Article

An Analytic Approach for Pricing American Options with Regime Switching

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Abstract: This paper investigates the American option price in a two-state regime-switching model. The dynamics of underlying are driven by a Markov-modulated Geometric Wiener process. That means the interest rate, the appreciation rate, and the volatility of underlying rely on hidden states of the economy which can be interpreted in terms of Markov chains. By means of the homotopy analysis method, an explicit formula for pricing two-state regime-switching American options is presented.

Keywords: option pricing; free boundary problem; regime switching; homotopy analysis method

1. Introduction

Pricing financial derivative is an ative field of research in mathematical finance. The pioneering work of Black and Scholes (1973) and Merton (1973) laid the foundations of field and gave rise tto a branch of the research in option pricing theory. The most options traded in the financial markets are American-style options. An American option grants its holder with the right to exercise the option before or on its expiry date. For pricing American options, one needs to determine the option price as well as its optimal exercise boundary. In the Black–Scholes framework, there is an analytical solution for pricing finite-horizon American options, which was first presented by Zhu (2006). Prior to the work of Zhu (2006), McKean (1965) and Merton (1973) pointed out that the problem of pricing American option is equivalent to a free-boundary problem. There are many methods, including analytical and numerical approximations, to solve this kind of free boundary problem. For instance, analytical approximation works include Johnson (1983); MacMillan (1986); Omberg (1987); Barone-Adesi and Whaley (1987); Barone-Adesi and Elliott (1991); Carr (1998) and Bunch and Johnson (2000). While numerical approximation work include Brennan and Schwartz (1977); Geske and Shastri (1985); Geske and Johnson (1984); Carr and Faguet (1996), and Hon and Mao (1997). An alternative approach aparting from a free boundary problem is called the decomposition method, see Jacka (1991); Kim (1990); Carr et al. (1992). Using the decomposition method, the price of an American option can be written as the sum of its European counterpart and the early exercise premium. Carr et al. (1992) proposed a variant decomposition approach, in which the price of American option can be written as its intrinsic value plus its time value. The early exercise premium is expressed in an integral form, and it implicitly relies on the early exercise boundary. The only known explicit solution for the optimal exercise boundary was found by Zhu (2006) in the form of infinite series. There are many approximation methods in the literature. In particular, the asymptotic behaviour of the optimal exercise boundary near maturity has attracted lots of attention as a promising way to derive an explicit formula, see Alobaidi and Mallier (2001); Chevalier (2005); Evans et al. (2002). However, these short-maturity asymptotic approximations of the optimal exercise boundary are not sufficiently precise under realistic model parameters, see Mallier (2002). The other numerical schemes for calculating the early exercise premium include Huang et al. (1996) and Ju (1998).

Recently, there is active research in the applications of regime switching models followed by an unobservable Markov chain to a variety of financial problems. For an overview of unobservable Markov chains and their financial applications, we refer to Elliott et al. (1994) and Elliott and Kopp (1999). Some financial applications of regime switching include Elliott and van der Hoek (1997) for asset allocation, Elliott et al. (2001) for interest rate models, Buffington and Elliott (2002a, 2002b) for pricing European and American options, Elliott et al. (2003) for volatility estimation, Guo (2001a) for option pricing, Elliott et al. (2005) for option pricing via an Esscher transformation. In addition, Guo (2001b) investigated Russian options in the regime switching model. Guo and Shepp (2001) investigated the perpetual American lookback options. Guo and Zhang (2004) priced the perpetual American options with regime switching. Jobert and Rogers (2006) priced the finite maturity American put and barrier options with regime switching using a lattice approach. Huang (2011) formulated the regime-switching American problem as an optimal control problem and used discretization methods to solve it. Lu and Putri (2020) applied Laplace transform to regime-switch the American option and obtained a semi-analytical solution. Egami and Kevkhishvili (2020) reduced an optimal stopping problem with an arbitrary value function in a two-regime environment to a pair of optimal stopping problems without regime switching. So far, there is no analytical formula for the finite-horizon regime switching American option.

Regime switching models have been used in finance practice and have recently started gaining increasing popularity. One of the key issues is how to identify the states of economy in regime switching models. In this regard, two recent publications have discussed this issue (see He and Zhu (2018) and He and Zhu (2021), in addition to using real market data in calibration, so that a practical use of regime switching models is clearly demonstrated through convincing empirical evidence. Another example that has clearly demonstrated the use of regime switching models in finance practice is the two earlier papers that were published by Elliott et al. (2003) and Elliott and Royal (2007), who used filtering methods to estimate the state of the economy from historical stock market data. In these two publications, a log-normal dynamics with Markov-switching volatility and a regime switching variance Gamma dynamics were adopted in their regime switching models, respectively.

We will extend the work of Zhu (2006) to obtain an explicit formula for pricing finite-horizon regime switching American options by means of the homotopy analysis method (HAM). HAM was first proposed by Ortega and Rheinboldt (1970) and has been successfully applied to solve a variety of heat transfer problems, see Liao (1997); Liao and Zhu (1999); and fluid-flow problems, see Liao and Zhu (1996); Liao and Campo (2002). Zhu (2006) applied HAM to derive an explicit pricing formula for American options in the Black–Scholes–Merton framework. Gounden and O'Hara (2010) extended the work of Zhu to pricing an American-style Asian option of floating strike type in the Black–Scholes–Merton model. Leung (2013) applied HAM to obtain an explicit formula for lookback options under a stochastic volatility model. Chan and Zhu (2015) used HAM to derive an explicit formula for American-style convertible bonds with regime switching.

In this paper, we derive an explicit formula for pricing American options with regime switching. The paper is organized, as follows. Section 2 describes the model and formulates the problem. Section 3 presents the pricing of American options based on the HAM approach. The final section concludes.

2. The Model Formulation

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, where $\mathcal{P}$ is an actual probability measure. Let $\mathcal{T}$ be the time index set $[0, T]$ of the model. Let $\{W_t\}_{t \in \mathcal{T}}$ be a standard Wiener process on $(\Omega, \mathcal{F}, \mathcal{P})$. We consider a continuous-time Markov chain $\{X_t\}_{t \in \mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{X} := (x_1, x_2, \ldots, x_N)$. Without a loss of generality, we can identify the state space of $\{X_t\}_{t \in \mathcal{T}}$ to be a finite set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathcal{R}^{\mathcal{N}}$. We assume that $\{X_t\}_{t \in \mathcal{T}}$ and $\{W_t\}_{t \in \mathcal{T}}$ are independent.

Denote A for the generator $[a_{ij}]_{i,j=1,2,\ldots,N}$. Subsequently, according to Elliott et al. (1994), the semi-martingale representation theorem is in the form of $\{X_t\}_{t\in\mathcal{T}}$:

$$X_t = X_0 + \int_0^t AX_s ds + M_t . \tag{1}$$

Here, $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathcal{R}^N$-valued martingale increment process with respect to the filtration that is generated by $\{X_t\}_{t\in\mathcal{T}}$.

The interest rate $\{r(t,X_t)\}_{t\in\mathcal{T}}$ of the money market account is given by:

$$r_t := r(t,X_t) =< r, X_t > , \tag{2}$$

where $r := (r_1, r_2, \ldots, r_N)$ with $r_i > 0$ for each $i = 1, 2, \ldots, N$ and $< \cdot, \cdot >$ denotes the inner product in $\mathcal{R}^N$.

In this case, the dynamics of underlying $\{B_t\}_{t\in\mathcal{T}}$ for the money market account are given by:

$$dB_t = r_t B_t dt , \quad B_0 = 1 . \tag{3}$$

We suppose that the stock appreciation rate $\{\mu_t\}_{t\in\mathcal{T}}$ and the volatility $\{\sigma_t\}_{t\in\mathcal{T}}$ of S also rely on $\{X_t\}_{t\in\mathcal{T}}$ and they are given by:

$$\mu_t := \mu(t,X_t) =< \mu, X_t > , \quad \sigma_t := \sigma(t,X_t) =< \sigma, X_t > . \tag{4}$$

Here, $\mu := (\mu_1, \mu_2, \ldots, \mu_N)$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)$ with $\sigma_i > 0$ for each $i = 1, 2, \ldots, N$.

The dynamics of underlying $\{S_t\}_{t\in\mathcal{T}}$ are then given by the following Markov-modulated geometric Wiener process:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t , \quad S_0 = s . \tag{5}$$

Here, $W := \{W_t\}$ is a standard Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the filtration $\{\mathcal{F}_t^W\}$, which is the $\mathcal{P}$-augmentation of the natural filtration that is generated by W.

Free Boundary Problem

Let us consider an American put option with expiry T and exercise price K. In this subsection, we consider a special case with the number of regimes N, being 2 in order to simplify our discussion. We adopt the approach in Buffington and Elliott (2002b) to derive the results in this section. Given that $S_t = S$ and that $X_t = X$, the price of the American put option at time t is given by:

$$V(t,T,S,X) = \sup_{\tau \in [t,T]} E_{\mathcal{Q}} \left[\exp\left(-\int_t^T r_u du \right) (K - S)^+ \Big| S_t = S, X_t = X \right] , \tag{6}$$

where the supremum is taken over the set of stopping times τ taking values in $[t, T]$.

Let $\mathbf{V} := \mathbf{V}(t,T,S) = (V(t,T,S,e_1), V(t,T,S,e_2)) = (V_1, V_2)$. When $X_t = e_i$ ($i = 1, 2$), the continuation region is given by:

$$\mathcal{C}^i = \{(S,t) \in \mathcal{R}^+ \times [0,T] | V(t,T,S,e_i) > (K-S)^+\} , \tag{7}$$

and the stopping region is given by:

$$\mathcal{S}^i = \{(S,t) \in \mathcal{R}^+ \times [0,T] | V(t,T,S,e_i) = (K-S)^+\} . \tag{8}$$

Follow the lines of Buffington and Elliott (2002b), we write $\mathcal{C}_t^i$ for the t-section of $\mathcal{C}^i$, for each $i = 1, 2$ and $t \in [0,T]$. Let $S_{f_i}(t) = S_f(t, e_i)$ be the critical price of the American

put when $X_t = e_i$ ($i = 1, 2$); that is, when the state $X_t = e_i$ and the price of the underlying at time t falls below $S_{f_i}(t)$, it is rational for the holder of the American put to exercise the option at time t. Note that $S_{f_i}(T) = K$, for each $i = 1, 2$. Subsequently, by Elliott and Kopp (1999) and Buffington and Elliott (2002b),

$$\mathcal{C}_t^i = (S_{f_i}(t), \infty), \quad i = 1, 2 . \tag{9}$$

We shall consider the case $N = 2$.

Yi (2008) showed that, if $\sigma_1 > \sigma_2$ then $V_1 \geq V_2$ and $S_{f_1}(t) \leq S_{f_2}(t)$ for each $t \in [0, T]$ in a two-state regime switching model. When $S > S_{f_2}(t)$, (S, t) is in the common continuation region $\mathcal{C}^1 \cup \mathcal{C}^2$ and $\mathbf{V} = (V_1, V_2)$ satisfies the following pair of coupled P.D.E.s:

$$\frac{\partial V_1}{\partial t} + r_1 S \frac{\partial V_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} - r_1 V_1 + < \mathbf{V}, Ae_1 > = 0 ,$$

$$\frac{\partial V_2}{\partial t} + r_2 S \frac{\partial V_2}{\partial S} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} - r_2 V_2 + < \mathbf{V}, Ae_2 > = 0 . \tag{10}$$

When $S \leq S_{f_1}(t)$,

$$V_1 = V(t, T, S, e_1) = K - S . \tag{11}$$

When $S \leq S_{f_2}(t)$,

$$V_2 = V(t, T, S, e_2) = K - S . \tag{12}$$

On the optimal curve $S_{f_i}(t)$ ($i = 1, 2$), it is required that V_i satisfies a high-contact condition:

$$V_i(t, T, S_{f_i}(t)) = K - S_{f_i}(t) , \tag{13}$$

and a smooth-pasting condition:

$$\frac{\partial V_i}{\partial S}(t, T, S_{f_i}(t)) = -1 . \tag{14}$$

When S is in the transition region $\{(S, t) | S_{f_1}(t) \leq S \leq S_{f_2}(t)\}$, $V_2 = K - S$ and V_1 satisfies the following P.D.E.:

$$\frac{\partial V_1}{\partial t} + r_1 S \frac{\partial V_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} - r_1 V_1 + a_{11} V_1 - a_{11}(K - S) = 0 , \tag{15}$$

and boundary conditions

$$V_1(t, T, S_{f_1}(t)) = K - S_{f_1}(t) , \tag{16}$$

$$\frac{\partial V_1}{\partial S}(t, T, S_{f_1}(t)) = -1 . \tag{17}$$

$$V_1(T, T, S) = V(T, T, S, e_1) = \max\{K - S, 0\}. \tag{18}$$

Clearly, there are two free boundary functions, $S_{f_1}(t)$ and $S_{f_2}(t)$, and two option prices, $V_1(S, t)$ and $V_2(S, t)$, as listed in the above PDE systems. Which one of these is used to determine if an option should be exercised or continuously held has not been clearly stated in the literature. A flow chart could be useful for financial practitioners who wish to use our solutions to price American put options in a two-state regime switching setting.

Assuming that all of the model parameters, including σ_1 and σ_2, have been extracted from market data and assuming that one knows the current status of economy, i.e., we are either in State 1 or State 2, before one decides to use a regime-switching model to price an American option. Subsequently, the actual pricing exercise needs be divided into two stages, relying on the current status of economy when a pricing task is conducted. If we are in a higher status of economy, i.e., in State 2, using the $S_{f_2}^{(1)}$ obtained in Section 3.1 would be sufficient, because, in this case, once S drops below $S_{f_2}^{(1)}$ from the above, the put option should be exercised as a result of the fact that $S_{f_1}^{(1)}(t) \leq S_{f_2}^{(1)}(t)$. On the other hand, if we are in a lower status of economy, i.e., in State 1, an additional PDE system may need to be solved relying on the current underlying value. For the range of $S_{f_1}^{(1)}(t) \leq S \leq S_{f_2}^{(1)}(t)$, the solution that is shown in Section 3.1 can no longer be used and one needs to use the solution shown in Section 3.2, in which V_2 has already "landed" on the payoff function. This pricing procedure can be better demonstrated with the following flow chart, see Figure 1.

Note that we use superscripts in the flow chart to denote the free boundaries that were obtained in the corresponding stage. For simplicity, they will all be removed in the next section without any ambiguity, as we have denoted each stage by a separate subsection.
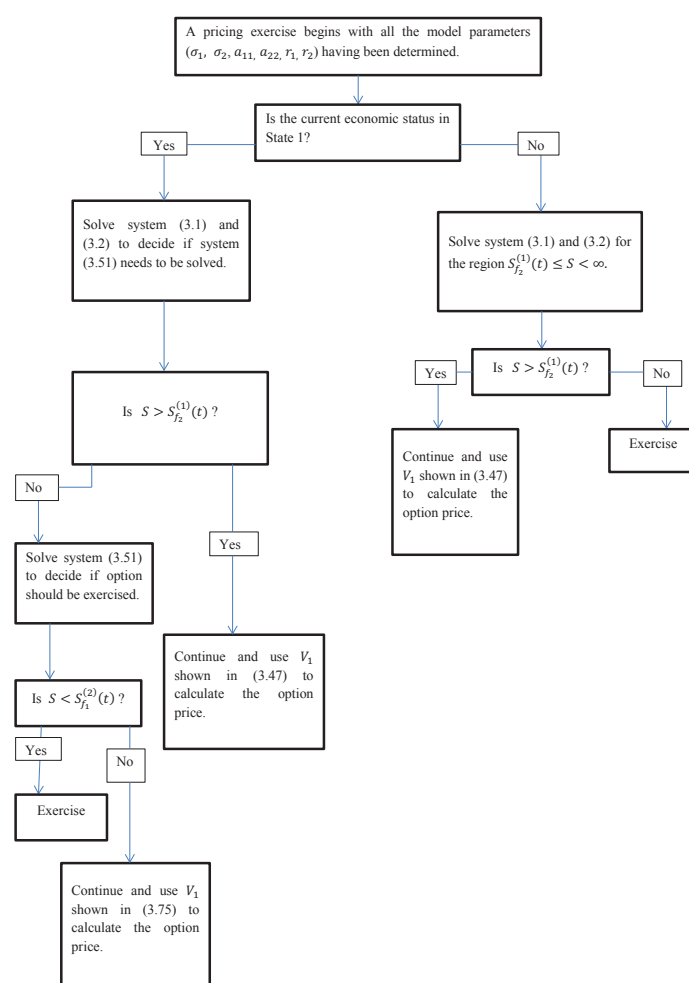

Figure 1. An illustrative flow chart for the usage of our solutions.

3. HAM Approach

It was noted earlier that American options can be formulated as free boundary problems. Closed-form solutions of free boundary problems are notoriously difficult to derive. However, by employing the homotopy analysis method, an exact and closed-form solution can be derived, in the form of infinite series. We have defined the common continuation region and the transition region in Section 2. Each problem in these regions is solved below.

3.1. The Common Continuation Region

Firstly, let us discuss the American put value function in the common continuation region where $S > S_{f_2}(t)$ and V_1, V_2 satisfy the equations:

$$
\begin{cases}
\frac{\partial V_1}{\partial t} + r_1 S \frac{\partial V_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} - r_1 V_1 + \langle \mathbf{V}, A e_1 \rangle = 0 \\
V_1(t, T, S_{f_1}(t)) = K - S_{f_1}(t) \\
\frac{\partial V_1}{\partial S}(t, T, S_{f_1}(t)) = -1 \\
V_1(T, T, S) = V(T, T, S, e_1) = \max\{K - S, 0\} \\
\lim_{S \to \infty} V_1(t, T, S) = 0
\end{cases}
\tag{19}
$$

$$
\begin{cases}
\frac{\partial V_2}{\partial t} + r_2 S \frac{\partial V_2}{\partial S} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} - r_2 V_2 + \langle \mathbf{V}, A e_2 \rangle = 0 \\
V_2(t, T, S_{f_2}(t)) = K - S_{f_2}(t) \\
\frac{\partial V_2}{\partial S}(t, T, S_{f_2}(t)) = -1 \\
V_2(T, T, S) = V(T, T, S, e_2) = \max\{K - S, 0\} \\
\lim_{S \to \infty} V_2(t, T, S) = 0
\end{cases}
\tag{20}
$$

Note that V_i and S_{f_i} are coupled. To solve these systems effectively, we shall make all variables dimensionless by introducing dimensionless variables

$$
V_i' = \frac{V_i}{K}, \quad S' = \frac{S}{K}, \quad \tau_i' = \frac{\sigma_i^2}{2}(T - t) \quad \text{for} \quad i = 1, 2.
$$

With all of the primes dropped from now on, the dimensionless systems read

$$
\begin{cases}
-\frac{\partial V_1}{\partial \tau_1} + \gamma_1 S \frac{\partial V_1}{\partial S} + S^2 \frac{\partial^2 V_1}{\partial S^2} - \gamma_1 V_1 = \lambda_1(V_2 - V_1) \\
V_1(\tau_1, S_{f_1}(\tau_1)) = 1 - S_{f_1}(\tau_1) \\
\frac{\partial V_1}{\partial S}(\tau_1, S_{f_1}(\tau_1)) = -1 \\
V_1(0, S) = V(0, S, e_1) = \max\{1 - S, 0\} \\
\lim_{S \to \infty} V_1(\tau_1, S) = 0
\end{cases}
\tag{21}
$$

$$
\begin{cases}
-\frac{\partial V_2}{\partial \tau_2} + \gamma_2 S \frac{\partial V_2}{\partial S} + S^2 \frac{\partial^2 V_2}{\partial S^2} - \gamma_2 V_2 = \lambda_2(V_1 - V_2) \\
V_2(\tau_2, S_{f_2}(\tau_2)) = 1 - S_{f_2}(\tau_2) \\
\frac{\partial V_2}{\partial S}(\tau_2, S_{f_2}(\tau_2)) = -1 \\
V_2(0, S) = V(0, S, e_2) = \max\{1 - S, 0\} \\
\lim_{S \to \infty} V_2(\tau_2, S) = 0
\end{cases}
\tag{22}
$$

Here, $\lambda_i = \frac{2a_{ii}}{\sigma_i^2}, i = 1, 2$ and $\gamma_i = \frac{2r_i}{\sigma_i^2}, i = 1, 2$, which can be viewed as an interest rate relative to the volatility of the underlying asset price.

Take the Laudau transformation

$$
x_i = \ln\left(\frac{S}{S_{f_i}(\tau_i)}\right)
$$

to change the moving boundary to fixed boundary conditions, thus yielding:

$$
\begin{cases}
\frac{\partial V_1}{\partial \tau_1} - (\gamma_1 - 1)\frac{\partial V_1}{\partial x_1} - \frac{\partial^2 V_1}{\partial x_1^2} + \gamma_1 V_1 = \lambda_1(V_1 - V_2) + \frac{1}{S_{f_1}(\tau_1)}\frac{dS_{f_1}(\tau_1)}{d\tau_1}\frac{\partial V_1}{\partial x_1} \\
V_1(\tau_1, 0) = 1 - S_{f_1}(\tau_1) \\
\frac{\partial V_1}{\partial x_1}(\tau_1, 0) = -S_{f_1}(\tau_1) \\
V_1(0, x_1) = V(0, x_1, e_1) = 0 \\
\lim_{x_1 \to \infty} V_1(\tau_1, x_1) = 0
\end{cases}
\tag{23}
$$

$$
\begin{cases}
\frac{\partial V_2}{\partial \tau_2} - (\gamma_2 - 1)\frac{\partial V_2}{\partial x_2} - \frac{\partial^2 V_2}{\partial x_2^2} + \gamma_2 V_2 = \lambda_2(V_2 - V_1) + \frac{1}{S_{f_2}(\tau_2)}\frac{dS_{f_2}(\tau_2)}{d\tau_2}\frac{\partial V_2}{\partial x_2} \\
V_2(\tau_2, 0) = 1 - S_{f_2}(\tau_2) \\
\frac{\partial V_2}{\partial x_2}(\tau_2, 0) = -S_{f_2}(\tau_2) \\
V_2(0, x_2) = V(0, x_2, e_2) = 0 \\
\lim_{x_2 \to \infty} V_2(\tau_2, x_2) = 0
\end{cases}
\tag{24}
$$

Now, to solve (23) and (24) by the homotopy analysis method, we introduce an embedding parameter $p \in [0, 1]$ and construct unknown functions $\bar{V}_i(\tau_i, x_i, p)$ and $\bar{S}_{f_i}(\tau_i, p), i = 1, 2$ that satisfy the following differential systems:

$$
\begin{cases}
(1-p)\mathcal{L}_1[\bar{V}_1(\tau_1, x_1, p) - \bar{V}_1^0(\tau_1, x_1)] = -p\left\{\mathcal{A}_1[\bar{V}_1(\tau_1, x_1, p), \bar{S}_{f_1}(\tau_1, p)]\right\} \\
\bar{V}_1(0, x_1, p) = (1-p)\bar{V}_1^0(0, x_1) \\
\bar{V}_1(\tau_1, 0, p) + \bar{S}_{f_1}(\tau_1, p) = 1 \\
\frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, 0, p) + \bar{S}_{f_1}(\tau_1, p) = (1-p)\left[1 + \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - \bar{V}_1^0(\tau_1, 0)\right] \\
\lim_{x_1 \to \infty} \bar{V}_1(\tau_1, x_1, p) = 0
\end{cases}
\tag{25}
$$

$$
\begin{cases}
(1-p)\mathcal{L}_2[\bar{V}_2(\tau_2, x_2, p) - \bar{V}_2^0(\tau_2, x_2)] = -p\left\{\mathcal{A}_2[\bar{V}_2(\tau_2, x_2, p), \bar{S}_{f_2}(\tau_2, p)]\right\} \\
\bar{V}_2(0, x_2, p) = (1-p)\bar{V}_2^0(0, x_2) \\
\bar{V}_2(\tau_2, 0, p) + \bar{S}_{f_2}(\tau_2, p) = 1 \\
\frac{\partial \bar{V}_2}{\partial x_2}(\tau_2, 0, p) + \bar{S}_{f_2}(\tau_2, p) = (1-p)\left[1 + \frac{\partial \bar{V}_2^0}{\partial x_2}(\tau_2, 0) - \bar{V}_2^0(\tau_2, 0)\right] \\
\lim_{x_2 \to \infty} \bar{V}_2(\tau_2, x_2, p) = 0
\end{cases}
\tag{26}
$$

Here, $\mathcal{L}_i$ is a differential operator that is defined as

$$
\mathcal{L}_i = \frac{\partial}{\partial \tau_i} - (\gamma_i - 1)\frac{\partial}{\partial x_i} - \frac{\partial^2}{\partial x_i^2} + \gamma_i,
\tag{27}
$$

and $\mathcal{A}_i$ is a functional defined as

$$
\begin{aligned}
&\mathcal{A}_1[\bar{V}_1(\tau_1, x_1, p), \bar{S}_{f_1}(\tau_1, p)] \\
&= \mathcal{L}_1(\bar{V}_1) - \lambda_1(\bar{V}_1 - \bar{V}_2) - \frac{1}{\bar{S}_{f_1}(\tau_1, p)}\frac{d\bar{S}_{f_1}(\tau_1, p)}{d\tau_1}\frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, x_1, p)
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&\mathcal{A}_2[\bar{V}_2(\tau_2, x_2, p), \bar{S}_{f_2}(\tau_2, p)] \\
&= \mathcal{L}_2(\bar{V}_2) - \lambda_2(\bar{V}_2 - \bar{V}_1) - \frac{1}{\bar{S}_{f_2}(\tau_2, p)}\frac{d\bar{S}_{f_2}(\tau_2, p)}{d\tau_2}\frac{\partial \bar{V}_2}{\partial x_2}(\tau_2, x_2, p)
\end{aligned}
\tag{29}
$$

Equations (25) and (26) are known as the zeroth order deformation equations and they determine the continuous deformation of $\bar{V}_i(\tau_i, x_i, p)$. To construct Equations (25) and (26),

we want to find a continuous map, such that the solution we are seeking becomes the result of a continuous deformation from an initial and known function. Here, the initial state of the continuous deformation is at $p = 0$. With $p = 0$, we have, from Equations (25) and (26),

$$\begin{cases} \mathcal{L}_1[\bar{V}_1(\tau_1, x_1, 0)] = \mathcal{L}_1[\bar{V}_1^0(\tau_1, x_1)] \\ \bar{V}_1(0, x_1, 0) = \bar{V}_1^0(0, x_1) \\ \bar{V}_1(\tau_1, 0, 0) + \bar{S}_{f_1}(\tau_1, 0) = 1 \\ \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, 0, 0) + \bar{S}_{f_1}(\tau_1, 0) = \left[1 + \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - \bar{V}_1^0(\tau_1, 0)\right] \\ \lim_{x_1 \to \infty} \bar{V}_1(\tau_1, x_1, 0) = 0 \end{cases} \tag{30}$$

$$\begin{cases} \mathcal{L}_2[\bar{V}_2(\tau_2, x_2, 0)] = \mathcal{L}_2[\bar{V}_2^0(\tau_2, x_2)] \\ \bar{V}_2(0, x_2, 0) = \bar{V}_2^0(0, x_2) \\ \bar{V}_2(\tau_2, 0, 0) + \bar{S}_{f_2}(\tau_2, 0) = 1 \\ \frac{\partial \bar{V}_2}{\partial x_2}(\tau_2, 0, 0) + \bar{S}_{f_2}(\tau_2, 0) = \left[1 + \frac{\partial \bar{V}_2^0}{\partial x_2}(\tau_2, 0) - \bar{V}_2^0(\tau_2, 0)\right] \\ \lim_{x_2 \to \infty} \bar{V}_2(\tau_2, x_2, 0) = 0 \end{cases} \tag{31}$$

Clearly, the solutions of the differential systems (30) and (31) are in forms of

$$\begin{cases} \bar{V}_1(\tau_1, x_1, 0) = \bar{V}_1^0(\tau_1, x_1) \\ \bar{S}_{f_1}(\tau_1, 0) = 1 - \bar{V}_1^0(\tau_1, 0) = \bar{S}^0(\tau_1) \end{cases} \tag{32}$$

$$\begin{cases} \bar{V}_2(\tau_2, x_2, 0) = \bar{V}_2^0(\tau_2, x_2) \\ \bar{S}_{f_2}(\tau_2, 0) = 1 - \bar{V}_2^0(\tau_2, 0) = \bar{S}^0(\tau_2) \end{cases} \tag{33}$$

so long as the initial guess $\bar{V}_i^0(\tau_i, x_i)$ satisfies the condition

$$\lim_{x_i \to \infty} \bar{V}_i^0(\tau_i, x_i) = 0. \tag{34}$$

Note that, other than this condition, theoretically, there are no other requirements for the initial guess $\bar{V}_i^0(\tau_i, x_i)$ to satisfy. However, a faster convergence of the series is expected if we select a function that has already satisfied an additional condition $\mathcal{L}_i[\bar{V}^0(\tau_i, x_i)] = 0$.

With $p = 1$, we have

$$\begin{cases} \mathcal{L}_1[\bar{V}_1(\tau_1, x_1, 1)] = \lambda_1(\bar{V}_1(\tau_1, x_1, 1) - \bar{V}_2(\tau_2, x_2, 1)) + \frac{1}{\bar{S}_{f_1}(\tau_1, 1)} \frac{d\bar{S}_{f_1}(\tau_1, 1)}{d\tau_1} \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, x_1, 1) \\ \bar{V}_1(0, x_1, 1) = 0 \\ \bar{V}_1(\tau_1, 0, 1) = 1 - \bar{S}_{f_1}(\tau_1, 1) \\ \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, 0, 1) = -\bar{S}_{f_1}(\tau_1, 1) \\ \lim_{x_1 \to \infty} \bar{V}_1(\tau_1, x_1, 1) = 0 \end{cases}$$

$$\begin{cases} \mathcal{L}_2[\bar{V}_2(\tau_2, x_2, 1)] = \lambda_2(\bar{V}_2(\tau_2, x_2, 1) - \bar{V}_1(\tau_1, x_1, 1)) + \frac{1}{\bar{S}_{f_2}(\tau_2, 1)} \frac{d\bar{S}_{f_2}(\tau_2, 1)}{d\tau_2} \frac{\partial \bar{V}_2}{\partial x_2}(\tau_2, x_2, 1) \\ \bar{V}_2(0, x_2, 1) = 0 \\ \bar{V}_2(\tau_2, 0, 1) = 1 - \bar{S}_{f_2}(\tau_2, 1) \\ \frac{\partial \bar{V}_2}{\partial x_2}(\tau_2, 0, 1) = -\bar{S}_{f_2}(\tau_2, 1) \\ \lim_{x_2 \to \infty} \bar{V}_2(\tau_2, x_2, 1) = 0 \end{cases}$$

Comparing with (23) and (24), it is easy to see that the solution we seek is nothing but

$$\begin{cases} V_1(\tau_1, x_1) = \bar{V}_1(\tau_1, x_1, 1) \\ S_{f_1}(\tau_1) = \bar{S}_{f_1}(\tau_1, 1) \end{cases} \tag{35}$$

$$\begin{cases} V_2(\tau_2, x_2) = \bar{V}_2(\tau_2, x_2, 1) \\ S_{f_2}(\tau_2) = \bar{S}_{f_2}(\tau_2, 1) \end{cases} \tag{36}$$

In order to determine the values of $\bar{V}_i(\tau_i, x_i, 1)$ and $\bar{S}_{f_i}(\tau_i, 1)$, $i = 1, 2$, we can apply a Taylor's series expansion of p to the functions $\bar{V}_i(\tau_i, x_i, p)$ and $\bar{S}_{f_i}(\tau_i, p)$

$$\begin{cases} \bar{V}_1(\tau_1, x_1, p) = \sum_{n=0}^{\infty} \frac{\bar{V}_1^n(\tau_1, x_1)}{n!} p^n \\ \bar{S}_{f_1}(\tau_1, p) = \sum_{n=0}^{\infty} \frac{\bar{S}^n(\tau_1)}{n!} p^n \end{cases} \tag{37}$$

$$\begin{cases} \bar{V}_2(\tau_2, x_2, p) = \sum_{n=0}^{\infty} \frac{\bar{V}_2^n(\tau_2, x_2)}{n!} p^n \\ \bar{S}_{f_2}(\tau_2, p) = \sum_{n=0}^{\infty} \frac{\bar{S}^n(\tau_2)}{n!} p^n \end{cases} \tag{38}$$

where

$$\begin{cases} \bar{V}_1^n(\tau_1, x_1) = \frac{\partial^n}{\partial p^n} \bar{V}_1(\tau_1, x_1, p)|_{p=0} \\ \bar{S}^n(\tau_1) = \frac{\partial^n}{\partial p^n} \bar{S}_{f_1}(\tau_1, p)|_{p=0} \end{cases} \tag{39}$$

$$\begin{cases} \bar{V}_2^n(\tau_2, x_2) = \frac{\partial^n}{\partial p^n} \bar{V}_2(\tau_2, x_2, p)|_{p=0} \\ \bar{S}^n(\tau_2) = \frac{\partial^n}{\partial p^n} \bar{S}_{f_2}(\tau_2, p)|_{p=0} \end{cases} \tag{40}$$

To determine the coefficients in the above Taylor series expansion, we derive a set of governing equations by differentiating the zeroth-order (25) and (26), with respect to p and then set $p = 0$. After this process, we get

$$\begin{cases} \mathcal{L}_1[\bar{V}_1^1(\tau_1, x_1)] = -\mathcal{L}_1[\bar{V}_1^0(\tau_1, x_1)] + \tilde{\mathcal{A}}_1(\tau_1, x_1, 0) \\ \bar{V}_1^1(0, x_1) = -\bar{V}_1^0(0, x_1) \\ \bar{V}_1^1(\tau_1, 0) + \bar{S}^1(\tau_1) = 0 \\ \frac{\partial \bar{V}_1^1}{\partial x_1}(\tau_1, 0) + \bar{S}^1(\tau_1) = \bar{V}_1^0(\tau_1, 0) - \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - 1 \\ \lim_{x_1 \to \infty} \bar{V}_1^1(\tau_1, x_1) = 0 \end{cases} \tag{41}$$

and

$$\begin{cases} \mathcal{L}_1[\bar{V}_1^n(\tau_1, x_1)] = n \frac{\partial^{n-1} \tilde{\mathcal{A}}_1}{\partial p^{n-1}}|_{p=0} \\ \bar{V}_1^n(0, x_1) = 0 \\ \bar{V}_1^n(\tau_1, 0) + \bar{S}^n(\tau_1) = 0, \quad \text{if} \quad n \geq 2 \\ \frac{\partial \bar{V}_1^n}{\partial x_1}(\tau_1, 0) + \bar{S}^n(\tau_1) = 0 \\ \lim_{x_1 \to \infty} \bar{V}_1^n(\tau_1, x_1) = 0 \end{cases} \tag{42}$$

Additionally,

$$\begin{cases} \mathcal{L}_2[\bar{V}_2^1(\tau_2, x_2)] = -\mathcal{L}_2[\bar{V}_2^0(\tau_2, x_2)] + \tilde{\mathcal{A}}_2(\tau_2, x_2, 0) \\ \bar{V}_2^1(0, x_2) = -\bar{V}_2^0(0, x_2) \\ \bar{V}_2^1(\tau_2, 0) + \bar{S}^1(\tau_2) = 0 \\ \frac{\partial \bar{V}_2^1}{\partial x_2}(\tau_2, 0) + \bar{S}^1(\tau_2) = \bar{V}_2^0(\tau_2, 0) - \frac{\partial \bar{V}_2^0}{\partial x_2}(\tau_2, 0) - 1 \\ \lim_{x_2 \to \infty} \bar{V}_2^1(\tau_2, x_2) = 0 \end{cases} \tag{43}$$

and

$$
\begin{cases}
\mathcal{L}_2[\bar{V}_2^n(\tau_2, x_2)] = n\frac{\partial^{n-1}\tilde{\mathcal{A}}_2}{\partial p^{n-1}}\big|_{p=0} \\
\bar{V}_2^n(0, x_2) = 0 \\
\bar{V}_2^n(\tau_2, 0) + \bar{S}^n(\tau_2) = 0, \quad \text{if} \quad n \geq 2 \\
\frac{\partial \bar{V}_2^n}{\partial x_2}(\tau_2, 0) + \bar{S}^n(\tau_2) = 0 \\
\lim_{x_2 \to \infty} \bar{V}_2^n(\tau_2, x_2) = 0
\end{cases}
\tag{44}
$$

Here, $\tilde{\mathcal{A}}_i = \mathcal{L}_i[\bar{V}_i] - \mathcal{A}_i, i = 1, 2$. After eliminating $\bar{S}^n(\tau_i)$ from the two boundary conditions at $x_i = 0$ in the systems, then the systems can be written in a general form,

$$
\begin{cases}
\mathcal{L}_1[\bar{V}_1^n(\tau_1, x_1)] = f_1^n(\tau_1, x_1) \\
\bar{V}_1^n(0, x_1) = \psi_1^n(x_1) \\
\frac{\partial \bar{V}_1^n}{\partial x_1}(\tau_1, 0) - \bar{V}_1^n(\tau_1, 0) = \phi_1^n(\tau_1) \\
\bar{V}_1^n(\tau_1, \infty) = 0
\end{cases}
\tag{45}
$$

where

$$
f_1^n(\tau_1, x_1) = \begin{cases}
-\mathcal{L}_1[\bar{V}_1^0(\tau_1, x_1)] + \tilde{\mathcal{A}}_1(\tau_1, x_1, 0), & \text{if} \quad n = 1 \\
n\frac{\partial^{n-1}\tilde{\mathcal{A}}_1}{\partial p^{n-1}}\big|_{p=0}, & \text{if} \quad n \geq 2
\end{cases}
\tag{46}
$$

$$
\psi_1^n(x_1) = \begin{cases}
-\bar{V}_1^0(0, x_1), & \text{if} \quad n = 1 \\
0, & \text{if} \quad n \geq 2
\end{cases}
\tag{47}
$$

$$
\phi_1^n(\tau_1) = \begin{cases}
\bar{V}_1^0(\tau_1, 0) - \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - 1, & \text{if} \quad n = 1 \\
0, & \text{if} \quad n \geq 2
\end{cases}
\tag{48}
$$

Additionally,

$$
\begin{cases}
\mathcal{L}_2[\bar{V}_2^n(\tau_2, x_2)] = f_2^n(\tau_2, x_2) \\
\bar{V}_2^n(0, x_2) = \psi_2^n(x_2) \\
\frac{\partial \bar{V}_2^n}{\partial x_2}(\tau_2, 0) - \bar{V}_2^n(\tau_2, 0) = \phi_2^n(\tau_2) \\
\bar{V}_2^n(\tau_2, \infty) = 0
\end{cases}
\tag{49}
$$

where

$$
f_2^n(\tau_2, x_2) = \begin{cases}
-\mathcal{L}_2[\bar{V}_2^0(\tau_2, x_2)] + \tilde{\mathcal{A}}_2(\tau_2, x_2, 0), & \text{if} \quad n = 1 \\
n\frac{\partial^{n-1}\tilde{\mathcal{A}}_2}{\partial p^{n-1}}\big|_{p=0}, & \text{if} \quad n \geq 2
\end{cases}
\tag{50}
$$

$$
\psi_2^n(x_2) = \begin{cases}
-\bar{V}_2^0(0, x_2), & \text{if} \quad n = 1 \\
0, & \text{if} \quad n \geq 2
\end{cases}
\tag{51}
$$

$$
\phi_2^n(\tau_2) = \begin{cases}
\bar{V}_2^0(\tau_2, 0) - \frac{\partial \bar{V}_2^0}{\partial x_2}(\tau_2, 0) - 1, & \text{if} \quad n = 1 \\
0, & \text{if} \quad n \geq 2
\end{cases}
\tag{52}
$$

Now, let us apply the following transformation

$$
\bar{V}_i^n(\tau_i, x_i) = \exp\left\{ -(\gamma_i - 1)x_i/2 - (\gamma_i + 1)^2 \tau_i/4 \right\} \bar{U}_i^n(\tau_i, x_i),
\tag{53}
$$

so that (45) and (49) can be expressed in the form of a standard nonhomogeneous diffusion system:

$$
\begin{cases}
\frac{\partial \bar{U}_1^n}{\partial \tau_1} - \frac{\partial^2 \bar{U}_1^n}{\partial x_1^2} = \exp\left\{(\gamma_1 - 1)x_1/2 + (\gamma_1 + 1)^2 \tau_1/4\right\} f_1^n(\tau_1, x_1) \\
\bar{U}_1^n(0, x_1) = \exp\{(\gamma_1 - 1)x_1/2\} \psi_1^n(x_1) \\
\frac{\partial \bar{U}_1^n}{\partial x_1}(\tau_1, 0) - \frac{1}{2}(\gamma_1 + 1)\bar{U}_1^n(\tau_1, 0) = \exp\{(\gamma_1 + 1)^2 \tau_1/4\} \phi_1^n(\tau_1) \\
\lim_{x_1 \to \infty} \bar{U}_1^n(\tau_1, x_1) = 0
\end{cases}
\tag{54}
$$

$$
\begin{cases}
\frac{\partial \bar{U}_2^n}{\partial \tau_2} - \frac{\partial^2 \bar{U}_2^n}{\partial x_2^2} = \exp\left\{(\gamma_2 - 1)x_2/2 + (\gamma_2 + 1)^2 \tau_2/4\right\} f_2^n(\tau_2, x_2) \\
\bar{U}_2^n(0, x_2) = \exp\{(\gamma_2 - 1)x_2/2\} \psi_2^n(x_2) \\
\frac{\partial \bar{U}_2^n}{\partial x_2}(\tau_2, 0) - \frac{1}{2}(\gamma_2 + 1)\bar{U}_2^n(\tau_2, 0) = \exp\{(\gamma_2 + 1)^2 \tau_2/4\} \phi_2^n(\tau_2) \\
\lim_{x_2 \to \infty} \bar{U}_2^n(\tau_2, x_2) = 0
\end{cases}
\tag{55}
$$

An explicit solution of Equations (54) and (55) at each order (i.e., with each n) can now be found by splitting the linear problem into three problems, a method that is commonly used in solving linear PDEs. For further details of the solution procedure we refer to Carslaw and Jaeger (1959). The final explicit solution at each order is then:

$$
\begin{aligned}
\bar{U}_1^n(\tau_1, x_1) &= \int_0^\infty \exp\{(\gamma_1 - 1)\xi_1/2\} \psi_1^n(\xi_1) G(x_1, \xi_1, \tau_1) d\xi_1 \\
&\quad - \int_0^{\tau_1} \exp\{(\gamma_1 + 1)^2 \tau_1/4\} \phi_1^n(\eta_1) G(x_1, 0, \tau_1 - \eta_1) d\eta_1 \\
&= \int_0^{\tau_1} \int_0^\infty \exp\left\{(\gamma_1 - 1)x_1/2 + (\gamma_1 + 1)^2 \tau_1/4\right\} f_1^n(\eta_1, \xi_1) G(x_1, \xi_1, \tau_1 - \eta_1) d\xi_1 d\eta_1,
\end{aligned}
\tag{56}
$$

where

$$
\begin{aligned}
G(x_1, \xi_1, \tau_1) &= \frac{1}{2\sqrt{\pi \tau_1}} \left\{ \exp\left[\frac{-(x_1 - \xi_1)^2}{4\tau_1}\right] + \exp\left[\frac{-(x_1 + \xi_1)^2}{4\tau_1}\right] \right. \\
&\quad - (\gamma_1 + 1)\sqrt{\pi \tau_1} \exp\left[\frac{(\gamma_1 + 1)^2}{4} + \frac{(\gamma_1 + 1)(x_1 + \xi_1)}{2}\right] \\
&\quad \left. \times \operatorname{erfc}\left(\frac{x_1 + \xi_1}{2\sqrt{\tau_1}} + \frac{1}{2}(\gamma_1 + 1)\sqrt{\tau_1}\right)\right] \right\}
\end{aligned}
\tag{57}
$$

Additionally,

$$
\begin{aligned}
\bar{U}_2^n(\tau_2, x_2) &= \int_0^\infty \exp\{(\gamma_2 - 1)\xi_2/2\} \psi_2^n(\xi_2) G(x_2, \xi_2, \tau_2) d\xi_2 \\
&\quad - \int_0^{\tau_2} \exp\{(\gamma_2 + 1)^2 \tau_2/4\} \phi_2^n(\eta_2) G(x_2, 0, \tau_2 - \eta_2) d\eta_2 \\
&= \int_0^{\tau_2} \int_0^\infty \exp\left\{(\gamma_2 - 1)x_2/2 + (\gamma_2 + 1)^2 \tau_2/4\right\} f_2^n(\eta_2, \xi_2) G(x_2, \xi_2, \tau_2 - \eta_2) d\xi_2 d\eta_2,
\end{aligned}
\tag{58}
$$

where

$$
\begin{aligned}
G(x_2, \xi_2, \tau_2) &= \frac{1}{2\sqrt{\pi \tau_2}} \left\{ \exp\left[\frac{-(x_2 - \xi_2)^2}{4\tau_2}\right] + \exp\left[\frac{-(x_2 + \xi_2)^2}{4\tau_2}\right] \right. \\
&\quad - (\gamma_2 + 1)\sqrt{\pi \tau_2} \exp\left[\frac{(\gamma_2 + 1)^2}{4} + \frac{(\gamma_2 + 1)(x_2 + \xi_2)}{2}\right] \\
&\quad \left. \times \operatorname{erfc}\left(\frac{x_2 + \xi_2}{2\sqrt{\tau_2}} + \frac{1}{2}(\gamma_2 + 1)\sqrt{\tau_2}\right)\right] \right\}
\end{aligned}
\tag{59}
$$

and erfc(x) denotes the complementary error function.

For the choice of an initial guess $V_i^0(\tau_i, x_i)$, $i = 1, 2$, any continuous functions satisfying the limiting condition can be used, we select the corresponding European option value as the initial guess with three apparent advantages given as discussed in Zhu (2006):

1. the boundary condition as $x_i \to \infty$ is automatically satisfied;
2. $\mathcal{L}_i[\bar{V}_i^0(\tau_i, x_i)]$ will become 0; which we expect will foster a faster convergence of the series and it also help to simply $f_i^1(\tau_i, x_i)$ as the first term of $f_i^1(\tau_i, x_i)$ will vanish; and,
3. also, as $\bar{V}_i^0$ satisfies the condition, $\bar{V}_i^0(0, x_i) = 0$ so $\psi_i^1(x_i) = 0$ and as $\psi_i^n(x_i) = 0$ for all $n \in \mathbb{N} \setminus \{1\}$, the integral containing $\psi_i^n(x_i)$ is eliminated.

Zhu et al. (2012) provides the analytic solution of a European put option with a two-state regime switching:

$$
\begin{aligned}
&V_i^0(t, S) \\
&= Ke^{-r_i(T-t)} + \frac{1}{4\pi\sqrt{2}}\sqrt{SK}e^{-\frac{1}{2}(r_i+a_{21}+a_{12}+\frac{\sigma_1^2+\sigma_2^2}{8})(T-t)} \int_0^\infty \frac{(-1)^{i-1}2\hat{f}_1(\rho)(a_{21}+a_{12})}{M(\rho)(\rho^4+\frac{1}{16})(\sigma_1^2-\sigma_2^2)} \\
&\times \left\{ e^{X_i(\rho)}\left[(2\rho^2-\frac{1}{2})\sin(\hat{f}_2(\rho)+\theta(\rho)-Y_i(\rho)) - (2\rho^2+\frac{1}{2})\cos(\hat{f}_2(\rho)+\theta(\rho)-Y_i(\rho))\right] \right. \\
&\left. - e^{-X_i(\rho)}\left[(2\rho^2-\frac{1}{2})\sin(\hat{f}_2(\rho)+\theta(\rho)+Y_i(\rho)) - (2\rho^2+\frac{1}{2})\cos(\hat{f}_2(\rho)+\theta(\rho)+Y_i(\rho))\right] \right\} \\
&+ \frac{2\hat{f}_1(\rho)}{M(\rho)}\left\{ e^{X_i(\rho)}\left[\sin(\hat{f}_2(\rho)+\theta(\rho)-Y_i(\rho)) + \cos(\hat{f}_2(\rho)+\theta(\rho)-Y_i(\rho))\right] \right. \\
&\left. - e^{-X_i(\rho)}\left[\sin(\hat{f}_2(\rho)+\theta(\rho)+Y_i(\rho)) + \cos(\hat{f}_2(\rho)+\theta(\rho)+Y_i(\rho))\right] \right\} \\
&+ \frac{\hat{f}_1(\rho)}{\rho^4+\frac{1}{16}}\left\{ e^{X_i(\rho)}\left[(2\rho^2-\frac{1}{2})\sin(\hat{f}_2(\rho)-Y_i(\rho)) - (2\rho^2+\frac{1}{2})\cos(\hat{f}_2(\rho)-Y_i(\rho))\right] \right. \\
&\left. + e^{-X_i(\rho)}\left[(2\rho^2-\frac{1}{2})\sin(\hat{f}_2(\rho)+Y_i(\rho)) - (2\rho^2+\frac{1}{2})\cos(\hat{f}_2(\rho)+Y_i(\rho))\right] \right\} d\rho,
\end{aligned}
$$

(60)

for $i = 1, 2$, where

$$
\tau = \frac{\sigma_1^2-\sigma_2^2}{4}(T-t), \quad \alpha = \frac{2(a_{12}-a_{21})}{\sigma_1^2-\sigma_2^2}, \quad \mu^2 = \frac{4a_{12}a_{21}}{(\sigma_1^2-\sigma_2^2)^2},
$$

$$
M(\rho) = \left\{ [(\frac{1}{4}+\alpha)^2 - \rho^4 + \mu^2]^2 + 4\rho^4(\frac{1}{4}+\alpha)^2 \right\}^{\frac{1}{4}},
$$

$$
\theta(\rho) = \frac{1}{2}\tan^{-1}\left[\frac{2\rho^2(\frac{1}{4}+\alpha)}{(\frac{1}{4}+\alpha)^2 - \rho^4 + \mu^2}\right],
$$

$$
X_i(\rho) = (-1)^{i-1}M(\rho)\tau\cos\theta(\rho), \quad Y_i(\rho) = (-1)^{i-1}M(\rho)\tau\sin\theta(\rho)
$$

and

$$
\hat{f}_1(\rho) = e^{-\frac{\rho}{\sqrt{2}}|\ln(\frac{S}{K})+r_1(T-t)|}, \quad \hat{f}_2(\rho) = \frac{\rho^2}{4}(\sigma_1^2+\sigma_2^2)(T-t) - \frac{\rho}{\sqrt{2}}|\ln(\frac{S}{K}) + r_1(T-t)|.
$$

The vanishing of $\psi_i^n(x_i)$ allows us to write the solution $\bar{U}_i^n(\tau_i, x_i)$ in a simple and closed form:

$$
\begin{aligned}
\bar{U}_1^n(\tau_1, x_1) &= -\frac{2}{\sqrt{\pi}} e^{(\gamma_1+1)^2 \tau_1/4} \int_0^\infty e^{-(\gamma_1+1)\eta_1/2} \int_{(x_1+\eta_1)/2\sqrt{\tau_1}}^\infty \phi_1^n\left(\tau_1 - \frac{(x_1+\eta_1)^2}{4\xi_1^2}\right) \\
&\times e^{-[(\gamma_1+1)(x_1+\eta_1)/4\xi_1]^2 - \xi_1^2} d\xi_1 d\eta_1 + \int_0^{\tau_1} \left\{ \frac{e^{(\gamma_1+1)^2\eta_1/4}}{\sqrt{\pi}} \left[e^{(\gamma_1-1)x_1/2} \right. \right. \\
&\times \int_{-x_1/2\sqrt{\tau_1-\eta_1}}^{x_1/2\sqrt{\tau_1-\eta_1}} f_1^n(\eta_1, 2\sqrt{\tau_1-\eta_1}\xi_1 + x_1) e^{(\gamma_1-1)\sqrt{\tau_1-\eta_1}\xi_1 - \xi_1^2} d\xi_1 \\
&+ \int_{x_1/2\sqrt{\tau_1-\eta_1}}^\infty \left[e^{(\gamma_1-1)x_1/2} f_1^n(\eta_1, 2\sqrt{\tau_1-\eta_1}\xi_1 + x_1) \right. \\
&+ \left. e^{-(\gamma_1-1)x_1/2} f_1^n(\eta_1, 2\sqrt{\tau_1-\eta_1}\xi_1 - x_1) \right] \times e^{(\gamma_1-1)\sqrt{\tau_1-\eta_1}\xi_1 - \xi_1^2} d\xi_1 \bigg] \\
&- (\gamma_1+1)\sqrt{\tau_1-\eta_1} e^{-(\gamma_1-1)x_1/2 + (\gamma_1+1)^2\tau_1/4} \int_{x_1/2\sqrt{\tau_1-\eta_1}}^\infty f_1^n(\eta_1, 2\sqrt{\tau_1-\eta_1}\xi_1 - x_1) \\
&\times e^{2\gamma_1\sqrt{\tau_1-\eta_1}\xi_1} \operatorname{erfc}\left(\xi_1 + \frac{(\gamma_1+1)}{2}\sqrt{\tau_1-\eta_1}\right) d\xi_1 \bigg\} d\eta_1,
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
\bar{U}_2^n(\tau_2, x_2) &= -\frac{2}{\sqrt{\pi}} e^{(\gamma_2+1)^2 \tau_2/4} \int_0^\infty e^{-(\gamma_2+1)\eta_2/2} \int_{(x_2+\eta_2)/2\sqrt{\tau_2}}^\infty \phi_2^n\left(\tau_2 - \frac{(x_2+\eta_2)^2}{4\xi_2^2}\right) \\
&\times e^{-[(\gamma_2+1)(x_2+\eta_2)/4\xi_2]^2 - \xi_2^2} d\xi_2 d\eta_2 + \int_0^{\tau_2} \left\{ \frac{e^{(\gamma_2+1)^2\eta_2/4}}{\sqrt{\pi}} \left[e^{(\gamma_2-1)x_2/2} \right. \right. \\
&\times \int_{-x_2/2\sqrt{\tau_2-\eta_2}}^{x_2/2\sqrt{\tau_2-\eta_2}} f_2^n(\eta_2, 2\sqrt{\tau_2-\eta_2}\xi_2 + x_2) e^{(\gamma_2-1)\sqrt{\tau_2-\eta_2}\xi_2 - \xi_2^2} d\xi_2 \\
&+ \int_{x_2/2\sqrt{\tau_2-\eta_2}}^\infty \left[e^{(\gamma_2-1)x_2/2} f_2^n(\eta_2, 2\sqrt{\tau_2-\eta_2}\xi_2 + x_2) \right. \\
&+ \left. e^{-(\gamma_2-1)x_2/2} f_2^n(\eta_2, 2\sqrt{\tau_2-\eta_2}\xi_2 - x_2) \right] \times e^{(\gamma_2-1)\sqrt{\tau_2-\eta_2}\xi_2 - \xi_2^2} d\xi_2 \bigg] \\
&- (\gamma_2+1)\sqrt{\tau_2-\eta_2} e^{-(\gamma_2-1)x_2/2 + (\gamma_2+1)^2\tau_2/4} \int_{x_2/2\sqrt{\tau_2-\eta_2}}^\infty f_2^n(\eta_2, 2\sqrt{\tau_2-\eta_2}\xi_2 - x_2) \\
&\times e^{2\gamma_2\sqrt{\tau_2-\eta_2}\xi_2} \operatorname{erfc}\left(\xi_2 + \frac{(\gamma_2+1)}{2}\sqrt{\tau_2-\eta_2}\right) d\xi_2 \bigg\} d\eta_2.
\end{aligned}
\tag{62}
$$

Once $\bar{V}_i^n(\tau_i, x_i)$, $i = 1, 2$ are found at each order by substituting Equations (61) and (62) back into Equation (53), $\bar{S}^n(\tau_i)$ can easily be found from the third equation of Equations (41)–(44), i.e.,

$$
\bar{S}^n(\tau_1) = -\bar{V}_1^n(\tau_1, 0),
\tag{63}
$$

$$
\bar{S}^n(\tau_2) = -\bar{V}_2^n(\tau_2, 0).
\tag{64}
$$

Upon finding the coefficients $\bar{V}_i^n(\tau_i, x_i)$ and $\bar{S}^n(\tau_i)$ from (43), (61)–(64), the final solution can be written in terms of a series of infinitely many terms, as

$$
\begin{cases}
V_1(\tau_1, x_1) = \bar{V}_1(\tau_1, x_1, 1) = \sum_{n=0}^\infty \frac{\bar{V}_1^n(\tau_1, x_1)}{n!} \\
S_{f_1}(\tau_1) = \bar{S}_{f_1}(\tau_1, 1) = \sum_{n=0}^\infty \frac{\bar{S}^n(\tau_1)}{n!}
\end{cases}
\tag{65}
$$

$$\begin{cases} V_2(\tau_2, x_2) = \bar{V}_2(\tau_2, x_2, 1) = \sum_{n=0}^{\infty} \frac{\bar{V}_2^n(\tau_2, x_2)}{n!} \\ S_{f_2}(\tau_2) = \bar{S}_{f_2}(\tau_2, 1) = \sum_{n=0}^{\infty} \frac{\bar{S}^n(\tau_2)}{n!} \end{cases} \qquad (66)$$

We also derive the exact, analytic, and closed-form formula for the optimal exercise boundaries $S_{f_i}(\tau_i), i = 1, 2$ in the form:

$$\begin{aligned} S_{f_1}(\tau_1) = {} & \frac{2}{\sqrt{\pi}} e^{-(\gamma_1 - 1)x_1/2 - (\gamma_1 + 1)^2 \tau_1/4} \sum_{n=0}^{\infty} \frac{1}{n!} \Big\{ e^{(\gamma_1 + 1)^2 \tau_1/4} \\ & \times \int_0^{\infty} e^{-(\gamma_1 + 1)\eta_1/2} \int_{\eta_1/2\sqrt{\tau_1}}^{\infty} \phi_1^n \left(\tau_1 - \frac{\eta_1^2}{4\xi_1^2} \right) \\ & \times e^{-[(\gamma_1 + 1)\eta_1/4\xi_1]^2 - \xi_1^2} d\xi_1 d\eta_1 \\ & + \int_0^{\tau_1} \Big[e^{(\gamma_1 + 1)^2 \eta_1/4} \int_0^{\infty} f_1^n(\eta_1, 2\sqrt{\tau_1 - \eta_1}\xi_1) e^{(\gamma_1 - 1)\sqrt{\tau_1 - \eta_1}\xi_1 - \xi_1^2} d\xi_1 \\ & - \frac{\sqrt{\pi}}{2}(\gamma_1 + 1)\sqrt{\tau_1 - \eta_1} e^{(\gamma_1 + 1)^2 \tau_1/4} \int_0^{\infty} f_1^n(\eta_1, 2\sqrt{\tau_1 - \eta_1}\xi_1) \\ & \times e^{2\gamma_1\sqrt{\tau_1 - \eta_1}\xi_1} \operatorname{erfc}\left(\xi_1 + \frac{(\gamma_1 + 1)}{2}\sqrt{\tau_1 - \eta_1} \right) d\xi_1 \Big] d\eta_1 \Big\}, \end{aligned}$$

$$(67)$$

and

$$\begin{aligned} S_{f_2}(\tau_2) = {} & \frac{2}{\sqrt{\pi}} e^{-(\gamma_2 - 1)x_2/2 - (\gamma_2 + 1)^2 \tau_2/4} \sum_{n=0}^{\infty} \frac{1}{n!} \Big\{ e^{(\gamma_2 + 1)^2 \tau_2/4} \\ & \times \int_0^{\infty} e^{-(\gamma_2 + 1)\eta_2/2} \int_{\eta_2/2\sqrt{\tau_2}}^{\infty} \phi_2^n \left(\tau_2 - \frac{\eta_2^2}{4\xi_2^2} \right) \\ & \times e^{-[(\gamma_2 + 1)\eta_2/4\xi_2]^2 - \xi_2^2} d\xi_2 d\eta_2 \\ & + \int_0^{\tau_2} \Big[e^{(\gamma_2 + 1)^2 \eta_2/4} \int_0^{\infty} f_2^n(\eta_2, 2\sqrt{\tau_2 - \eta_2}\xi_2) e^{(\gamma_2 - 1)\sqrt{\tau_2 - \eta_2}\xi_2 - \xi_2^2} d\xi_2 \\ & - \frac{\sqrt{\pi}}{2}(\gamma_2 + 1)\sqrt{\tau_2 - \eta_2} e^{(\gamma_2 + 1)^2 \tau_2/4} \int_0^{\infty} f_2^n(\eta_2, 2\sqrt{\tau_2 - \eta_2}\xi_2) \\ & \times e^{2\gamma_2\sqrt{\tau_2 - \eta_2}\xi_2} \operatorname{erfc}\left(\xi_2 + \frac{(\gamma_2 + 1)}{2}\sqrt{\tau_2 - \eta_2} \right) d\xi_2 \Big] d\eta_2 \Big\}. \end{aligned}$$

$$(68)$$

3.2. Transition Region

Let us consider the transition region between the two stopping curves:

$$\Gamma : \{ (S, t) : S_{f_1}(t) \leq S \leq S_{f_2}(t) \}$$

In this region $V_2 = V(t, T, S, e_2) = K - S$ and V_1 satisfies the Black–Scholes–Merton equation

$$\frac{\partial V_1}{\partial t} + r_1 S \frac{\partial V_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} - r_1 V_1 + a_{11} V_1 - a_{11}(K - S) = 0, \qquad (69)$$

with a terminal condition

$$V_1(T, T, S) = V(T, T, S, e_1) = \max\{K - S, 0\}.$$

Additionally, continuity on $S_{f_1}(t)$ gives:

$$V_1(t, T, S) = K - S_{f_1}(t)$$

and smoothness on $S_{f_1}(t)$ gives:

$$\frac{\partial V_1}{\partial S}(t, T, S_{f_1}(t)) = -1.$$

(70)

To solve this system effectively, we shall make all of the variables dimensionless by introducing dimensionless variables

$$V_1' = \frac{V_1}{K}, \quad S' = \frac{S}{K}, \quad \tau_1' = \frac{\sigma_1^2}{2}(T - t)$$

and also make the Laudau transformation

$$x_1 = \ln\left(\frac{S'}{S_{f_1}(\tau_1)}\right).$$

With all of the primes dropped from now on, the dimensionless systems with fixed boundary conditions read

$$\begin{cases} \frac{\partial V_1}{\partial \tau_1} - (\gamma_1 - 1)\frac{\partial V_1}{\partial x_1} - \frac{\partial^2 V_1}{\partial x_1^2} + \gamma_1 V_1 = \lambda_1(K - e^{x_1}S_{f_1}(\tau_1) - V_1) + \frac{1}{S_{f_1}(\tau_1)}\frac{dS_{f_1}(\tau_1)}{d\tau_1}\frac{\partial V_1}{\partial x_1} \\ V_1(\tau_1, 0) = 1 - S_{f_1}(\tau_1) \\ \frac{\partial V_1}{\partial x_1}(\tau_1, 0) = -S_{f_1}(\tau_1) \\ V_1(0, x_1) = 0 \\ \lim_{x_1 \to \infty} V_1(\tau_1, x_1) = 0 \end{cases}$$

(71)

We will use the homotopy analysis method to solve this nonlinear system of equations. We introduce an embedding parameter $p \in [0, 1]$ and construct unknown functions $\bar{V}_1(\tau_1, x_1, p)$ and $\bar{S}_{f_1}(\tau_1, p)$. The zeroth-order deformation is given by

$$\begin{cases} (1 - p)\mathcal{L}_1[\bar{V}_1(\tau_1, x_1, p) - \bar{V}_1^0(\tau_1, x_1)] = -p\left\{\hat{\mathcal{A}}_1[\bar{V}_1(\tau_1, x_1, p), \bar{S}_{f_1}(\tau_1, p)]\right\} \\ \bar{V}_1(0, x_1, p) = (1 - p)\bar{V}_1^0(0, x_1) \\ \bar{V}_1(\tau_1, 0, p) + \bar{S}_{f_1}(\tau_1, p) = 1 \\ \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, 0, p) + \bar{S}_{f_1}(\tau_1, p) = (1 - p)\left[1 + \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - \bar{V}_1^0(\tau_1, 0)\right] \\ \lim_{x_1 \to \infty} \bar{V}_1(\tau_1, x_1, p) = 0 \end{cases}$$

(72)

Here, $\mathcal{L}_1$ is a differential operator that is defined as

$$\mathcal{L}_1 = \frac{\partial}{\partial \tau_1} - (\gamma_1 - 1)\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_1^2} + \gamma_1,$$

(73)

and $\hat{\mathcal{A}}_1$ is a functional defined as

$$\hat{\mathcal{A}}_1[\bar{V}_1(\tau_1, x_1, p), \bar{S}_{f_1}(\tau_1, p)]$$
$$= \mathcal{L}_1(\bar{V}_1) - \lambda_1(\bar{V}_1 + e^{x_1}\bar{S}_{f_1}(\tau_1, p) - K) - \frac{1}{\bar{S}_{f_1}(\tau_1, p)} \frac{d\bar{S}_{f_1}(\tau_1, p)}{d\tau_1} \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, x_1, p).$$

(74)

With $p = 1$, we have

$$\begin{cases} \mathcal{L}_1(\bar{V}_1) - \lambda_1(\bar{V}_1 + e^{x_1}\bar{S}_{f_1}(\tau_1, 1) - K) - \frac{1}{\bar{S}_{f_1}(\tau_1, 1)} \frac{d\bar{S}_{f_1}(\tau_1, 1)}{d\tau_1} \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, x_1, 1) = 0 \\ \bar{V}_1(0, x_1, 1) = 0 \\ \bar{V}_1(\tau_1, 0, 1) = 1 - \bar{S}_{f_1}(\tau_1, 1) \\ \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, 0, 1) = -\bar{S}_{f_1}(\tau_1, 1) \\ \lim_{x_1 \to \infty} \bar{V}_1(\tau_1, x_1, 1) = 0 \end{cases}$$

(75)

When comparing with (71), it is easy to see that the solution we seek is nothing but

$$\begin{cases} V_1(\tau_1, x_1) = \bar{V}_1(\tau_1, x_1, 1) \\ S_{f_1}(\tau_1) = \bar{S}_{f_1}(\tau_1, 1) \end{cases}$$

(76)

If $p = 0$, then we have

$$\begin{cases} \mathcal{L}_1[\bar{V}_1(\tau_1, x_1, 0)] = \mathcal{L}_1[\bar{V}_1^0(\tau_1, x_1)] \\ \bar{V}_1(0, x_1, 0) = \bar{V}_1^0(0, x_1) \\ \bar{V}_1(\tau_1, 0, 0) + \bar{S}_{f_1}(\tau_1, 0) = 1 \\ \frac{\partial \bar{V}_1}{\partial x_1}(\tau_1, 0, 0) + \bar{S}_{f_1}(\tau_1, 0) = \left[1 + \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - \bar{V}_1^0(\tau_1, 0)\right] \\ \lim_{x_1 \to \infty} \bar{V}_1(\tau_1, x_1, 0) = 0 \end{cases}$$

(77)

Clearly, the solution of the differential system (77) is

$$\begin{cases} \bar{V}_1(\tau_1, x_1, 0) = \bar{V}_1^0(\tau_1, x_1) \\ \bar{S}_{f_1}(\tau_1, 0) = 1 - \bar{V}_1^0(\tau_1, 0) = \bar{S}^0(\tau_1) \end{cases}$$

(78)

so long as the initial guess $\bar{V}_1^0(\tau_1, x_1)$ satisfies the condition

$$\lim_{x_1 \to \infty} \bar{V}_1^0(\tau_1, x_1) = 0.$$

(79)

The initial guess $\bar{V}_1^0(\tau_1, x_1)$ is given in Equation (60). To determine the values of $\bar{V}_1(\tau_1, x_1, 1)$ and $\bar{S}_{f_1}(\tau_1, 1)$, we can apply a Taylor's series expansion of p to the functions $\bar{V}_1(\tau_1, x_1, p)$ and $\bar{S}_{f_1}(\tau_1, p)$

$$\begin{cases} \bar{V}_1(\tau_1, x_1, p) = \sum_{n=0}^{\infty} \frac{\bar{V}_1^n(\tau_1, x_1)}{n!} p^n \\ \bar{S}_{f_1}(\tau_1, p) = \sum_{n=0}^{\infty} \frac{\bar{S}^n(\tau_1)}{n!} p^n \end{cases}$$

(80)

where

$$\begin{cases} \bar{V}_1^n(\tau_1, x_1) = \frac{\partial^n}{\partial p^n} \bar{V}_1(\tau_1, x_1, p)|_{p=0} \\ \bar{S}^n(\tau_1) = \frac{\partial^n}{\partial p^n} \bar{S}_{f_1}(\tau_1, p)|_{p=0} \end{cases}$$

(81)

In order to determine the coefficients in the above Taylor series expansion, we derive a set of governing equations by differentiating the zeroth-order (72), with respect to p and then setting $p = 0$. After this process, we obtain

$$\begin{cases} \mathcal{L}_1[\bar{V}_1^1(\tau_1, x_1)] = -\mathcal{L}_1[\bar{V}_1^0(\tau_1, x_1)] + \bar{\mathcal{A}}_1(\tau_1, x_1, 0) \\ \bar{V}_1^1(0, x_1) = -\bar{V}_1^0(0, x_1) \\ \bar{V}_1^1(\tau_1, 0) + \bar{S}^1(\tau_1) = 0 \\ \frac{\partial \bar{V}_1^1}{\partial x_1}(\tau_1, 0) + \bar{S}^1(\tau_1) = \bar{V}_1^0(\tau_1, 0) - \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - 1 \\ \lim_{x_1 \to \infty} \bar{V}_1^1(\tau_1, x_1) = 0 \end{cases} \quad (82)$$

and

$$\begin{cases} \mathcal{L}_1[\bar{V}_1^n(\tau_1, x_1)] = n \frac{\partial^{n-1} \bar{\mathcal{A}}_1}{\partial p^{n-1}}\big|_{p=0} \\ \bar{V}_1^n(0, x_1) = 0 \\ \bar{V}_1^n(\tau_1, 0) + \bar{S}^n(\tau_1) = 0, \quad \text{if} \quad n \geq 2 \\ \frac{\partial \bar{V}_1^n}{\partial x_1}(\tau_1, 0) + \bar{S}^n(\tau_1) = 0 \\ \lim_{x_1 \to \infty} \bar{V}_1^n(\tau_1, x_1) = 0 \end{cases} \quad (83)$$

Here, $\bar{\mathcal{A}}_1 = \mathcal{L}[\bar{V}_1] - \hat{\mathcal{A}}_1$. After eliminating $\bar{S}^n(\tau_1)$ from the two boundary conditions at $x_1 = 0$ in the systems, then the systems can be written in a general form,

$$\begin{cases} \mathcal{L}_1[\bar{V}_1^n(\tau_1, x_1)] = \tilde{f}_1^n(\tau_1, x_1) \\ \bar{V}_1^n(0, x_1) = \psi_1^n(x_1) \\ \frac{\partial \bar{V}_1^n}{\partial x_1}(\tau_1, 0) - \bar{V}_1^n(\tau_1, 0) = \phi_1^n(\tau_1) \\ \bar{V}_1^n(\tau_1, \infty) = 0 \end{cases} \quad (84)$$

where

$$\tilde{f}_1^n(\tau_1, x_1) = \begin{cases} -\mathcal{L}_1[\bar{V}_1^0(\tau_1, x_1)] + \bar{\mathcal{A}}_1(\tau_1, x_1, 0), & \text{if} \quad n = 1 \\ n \frac{\partial^{n-1} \bar{\mathcal{A}}_1}{\partial p^{n-1}}\big|_{p=0}, & \text{if} \quad n \geq 2 \end{cases} \quad (85)$$

$$\psi_1^n(x_1) = \begin{cases} -\bar{V}_1^0(0, x_1), & \text{if} \quad n = 1 \\ 0, & \text{if} \quad n \geq 2 \end{cases} \quad (86)$$

$$\phi_1^n(\tau_1) = \begin{cases} \bar{V}_1^0(\tau_1, 0) - \frac{\partial \bar{V}_1^0}{\partial x_1}(\tau_1, 0) - 1, & \text{if} \quad n = 1 \\ 0, & \text{if} \quad n \geq 2 \end{cases} \quad (87)$$

Now, we apply the following transformation

$$\bar{V}_1^n(\tau_1, x_1) = \exp\left\{ -(\gamma_1 - 1)x_1/2 - (\gamma_1 + 1)^2 \tau_1/4 \right\} \bar{U}_1^n(\tau_1, x_1), \quad (88)$$

so that (84) can be written in the form of a standard nonhomogeneous diffusion system:

$$\begin{cases} \frac{\partial \bar{U}_1^n}{\partial \tau_1} - \frac{\partial^2 \bar{U}_1^n}{\partial x_1^2} = \exp\left\{ (\gamma_1 - 1)x_1/2 + (\gamma_1 + 1)^2 \tau_1/4 \right\} \tilde{f}_1^n(\tau_1, x_1) \\ \bar{U}_1^n(0, x_1) = \exp\{(\gamma_1 - 1)x_1/2\} \psi_1^n(x_1) \\ \frac{\partial \bar{U}_1^n}{\partial x_1}(\tau_1, 0) - \frac{1}{2}(\gamma_1 + 1)\bar{U}_1^n(\tau_1, 0) = \exp\{(\gamma_1 + 1)^2 \tau_1/4\} \phi_1^n(\tau_1) \\ \lim_{x_1 \to \infty} \bar{U}_1^n(\tau_1, x_1) = 0 \end{cases} \quad (89)$$

The final explicit solution at each order is then:

$$
\begin{aligned}
\bar{U}_1^n(\tau_1, x_1) \;=\;& \int_0^\infty \exp\{(\gamma_1 - 1)\xi_1/2\}\psi_1^n(\xi_1)G(x_1, \xi_1, \tau_1)d\xi_1 \\
&- \int_0^{\tau_1} \exp\{(\gamma_1 + 1)^2\tau_1/4\}\phi_1^n(\eta_1)G(x_1, 0, \tau_1 - \eta_1)d\eta_1 \\
\;=\;& \int_0^{\tau_1}\int_0^\infty \exp\left\{(\gamma_1 - 1)x_1/2 + (\gamma_1 + 1)^2\tau_1/4\right\}\tilde{f}_1^n(\eta_1, \xi_1)G(x_1, \xi_1, \tau_1 - \eta_1)d\xi_1 d\eta_1,
\end{aligned}
\tag{90}
$$

where

$$
\begin{aligned}
G(x_1, \xi_1, \tau_1) \;=\;& \frac{1}{2\sqrt{\pi\tau_1}}\left\{\exp\left[\frac{-(x_1 - \xi_1)^2}{4\tau_1}\right] + \exp\left[\frac{-(x_1 + \xi_1)^2}{4\tau_1}\right]\right. \\
&- (\gamma_1 + 1)\sqrt{\pi\tau_1}\exp\left[\frac{(\gamma_1 + 1)^2}{4} + \frac{(\gamma_1 + 1)(x_1 + \xi_1)}{2}\right] \\
&\left.\times \;\; \mathrm{erfc}\left(\frac{x_1 + \xi_1}{2\sqrt{\tau_1}} + \frac{1}{2}(\gamma_1 + 1)\sqrt{\tau_1}\right)\right]\right\}.
\end{aligned}
\tag{91}
$$

Once $\bar{V}_1^n(\tau_1, x_1)$, are found at each order by substituting Equations (90) back into Equation (88), $\bar{S}^n(\tau_1)$ can easily be found from the third equation of Equations (82) and (83), i.e.,

$$
\bar{S}^n(\tau_1) = -\bar{V}_1^n(\tau_1, 0).
\tag{92}
$$

Upon finding the coefficients $\bar{V}_1^n(\tau_1, x_1)$ and $\bar{S}^n(\tau_1)$ from (88), (90), and (92), the final solution can be written in terms of a series of infinitely many terms as

$$
\begin{cases}
V_1(\tau_1, x_1) = \bar{V}_1(\tau_1, x_1, 1) = \sum_{n=0}^\infty \frac{\bar{V}_1^n(\tau_1, x_1)}{n!}, \\
S_{f_1}(\tau_1) = \bar{S}_{f_1}(\tau_1, 1) = \sum_{n=0}^\infty \frac{\bar{S}^n(\tau_1)}{n!}.
\end{cases}
\tag{93}
$$

4. Conclusions

By means of the homotopy analysis method, an exact analytical pricing formula for the American option under a two-state regime-switching economy is derived. It is shown that the optimal exercise boundary, which is the key difficulty in the valuation of American options, can be explicitly expressed in the form of infinite series. In addition, we provide a flow chart to show when a particular formula should be used for the easiness of using our solutions to price American put options under a two-state regime switching model.

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