1$ that adds to the linear components where ϕ^1 is the vector of the reduced parameters of regime 1.

Proposition 3 allows us to identify between a change in the monetary rule due to a change in the monetary regime i.e., change in (λ, β) or a change in the structural coefficient σk which affect Equations (1) and (2). We estimate Equation (19) and compare the two monetary regimes to study whether the monetary rule changed due to the central bank's preferences and the discount rate or more due to changes in the non behavioral parameters of the economy or both.

3. Estimation of the Optimal Two Regime Monetary Rule

Equation (19) generates a non linear optimal monetary rule due to G_t . This variable is assumed to depend on a transition or delay variable S_t , a cutoff point c and speed of adjustment γ i.e., $G_t = G(S_t, c, \gamma)$. The delay variable S_t can be a lag of the dependent variable Teräsvirta (1994), an exogenous variable Dijk et al. (2002) or a linear combination of variables. Importantly, the delay variable has to be a stationary time series variable in its own right. The threshold c is a cutoff value that makes the transition variable change when S_t is above or below c while γ governs the speed of transition towards the regime.

The empirical specification in Equation (19) generates a smooth transition autoregressive (STAR) model where the requirement is that all variables in the vector $\mathbf{z}_t$ are stationary time series and that G_t is defined parametrically for a given delay variable where (γ, c) are estimated as well. Interestingly a STAR model can be viewed as a regime switching model with two regimes which are associated with the extreme values of the transition function $G(S_t, \gamma, c) = 0$ and $G(S_t, \gamma, c) = 1$, such that the transition from one regime to the other is smooth. If γ is sufficiently large a STAR model approximates a Markov switching model. We rule out a Markov switching model in this article since under interest rate smoothing a central bank would not want to jump from one regime to the other abruptly due to the loss function in (3), but more reasonably in a continuous and smooth manner.

In the literature there are two specifications for the transition function G_t that give rise to different empirical models

$$G_t = G(S_t, \gamma, c) = \begin{cases} \frac{1}{1+e^{-\gamma(S_t-c)}} & \gamma > 0 \quad \text{LSTAR} \\ 1 - e^{-\gamma(S_t-c)^2} & \gamma > 0 \quad \text{ESTAR} \end{cases}.$$

where LSTAR uses a logistic transition function while ESTAR uses an exponential transition function.

The asymmetric or symmetric behavior for G_t could make sense for certain delay variables but not for others. To make the point if the delay variable is a lagged output gap variable then a LSTAR specification makes more sense than a ESTAR specification. The reason is that a central bank would react differently when a recessionary state of the economy arises than when an overheating state arises. On the other hand, if the delay variable is a lag of the inflation gap then an ESTAR specification might make more sense than a LSTAR specification. The reason is that a central bank might want to react symmetrically if the inflation gap is too far from zero either above or below.

The standard NKM model of inflation does not specify a delay variable that could be considered as S_t . Nonetheless, we use statistical tests to determine the S_t that generates a non linear monetary rule, which assumes G_t to be either logistic or exponential, from the following list of candidate variables in the following set:

$$\Omega = \left\{ \begin{array}{l} x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4}, \pi_{t-1}, \pi_{t-1}, \\ \pi_{t-2}, \pi_{t-3}, \pi_{t-4}, \Delta i_{t-1}, \Delta i_{t-2}, \Delta i_{t-3}, \Delta i_{t-4} \end{array} \right\}.$$

Since in principle there can be several candidates for a delay variable we consider a list of them included in set Ω and leave open the specification of LSTAR and ESTAR for each of them according to empirical specification tests used in the literature to find the appropriate non linear specification, if one exists. We search within this set of a priori reasonable transition or delay variables given by a list four lagged output and inflation gaps as well as four lags of the first difference of the interest rate.

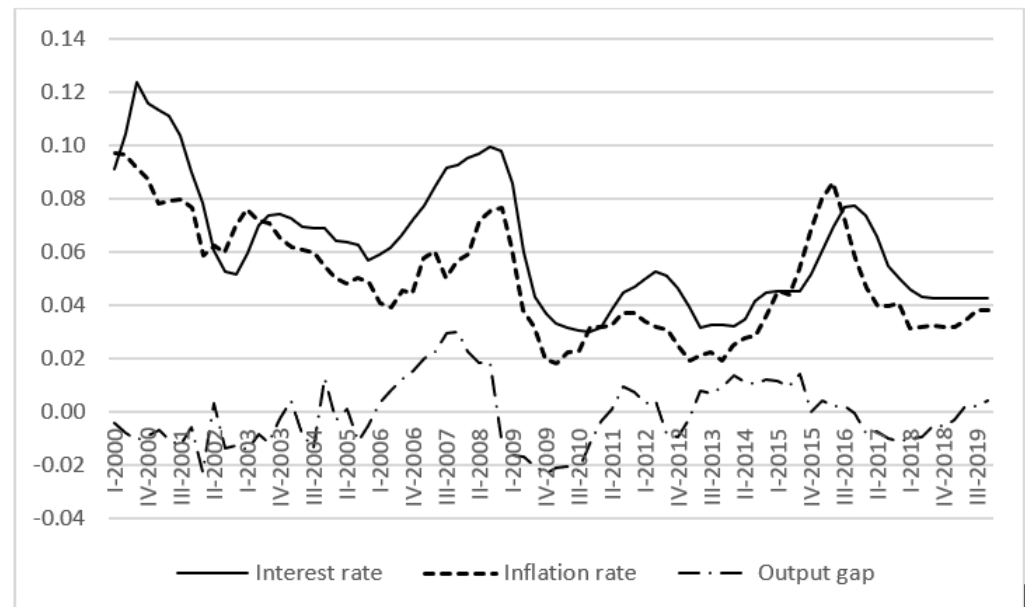
The economic intuition for the list is that Banrep's constitutional mandate is to control first and foremost inflation in the country's economy which justifies the inflation gap and its lags as a relevant delay variable. On the other hand, the output gaps can generate also different monetary regimes due to the stabilization of the output in recessions and/or expansions which also justifies the inclusion of four lags of it in the list. Finally we include lags of the first difference of the interest rate since it captures the possibility that interest rate smoothing could generate different monetary regimes.

In sum, we have laid out a framework in which the optimal monetary rule can be non linear because structural coefficients can be state dependent and therefore the modified Taylor principle could be satisfied in one regime even if it does not hold in a single monetary regime consistent with a linear specification of the monetary rule.

4. Data and Results

4.1. Data and Unit Root Tests

Descriptive statistics of the time series we use are reported in the Appendix A under Table A1. The data is publicly available in the webpage of Banrep at www.banrep.gov.co/es. Accessed on 31 May 2020. Figure 1 plots the policy interest rate (overnight interest rate), inflation rate and output gap for the period 2000.I to 2019.IV annualized.



Source: Banco de la República and author's calculations

Figure 1. Time series.

Interest and inflation rates are observed monthly. However, since GDP is observed quarterly, we use quarterly averages of the monthly data to align with the GDP's time frequency. All series are annualized. While interest and inflation rates are directly observed, the output gap is not. Estimating the output gap requires knowledge of potential GDP, which is unobserved.

Following standard practice, we estimate the output gap using the Hodrick-Prescott filter on quarterly GDP data in millions of dollars at constant prices. This filter has well-known limitations. It is not derived from theoretical knowledge of the data generating process and is sensitive to the time series' starting and ending values. Consequently, the output gap series may contain measurement error, which is consistent with our assumption that $x_t = y_t(1 - \zeta)$ for all t where $\zeta \in \mathbb{R}$.

For estimation purposes, all time series need to be stationary. Table 1 reports ADF tests for these time series, where we are able to reject the null of a unit root in each case. We use ADF tests as they provide the greatest statistical power under the alternative hypothesis of stationarity in the data-generating process for small samples (Choi and Moh 2007). Notably, the interest rate is trend-stationary with a negative trend. Descriptive statistics for the time series are reported in the Appendix A.

Table 1. Unit root tests.

Period I.2000 to IV.2019 ¹	ADF Test	Critical Value 10%	Decision
Inflation rate, exogenous variables: constant, Lag length 1	−3.025	−2.586	Reject null of Unit root
Interest rate, exogenous variables: constant, linear trend, Lag length 1	−4.446	−3.161	Reject null of Unit root
Output gap, exogenous variable: none, Lag length 4	−3.549	−1.614	Reject null of Unit root

¹ Source: author's calculations.

4.2. Empirical Results

4.2.1. Linear Monetary Rules

Table 2 presents different empirical specifications of the monetary rules we consider. The first two columns show the ad-hoc Taylor rules commonly used in empirical and theoretical literature. The first column reports the estimation of Equation (8), viewed as a mis-specification of the optimal monetary rule. This shows an LS estimate of the coefficient for the inflation rate close to one, reflecting the so-called empirical Taylor principle, in line with a positively inconsistent estimator.

Our results align with findings in the empirical literature for other countries using this ad-hoc Taylor rule, providing weak evidence that Banrep met the original Taylor principle (see Petersen (2007) for the U.S. case). We also find serial correlation in this specification. This is consistent with our earlier argument that such correlation might appear if the true model is the optimal monetary rule in (5) or (6), suggesting the specification is dynamically incomplete.

We also find similar results for Equation (9) which is reported in column 2 of Table 2. We find evidence consistent with the literature for an interest rate smoothing Taylor rule (Dery and Serletis 2024; Filippou and Taylor 2023; Kuzin 2006), where the coefficient associated with the lag of the interest rate is statistically significant. We also find serial correlation consistent with the view of an empirical mis-specification compared the assumed true model as represented in (5) or (6). This result would be in accordance with a positively inconsistent LS estimator associated with the inflation rate if the two follow the same negative trend. From Figure 1 it seems plausible that a common negative trend could arise between the interest and inflation rate time series.

The third and fourth columns of Table 2 report the empirical estimation of the optimal monetary rules in Equations (5) and (6). For these two empirical specifications we do not find serial correlation which is consistent with the NKM model developed. Moreover, we want to know which of these two empirical specifications describes better the behavior of Banrep under IT. To do so we set $\alpha = 0.1$, given that we have only 76 observations, and then consistent with our approach, described above, we test the null $\delta_6 = 0$ (associated with Δi_{t-3} in Table 2) which we are not able to reject at the 10% level. This implies that either $\xi = 0$ and/or $\eta = 0$. We then test the null $\delta_3 = 0$ (associated with x_{t-2} in Table 2), where we fail to reject at the 10% level which implies either $\xi = 0$ or $\zeta = 1$ according to the theory, given that $\Psi(1) > 0$. Moreover, we test the null $\delta_2 = 0$ (associated with x_{t-1} in Table 2) where we reject it at the 10% level which implies that $\zeta \neq 1$. Hence we conclude that $\xi = \eta = 0$ and $\zeta \neq 1$. This implies that the linear optimal monetary rule under IT describes better the behavior of Banrep as in Equation (6) consistent with the fourth specification of Table 2.

Table 2. Estimation results.

Dependent Variable *	Int. Rate	Int. Rate	Diff. Int. Rate	Diff. Int. Rate	Diff. Int. Rate (Linear Part)	Diff. Int. Rate (Non-Linear Part)
Inflation rate	0.9568 * (0.0689)	0.2995 * (0.0435)	—	—	—	—
Inflation rate −1	—	—	−0.0087 (0.0264)	−0.0401 * (0.0281)	−0.0714 * (0.0027)	−0.0658 * (0.0051)
Output gap	0.1846 (0.1160)	0.2212 * (0.0453)	—	—	—	—
Output gap −1	—	—	0.1752 * (0.0644)	0.1335 * (0.0476)	0.1123 * (0.0013)	0.0559 * (0.0175)
Output gap −2	—	—	−0.1028 (0.0631)	—	—	—
Interest rate −1	—	0.7582 * (0.0371)	—	—	—	—
Diff. Interest rate −1	—	—	0.8291 * (0.1051)	0.7121 * (0.1055)	0.5075 * (0.0025)	1.0518 * (0.0164)
Diff. Interest rate −2	—	—	−0.1678 (0.0991)	−0.2404 * (0.1090)	−0.0224 * (0.0064)	−0.9769 * (0.0016)
Diff. Interest rate −3	—	—	−0.0974 (0.0991)	—	—	—
Constant	0.0145 * (0.0037)	−0.0001 (0.0016)	0.0002 (0.0013)	0.0014 (0.0014)	0.0034 * (0.0001)	−0.0047 * (0.0007)
Threshold c	—	—	—	—	—	0.0023 * (0.0006)
Velocity gamma	—	—	—	—	—	7.3025 * (3.9332)
Number of obs.	80	79	76	77	76	76
R squared	0.7171	0.9571	0.6447	0.5452	—	—
p-value joint sig.	0.0001	0.0001	0.0001	0.0001	—	—
Serial correlation	Yes	Yes	No	No	—	—

Source: author's calculations. Std errors in parenthesis; * significance at 10%.

We use proposition 3 and the reduced form coefficients $\hat{\rho}_3 = 0.721$ and $\hat{\rho}_4 = -0.240$ to identify β to be 3.0042. Since time frequency is quarterly and the data is annualized then the discount rate needs to be divided by four, since the value 3.0042 would be an annual discount rate. This yields $\hat{\beta} = 0.751$. Since it has been established that $\xi = \eta = 0$ then proposition 3 implies that the estimates $\hat{\rho}_1 = -0.04$ and $\hat{\rho}_3 = 0.721$ allow us to obtain $\hat{\sigma k} = 0.315$ and $\frac{\hat{\lambda}_\pi}{\hat{\lambda}_i} = 0.058$. These results show that the invertibility condition of proposition 1 i.e., $0 < \frac{\sigma k}{\beta} < 1$ holds. Moreover, the condition $0 < \sigma k < \frac{\beta^2}{1+\beta}$ also holds since $\frac{\hat{\beta}^2}{1+\hat{\beta}} = 0.32143$. Importantly, these estimates verify the non ambiguous signs in Equation (14)

$$\hat{\rho}_1 = -0.04, \hat{\rho}_3 = 0.721, \hat{\rho}_4 = -0.240$$

We find $\hat{\rho}_2 = 0.1335$ which is positive and should be negative unless there is measurement error. Hence we conclude that the output gap is measured with sizable measurement error. Moreover, for this specification we find no empirical evidence that supports the modified Taylor principal discussed above since $\frac{\hat{\lambda}_\pi}{\hat{\lambda}_i} = 0.058 < 1$. According to our conceptual framework this is consistent with Banrep having higher preference for interest rate smoothing than aversion to the inflation gap.

4.2.2. Non Linear Monetary Rule

The last column of Table 2 reports the LSTAR estimation of the non linear optimal monetary rule according to Equation (19) with delay variable $S_t = x_{t-3}$. This delay variable emerged as the variable that satisfies the statistical tests of Teräsvirta (1994) and Escríbano (1999) at the 10% significance level, which gives empirical support to the existence of the non linearity in Equation (19). Appendix A reports these statistical specification tests as well as tests for serial correlation, and normality of the residuals for the non linear LSTAR specification with delay variable $S_t = x_{t-3}$.

Statistical tests show that an LSTAR with $S_t = x_{t-3}$ is consistent with a non linear monetary rule while an ESTAR specification is rejected. Moreover, we are able to reject the existence of serial correlation for the LSTAR specification with $S_t = x_{t-3}$, while not being able to reject normality at the 10% significance level. Figure 2 reports the time series of the estimated transition function G_t in the period under study.

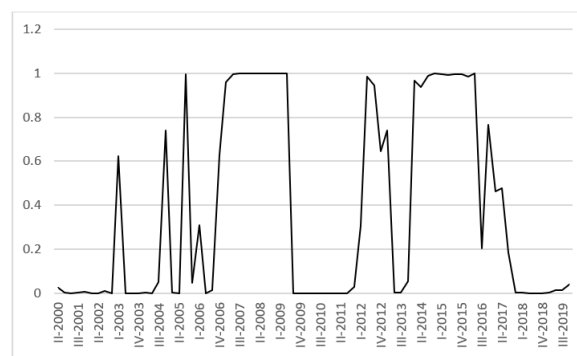


Figure 2. Quarterly time series of estimated G_t .

Figure 3 shows evidence that the LSTAR non linear monetary rule has two distinct monetary regimes. One regime, labeled regime 0, arises when the delay variable x_{t-3} is below the threshold $c = 0.0023$, while another regime, labeled regime 1, arises when the delay variable is above this threshold. The interpretation is straightforward, Banrep transitioned to regime 0, the monetary regime for a recessionary economy, since the output gap (delayed three quarters) is below a critical threshold value close to zero. On the other hand, Banrep transitioned to regime 1, the monetary regime for an overheating economy, since the delayed output gap is above the threshold value. The transitioning speed from one regime to the other, reported in Table 2, is estimated to be $\hat{\gamma} = 7.3025$. Figures 2 and 3 reveal that most of the observations in the period are either in one of the two regimes according to the delay variable while few observations are in the transition phase.

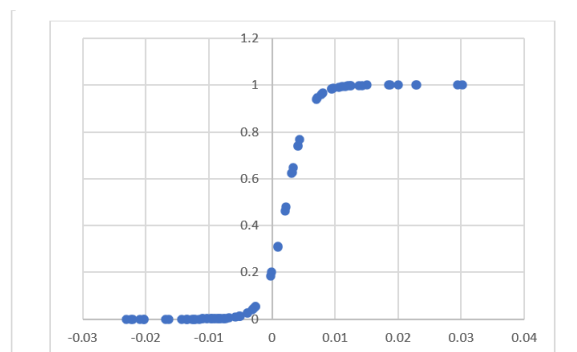


Figure 3. LSTAR transition function G_t with $c = 0.0023$.

The estimated non linear LSTAR monetary rule with $S_t = x_{t-3}$ that corresponds to Equation (20) is the following

$$\begin{aligned}\hat{\Delta}i_t = & \hat{\rho}_0^0 + \hat{\rho}_1^0\pi_{t-1} + \hat{\rho}_2^0x_{t-1} + \hat{\rho}_3^0\Delta i_{t-1} + \hat{\rho}_4^0\Delta i_{t-2} \\ & + (\hat{\psi}_0 + \hat{\psi}_1\pi_{t-1} + \hat{\psi}_2x_{t-1} + \hat{\psi}_3\Delta i_{t-1} + \hat{\psi}_4\Delta i_{t-2})G_t(S_t)\end{aligned}$$

where reduced form parameters for regime 0 are

$$\begin{aligned}\hat{\rho}_0^0 &= 0.0034 & \hat{\psi}_0 &= -0.0047 \\ \hat{\rho}_1^0 &= -0.0714 & \hat{\psi}_1 &= 0.0658 \\ \hat{\rho}_2^0 &= 0.1123 & \hat{\psi}_2 &= 0.0559 \\ \hat{\rho}_3^0 &= 0.5075 & \hat{\psi}_3 &= 1.0518 \\ \hat{\rho}_4^0 &= -0.0224 & \hat{\psi}_4 &= -0.9769\end{aligned}$$

Using the definition $\psi \equiv \phi^1 - \phi^0$ when $G_t(S_t) = 1$ we can obtain estimates of the reduced form parameters for regime 1

$$\begin{aligned}\hat{\rho}_0^1 &= -0.0047 + 0.0034 = -0.0013 \\ \hat{\rho}_1^1 &= 0.0658 - 0.0714 = -0.0056 \\ \hat{\rho}_2^1 &= 0.0559 + 0.1123 = 0.1682 \\ \hat{\rho}_3^1 &= 1.0518 + 0.5075 = 1.5593 \\ \hat{\rho}_4^1 &= -0.9769 - 0.0224 = -0.9993\end{aligned}$$

The last column of fifth specification of Table 2 reports the standard errors for the estimates of $\hat{\psi}$. We can test the null hypothesis $\psi = 0$ vs $\psi \neq 0$ which is equivalent to testing the null $\phi^1 = \phi^0$ vs $\phi^1 \neq \phi^0$ given the definition $\psi \equiv \phi^1 - \phi^0$. We reject at the 10% statistical significance level each of the individual hypotheses which gives us evidence that the two monetary regimes are statistically different from each other.

Note that the signs of the estimates $\hat{\phi}^1, \hat{\phi}^0$ are the same across the two monetary regimes except for the intercept. This suggests the existence of two regimes that implement the monetary policy under IT that differ in their relative weight on concerning issues by Banrep without being opposite in nature. To assess this better, we can use the estimates above in order to estimate the structural coefficients for each monetary regime as we did for the optimal linear monetary rule. From proposition 3 and the reduced coefficients for regime 0 and 1 we obtain

Regime 0	Regime 1
$\hat{\beta}^0 = 5.66$	$\hat{\beta}^1 = 0.39$
$\hat{\sigma}k^0 = 1.91$	$\hat{\sigma}k^1 = 0.24$
$\frac{\hat{\lambda}_\pi}{\hat{\lambda}_i} = 0.14$	$\frac{\hat{\lambda}_\pi}{\hat{\lambda}_i} = 0.004$
$\frac{(\hat{\beta}^0)^2}{1+\hat{\beta}^0} = 4.81$	$\frac{(\hat{\beta}^1)^2}{1+\hat{\beta}^1} = 0.11$

As the theoretical model implies both regimes seem to have different parameters but still satisfy the key condition $\frac{\beta^2}{1+\beta} > \sigma k$ across regimes. Even though the empirical evidence supports the conclusion that the two regimes are still different where the estimate values of $\hat{\beta}^0$ and $\hat{\beta}^1$ are quite different to each other, where the former is above significantly of the estimate for the linear monetary rule $\hat{\beta} = 0.751$ and the latter is below it. These values show evidence that the discount factor is much higher for regime zero than regime one. The values for the discount rates were adjusted to quarterly frequency by dividing the annualized estimated value by four. The estimate value of $\hat{\beta}^0$ is greater than one which should be considered with caution since theoretically it should not be greater than one.

Given that the estimate values of $\frac{\hat{\lambda}_\pi}{\hat{\lambda}_i}^0$ and $\frac{\hat{\lambda}_\pi}{\hat{\lambda}_i}^1$ are also quite different from each other we conclude that the non linear optimal monetary rule changes in accordance to changes in the monetary preferences of the central bank. Moreover, since the weight ratio

$\frac{\lambda_\pi}{\lambda_i} = \frac{1-\lambda_x}{\lambda_i} - 1$ decreases in regime one compared to regime zero then it must be that λ_i and/or λ_x must be greater in regime one than in regime zero, consistent with a monetary regime for an overheating economy, which increases the preference weight on λ_x and/or λ_i relative to a monetary regime for a recessionary economy. Furthermore, the estimate values of $\widehat{\sigma k}^0$ and $\widehat{\sigma k}^1$ are quite different from each other since the value in regime zero is greater than one while the value for regime one is less than one. This finding suggests that the two monetary regimes are distinct due to the change in preferences and discount rate of the central bank as well as to the structural parameters that govern the aggregate supply and demand in Equations (1) and (2).

Our findings allow us to conclude that the reduced form monetary rule switch across regimes under IT would have happened due to a change in both preferences and non behavioral coefficients in the economy, and not only due to change in preferences. Moreover, both monetary regimes under IT do not satisfy the modified Taylor principle since across regimes the values of $\frac{\lambda_\pi}{\lambda_i}$ are positive but less than one. This suggests that Banrep was much more responsive to inflation gap shocks in regime zero than in regime one.

5. Discussion

The literature in the estimation of Taylor rules, after Taylor (1993) seminal article, has been extensive and varied but usually focused on linear rules that have constant parameters (Taylor 1999a). Later studies extended the original linear Taylor rule (Ball 2002; Clarida et al. 1998; Ghatak and Moore 2011; Shortland and Stasavage 2004; Svensson 1999; Taylor 1999b) and emphasized possible nonlinearities in the reaction function of central banks (Caglayan et al. 2016; Martin and Milas 2013; Taylor and Davradakis 2006).

Several studies have employed Markov switching models to estimate Taylor rules, such as those by Kuzin (2006); Bae et al. (2012); Gonzalez-Astudillo (2014); Murray et al. (2015); Giraldo et al. (2012), and Villa et al. (2014). However, these models typically do not accommodate smooth transitions between policy regimes. Earlier works by Martin and Milas (2004); Dolado et al. (2005); Surico (2007), and Petersen (2007) offered early approaches that address this limitation.

More recently, there has been an increase in studies indicating that nonlinear monetary Taylor rules better describe the behavior of central banks across several economies, like Castro (2011) for the European Central Bank and the Bank of England, and Caporale et al. (2018) for Indonesia, Israel, South Korea, Thailand, and Turkey. This literature has shifted towards nonlinear smooth transition models that estimate coefficients based on the state of the economy at a given moment in time, utilizing a threshold that triggers an asymmetric response from central banks.

In terms of the utilization of LSTAR models, Huh et al. (2009) model the U.S. Phillips curve with an LSTAR specification and different non linearities for the Phillips curve, and estimates the NAIRU. Petersen (2007) uses an LSTAR model and finds that the Fed followed a non-linear Taylor rule during the great moderation period (1985-2005) and a linear Taylor rule in the dark age of monetary policy, 1960-1979. This author concludes that good monetary policy for the Fed is coupled with a non-linear Taylor rule and not a linear Taylor rule satisfying the Taylor principle. Alcidi et al. (2011) estimates a LSTAR model for a Taylor rule with interest rate smoothing. They intuitively use the policy-makers' judgment as the driver of transition among states. They show that ignoring that judgment may induce significant costs in terms of fit.

Moura and de Carvalho (2010) also estimate LSTAR models for backward and forward-looking traditional Taylor rules assuming asymmetry on the response of policy-makers using data from 16 countries in Latin America. Their findings go in the same direction as articles in which non linearities are explained by asymmetric responses of central banks. Using US data, Bunzel and Enders (2010) estimate Taylor rules with a threshold process in which the Fed reacted asymmetrically in some circumstances while in others it did not. The Fed seemed to act more aggressively when inflation was high than when it was low. They concluded that "opportunistic" monetary policy behavior explains the Fed's behavior.

Brüggemann and Riedel (2011) also estimate reaction functions for the UK finding that interest rate movements are better explained by non linear reaction functions.

In the same tradition of non linear estimation procedures but using an instrumental variable approach, Koustas and Lamarche (2012) estimate a Taylor rule that captures asymmetric responses of the Bank of England and the regressors are endogenous. Akdoğan (2015) also estimates an ESTAR to forecast the inflation for a set of developing and developed countries. As in this branch of the literature, these authors show that the inflation rate behave in an asymmetric fashion. Using real-time data and following the tradition of Taylor type rules Ahmad (2016) estimates a smoothing Taylor rule with a Multiple Regime Smooth Transition model (MSTAR) to capture asymmetries and “opportunistic” behavior of monetary policy.

Agliari et al. (2017) and Ma et al. (2018) have estimated nonlinear reaction functions but including exchange rates as part of the explanatory variables. The former authors model the economy in a pure exchange overlapping generations model with an endogenous money growth rule. The latter estimate non-linear Taylor-type rules by including principal component analysis. Their results go in the same line as the literature on nonlinear reaction functions.

To summarize, the results of this literature suggest that nonlinear Taylor-type rules more accurately describe the behavior of central banks. These findings indicate that asymmetric behavior arises when central banks intervene in monetary markets, as they tend to be more aggressive when either the inflation gap or the output gap exhibits positive values compared to negative ones.

Our results align with this literature, as we find that Banrep’s behavior during the period of IT, prior to COVID-19, is consistent with a nonlinear policy framework in which two policy regimes coexist and emerge depending on the state of the economy, whether it is experiencing a recession or an inflationary phase. These two regimes suggest that Banrep had an asymmetric response to inflation shocks based on the economic context. Under conditions of credibility, we were able to partially identify some of the structural parameters of the modified NKM of inflation, where the optimal monetary rule is empirically described as a nonlinear LSTAR model with two regimes, and the transition between them occurs in accordance with the output gap lagged by three periods, serving as the delay transition variable.

The intuition behind our results can be explained in a straightforward manner. The asymmetric behavior of policy across monetary regimes can be rationalized through asymmetric price stickiness. Specifically, for positive output gaps that correspond to a monetary regime indicative of an overheating economy, weak upward price rigidity necessitates a more robust response from Banrep to the output gap in order to avert future inflationary pressures. This is reasonable because economic agents are less resistant to price increases in an overheating economic context. In contrast, for negative output gaps that characterize a monetary regime aligned with a recessionary state, strong downward price rigidity among the public allows Banrep to respond more significantly to the inflation gap to prevent deflation.

For an emerging economy’s central bank considering IT as its monetary policy framework, establishing the credibility of its inflation target is essential to anchor forward-looking expectations. Our results suggest that IT should be implemented with flexibility, adapting to the economic cycle. In periods of economic expansion, where inflation expectations may rise, IT should respond decisively, potentially with aggressive interest rate adjustments or other monetary tools, to counteract inflationary pressures. Conversely, in economic downturns, where disinflationary or deflationary pressures emerge, a more moderate approach may be warranted to stabilize expectations without exacerbating the slowdown. This flexible, context-sensitive approach helps to ensure that central bank actions effectively anchor inflation expectations over time.

6. Conclusions

We develop a modified New Keynesian model of inflation with rational expectations, allowing the monetary framework to have state-contingent parameters. We argue that this framework can describe an inflation-targeting (IT) monetary policy, such as that implemented by Colombia's Banrep from Q4 2000 to Q4 2019, which is flexible enough to offset the pressures of forward-looking expectations by the public in the presence of demand and/or supply shocks. The model shows that IT, under credibility, can be characterized as a two-regime monetary framework, where a delay transition variable determines the central bank's state preferences (e.g., the lagged output gap for Banrep) depending on whether the economy is experiencing a recessionary or inflationary state.

We show that this setup is consistent with an optimal monetary rule, where both the central bank's preferences and the economy's non-behavioral parameters can be state-dependent, leading to non-linearity in the monetary rule. We argue that a modified Taylor principle does not necessarily need to be satisfied for the non-linear monetary rule to be stable and dynamically consistent under rational expectations.

The theoretical model leads to an empirical smooth transition autoregressive model (STAR) that can be tested using data. We perform statistical specification tests to assess the appropriate non-linear specification, which is determined to be an LSTAR specification with a transition variable that uses the output gap lagged by three periods.

We find empirical evidence that the optimal LSTAR monetary rule better describes Banrep's monetary policy under IT compared to an otherwise optimal linear rule with a single monetary regime. We then use the reduced-form estimates in each regime to identify some key structural parameters and to test whether the two monetary regimes differ due to changes in the central bank's preferences and/or changes in the economy's structural parameters. We find empirical evidence supporting a monetary rule that switches between the two regimes due to changes in both preference parameters and structural parameters in the Colombian case study. Under the key assumption of credibility of IT, we were able to identify separately the changes in both preference parameters of Banrep and some key parameters of the economy related to the IS aggregate demand and Phillips supply curves.

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Appendix A

Proof of Proposition 1. The fundamental problem to solve is

$$\mathcal{L} = E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\lambda_{\pi}(\pi_t - \bar{\pi})^2 + \lambda_x x_t^2 + \lambda_i(i_t - i_{t-1})^2}{2} + \phi_{1,t}[x_t - \eta x_{t+1} + \sigma(i_t - \pi_{t+1} - r^n) - \varepsilon_{1t}] \right. \right. \\ \left. \left. + \phi_{2,t}[\pi_t - \zeta \pi_{t+1} - kx_t - \varepsilon_{2t}] \right\} \right\} \quad (\text{A1})$$

where $\phi_{1,t}$ and $\phi_{2,t}$ are the Lagrangian multipliers associated with (1) and (2). First order conditions, which are necessary and sufficient, given that L_t is a quadratic function in the control variables, that solve the optimization program are

$$(\pi_t) : \beta^t[\lambda_{\pi}(\pi_t - \bar{\pi}) + \phi_{2,t}] - \beta^{t-1}(\sigma\phi_{1,t-1} + \zeta\phi_{2,t-1}) = 0$$

$$(x_t) : \beta^t[\lambda_x x_t + \phi_{1,t} - k\phi_{2,t}] - \beta^{t-1}\eta\phi_{1,t-1} = 0$$

$$(i_t) : \beta^t[\lambda_i(i_t - i_{t-1}) + \sigma\phi_{1,t}] - \beta^{t+1}[\lambda_i(E_t i_{t+1} - i_t)] = 0$$

$$(\phi_{1,t}) : x_t - \eta E_t x_{t+1} + \sigma(i_t - E_t \pi_{t+1} - r^n) - \varepsilon_{1t} = 0 \quad (\text{A2})$$

$$(\phi_{2,t}) : \pi_t - \zeta E_t \pi_{t+1} - kx_t - \varepsilon_{2t} = 0. \quad (\text{A3})$$

The first three equations can be simplified to get

$$\lambda_{\pi}(\pi_t - \bar{\pi}) + \phi_{2,t} - \frac{\sigma\phi_{1,t-1} + \zeta\phi_{2,t-1}}{\beta} = 0 \quad (\text{A4})$$

$$\lambda_x x_t + \phi_{1,t} - k\phi_{2,t} - \frac{\eta}{\beta}\phi_{1,t-1} = 0 \quad (\text{A5})$$

$$\lambda_i(i_t - i_{t-1}) + \sigma\phi_{1,t} - \beta\lambda_i(E_t i_{t+1} - i_t) = 0 \quad (\text{A6})$$

From Equation (A6) we get

$$\phi_{1,t} = \frac{\beta\lambda_i(E_t i_{t+1} - i_t) - \lambda_i(i_t - i_{t-1})}{\sigma} \quad (\text{A7})$$

$$\phi_{1,t-1} = \frac{\beta\lambda_i(E_{t-1} i_t - i_{t-1}) - \lambda_i(i_{t-1} - i_{t-2})}{\sigma} \quad (\text{A8})$$

We replace Equations (A7) and (A8) in Equation (A5) which yields

$$\phi_{2,t} = \frac{\lambda_x}{k}x_t + \frac{\lambda_i}{\sigma k}(\beta(E_t i_{t+1} - i_t) - (i_t - i_{t-1})) \\ - \frac{\eta\lambda_i}{\beta\sigma k}(\beta(E_{t-1} i_t - i_{t-1}) - (i_{t-1} - i_{t-2})) \quad (\text{A9})$$

as well as

$$\phi_{2,t-1} = \frac{\lambda_x}{k}x_{t-1} + \frac{\lambda_i}{\sigma k}(\beta(E_{t-1} i_t - i_{t-1}) - (i_{t-1} - i_{t-2})) \\ - \frac{\eta\lambda_i}{\beta\sigma k}(\beta(E_{t-2} i_{t-1} - i_{t-2}) - (i_{t-2} - i_{t-3})) \quad (\text{A10})$$

We replace Equations (A7), (A8), (A9), and (A10) in Equation (A4) to get after rearranging

$$\begin{aligned} E_t(i_{t+1} - i_t) = & -\frac{\sigma k \lambda_\pi}{\beta \lambda_i}(\pi_t - \bar{\pi}) - \frac{\sigma \lambda_x}{\beta \lambda_i}x_t + \frac{\sigma \xi \lambda_x}{\beta^2 \lambda_i}x_{t-1} \\ & + \left(\frac{\sigma k + \eta + \xi}{\beta}\right)(E_{t-1}(i_t - i_{t-1})) + \frac{1}{\beta}(i_t - i_{t-1}) \\ & - \left(\frac{\sigma k + \eta + \xi}{\beta^2}\right)(i_{t-1} - i_{t-2}) - \frac{\xi \eta}{\beta^2}(E_{t-2}(i_{t-1} - i_{t-2})) \\ & + \frac{\xi \eta}{\beta^3}(i_{t-2} - i_{t-3}) \end{aligned} \quad (\text{A11})$$

Define $\Delta i_t^e \equiv E_{t-1}(i_t - i_{t-1})$ for all t where we suppose that rational expectations hold from the central bank when taking decisions. Hence the optimal monetary rule that is consistent with rational expectations would satisfy

$$\begin{aligned} \left(L^{-1} - \frac{\sigma k + \eta + \xi}{\beta} + L \frac{\xi \eta}{\beta^2}\right) \Delta i_t^e = & -\frac{\sigma k \lambda_\pi}{\beta \lambda_i}(\pi_t - \bar{\pi}) - \frac{\sigma \lambda_x}{\beta \lambda_i}x_t + \frac{\sigma \xi \lambda_x}{\beta^2 \lambda_i}x_{t-1} \\ & + \frac{1}{\beta} \Delta i_t - \left(\frac{\sigma k + \eta + \xi}{\beta^2}\right) \Delta i_{t-1} + \frac{\xi \eta}{\beta^3} \Delta i_{t-2} \end{aligned}$$

we use the lag and forward operator's namely $Lv_t = v_{t-1}$ and $L^{-1}v_t = v_{t+1}$ for any random variable v_t . This equation can be written in the following way lagging on both sides

$$\begin{aligned} \Gamma(L) \Delta i_t^e = & -\frac{\sigma k \lambda_\pi}{\beta \lambda_i}(\pi_{t-1} - \bar{\pi}) - \frac{\sigma \lambda_x}{\beta \lambda_i}x_{t-1} + \frac{\sigma \xi \lambda_x}{\beta^2 \lambda_i}x_{t-2} \\ & + \frac{1}{\beta} \Delta i_{t-1} - \frac{\sigma k + \eta + \xi}{\beta^2} \Delta i_{t-2} + \frac{\xi \eta}{\beta^3} \Delta i_{t-3} \end{aligned} \quad (\text{A12})$$

where $\Gamma(L) \equiv 1 - \frac{\sigma k + \eta + \xi}{\beta}L + \frac{\xi \eta}{\beta^2}L^2$ and needs to be invertible. To see what is required note that the characteristic equation is

$$1 - \frac{\sigma k + \eta + \xi}{\beta}z + \frac{\xi \eta}{\beta^2}z^2 = 0 \quad (\text{A13})$$

where the two roots are given by

$$\begin{aligned} z_1 &= \frac{\beta}{2\xi\eta} \left(\xi + \eta + \sigma k - \sqrt{k^2\sigma^2 + 2\sigma k(\xi + \eta) + (\xi - \eta)^2} \right) \\ z_2 &= \frac{\beta}{2\xi\eta} \left(\xi + \eta + \sigma k + \sqrt{k^2\sigma^2 + 2\sigma k(\xi + \eta) + (\xi - \eta)^2} \right) \end{aligned}$$

where it can be shown that $z_1 > 0$ and $z_2 > 0$ for any set of values of the parameters. For invertibility of $\Gamma(L)$ the roots of the characteristic equation have to be outside the unit circle i.e., $|z_i| > 1$ for $i = 1, 2$ but since $z_i > 0$ for $i = 1, 2$ then sufficient conditions are the following

$$\begin{aligned} z_1 &> 1 \iff \sigma k + (\xi + \eta) < \beta + \frac{\xi \eta}{\beta} \\ z_2 &> 1 \iff \beta > \frac{2\xi\eta}{\xi + \eta + \sigma k + \sqrt{k^2\sigma^2 + 2\sigma k(\xi + \eta) + (\xi - \eta)^2}} \end{aligned}$$

or the sufficient condition $\bar{\beta} < \beta$ where

$$\bar{\beta} \equiv \max \left\{ \frac{2\zeta\eta}{\zeta + \eta + \sigma k + A}, \frac{\zeta + \eta + \sigma k + A}{2} \right\}$$

where $A \equiv \sqrt{k^2\sigma^2 + 2\sigma k(\zeta + \eta) + (\zeta - \eta)^2}$. For $\bar{\beta} \in (0, 1)$ to be the case it is required that for a given value σk the values of ζ and η are not too close to one. Note that $\Gamma(L)$ can be written as $\Gamma(L) = (1 - G_1L)(1 - G_2L)$ where $G_i \equiv \frac{1}{z_i} \in (0, 1)$ for $i = 1, 2$ which implies that

$$\begin{aligned} (1 - G_iL)^{-1} &= 1 + G_iL + G_i^2L^2 + G_i^3L^3 + \dots \\ \text{for } i &= 1, 2 \end{aligned}$$

and therefore

$$\Gamma(L)^{-1} \equiv \Psi(L) = (1 + G_1L + G_1^2L^2 + G_1^3L^3 + \dots)(1 + G_2L + G_2^2L^2 + G_2^3L^3 + \dots)$$

or equivalently

$$\begin{aligned} \Psi(L) &= 1 + (G_1 + G_2)L + (G_1^2 + G_1G_2 + G_2^2)L^2 \\ &\quad + (G_1^3 + G_1G_2^2 + G_1^2G_2 + G_2^3)L^3 + \dots \\ &= \sum_{h=0}^{\infty} \Psi_h L^h \end{aligned}$$

where

$$\begin{aligned} \Psi_0 &= 1 \\ \Psi_1 &= G_1 + G_2 \\ \Psi_2 &= G_1^2 + G_1G_2 + G_2^2 \\ &\dots \\ \Psi_h &= A_1G_1^h + A_2G_2^h \end{aligned}$$

for some constants A_1 and A_2 . These constants can be found by setting $k = 0$ and $k = 1$ such that A_1 and A_2 satisfy

$$\begin{aligned} 1 &= A_1 + A_2 \\ G_1 + G_2 &= A_1G_1 + A_2G_2 \end{aligned}$$

or equivalently

$$\begin{pmatrix} 1 \\ G_1 + G_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ G_1 & G_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & 1 \\ G_1 & G_2 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{G_2}{G_1 - G_2} & \frac{1}{G_1 - G_2} \\ \frac{G_1}{G_1 - G_2} & -\frac{1}{G_1 - G_2} \end{pmatrix}$ then $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \frac{G_1}{G_1 - G_2} \\ -\frac{G_2}{G_1 - G_2} \end{pmatrix}$. Hence

$$\Psi_h = \frac{G_1^{h+1} - G_2^{h+1}}{G_1 - G_2} \text{ for } h \in \{0, 1, 2, \dots\}$$

Note that $\Psi_h > 0$ for all $h \in \{0, 1, 2, \dots\}$ such that they decrease for larger h given that $G_i \in (0, 1)$ for $i = 1, 2$ and $G_1 > G_2$ or $G_1 < G_2$. Multiply Equation (A12) by $\Gamma(L)^{-1} = \Psi(L)$ on both sides to get

$$\begin{aligned} \Delta i_t^e = & -\frac{\sigma k \lambda_\pi}{\beta \lambda_i} \Psi(L) (\pi_{t-1} - \bar{\pi}) - \frac{\sigma \lambda_x}{\beta \lambda_i} \Psi(L) x_{t-1} + \frac{\sigma \xi \lambda_x}{\beta^2 \lambda_i} \Psi(L) x_{t-2} \\ & + \frac{\sigma k + \eta + \xi}{\beta} \Psi(L) \Delta i_{t-1} - \frac{\sigma k + \eta + \xi}{\beta^2} \Psi(L) \Delta i_{t-2} + \frac{\xi \eta}{\beta^3} \Psi(L) \Delta i_{t-3} \end{aligned} \quad (\text{A14})$$

Once $x_{t-1} = (1 - \zeta)y_{t-1}$ and $x_{t-2} = (1 - \zeta)y_{t-2}$ are replaced in the last equation we get Equation (5). $\square$

Proof of Corollary 1. When $\xi = \eta = 0$, then the characteristic function in Equation (A13) is $1 - \frac{\sigma k}{\beta} z = 0$ and the unique root is $z = \frac{\beta}{\sigma k} > 0$. Hence invertibility of $\Gamma(L)$ requires the condition $z > 1$ or $\frac{\sigma k}{\beta} < 1$. $\square$

Proof of Proposition 2. Equation (6) under perfect foresight $\Delta i_t^e = \Delta i_t$ for all t can be written as

$$\Lambda(L) \Delta i_t = \frac{\lambda_\pi}{\lambda_i} (\pi_{t-1} - \bar{\pi}) + \frac{(1 - \zeta) \lambda_x}{k \lambda_i} y_{t-1} \quad (\text{A15})$$

where $\Lambda(L) \equiv 1 - \frac{\beta}{\sigma k} + L - \frac{1}{\beta} L^2$ needs to be invertible in order to find the long run impact on the difference of the nominal interest rate due to a change in the inflation or output gap. To see what is required note that the characteristic equation is

$$1 - \frac{\beta}{\sigma k} + z - \frac{1}{\beta} z^2 = 0 \quad (\text{A16})$$

where the two roots are given by

$$\begin{aligned} z_1 &= \frac{1}{2} \beta + \frac{1}{2} \sqrt{\beta(4 + \beta) - \frac{4\beta^2}{\sigma k}} \\ z_2 &= \frac{1}{2} \beta - \frac{1}{2} \sqrt{\beta(4 + \beta) - \frac{4\beta^2}{\sigma k}} \end{aligned}$$

The roots are complex numbers (resp. real numbers) if $\sigma k < \frac{4\beta}{4+\beta}$ (resp. $\sigma k > \frac{4\beta}{4+\beta}$). Nonetheless if $\beta \in (\frac{1}{2}, 1)$ then the roots outside the unit circle cannot be real numbers. To see this suppose z_1 is a real number greater than one. This implies that $\sigma k > \frac{\beta^2}{2\beta-1}$ must be satisfied and since $\frac{\beta^2}{2\beta-1} > \beta$ for $\beta \in (\frac{1}{2}, 1)$ then $\sigma k > \beta$, which contradicts the invertibility condition of corollary 1. We conclude that if $\beta \in (\frac{1}{2}, 1)$ then z_i for $i = 1, 2$ are complex numbers. The criterion for complex roots to be outside the unit circle is that the modulus of the complex roots has to be greater than one. If $\sigma k < \frac{4\beta}{4+\beta}$ then $z_1 = \frac{1}{2} \beta + \frac{1}{2} \sqrt{\frac{4\beta^2}{\sigma k} - \beta(4 + \beta)} i$ and $z_2 = \frac{1}{2} \beta - \frac{1}{2} \sqrt{\frac{4\beta^2}{\sigma k} - \beta(4 + \beta)} i$ are the complex roots where $i = \sqrt{-1}$. The invertibility condition then is that modulus for the complex number is greater than one¹ i.e., $\sqrt{\frac{\beta^2}{\sigma k} - \beta} > 1$ or equivalently $\sigma k < \frac{\beta^2}{1+\beta}$. Since $\beta > \frac{\beta^2}{1+\beta}$ then a sufficient condition for invertibility is that σk is sufficiently close to zero such that $\sigma k < \frac{\beta^2}{1+\beta}$

for a given $\beta \in (0, 1)$. Note that $\Lambda(L)$ can be written as $\Lambda(L) = (1 - G_1L)(1 - G_2L)$ where $G_i \equiv \frac{1}{z_i} \in (-1, 1)$ for $i = 1, 2$ which implies that

$$(1 - G_iL)^{-1} = 1 + G_iL + G_i^2L^2 + G_i^3L^3 + \dots$$

for $i = 1, 2$

and therefore

$$\begin{aligned} \Lambda(L)^{-1} &\equiv \Phi(L) = 1 + (G_1 + G_2)L + (G_1^2 + G_1G_2 + G_2^2)L^2 \\ &+ (G_1^3 + G_1G_2^2 + G_1^2G_2 + G_2^3)L^3 + \dots = \sum_{h=0}^{\infty} \Phi_h L^h \end{aligned}$$

where by a similar reasoning as in the Proof of Proposition 1 we have

$$\begin{aligned} \Phi_0 &= 1 \\ \Phi_1 &= G_1 + G_2 \\ \Phi_2 &= G_1^2 + G_1G_2 + G_2^2 \\ &\dots \\ \Phi_h &= \frac{G_1^{h+1} - G_2^{h+1}}{G_1 - G_2} \end{aligned}$$

Multiply Equation (A15) by $\Lambda(L)^{-1} = \Phi(L)$ on both sides to get

$$\Delta i_t = \frac{\lambda\pi}{\lambda_i} \Phi(L)(\pi_{t-1} - \bar{\pi}) + \frac{(1 - \zeta)\lambda_x}{k\lambda_i} \Phi(L)y_{t-1}$$

which is Equation (7) reported above. $\square$

Table A1. Descriptive statistics ¹.

Variable	Interest Rate	Inflation Rate	Output Gap
Mean	0.0620	0.0499	−0.0009
Std Error	0.0026	0.0023	0.0014
Median	0.0593	0.0464	−0.0027
Range	0.0932	0.0789	0.0533
Min	0.0303	0.0184	−0.0231
Max	0.1235	0.0973	0.0302

¹ Source: Banco de La Republica; author's calculations.

Table A2. Terasvirta's tests.

Test with Delay Variable	LM2 (<i>p</i> -Value, Decision)	LMR1 (<i>p</i> -Value, Decision)	LMR2 (<i>p</i> -Value, Decision)	LMR3 (<i>p</i> -Value, Decision)
Inflation rate $t - 1$	0.483 Linear	—	—	—
Inflation rate $t - 2$	0.001 Non linear	0.047 Reject ESTAR	0.100 Non Informative	0.0002 Do not reject LSTAR
Inflation rate $t - 3$	0.026 Non linear	0.251 Could be LSTAR	0.139 Non informative	0.024 Do not reject LSTAR
Inflation rate $t - 4$	0.104 Linear	—	—	—

Table A2. Cont.

Test with Delay Variable	LM2 (<i>p</i> -Value, Decision)	LMR1 (<i>p</i> -Value, Decision)	LMR2 (<i>p</i> -Value, Decision)	LMR3 (<i>p</i> -Value, Decision)
Output gap $t - 1$	0.079 Linear	0.242 Could be LSTAR	0.460 Non informative	0.0402 Do not reject LSTAR
Output gap $t - 2$	0.272 Linear	—	—	—
Output gap $t - 3$	0.056 Non linear	0.063 Reject ESTAR	0.875 Non informative	0.0386 Do not reject LSTAR
Output gap $t - 4$	0.014 Non linear	0.006 Reject ESTAR	0.384 Non informative	0.1479 Could be ESTAR
Diff. Interest rate $t - 1$	0.002 Non linear	0.082 Could be LSTAR	0.147 Non informative	0.0009 Do not reject LSTAR
Diff. Interest rate $t - 2$	0.006 Linear	—	—	—
Diff. Interest rate $t - 3$	0.701 Linear	—	—	—
Diff. Interest rate $t - 4$	0.601 Linear	—	—	—

Source: author's calculations.

Table A3. Jorda-Escribano's tests.

Test with Delay Variable	H0L <i>p</i> -Value	H0E <i>p</i> -Value	Decision	Non Linear Test NL4 (<i>p</i> -Value, Decision)
Inflation rate $t - 1$	—	—	—	0.553 Linear
Inflation rate $t - 2$	0.0002	0.0003	ESTAR	0.001 Non Linear
Inflation rate $t - 3$	0.1633	0.2836	ESTAR	0.015 Non Linear
Inflation rate $t - 4$	—	—	—	0.251 Linear
Output gap $t - 1$	—	—	—	0.130 Linear
Output gap $t - 2$	—	—	—	0.149 Linear
Output gap $t - 3$	0.0968	0.0116	LSTAR	0.016 Non linear
Output gap $t - 4$	0.7539	0.0107	LSTAR	0.033 Non linear
Diff. Interest rate $t - 1$	0.0875	0.0167	LSTAR	0.004 Non linear
Diff. Interest rate $t - 2$	—	—	—	0.054 Linear
Diff. Interest rate $t - 3$	0.0009	0.0009	LSTAR	0.012 Non linear
Diff. Interest rate $t - 4$	—	—	—	0.874 Linear

Source: author's calculations.

Table A4. Autocorrelation tests.

Lags	<i>p</i> -Value
1	0.0898
4	0.103
8	0.156

Source: author's calculations.

Table A5. Normality tests.

Statistic	p-Value
Shapiro-Wilk	0.1334
Kolmogorov-Smirnov	0.06
Cramer-Von Mises	0.0436
Anderson-Darling	0.0664

Source: author's calculations.

Note

¹ Let $z = u + vi$ be a complex number with u, v real numbers. The modulus of z is $\sqrt{u^2 + v^2}$.

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