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# Nonparametric Estimation of the Ruin Probability in the Classical Compound Poisson Risk Model

**Yuan Gao [1,†], Lingju Chen [2,†], Jiancheng Jiang [3,*,†] and Honglong You [4,†]**

1 School of Mathematical Sciences, Qufu Normal University, Shandong 273165, China; gyyql413@163.com
2 College of Mathematics and Data Science, Minjiang University, Fuzhou 350108, China; 1517@mju.edu.cn
3 Department of Mathematics and Statistics & School of Data Science, University of North Carolina, Charlotte, NC 28223, USA
4 School of Statistics, Qufu Normal University , Shandong 273165, China; yougaoyou815@qfnu.edu.cn
* Correspondence: jjiang1@uncc.edu
† These authors contributed equally to this work.

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**Abstract:** In this paper we study estimating ruin probability which is an important problem in insurance. Our work is developed upon the existing nonparametric estimation method for the ruin probability in the classical risk model, which employs the Fourier transform but requires smoothing on the density of the sizes of claims. We propose a nonparametric estimation approach which does not involve smoothing and thus is free of the bandwidth choice. Compared with the Fourier-transformation-based estimators, our estimators have simpler forms and thus are easier to calculate. We establish asymptotic distributions of our estimators, which allows us to consistently estimate the asymptotic variances of our estimators with the plug-in principle and enables interval estimates of the ruin probability.

**Keywords:** classical risk model; nonparametric estimation; ruin probability

## 1. Introduction

It is well-known that estimating the ruin probability is an important problem in economics and insurance. There are many works on this topic in the literature, and they can be divided into two groups, parametric and nonparametric. See (Baumgartner and Gatto 2010); (Croux and Veraverbeke 1990), (Feng et al. 2020); (Frees 1986); (Mnatsakanov et al. 2008); (Pitts 1994); (Politis 2003); (You et al. 2020) and (Zhang et al. 2014), among others. In this paper we propose a nonparametric approach to estimating the ruin probability. Unlike the existing nonparametric methods, our procedure does not involve smoothing. To illustrate our idea, we begin with the classical risk model

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \tag{1}$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the constant premium rate, and $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$, and $X_1, X_2, X_3, ...$ are independent and identically distributed random variables with density function $f$ and distribution function $F$ supported on $(0, \infty)$. The corresponding process $N = \{N_t, t \geq 0\}$ is called the claim number process, and $X = \{X_i, i = 1, 2, 3...\}$ are the sizes of claims. Further, $N$ and $\{X_i\}_{i=1,2,...}$ are independent. Suppose the mean and variance of claim sizes are finite, i.e. $\mu = \int_0^\infty xF(dx) < \infty, \sigma^2 = \int_0^\infty x^2 F(dx) - \mu^2 < \infty$.

Let $\tau = \inf\{t > 0 : U_t < 0\}$, which is the first time when $U_t < 0$. Then the ruin probability is

$$\Psi(u) \quad = \quad \mathbf{P}\left(\tau < \infty | U_0 = u\right).$$

In order to prevent ruin happening with probability one, we assume that the risk process has a relative safety loading (Gerber 1979) and (Bower et al. 1997), that is,

$$\theta = c/(\lambda\mu) - 1 > 0. \tag{2}$$

As illustrated in (Asmussen and Albrecher 2010) and (Landriault and Willmot 2008), there are many methods for estimating the ruin probability. For example, the integro-differential equation technique, renewal theory, Laplace transform, martingale theory, and so on. Since explicit formulas for ruin probabilities are usually not available, some authors roughly estimate some bounds of ruin probabilities (Cai and Wu 1997) and (Dickson 1994). However, these methods assume knowing the inter-claim distribution and the claim size distribution.

Naturally, some authors seek to nonparametrically estimate the ruin probability which does not require knowing the distribution of the claim size. Examples include but not limited to the sample reuse method in (Frees 1986), which was extended by (Croux and Veraverbeke 1990) by virtue of the Pollaczeck–Khinchin formula and U-statistics, the Laplace transform method in (Mnatsakanov et al. 2008), and the inverse Fourier method in (Zhang et al. 2014). These approaches assumed that the distribution of the individual claim size $F$ is unknown, but the Poisson intensity $\lambda$ is known. Other authors contributed their works with both $F$ and $\lambda$ estimated from the data, for example, (Pitts 1994); (Politis 2003); (Baumgartner and Gatto 2010) and (You et al. 2020).

In this paper, we make our contribution to statistical inference for $\Psi(u)$. Our approach is based on the Fourier inversion in (Zhang et al. 2014) and (You et al. 2020). Using the Fourier inversion and kernel density, (Zhang et al. 2014) obtained an explicit expression of $\Psi(u)$, with which they proposed a nonparametric estimator of the ruin probability based on the plug-in principle (see Section 2 for details).

The problem of estimating the probability of ruin when the Poisson intensity $\lambda$ are unknown was not studied in (Zhang et al. 2014). Based on (Zhang et al. 2014) and (Politis 2003), (You et al. 2020) established asymptotically normal distributions of the nonparametric estimators of $\Psi(u)$ with known $\lambda$ and with unknown $\lambda$. Both (Zhang et al. 2014) and (You et al. 2020) employed the kernel density estimation of $f(x)$, which requires choice of the smoothing parameter. Specially, given a sample of claims $X = \{X_i, i = 1, 2, ..., n\}$, (You et al. 2020) and (Zhang et al. 2014) estimated the claim size density $f$ by the kernel density estimator

$$\hat{f}_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} K(\frac{x - X_j}{h_n}),$$

where $K$ is a kernel function and $h_n$ is the bandwidth. Instead of estimating $f(x)$, in this paper we estimate the distribution function $F$ by the empirical distribution function

$$F_n(x) = n^{-1} \sum_{j=1}^{n} I_{\{X_j \le x\}},$$

where $I_{\{\cdot\}}$ is the indicator function. Therefore, our estimator does not involve nonparametric smoothing and avoids choice of the bandwidth $h_n$. Given the expression of

$$\Psi(u) = \frac{\lambda\mu}{c} - (1 - \frac{\lambda\mu}{c})\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1 - e^{-isu}}{is} \frac{\lambda \frac{\int_0^\infty e^{isu} f(u)\, du - 1}{ics}}{1 - \lambda \frac{\int_0^\infty e^{isu} f(u)\, du - 1}{ics}} ds \tag{3}$$

from (Zhang et al. 2014), we will construct an estimator of $\Psi(u)$ by the plug-in device, which replaces $\lambda$ and $\mu$ by their empirical estimators and $f(u)\,du$ by $dF_n(u)$. Similar to (You et al. 2020), we will establish asymptotically normal distributions of the nonparametric estimators of $\Psi(u)$ with known $\lambda$ and with unknown $\lambda$. Since our estimators do not involve the kernel smoothing, they are easier to calculate than those in (Zhang et al. 2014) and (You et al. 2020). Furthermore, since there is no bandwidth choice for our procedure, our procedure is expected to have more stable performance. The asymptotic distributions of our estimators allow us to consistently estimate the asymptotic variances of our estimators with the plug-in principle and thus enable interval estimates of the ruin probability.

## 2. Main Results

The ruin probability $\Psi(u)$ in (3) can be rewritten as

$$\Psi(u) = \frac{\lambda\mu}{c} - (1 - \frac{\lambda\mu}{c})\varphi(u), \tag{4}$$

where

$$\varphi(u) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1 - e^{-isu}}{is} \frac{\lambda\phi_f^*(s)}{1 - \lambda\phi_f^*(s)} ds, \tag{5}$$

with $\phi_f^*(s) = \frac{\phi_f(s) - 1}{ics}$ and $\phi_f(s) = \int_0^\infty e^{isu} f(u)\, du$ being the inverse Fourier transform of density $f$. In the following we consider estimating the ruin probability when $\lambda$ is known or unknown.

### 2.1. Estimation with Known Intensity $\lambda$

Suppose that a sample $\{X_1, X_2, ..., X_n\}$ of the claim size are observed. When $\lambda$ is known, a natural estimation method for $\Psi(u)$ is to use the plug-in device, based on (4) and (5). This requires us to estimate the inverse Fourier transform $\phi_f(\cdot)$ of density $f(\cdot)$.

Note that $\phi_f(s) = \int_0^\infty e^{isu}\, dF(u)$. It follows that an empirical type estimator of is given by $\phi_{emp}(s) = \int_0^\infty e^{isx} dF_n(s) = \frac{1}{n}\sum_{j=1}^n e^{isX_j}$. By (5), a plug-in estimator of $\varphi(u)$ is

$$\tilde{\varphi}_n(u) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1 - e^{-isu}}{is} \frac{\lambda\phi_n^*(s)}{1 - \lambda\phi_n^*(s)} ds, \tag{6}$$

where $\phi_n^*(s) = \frac{\phi_{emp}(s) - 1}{ics}$. It is easy to see that $\tilde{\varphi}_n(u)$ must be real, because the integrand is conjugate symmetric. Since the integral in (6) is possibly infinite, following (Zhang et al. 2014) we estimate $\Psi(u)$ by

$$\hat{\Psi}_n(u) = \frac{\lambda\hat{\mu}_n}{c} - (1 - \frac{\lambda\hat{\mu}_n}{c})\hat{\varphi}_n(u), \quad u \geq 0, \tag{7}$$

where $\hat{\mu}_n = \frac{1}{n}\sum_{j=1}^n X_j$, $\hat{\varphi}_n(u) = \max\{\min(M, \tilde{\varphi}_n(u)), -M\}$, and $M$ is a large constant. Since the above estimate does not involve the kernel density estimate $\hat{f}_n(x)$, it is simpler and easy to calculate than that of (Zhang et al. 2014). In particular, we do not need to select the bandwidth $h_n$, and thus it is expected that our estimator is more stable than Zhang et al.'s and You et al.'s.

In the following we illustrate the above estimation method is feasible through its asymptotic distribution. To this end, we propose the following assumptions:

(A1) The density function $f(x)$ satisfies that

(1) function $f(x)$ is right-continuous at zero and continuously differentiable in $(0, \infty)$, and $f''(x)$ exists almost everywhere. Further, $\int_0^\infty |f'(x)| dx < \infty$, $\int_0^\infty |f''(x)| dx < \infty$, and $\int_0^\infty (f'(x))^2 dx < \infty$;

(2) $\mathbf{E}(X^4) < \infty$, and for some $0 < \alpha < \frac{1}{2}$, $x^{1+\alpha}\bar{F}(x^\alpha) \to 0$ as $x \to \infty$, where $\bar{F}(x) = 1 - F(x)$.

(A2) There exists a constant $\rho > 0$ such that $1 - \lambda\mu c^{-1} \geq \rho$.

The above assumptions are wild and were used in (Zhang et al. 2014) and (You et al. 2020). The following result demonstrates that $\hat{\Psi}_n(u)$ is $\sqrt{n}$-consistent and asymptotically normal.

**Theorem 1.** *Suppose that Assumptions (A1)–(A2) hold, then*

$$\sqrt{n}\{\hat{\Psi}_n(u) - \Psi(u)\} \xrightarrow{\mathbf{D}} \mathcal{N}(0, S^2), \tag{8}$$

*where* $S^2 = [\frac{\lambda}{c}\{1 + \varphi(u)\}]^2\sigma^2 + (\frac{\lambda}{\mu} - 1)^2 D^2 + C_0$, *with* $C_0 = \frac{1}{\pi}\frac{\lambda^2}{c^2}\{1 + \varphi(u)\}(\frac{\lambda}{c}\mu - 1)\int_{-\infty}^{\infty}\frac{(1-e^{-isu})\{-i\phi_f'(s) - \mu\phi_f(s)\}}{-s^2\{1-\lambda\phi_f^*(s)\}^2}ds$ *and* $D^2 = \frac{\lambda^2}{4\pi^2c^2}E[\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{-s^2}\frac{\phi_f(s)-e^{isX_j}}{(1-\lambda\phi_f^*(s))^2}ds]^2$.

**Remark 1.** *Compared with Theorem 1 in (You et al. 2020), we find that the difference of asymptotic variances between our estimator and their estimator is*

$$n^{-1}(\frac{\lambda}{\mu} - 1)^2(D^2 - D^{*2}),$$

*where* $D^{*2} = \frac{\lambda^2}{c^2}\{\sum_{j=1}^3 Q_j(u) + 2\sum_{j=4}^6 Q_j(u)\}$ *(for details, see page 11 of (You et al. 2020)). Hence, the asymptotic variance of our estimator has a much simpler form, which allows us a plug-in estimate, but this is contrary to that of (You et al. 2020) which is hard to estimate directly. It fact, they proposed ro estimate the variance with resampling methods such as the bootstrap.*

### 2.2. Estimation with Unknown Intensity $\lambda$

Theorem 1 is useful if $\lambda$ is known. However, the true value of $\lambda$ is not available in practice. Suppose there are $\kappa_n$ contiguous observational time intervals with same length of $d$. For $j = 1, \ldots, \kappa_n$, let $\{N_{jd}\}$ be the total number of claims from the starting time to the $j$th observational time interval with length $d > 0$. Then an unbiased estimator of Poisson intensity parameter $\lambda$ is

$$\hat{\lambda}_{\kappa_n} = N_{\kappa_n d}/(\kappa_n d), \tag{9}$$

where $N_{0d} = 0$. By (7), a natural estimator of $\Psi(u)$ (see Remark 1 of (Zhang et al. 2014)) is

$$\hat{\Psi}_n^*(u) = \frac{\hat{\lambda}_{\kappa_n}\hat{\mu}_n}{c} - (1 - \frac{\hat{\lambda}_{\kappa_n}\hat{\mu}_n}{c})\hat{\varphi}_n^*(u), \quad u \geq 0, \tag{10}$$

where $\hat{\varphi}_n^*(u) = (M^* \wedge \tilde{\varphi}_n^*(u)) \vee (-M^*)$ for a large constant $M^*$, and

$$\tilde{\varphi}_n^*(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1 - e^{-isu}}{is}\frac{\hat{\lambda}_{\kappa_n}\phi_n^*(s)}{1 - \hat{\lambda}_{\kappa_n}\phi_n^*(s)}ds. \tag{11}$$

Additionally, we need the following assumption for establishing the asymptotic distribution of $\hat{\Psi}_n^*(u)$.

(A3) For integer $m \geq 2$, $E|X|^m \leq m!H^{m-2}\sigma^2/2$, where $H$ is a positive constant.

Assumption (A3) is the classical Cramér condition for all moments of $X$. It was used in (You et al. 2020). The following theorem indicates $\hat{\Psi}_n^*(u)$ converges to the ruin probability at the best parametric rate of $\sqrt{n}$.

**Theorem 2.** *Suppose Assumptions (A1)–(A3) hold. If* $0 \leq \lambda^* \equiv \lim_{n\to\infty} n/\kappa_n < \infty$, *then*

$$\sqrt{n}\{\hat{\Psi}_n^*(u) - \Psi(u)\} \xrightarrow{\mathbf{D}} \mathcal{N}(0, S^{*2}), \tag{12}$$

*where* $S^{*2} = S^2 + \left\{\frac{\mu}{c}(1+\varphi(u)) - (1-\frac{\lambda\mu}{c})a(f,\lambda)\right\}^2 \frac{\lambda\lambda^*}{d}$ *and*

$$a(f,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-e^{-isu}}{is} \frac{\phi_f^*(s)}{\{1-\lambda\phi_f^*(s)\}^2} ds.$$

**Remark 2.** *Both* $\hat{\Psi}_n(u)$ *and* $\hat{\Psi}_n^*(u)$ *are* $\sqrt{n}$*-consistent, but the latter has larger variance, which is understandable because estimating* $\lambda$ *brings an additional variation.*

Given an initial surplus $u$, Theorem 2 allows us to construct an $100(1-\alpha)\%$ confidence interval for $\Psi(u)$:

$$\hat{\Psi}^*(u) \pm n^{-1/2} z_{1-\alpha/2} \hat{S}^*,$$

where $z_{1-\alpha/2}$ is the $100(1-\alpha/2)$th percentile of the standard normal distribution, and $\hat{S}^*$ is a consistent estimator of $S^*$, for example, from the plug-in device.

### 2.3. Computation on Ruin Probability

To obtain the estimate $\hat{\Psi}_n^*(u)$ of ruin probability in (10), one needs to calculate $\tilde{\varphi}_n^*(u)$ in (11). Let $G_n^*(s) = \frac{\hat{\lambda}_{\kappa_n}\phi_n^*(s)}{is\{1-\hat{\lambda}_{\kappa_n}\phi_n^*(s)\}}$. Then

$$\tilde{\varphi}_n^*(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_n^*(s)\,ds - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isu} G_n^*(s)\,ds.$$

The above equation is well defined, since the integrals are finite. Let $H_n^*(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isu} G_n^*(s)\,ds$ be the Fourier transform of $G_n^*(s)$. Then $\tilde{\varphi}_n^*(u) = H_n^*(0) - H_n^*(u)$. Therefore, our major task is to calculate $H_n^*(u)$, which is easy to implement in standard software.

### 2.4. Proofs of Theorems

To facilitate the proofs, we first introduce some technical lemmas. Then we give the proofs of theorems.

**Lemma 1.** *If Assumption (A1) holds. Then, for any* $\varepsilon > 0$, $P(\sup_s |\frac{\phi_f(s)-\phi_{emp}(s)}{is}| > \varepsilon) = o(n^{-1})$.

**Proof of Lemma 1.** By the inequality $|e^{isx}-1| \le |sx|$, we have

$$\begin{aligned}
\left|\frac{\phi_f - \phi_{emp}(s)}{is}\right| &= \left|\frac{\int_0^\infty (e^{isx}-1)dF(x) - \int_0^\infty (e^{isx}-1)dF_n(x)}{is}\right| \\
&= \left|\frac{\int_0^\infty \int_0^x e^{isy}dydF(x) - \int_0^\infty \int_0^x e^{isy}dydF_n(x)}{is}\right| \\
&= \left|\frac{\int_0^\infty (1-F(x))e^{isx}dx - \int_0^\infty (1-F_n(x))e^{isx}dx}{is}\right|.
\end{aligned}$$

Hence, for $\varepsilon > 0$, by Lemma 1 of (Zhang et al. 2014), we obtain that

$$P\Big(\sup_s\left|\frac{\phi_f - \phi_{emp}(s)}{is}\right| > \varepsilon\Big) \le P\Big(\int_0^\infty |F(x)-F_n(x)|dx > \varepsilon\Big) = o(n^{-1}).$$

□

**Lemma 2.** *Let* $A_n(s) = \frac{\lambda}{c}\frac{\phi_f(s)-\phi_{emp}(s)}{is}$. *If Assumptions (A1)-(A2) hold, then* $E\int_{-\infty}^{\infty}|A_n(s)|^2 ds = O(n^{-1})$ and $E\int_{-\infty}^{\infty}|A_n(s)|^4 ds = O(n^{-2})$.

**Proof of Lemma 2.** Note that $\phi_{emp}(s) = \frac{1}{n}\sum_{j=1}^{n} e^{isX_j}$, it follows that

$$\frac{\phi_f(s) - \phi_{emp}(s)}{is} = \frac{1}{n}\sum_{j=1}^{n} \frac{\phi_f(s) - e^{isX_j}}{is}.$$

Taking expectation, we obtain that $E\{\phi_{emp}(s) - \phi_f(s)/(is)\} = 0$ and

$$Var\{\phi_{emp}(s) - \phi_f(s)/(is)\} = n^{-1}Var\{\frac{\phi_f(s) - e^{isX_j}}{is}\} = \frac{\phi_f^2(s) - \phi_f(2s)}{ns^2}.$$

By Fubini's theorem, we have

$$\begin{aligned} E\int_{-\infty}^{\infty} |A_n(s)|^2 ds &= \int_{-\infty}^{\infty} E\Big|\frac{\lambda}{c}\frac{\phi_f(s) - \phi_{emp}(s)}{is}\Big|^2 ds \\ &= (\frac{\lambda}{c})^2\frac{1}{n}\int_{-\infty}^{\infty} \frac{\phi_f^2(s) - \phi_f(2s)}{s^2} ds \\ &= O(n^{-1}). \end{aligned}$$

Again, using Fubini's theorem, we establish that

$$\begin{aligned} E\int_{-\infty}^{\infty} |A_n(s)|^4 ds &= E\int_{-\infty}^{\infty} \Big|\frac{\lambda}{c}\frac{\phi_f(s) - \phi_{emp}(s)}{is}\Big|^4 ds \\ &= \int_{-\infty}^{\infty} \Big[Var(\frac{\lambda}{c}\frac{\phi_f(s) - \phi_{emp}(s)}{is})^2 + (Var(\frac{\lambda}{c}\frac{\phi_f(s) - \phi_{emp}(s)}{is}))^2\Big] ds \\ &= \int_{-\infty}^{\infty} (\frac{\lambda}{c})^4 \frac{1}{n^2}\Big[Var(\frac{\phi_f(s) - e^{isX_j}}{is})^2 + (Var(\frac{\phi_f(s) - e^{isX_j}}{is}))^2\Big] ds. \end{aligned}$$

By independence of the sample points, we have

$$\begin{aligned} E\int_{-\infty}^{\infty} |A_n(s)|^4 ds &= \int_{-\infty}^{\infty} (\frac{\lambda}{c})^4 \frac{1}{n^2} E\Big[\frac{\phi_f(s) - e^{isX_j}}{is}\Big]^4 ds \\ &= (\frac{\lambda}{c})^4 \frac{1}{n^2}\int_{-\infty}^{\infty} \frac{6\phi_f^2(s)\phi_f(2s) - 4\phi_f(s)\phi_f(3s) + \phi_f(4s) - 3\phi_f^4(s)}{s^4} ds \\ &= O(n^{-2}). \end{aligned}$$

□

**Lemma 3.** *Suppose that Assumptions (A1)–(A2) hold. Then*

(i) *(Consistency)* $\hat{\mu}_n - \mu \xrightarrow{\mathbf{P}} 0$, $\hat{\varphi}_n(u) - \varphi(u) \xrightarrow{\mathbf{P}} 0$, and $\hat{\lambda}_{\kappa_n} - \lambda \xrightarrow{\mathbf{P}} 0$.

(ii) *(Asymptotic normality)* $\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{\mathbf{D}} \mathbf{N}(0, \sigma^2)$, $\sqrt{n}(\hat{\lambda}_{\kappa_n} - \lambda) \xrightarrow{\mathbf{D}} \mathbf{N}(0, \frac{\lambda}{d})$, and $\sqrt{n}(\hat{\varphi}_n(u) - \varphi(u)) \xrightarrow{\mathbf{D}} \mathbf{N}(0, D^2)$.

**Proof of Lemma 3.** By the property of the Poisson process $N$ and the size of claim $X$, it is easy to obtain $\hat{\mu}_n - \mu \xrightarrow{\mathbf{P}} 0$, $\hat{\lambda}_{\kappa_n} - \lambda \xrightarrow{\mathbf{P}} 0$, $\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{\mathbf{D}} \mathbf{N}(0, \sigma^2)$, and $\sqrt{n}(\hat{\lambda}_{\kappa_n} - \lambda) \xrightarrow{\mathbf{D}} \mathbf{N}(0, \frac{\lambda}{d})$. In the following we show that $\hat{\varphi}_n(u) - \varphi(u) \xrightarrow{\mathbf{P}} 0$.

First, combining (5) and (6) leads to

$$\begin{aligned}\tilde{\varphi}_n(u)-\varphi(u) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{is}\frac{A_n(s)}{(1-\lambda\phi_f^*(s))^2}ds \\ &\quad +\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{is}\frac{A_n^2(s)}{(1-\lambda\phi_f^*(s)+A_n(s))(1-\lambda\phi_f^*(s))^2}ds \\ &\equiv -\mathrm{I}_n(u)+\mathrm{II}_n(u). \end{aligned} \tag{13}$$

Let $\epsilon_1$ and $\epsilon_2$ be two small positive constants such that $\varphi(u)+\epsilon_1+\epsilon_2<M$. Define $V_n=\{|\mathrm{I}_n(u)|<\epsilon_1, |\mathrm{II}_n(u)|<\epsilon_2\}$ and its complementary set $\bar{V}_n$. Then

$$\begin{aligned}\hat{\varphi}_n-\varphi(u) &= \{\hat{\varphi}_n-\varphi(u)\}I_{V_n}+\{\hat{\varphi}_n-\varphi(u)\}I_{\bar{V}_n} \\ &= \{\tilde{\varphi}_n-\varphi(u)\}I_{V_n}+\{\hat{\varphi}_n-\varphi(u)\}I_{\bar{V}_n} \\ &= \sum_{i=1}^{4}H_{n,i}(u), \end{aligned} \tag{14}$$

where $H_{n,1}(u)=-\mathrm{I}_n(u)$, $H_{n,2}(u)=\mathrm{II}_n(u)I_{V_n}$, $H_{n,3}=\mathrm{I}_n(u)I_{\bar{V}_n}$ and $H_{n,4}=(\hat{\varphi}_n(u)-\varphi(u))I_{\bar{V}_n}$. Using the inequality $|\phi_f-1|\le|i\mu s|$, Hölder's inequality and Assumption (A2), we obtain that

$$\begin{aligned}|\mathrm{I}_n(u)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}|\frac{1-e^{-isu}}{is}|\frac{|A_n(s)|}{|1-\lambda\phi_f^*(s)|^2}ds \\ &\le \frac{1}{2\rho^2\pi}\int_{-\infty}^{\infty}|\frac{1-e^{-isu}}{is}||A_n(s)|ds \\ &\le \frac{1}{2\rho^2\pi}(\int_{-\infty}^{\infty}|\frac{1-e^{-isu}}{is}|^{\frac{4}{3}}ds)^{\frac{3}{4}}(\int_{-\infty}^{\infty}|A_n(s)|^4ds)^{\frac{1}{4}} \\ &\le C(\int_{-\infty}^{\infty}|A_n(s)|^4ds)^{\frac{1}{4}}. \end{aligned}$$

By Markov's inequality and Lemma 2, we have

$$P(|\mathrm{I}_n(u)|\ge\epsilon_1)\le\frac{E|\mathrm{I}_n(u)|^4}{\epsilon_1^4}\le C\cdot E\int_{-\infty}^{\infty}|A_n(s)|^4ds=O(\frac{1}{n^2}). \tag{15}$$

For $0<\delta<\rho$, let $B_{n,\delta}=\{\sup_s|A_n(s)|\le\delta\}$, and let $\overline{B}_{n,\delta}$ be its complementary set. On the set $B_{n,\delta}$, we have

$$\begin{aligned}|\mathrm{II}_n(u)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}|\frac{1-e^{-isu}}{is}|\frac{A_n^2(s)}{(1-\lambda\phi_f^*(s)+A_n(s))(1-\lambda\phi_f^*(s))^2}ds \\ &\le C\int_{-\infty}^{\infty}\left|\frac{1-e^{-isu}}{is}\right||A_n(s)|^2ds. \end{aligned} \tag{16}$$

Then, the Cauchy-Schwarz inequality, $|\mathrm{II}_n(u)|\le C\int_{-\infty}^{\infty}|A_n(s)|^2ds$ and $|\mathrm{II}_n(u)|\le C(\int_{-\infty}^{\infty}|A_n(s)|^4ds)^{\frac{1}{2}}$. Therefore,

$$\begin{aligned}E|H_{n,2}| &= E[|H_{n,2}|I(V_n\cap B_{n,\delta})]+E[|H_{n,2}|I(V_n\cap\bar{B}_{n,\delta})] \\ &\le C\cdot E\int_{-\infty}^{\infty}|A_n(s)|^2ds+\varepsilon_2P(\bar{B}_{n,\delta})=O(n^{-1}). \end{aligned} \tag{17}$$

In order to get the bounds of $H_{n,3}$ and $H_{n,4}$, we need to study probability $P(\bar{V}_n)$. By Markov's inequality, we have

$$P(\{|\mathrm{II}_n(u)| \geq \varepsilon_2\} \cap B_{n,\delta}) \leq \frac{E[|\mathrm{II}_n(u)|^2 I(B_{n,\delta})]}{\varepsilon_2^2} \leq C \cdot E \int_{-\infty}^{\infty} |A_n(s)|^4 ds = O(\frac{1}{n^2}),$$

which, combined with Lemma 1, leads to

$$\begin{aligned} P(|\mathrm{II}_n(u)| \geq \varepsilon_2) &= P(\{|\mathrm{II}_n(u)| \geq \varepsilon_2\} \cap B_{n,\delta}) + P(\{|\mathrm{II}_n(u)| \geq \varepsilon_2\} \cap \bar{B}_{n,\delta}) \\ &\leq P(\{|\mathrm{II}_n(u)| \geq \varepsilon_2\} \cap B_{n,\delta}) + P(\bar{B}_{n,\delta}) \\ &= O(\frac{1}{n^2}) + o(\frac{1}{n}). \end{aligned} \tag{18}$$

This, combined with (15) and (18), gives us that

$$\begin{aligned} P(\bar{V}_n) &\leq P(|\mathrm{I}_n(u)| \geq \varepsilon_1) + P(|\mathrm{II}_n(u)| \geq \varepsilon_2) \\ &\leq P(|\mathrm{I}_n(u)| \geq \varepsilon_1) + P(\{|\mathrm{II}_n(u)| \geq \varepsilon_2\} \cap B_{n,\delta}) + P(\bar{B}_{n,\delta}) \\ &= o(\frac{1}{n}) + O(\frac{1}{n^2}). \end{aligned} \tag{19}$$

Hence, applying Hölder's inequality and Lemma 2, we obtain that

$$\begin{aligned} E|H_{n,3}(u)| &\leq E[|\mathrm{I}_n(u)| I_{\bar{V}_n}|] \\ &= (E|\mathrm{I}_n(u)|^4)^{\frac{1}{4}} P(\bar{V}_n)^{\frac{3}{4}} \\ &= o(\frac{1}{n}) + O(\frac{1}{n^2}). \end{aligned} \tag{20}$$

By (19), we get

$$E|H_{n,4}(u)| \leq (M + \varphi(u)) P(\bar{V}_n) = o(\frac{1}{n}) + O(\frac{1}{n^2}). \tag{21}$$

It follows from (14), (15), (17), (20) and (21) and Markov's inequality that $\hat{\varphi}_n(u) - \varphi(u) \xrightarrow{\mathbf{P}} 0$.

Next, we show that $\sqrt{n}(\hat{\varphi}_n(u) - \varphi(u)) \xrightarrow{\mathbf{D}} \mathbf{N}(0, D^2)$. By (14), $\sqrt{n}\{\hat{\varphi}_n(u) - \varphi(u)\} = \sqrt{n} \sum_{i=1}^4 H_{n,i}$. It follows from Markov's inequality that for any $\varepsilon > 0$,

$$P(|\sqrt{n} \sum_{i=2}^4 H_{n,i}| > \varepsilon) \leq \frac{\sqrt{n} \sum_{i=2}^4 E|H_{n,i}|}{\varepsilon},$$

which together with (15), (17), (20) and (21) implies that $\sqrt{n} \sum_{i=2}^4 H_{n,i} \xrightarrow{\mathbf{P}} 0$. Therefore,

$$\begin{aligned} \sqrt{n}\{\hat{\varphi}_n(u) - \varphi(u)\} &= \sqrt{n} H_{n,1} + o_P(1) \\ &= -\sqrt{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-isu}}{is} \frac{A_n(s)}{(1 - \lambda \phi_f^*(s))^2} ds + o_P(1) \\ &= -\frac{\lambda}{2c\sqrt{n}\pi} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \frac{1 - e^{-isu}}{-s^2} \frac{\phi_f(s) - e^{isX_j}}{(1 - \lambda \phi_f^*(s))^2} ds + o_P(1). \end{aligned} \tag{22}$$

Applying Fubini's theorem, we establish that

$$
\begin{aligned}
E[\sqrt{n}H_{n,1}] &= E\Big[-\frac{1}{2\sqrt{n}\pi}\frac{\lambda}{c}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{-s^2}\frac{\phi_f(s)-e^{isX_j}}{(1-\lambda\phi_f^*(s))^2}\,ds\Big] \\
&= -\frac{1}{2\sqrt{n}\pi}\frac{\lambda}{c}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{-s^2}\frac{E[\phi_f(s)-e^{isX_j}]}{(1-\lambda\phi_f^*(s))^2}ds \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
Var(\sqrt{n}H_{n,1}) &= (\frac{1}{2\sqrt{n}\pi}\frac{\lambda}{c})^2\sum_{j=1}^{n}Var\Big(\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{-s^2}\frac{\phi_f(s)-e^{isX_j}}{(1-\lambda\phi_f^*(s))^2}\,ds\Big) \\
&= \frac{\lambda^2}{4\pi^2c^2}Var\Big(\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{-s^2}\frac{\phi_f(s)-e^{isX_j}}{(1-\lambda\phi_f^*(s))^2}\,ds\Big) \\
&= \frac{\lambda^2}{4\pi^2c^2}E\Big[\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{-s}\frac{\phi_f(s)-e^{isX_j}}{s(1-\lambda\phi_f^*(s))^2}\,ds\Big]^2 \\
&\le \frac{\lambda^2}{4\pi^2c^2}E\Big\{\int_{-\infty}^{\infty}\Big(\frac{1-e^{-isu}}{-s}\Big)^2ds\int_{-\infty}^{\infty}\Big[\frac{\phi_f(s)-e^{isX_j}}{s(1-\lambda\phi_f^*(s))^2}\Big]^2ds\Big\},
\end{aligned}
$$

where the last inequality is from the Hölder inequality. Therefore,

$$
\begin{aligned}
Var(\sqrt{n}H_{n,1}) &\le \frac{\lambda^2}{4\pi^2c^2}\int_{-\infty}^{\infty}\Big(\frac{1-e^{-isu}}{-s}\Big)^2ds\int_{-\infty}^{\infty}\frac{1}{s^2(1-\lambda\phi_f^*(s))^4}E[\phi_f(s)-e^{isX_j}]^2ds \\
&= \frac{\lambda^2}{4\pi^2c^2}\int_{-\infty}^{\infty}\frac{(1-e^{-isu})^2}{s^2}\,ds\int_{-\infty}^{\infty}\frac{\phi_f(2s)-\phi_f^2(s)}{s^2(1-\lambda\phi_f^*(s))^4}\,ds.
\end{aligned}
$$

Using the inequality $|\phi_f(s)-1|\le|i\mu s|$ and Assumption (A2), we obtain that

$$
(1-\lambda\phi_f^*(s))^4\ge(1-|\lambda\phi_f^*(s)|)^4=(1-|\frac{\lambda}{c}\frac{\phi_f(s)-1}{is}|)^4\ge(1-\frac{\mu\lambda}{c})^4\ge\rho^4.
$$

Then

$$
\begin{aligned}
Var(\sqrt{n}H_{n,1}) &\le \frac{\lambda^2}{4\rho^4\pi^2c^2}\int_{-\infty}^{\infty}\frac{(1-e^{-isu})^2}{s^2}ds\int_{-\infty}^{\infty}\frac{\phi_f(2s)-\phi_f^2(s)}{s^2}ds \\
&< \infty.
\end{aligned}
$$

Therefore, by the CLT and Slutsky's theorem, we obtain that

$$
\sqrt{n}(\hat{\varphi}_n(u)-\varphi(u))\to N(0,D^2),
$$

where $D^2=\frac{\lambda^2}{4\pi^2c^2}E[\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{-s^2}\frac{\phi_f(s)-e^{isX_j}}{(1-\lambda\phi_f^*(s))^2}ds]^2$. □

**Lemma 4.** *Let* $\xi_{n1}=(\hat{\lambda}_{\kappa_n}-\lambda)a(f,\lambda)$, $\xi_{n2}=n^{-1}\sum_{j=1}^{n}Z_j$ *and* $\xi_n=\xi_{n1}+\xi_{n2}$, *where*

$$
Z_j=-\frac{\lambda}{2\pi c}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{s^2[1-\lambda\phi_f^*(s)]^2}[e^{isX_j}-\phi_f(s)]\,ds.
$$

*Suppose that assumptions (A1)-(A3) hold. Then* $P\{|\xi_n| > C + o_p(n^{-1/2})\} = o(n^{-1/2})$*, where C is any given constant.*

**Proof of Lemma 4.** The proof is similar to Lemma A.2 in (You et al. 2020). □

**Proof of Theorem 1.** Our argument for the proof is to first establish the Bahardur representation of our estimator, based on its explicit formula in (7). Then the result of theorem follows from the Bahardur's representation. By (4) and (7), we have

$$\begin{aligned}\sqrt{n}\left\{\hat{\Psi}_n(u) - \Psi(u)\right\} &= \sqrt{n}\left[\frac{\lambda}{c}\{1+\varphi(u)\}(\hat{\mu}_n - \mu) + (\frac{\lambda}{c}\hat{\mu}_n - 1)\{\hat{\varphi}_n(u) - \varphi(u)\}\right] \\ &= \frac{\lambda}{c}\{1+\varphi(u)\}\sqrt{n}(\hat{\mu}_n - \mu) + (\frac{\lambda}{c}\mu - 1)\sqrt{n}\{\hat{\varphi}_n(u) - \varphi(u)\} \\ &\quad + \frac{\lambda}{c}\sqrt{n}(\hat{\mu}_n - \mu)\{\hat{\varphi}_n(u) - \varphi(u)\}. \end{aligned} \tag{23}$$

By Lemma 3 and Slutsky's theorem, the last term of (23) converges to zero in probability as $n \to \infty$. Thus,

$$\begin{aligned}\sqrt{n}\{\hat{\Psi}_n(u) - \Psi(u)\} &= \frac{\lambda}{c}\{1+\varphi(u)\}\sqrt{n}(\hat{\mu}_n - \mu) \\ &\quad + (\frac{\lambda}{c}\mu - 1)\sqrt{n}\{\hat{\varphi}_n(u) - \varphi(u)\} + o_p(1). \end{aligned} \tag{24}$$

This, combined with (22), leads to the following Bahardur representation:

$$\begin{aligned}\sqrt{n}\{\hat{\Psi}_n(u) - \Psi(u)\} &= \frac{\lambda}{c}\{1+\varphi(u)\}\sqrt{n}(\hat{\mu}_n - \mu) + (\frac{\lambda}{c}\mu - 1)\sqrt{n}H_{n,1} + o_p(1) \\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{\lambda}{c}\Big[(1+\varphi(u))(X_j - \mu) \\ &\quad + (\frac{\lambda}{c}\mu - 1)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1-e^{-isu})(e^{isX_j} - \phi_f(s))}{-s^2(1-\lambda\phi_f^*(s))^2}ds\Big] + o_p(1) \\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}B_j + o_p(1), \end{aligned} \tag{25}$$

where $B_j = \frac{\lambda}{c}\left[(1+\varphi(u))(X_j - \mu) + (\frac{\lambda}{c}\mu - 1)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1-e^{-isu})(e^{isX_j} - \phi_f(s))}{-s^2(1-\lambda\phi_f^*(s))^2}ds\right]$. Note that $\mathbf{E}[B_1] = \mathbf{E}[\frac{1}{n}\sum_{j=1}^{n}B_j] = (\frac{\lambda}{c}\mu - 1)\mathbf{E}[H_{n,1}] = 0$ and

$$\begin{aligned}\mathbf{Var}(B_1) &= \frac{\lambda^2}{c^2}\mathbf{Var}\Big(\{1+\varphi(u)\}(X_j - \mu)\Big) \\ &\quad + \frac{\lambda^2}{c^2}\mathbf{Var}\Big((\frac{\lambda}{c}\mu - 1)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1-e^{-isu})(e^{isX_j} - \phi_f(s))}{-s^2(1-\lambda\phi_f^*(s))^2}ds\Big) \\ &\quad + 2\frac{\lambda^2}{c^2}Cov\Big(\{1+\varphi(u)\}(X_j - \mu), (\frac{\lambda}{c}\mu - 1)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(1-e^{-isu})(e^{isX_j} - \phi_f(s))}{-s^2(1-\lambda\phi_f^*(s))^2}ds\Big) \\ &= (\frac{\lambda}{c}(1+\varphi(u)))^2\sigma^2 + n(\frac{\lambda}{c}\mu - 1)^2\mathbf{Var}\Big(-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{is}\frac{A_n(s)}{(1-\lambda\phi_f^*(s))^2}ds\Big) \\ &\quad + \frac{1}{\pi}\frac{\lambda^2}{c^2}(1+\varphi(u))(\frac{\lambda}{c}\mu - 1)E\Big[\int_{-\infty}^{\infty}\frac{(X_j - \mu)(1-e^{-isu})(e^{isX_j} - \phi_f(s))}{-s^2(1-\lambda\phi_f^*(s))^2}ds\Big]. \end{aligned}$$

Then, by simple algebra, we arrive at

$$\begin{aligned}
\mathbf{Var}(B_1) &= \Big[\frac{\lambda}{c}\{1+\varphi(u)\}\Big]^2\sigma^2 + (\frac{\lambda}{c}\mu - 1)^2\mathbf{Var}(\sqrt{n}H_{n,1}) \\
&\quad + \frac{1}{\pi}\frac{\lambda^2}{c^2}\{1+\varphi(u)\}(\frac{\lambda}{c}\mu - 1)\int_{-\infty}^{\infty}\frac{(1-e^{-isu})(-i\phi_f'(s) - \mu\phi_f(s))}{-s^2(1-\lambda\phi_f^*(s))^2}ds. \\
&= \Big[\frac{\lambda}{c}\{1+\varphi(u)\}\Big]^2\sigma^2 + (\frac{\lambda}{c}\mu - 1)^2 D^2 \\
&\quad + \frac{1}{\pi}\frac{\lambda^2}{c^2}\{1+\varphi(u)\}(\frac{\lambda}{c}\mu - 1)\int_{-\infty}^{\infty}\frac{(1-e^{-isu})(-i\phi_f'(s) - \mu\phi_f(s))}{-s^2(1-\lambda\phi_f^*(s))^2}ds = S^2.
\end{aligned}$$

It follow from the central limit theorem that $\frac{1}{\sqrt{n}}\sum_{j=1}^n B_j \to N(0,S^2)$. Applying Slutsky's theorem and (25), we obtain that $\sqrt{n}\{\hat{\Psi}_n(u) - \Psi(u)\} \xrightarrow{\mathbf{D}} \mathcal{N}(0,S^2)$, as $n \to \infty$. $\square$

**Proof of Theorem 2.** Our idea of the proof is to apply Lemma 4, and to follow the argument for the proof of Theorem 2.2 in (You et al. 2020). By (4) and (10), we have

$$\hat{\Psi}_n^*(u) - \Psi(u) = \Big\{\frac{\hat{\lambda}_{\kappa_n}}{c}(\hat{\mu}_n - \mu) + \frac{\mu}{c}(\hat{\lambda}_{\kappa_n} - \lambda)\Big\}\Big\{1+\varphi(u)\Big\} - (1 - \frac{\hat{\lambda}_{\kappa_n}\hat{\mu}_n}{c})\{\hat{\varphi}_n^*(u) - \varphi(u)\},$$

where $\hat{\varphi}_n^*(u) = (M^* \wedge \tilde{\varphi}_n^*(u)) \vee (-M^*)$. To establish the asymptotic distribution of $\hat{\Psi}_n^*(u) - \Psi(u)$, we need to study $\tilde{\varphi}_n^*(u)$. By (6) and (11),we have

$$\tilde{\varphi}_n^*(u) - \tilde{\varphi}_n(u) = (\hat{\lambda}_{\kappa_n} - \lambda)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{is}\frac{\phi_n^*(s)}{(1-\lambda\phi_n^*(s))(1-\hat{\lambda}_{\kappa_n}\phi_n^*(s))}ds.$$

Applying Lemmas 1 and 3, we obtain that $\sup_s|\phi_n^*(s) - \phi_f^*(s)| = o_p(1)$ and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{is}\frac{\phi_n^*(s)}{(1-\lambda\phi_n^*(s))(1-\hat{\lambda}_{\kappa_n}\phi_n^*(s))}ds = a(f,\lambda) + o_p(1),$$

where $a(f,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{is}\frac{\phi_f^*(s)}{\{1-\lambda\phi_f^*(s)\}^2}ds$. Therefore,

$$\tilde{\varphi}_n^*(u) - \tilde{\varphi}_n(u) = (\hat{\lambda}_{\kappa_n} - \lambda)a(f,\lambda) + o_p(n^{-1/2}). \tag{26}$$

Combining (13), (16) and (26) and Lemma 2 leads to

$$\tilde{\varphi}_n^*(u) - \varphi(u) = (\hat{\lambda}_{\kappa_n} - \lambda)a(f,\lambda) - \mathrm{I}_n(u) + r_n,$$

where $r_n = o_p(n^{-1/2})$. By (15), we see that $I_n(u) = O_p(n^{-2})$. Hence, $\tilde{\varphi}_n^*(u) - \varphi(u) = o_p(1)$. Let $\xi_n = (\hat{\lambda}_{\kappa_n} - \lambda)a(f,\lambda) - \mathrm{I}_n(u)$. Then

$$\begin{aligned}
\xi_n &= (\hat{\lambda}_{\kappa_n} - \lambda)a(f,\lambda) - \frac{\lambda}{2\pi c}n^{-1}\sum_{j=1}^n\int_{-\infty}^{\infty}\frac{1-e^{-isu}}{s^2[1-\lambda\phi_f^*(s)]^2}[e^{isX_j} - \phi_f(s)]\,ds \\
&\equiv \xi_{n1} + \xi_{n2},
\end{aligned}$$

$\tilde{\varphi}_n^*(u) = \xi_n + \varphi(u) + r_n$, and $P(|\tilde{\varphi}_n^*(u)| > M^*) \le P\{|\xi_n| > (M^* - |\varphi(u)| - |r_n|\}$. Then, by Lemma 4 and the proof of Theorem 2.2 in (You et al. 2020), it is easy to obtain $\sqrt{n}(\hat{\Psi}_n^*(u) - \Psi(u)) \xrightarrow{\mathbf{D}} \mathcal{N}(0,S^{*2})$. $\square$

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## References

Asmussen, Søren, and Hansjörg Albrecher. 2010. *Ruin Probabilities*, 2nd ed. Singapore: World Scientific.

Baumgartner, Benjamin, and Riccardo Gatto. 2010. A bootstrap test for the probability of ruin in the compound poisson risk process. *ASTIN Bulletin* 40: 241–55.

Bowers, Newton, Hans Gerber, James Hickmann, Donald Jones, and Cecil Nesbitt. 1997. *Actuarial Mathematics*, 2nd ed.; Schaumburg: The Society of Actuaries.

Cai, Jun, and Yanhong Wu. 1997. Some improvements on the lundberg bound for the ruin probability. *Statistics and Probability Letters* 33: 395–403.

Croux, Kristof, and Noel Veraverbeke. 1990. Nonparametric estimators for the probability of ruin. *Insurance: Mathematics and Economics* 9: 127–30.

Dickson, David. 1994. An upper bound for the probability of ultimate ruin. *Scandinavian Actuarial Journal* 2: 131–8.

Feng, Minyu, Liang-Jian Deng, Feng Chen, Matjaž Perc, and Jürgen Kurths. 2020. The accumulative law and its probability model: An extension of the Pareto distribution and the log-normal distribution. *Proceedings of the Royal Society A* 47620200019. doi:10.1098/rspa.2020.0019

Frees, Edward W. 1986. Nonparametric estimation of the probability of ruin. *ASTIN Bulletin* 16: 81–90.

Gerber, Hans U. 1979. An Introduction to Mathematical Risk Theory, no.8. In *Huebner Foundation Monograph Series*. Philadelphia: S. S. Huebner Foundation for Insurance Education.

Landriault, David, and Gordon Willmot. 2008. On the gerber-shiu discounted penalty function in the sparre andersen model with an arbitrary interclaim time distribution. *Insurance: Mathematics and Economics* 42: 600–8.

Mnatsakanov, Robert, L. L. Ruymgaart, and Frits H. Ruymgaart. 2008. Nonparametric estimation of ruin probabilities given a random sample of claims. *Mathematical Methods of Statistics* 17: 35–43.

Politis, Konstadinos. 2003. Semiparametric estimation for non-ruin probabilities. *Scandinavian Actuarial Journal* 1: 75–96.

Pitts, Susan M. 1994. Nonparametric estimation of compound distributions with applications in insurance. *Annals of the Institute of Statistical Mathematics* 46: 537–55.

You, Honglong, Junyi Guo, and Jiancheng Jiang. 2020. Interval estimation of the ruin probability in the classical compound Poisson risk model. *Computational Statistics and Data Analysis* 144: 106890.

Zhang, Zhimin, Hailiang Yang, and Hu Yang. 2014. On a nonparametric estimator for ruin probability in the classical risk model. *Scandinavian Actuarial Journal* 4: 309–38.