Journal of *Risk and Financial Management*

*Communication*

# $\nu$-Generalized Hyperbolic Distributions

**Lev Klebanov** [1,2,†] **and Svetlozar T. Rachev** [3,*,†]

1 Department of Probability and Statistics, MFF, Charles University, 18675 Prague, Czech Republic
2 Massaryk Institute ČVUT, 18690 Prague, Czech Republic
3 Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409, USA
* Correspondence: zari.rachev@ttu.edu
† These authors contributed equally to this work.

**Abstract:** A new class of probability distributions closely connected to generalized hyperbolic distributions is introduced. It is better adapted for studying the distributions of sums of a random number of random variables. The properties of these distributions are studied. It seems that this class may be useful for modeling asset returns.

**Keywords:** asset returns modeling; hyperbolic distributions; sums of random variables; random number of summands

check for updates

**Citation:** Klebanov, Lev, and Svetlozar T. Rachev. 2023. $\nu$-Generalized Hyperbolic Distributions. *Journal of Risk and Financial Management* 16: 251. https://doi.org/10.3390/jrfm16040251

Academic Editor: Zbigniew Palmowski

Received: 2 February 2023
Revised: 4 April 2023
Accepted: 17 April 2023
Published: 20 April 2023



## 1. Introduction

The family of generalized hyperbolic (GH) distributions is well-known in probability. It was introduced by Barndorff-Nielsen, who studied it in the context of the physics of wind-blown sand Barndorff-Nielsen (1977). These distributions are given by their probability density functions, and their properties have been rather well-studied. In particular, GH distributions are infinitely divisible Barndorff-Nielsen and Halgreen (1977). The connection of GH distributions with other distributions is given in Paolella (2007). One of the more interesting applications of GH distributions is to the field of Finance Eberlein and Keller (1995); Gentle et al. (2012).

On the other hand, there is a large amount of literature on the financial applications of stable and geometric stable distributions in finance (see, for example, Gentle et al. (2012)). Geometric stable distributions were introduced in Klebanov et al. (1987) and are analogues of classical stable distributions for the case of sums of a random number of random variables, where the number of summands has the geometric distribution. The case of a general distribution of the number of summands was studied in Klebanov and Rachev (1996) (see also Klebanov et al. (2006)). Because the number of transactions on a financial market is random, it is natural to consider sums of a random number of random variables as an element of a model for the distribution of asset returns. Our aim here is to introduce a variant of the GH distributions, connected to sums of a random number of random variables. We will refer to these distributions as $\nu$-GH.

Let us give precise definitions.

**Definition 1.** *Let $X, X_1, \dots, X_n, \dots$ be a sequence of independent identically distributed (i.i.d.) random variables. Suppose that $\{\nu_p,\ p \in \Delta\}$ is a family of random variables, independent of the sequence above, taking positive integer values, and such that $\mathbb{E}\nu_p = 1/p$ for all $p \in \Delta$. We shall say that the random variable $X$ has $\nu$-strictly Gaussian (or $\nu$-strictly normal) distribution if and only if*

*(a)* $X \stackrel{d}{=} p^{1/2} \sum_{j=1}^{\nu_p} X_j$ *for all* $p \in \Delta$,
*and*
*(b) the random variable $X$ has a finite second moment:* $\mathbb{E}X^2 < \infty$.

**Definition 2.** *Suppose that the sequence of random variables $X, X_1, \ldots, X_n, \ldots$ and the family $\{\nu_p,\ p \in \Delta\}$ are the same as in Definition 1. We shall say that the random variable $X$ has a $\nu$-strictly stable distribution with index $\alpha \in (0,2)$ if $X \stackrel{d}{=} p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j$ for all $p \in \Delta$.*

Let $\mathcal{P}_p(z),\ p \in \Delta$ be a family of probability generating functions (p.g.f.) of random variables $\nu_p$. In Klebanov and Rachev (1996), it was shown that a $\nu$-strictly Gaussian distribution exists if and only if the semigroup $\mathfrak{P}$, generated by the family $\mathcal{P}_p(z),\ p \in \Delta$ with superposition operation, is commutative. In the case of the commutativity of $\mathfrak{P}$, the system of functional equations

$$\varphi(t) = \mathcal{P}_p(\varphi(pt)),\ \forall p \in \Delta,\ t > 0 \tag{1}$$

has a solution with the initial values

$$\varphi(0) = 1,\ \ \varphi'(0) = -1. \tag{2}$$

This solution is unique, and may be represented as the Laplace transform of a distribution function $\mathcal{A}$ defined on non-negative semi-axes:

$$\varphi(t) = \int_0^\infty \exp(-tx) d\mathcal{A}(x). \tag{3}$$

We call this function the standard solution of the Poincare equation. In this case, the characteristic function of a non-degenerate $\nu$-strictly Gaussian distribution has the form $f(t) = \varphi(at^2)$, where $a > 0$ (see Klebanov and Rachev (1996); Klebanov et al. (2006)).

The function $\varphi$ may also be used to define a one-to-one map from the set of all infinitely divisible distributions onto the set of all $\nu$-infinite stable distributions. Namely, the following theorem holds (see Klebanov and Rachev (1996); Klebanov et al. (2006)).

**Theorem 1.** *If $\varphi$ is a standard solution of the Poincare equation, then the function*

$$g(t) = \varphi(-\log(f(t))),\ \ t \in \mathbb{R}^1, \tag{4}$$

*is a $\nu$-infinitely divisible characteristic function if and only if the characteristic function $f(t)$ is infinitely divisible in the classical sense.*

Theorem 1 is the main tool for the definition of $\nu$-GH distributions. Let us also note that all previous definitions as well as Theorem 1 remain true for random vectors taking values in Euclidean space $\mathbb{R}^d$. Many examples of $\nu$-stable distributions are given in Klebanov et al. (2012).

## 2. $\nu$-GH Distrbutions

As was mentioned in the previous section, GH distributions are infinitely divisible. Therefore, we may use their characteristic functions instead of the function $f(t)$ in Theorem 1. The function $g(t)$ obtained in such a way is a natural analog of the GH characteristic function for the case of sums of a random number of random summands.

Let us give some examples.

**Example 1.** *Let $\{\nu_p,\ p \in (0,1)\}$ be a family of random variables with geometric distribution:*

$$\mathbb{P}\{\nu_p = k\} = p(1-p)^{k-1},\ k = 1, 2, \ldots$$

*It is known (see Klebanov et al. 1987, 2006) that in this case the standard solution of Poincare equation has the form*

$$\varphi(t) = \frac{1}{1+t}.$$

*For the one-dimensional case, the characteristic function of a GH distribution has the form*

$$f(t) = \frac{e^{it\mu}\left(\sqrt{\alpha^2 - \beta^2}\delta\right)^{\lambda} K_\lambda\left[\sqrt{\alpha^2 - (it+\beta)^2}\delta\right]}{\left(\sqrt{\alpha^2 - (it+\beta)^2}\delta\right)^{\lambda} K_\lambda\left[\sqrt{\alpha^2 - \beta^2}\delta\right]},$$

*where $K_\lambda$ is the modified Bessel function of the second kind. Substituting this into Equation (4), we find that the function*

$$g(t) = 1/\left(1 - \log[\frac{e^{it\mu}\left(\sqrt{\alpha^2 - \beta^2}\delta\right)^{\lambda} K_\lambda\left[\sqrt{\alpha^2 - (it+\beta)^2}\delta\right]}{\left(\sqrt{\alpha^2 - (it+\beta)^2}\delta\right)^{\lambda} K_\lambda\left[\sqrt{\alpha^2 - \beta^2}\delta\right]}]\right)$$

*is the characteristic function of a geometric GH distribution (or geo-GH distribution).*

As was noted above, it is possible to substitute the multivariate GH characteristic function $f(t)$ into (4) and obtain the characteristic function of a multivariate geo-GH distribution. We will not do this substitution explicitly.

If our GH distribution does not coincide with the Student's $t$-distribution, then its characteristic function is analytic in a strip containing the real line. This means that this distribution has exponential tails. It is clear that in this situation, the function $g(t)$ is analytic in a strip, too. Therefore, the tails of a geo-GH distribution are exponential, too. If our GH distribution coincides with the Student's $t$-distribution, it has only a finite number of moments. It is clear that the same is true for a geo-Student distribution with characteristic function $g(t)$.

**Example 2.** *Let us consider the following family of random variables $\{\nu_p,\ p \in (1/n^2,\ n = 1, 2, \dots)\}$ with probability generating function*

$$\mathcal{P}_p(z) = \frac{1}{T_{1/\sqrt{p}}}(1/z), \tag{5}$$

*where $T_n(x)$ is the Chebyshev polynomial of the first kind. In Klebanov et al. (2012), it was proven that (5) really defines a family of probability generation functions. The standard solution of the Poincare equation has the form (see, Klebanov et al. (2012))*

$$\varphi(t) = \frac{1}{\cosh(\sqrt{2t})}.$$

*Using this function, we obtain the corresponding representation for $g(t)$:*

$$g(t) =$$

$$\sec\left[\sqrt{2}\log^{1/2}\left[\frac{e^{it\mu}\left(\sqrt{\alpha^2 + (t - i\beta)^2}\delta\right)^{-\lambda}\left(\sqrt{(\alpha-\beta)(\alpha+\beta)}\delta\right)^{\lambda} K_\lambda\left[\sqrt{\alpha^2 + (t - i\beta)^2}\delta\right]}{K_\lambda\left[\sqrt{\alpha^2 - \beta^2}\delta\right]}\right]\right]$$

We call a distribution with this characteristic function a Chebyshev-GH distribution. The behavior of the tails of a Chebyshev-GH distribution is the same as that for a GH distribution.

**Author Contributions:** Conceptualization L.K. and S.T.R.; methodology, L.K. and S.T.R.; investigation, L.K. and S.T.R.; writing original draft preparation, L.K. and S.T.R.; writing review and editing, L.K. and S.T.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** The work was partially supported by grant GA ČR 19-28231X (Lev Klebanov).

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

Barndorff-Nielsen, Ole. 1977. Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 353: 401–9.

Barndorff-Nielsen, Ole, and Christian Halgreen. 1977. Infinite Divisibility of the Hyperbolic and Generalized Inverse Gaussian Distributions. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 38: 309–11. [CrossRef]

Eberlein, Ernst, and Ulrich Keller. 1995. Hyperbolic distributions in finance. *Bernoulli* 1: 281–99. [CrossRef]

Gentle, James E., Wolfgang Karl Härdle, and Yuichi Mori, eds. 2012. *Handbook of Computational Statistics*. Concepts and Methods. Second Revised and Updated Edition. Berlin: Springer.

Klebanov, Lev. B., and Svetlozar T. Rachev. 1996. Sums of a random number of random variables an their approximations with $\nu$-accompanying infinitely divisible laws. *Serdica Mathematical Journal* 22: 471–96.

Klebanov, Lev. B., G. M. Manija, and I. A. Melamed. 1987. *$\nu_p$-Strictly Stable Laws and Estimation of Their Parameters.* Lecture Notes in Mathematics Vol. 1233. Berlin: Springer, pp. 23–31.

Klebanov, Lev B., Tomasz J. Kozubowski, and Svetlozar T. Rachev. 2006. *Ill-Posed Problems in Probability and Stability of Random Sums*. New York: Nova Scientific Publishers.

Klebanov, Lev B., Ashot V. Kakosyan, S. T. Rachev, and Grigory Temnov. 2012. On a class of distributions stable under random summation. *Journal of Applied Probability* 49: 303–18. [CrossRef]

Paolella, Marc S. 2007. *Intermediate Probability, A Computational Approach*. Chichester: Wiley.