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Determining Distribution for the Quotients of Dependent and Independent Random Variables by Using Copulas

Sel Ly [1], Kim-Hung Pho [1,*], Sal Ly [1] and Wing-Keung Wong [2,3,4]

1. Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 756636, Vietnam; lysel@tdtu.edu.vn (S.L.); shanlee5611@gmail.com (S.L.)
2. Department of Finance, Fintech Center, and Big Data Research Center, Asia University, Taichung 41354, Taiwan; wong@asia.edu.tw
3. Department of Medical Research, China Medical University Hospital, Taichung 40402, Taiwan
4. Department of Economics and Finance, Hang Seng University of Hong Kong, Shatin, Hong Kong 999077, China

* Correspondence: phokimhung@tdtu.edu.vn

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Abstract: Determining distributions of the functions of random variables is a very important problem with a wide range of applications in Risk Management, Finance, Economics, Science, and many other areas. This paper develops the theory on both density and distribution functions for the quotient $Y = \frac{X_1}{X_2}$ and the ratio of one variable over the sum of two variables $Z = \frac{X_1}{X_1+X_2}$ of two dependent or independent random variables X_1 and X_2 by using copulas to capture the structures between X_1 and X_2. Thereafter, we extend the theory by establishing the density and distribution functions for the quotients $Y = \frac{X_1}{X_2}$ and $Z = \frac{X_1}{X_1+X_2}$ of two dependent normal random variables X_1 and X_2 in the case of Gaussian copulas. We then develop the theory on the median for the ratios of both Y and Z on two normal random variables X_1 and X_2. Furthermore, we extend the result of median for Z to a larger family of symmetric distributions and symmetric copulas of X_1 and X_2. Our results are the foundation of any further study that relies on the density and cumulative probability functions of ratios for two dependent or independent random variables. Since the densities and distributions of the ratios of both Y and Z are in terms of integrals and are very complicated, their exact forms cannot be obtained. To circumvent the difficulty, this paper introduces the Monte Carlo algorithm, numerical analysis, and graphical approach to efficiently compute the complicated integrals and study the behaviors of density and distribution. We illustrate our proposed approaches by using a simulation study with ratios of normal random variables on several different copulas, including Gaussian, Student-t, Clayton, Gumbel, Frank, and Joe Copulas. We find that copulas make big impacts from different Copulas on behavior of distributions, especially on median, spread, scale and skewness effects. In addition, we also discuss the behaviors via all copulas above with the same Kendall's coefficient. The approaches developed in this paper are flexible and have a wide range of applications for both symmetric and non-symmetric distributions and also for both skewed and non-skewed copulas with absolutely continuous random variables that could contain a negative range, for instance, generalized skewed-t distribution and skewed-t Copulas. Thus, our findings are useful for academics, practitioners, and policy makers.

Keywords: copulas; dependence structures; quotient of random variables; density functions; distribution functions

1. Introduction

Determining distributions of the functions of random variables is a very crucial task and this problem has been attracted a number of researchers because there are numerous applications in Risk Management, Finance, Economics, Science, and, many other areas, see, for example, (Donahue 1964; Ly et al. 2016; Nadarajah and Espejo 2006; Springer 1979). Basically, the distributions of an algebraic combination of random variables including the sum, product, and quotient are focused on some common distributions along with the assumptions of independence or correlated through Pearson's coefficient or dependence via multivariate normal joint distributions (Arnold and Brockett 1992; Bithas et al. 2007; Cedilnik et al. 2004; Hinkley 1969; Macalos and Arcede 2015; Marsaglia 1965; Matović et al. 2013; Mekićet al. 2012; Nadarajah and Espejo 2006; Nadarajah and Kotz 2006a, 2006b; Pham-Gia et al. 2006; Pham-Gia 2000; Rathie et al. 2016; Sakamoto 1943). Regarding ratio, it often appears in the problems of constructing statistics used in hypothesis testing and estimating issues. Some well-known distributions are results of such quotients. For example, the quotient of a Gaussian random variable divided by a square root of an independent chi-distributed random variable follows the t-distribution while the F-distribution is derived via the ratio of two independent chi-squared distributed random variables. To relax independence assumption, it is necessary to develop a framework for modeling dependence structures of random vectors in more general sense. To do so, Dolati et al. (2017) develop the distribution for X/Y in which both X and Y are positive.

In our paper, we first extend the theory developed by Dolati et al. (2017) to relax the positive assumption for the variables by developing the theory on both density and distribution function (CDF) for the quotient $Y = \frac{X_1}{X_2}$ of two dependent or independent continuous random variables X_1 and X_2 in which X_1 and X_2 could be any real number. Thereafter, we develop a theory on both density and distribution function for the ratio of one variable over the sum of two variables $Z = \frac{X_1}{X_1+X_2}$ of two dependent or independent continuous random variables X_1 and X_2 by using copulas to capture the structures between X_1 and X_2.

Since the density and the CDF formula of the ratios of both Y and Z are in terms of integrals and are very complicated, we cannot obtain the exact forms of the densities and the CDFs. To circumvent the difficulty, in this paper, we propose to use a Monte Carlo algorithm, numerical analysis, and graphical approach to study behavior of density and distribution. We illustrate our proposed approaches by using a simulation study with ratios of standard normal random variables on several different copulas, including Gaussian, Student-t, Clayton, Gumbel, Frank, and Joe Copulas and we find that copulas make big impacts from different Copulas on behavior of distributions, especially on median, spread, skewness and scale effects. For instance, when X_1 and X_2 tend to be more co-monotonic indicated by increasing the parameters of copulas, then the median of Y is shifted to be higher and its shape tends to be more symmetric. In the meantime, the median of Z is equally unchanged one-half and the shape always has symmetry. We note that the approaches developed in this paper are flexible and have a wide range of applications for both symmetric and non-symmetric distributions and also for both skewed and non-skewed copulas with absolutely continuous random variables that could contain a negative range, for instance, generalized skewed-t distribution and skewed-t Copulas.[1] Thus, our findings are useful for academics, practitioners, and policy makers.

The rest of the paper is organized as follows. In Sections 2 and 3, we will briefly discuss the background theory and copula theory related to the theory developed in our paper. In Section 4, we provide main results on the quotients of dependent and independent random variables. Section 5 proposes using the Monte Carlo to deal with complex integrals and estimate some percentiles by using some special copulas, and investigate their effects on the behavior of ratios of two standard normal random variables. The last section provides the conclusions.

[1] We would like to thank the anonymous reviewer for giving us helpful comments so that we could draw this conclusion.

2. Background Theory

We first review some previous work on the weighted sum, for example, in constructing portfolio Y_1 that is composed of two dependent assets defined by

$$Y_1 = w_1 X_1 + w_2 X_2, \tag{1}$$

in which the random variables $X_{i,t}$ $(i = 1, 2)$ denotes the rate of return at time t for the asset defined in terms of the following random quotient:

$$X_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}},$$

where $P_{i,t}$ denotes the price of the ith asset at time t. Note that X_i is assumed to be absolutely continuous with the cumulative distribution functions (CDF) F_i. Suppose that (X_1, X_2) follows copula C, then the CDF, $F_{Y_1}(y)$, of Y_1 defined in (1) satisfies:

$$F_{Y_1}(y) = \mathbf{1}_{\{w_2<0\}} + \operatorname{sgn}(w_2) \int_0^1 \frac{\partial}{\partial u} C\left(u, F_2\left(\frac{y - w_1 F_1^{-1}(u)}{w_2}\right)\right) du, \tag{2}$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function such that

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ -1, & \text{if} \quad x < 0. \end{cases}$$

Then, the CDF, F_{Y_1}, can be used to estimate the distortion risk measure of the portfolio defined by

$$R_g[Y_1] = \int_0^{\infty} g(\overline{F}_{Y_1}(y))dy + \int_{-\infty}^{0} [g(\overline{F}_{Y_1}(y)) - 1]dy,$$

where g is a *distortion function* and $\overline{F}_{Y_1}(y) = 1 - F_{Y_1}(y)$ is a survival function of Y_1. Readers may refer to Ly et al. (2016) for more detailed information.

In the credit model, the total loss is defined as the aggregation of the product of risk factors. Thus, it is necessary to find the distribution for the product case, for instance, Y_2 given by

$$Y_2 = X_1 X_2. \tag{3}$$

Ly et al. (2019) show that the CDF of Y_2 can be determined by

$$F_{Y_2}(y) = F_1(0) + \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right) \frac{\partial}{\partial u} C\left(u, F_2\left(\frac{y}{F_1^{-1}(u)}\right)\right) du. \tag{4}$$

3. Copulas

In this section, we will briefly discuss the copula theory related to the theory developed in our paper. Readers may refer to (Cherubini et al. 2004; Joe 1997; Nelsen 2007; Tran et al. 2015, 2017) for more information. Let $\mathbb{I} = [0, 1]$ be the closed unit interval and $\mathbb{I}^2 = [0, 1] \times [0, 1]$ be the closed unit square interval. We first state the most basic definition of copula in two dimensions in the following:

Definition 1. ***(Copula)*** *A 2-copula (two-dimensional copula) is a function C: $\mathbb{I}^2 \to \mathbb{I}$ satisfying the following conditions:*

(i) $C(u, 0) = C(0, v) = 0$ *for any* $u, v \in \mathbb{I}$;
(ii) $C(u, 1) = u$ *and* $C(1, v) = v$ *for any* $u, v \in \mathbb{I}$; *and*

(iii) *for any* $u_1, u_2, v_1, v_2 \in \mathbb{I}$ *with* $u_1 \le u_2$ *and* $v_1 \le v_2$,

$$C(u_2, v_2) + C(u_1, v_1) - C(u_2, v_1) - C(u_1, v_2) \ge 0.$$

In copula theory, Sklar proposed a very important theorem in 1959 called Sklar's Theorem (Cherubini et al. (2004); Joe (1997); Nelsen (2007)), which plays the most important role in this theory. It tells us that given a random vector (X_1, X_2) with absolutely continuous marginal distribution functions F_{X_1} and F_{X_2}, respectively, and its joint distribution function denoted by H, and then there exists a unique copula C such that

$$\begin{aligned} H(x_1, x_2) &= C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right), \\ h(x_1, x_2) &= \frac{\partial^2}{\partial x_1 \partial x_2} H(x_1, x_2) = c\left(F_{X_1}(x_1), F_{X_2}(x_2) f_{X_1}(x_1) f_{X_2}(x_2)\right), \end{aligned} \tag{5}$$

where $c(u, v) := \frac{\partial^2}{\partial u \partial v} C(u, v)$ denotes density of copula C, f_{X_i} is probability density function (PDF) of $X_i, i = 1, 2$, and $h(x_1, x_2)$ is the joint density function of X_1 and X_2. Copula is used to combine several univariate distributions together into bivariate [multivariate] settings so as the copula C can capture the dependence structure of (X_1, X_2) $[(X_1, \cdots, X_n)]$. For any copula C, we have the bounds

$$W(u, v) \le C(u, v) \le M(u, v),$$

where the copula $W(u, v) := \max(u + v - 1, 0)$ captures counter-monotonicity structure; that is, $X_2 = f(X_1)$ a.s., where f is strictly decreasing, while the copula $M(u, v) := \min(u, v)$ is used to capture comonotonicity; that is, $X_2 = f(X_1)$ a.s., where f is strictly increasing. In case X_1 and X_2 are independent, they follow copula denoted by $\Pi(u, v) := uv$. Copulas can be used not only to model the dependence structure of the variables, but also capture the correlation between the variables. The Kendall's coefficient τ can be expressed in terms of copulas as shown in the following:

$$\tau(X_1, X_2) = \tau(C) = 4 \int\int_{\mathbb{I}^2} C(u, v) dC(u, v) - 1. \tag{6}$$

In the next section, we will derive the two main propositions regarding formulas that can be used to determine the probability density and probability distribution of the quotient of dependent random variables by using copulas. In addition, we will apply the results to derive some corollaries on PDFs, CDFs, and median of the ratios in case X_1 and X_2 are normal distributed and they follow the Gaussian copulas.

4. Theory

We now develop two propositions on both density and distribution functions for the quotient $Y := \frac{X_1}{X_2}$ and the ratio of one variable over the sum of two variables $Z := \frac{X_1}{X_1 + X_2}$ of two dependent random variables X_1 and X_2 by using copulas. We first develop the proposition on the density and distribution functions for the quotient $Y = \frac{X_1}{X_2}$ as stated in the following:

Proposition 1. *Supposing that* (X_1, X_2) *is a vector of two absolutely continuous random variables* X_1 *and* X_2 *with the marginal distributions* F_1 *and* F_2*, respectively, let* C *be an absolutely continuous copula modeling dependence structure of the random vector* (X_1, X_2)*, and define* Y *as*

$$Y := \frac{X_1}{X_2}. \tag{7}$$

Then, the density $f_Y(y)$ and distribution functions $F_Y(y)$ of Y are

$$f_Y(y) = \int_0^1 |F_2^{-1}(v)| c\Big(F_1\left(yF_2^{-1}(v)\right), v\Big) f_1\left(yF_2^{-1}(v)\right) dv, \tag{8}$$

$$F_Y(y) = F_2(0) + \int_0^1 \operatorname{sgn}\left(F_2^{-1}(v)\right) \frac{\partial}{\partial v} C\left(F_1\left(yF_2^{-1}(v)\right), v\right) dv, \tag{9}$$

respectively, where F_2^{-1} denotes the inverse function of F_2, c is the density of copula C, and $\operatorname{sgn}(\cdot)$ stands for a sign function such that

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ -1, & \text{if} \quad x < 0. \end{cases}$$

Proof. Letting

$$\begin{cases} Y_1 := \dfrac{X_1}{X_2}, \\ Y_2 := X_2. \end{cases}$$

We note that since X_2 is absolutely continuous, $P(X_2 = 0) = 0$; that is, $X_2 \neq 0$ almost surely. Hence, the transformation $Y_1 = \frac{X_1}{X_2}$ always exists with probability 1 and we can obtain its inverse transformation by using

$$\begin{cases} X_1 = Y_1 Y_2, \\ X_2 = Y_2, \end{cases}$$

and their corresponding Jacobian

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2.$$

Then, we obtain the joint density of Y_1 and Y_2 such that

$$\begin{aligned} h(y_1, y_2) &= f(y_1 y_2, y_2)\, |y_2| \\ &= |y_2|\, c\left(F_1(y_1 y_2), F_2(y_2)\right) f_1(y_1 y_2) f_2(y_2). \end{aligned}$$

This yields the density of Y_1:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} |y_2|\, c\left(F_1(y_1 y_2), F_2(y_2)\right) f_1(y_1 y_2) f_2(y_2)\, dy_2 \tag{10}$$

$$= \int_0^1 \left|F_2^{-1}(v)\right| c\Big(F_1\left(y_1 F_2^{-1}(v)\right), v\Big) f_1\left(y_1 F_2^{-1}(v)\right) dv. \tag{11}$$

As a result, the CDF of Y_1 is determined by

$$F_{Y_1}(t) = \int_0^1 \int_{-\infty}^{t} \left|F_2^{-1}(v)\right| c\Big(F_1\left(y_1 F_2^{-1}(v)\right), v\Big) f_1\left(y_1 F_2^{-1}(v)\right) dy_1 dv. \tag{12}$$

By changing variable, $u = F_1\left(y_1 F_2^{-1}(v)\right)$, we get $du = F_2^{-1}(v) f_1\left(y_1 F_2^{-1}(v)\right) dy_1$ and we note that

$$F_2^{-1}(v) \geq 0 \Longleftrightarrow v \in [0, F_2(0)], \text{ and } F_2^{-1}(v) \leq 0 \Longleftrightarrow v \in [F_2(0), 1].$$

This yields

$$
\begin{aligned}
F_{Y_1}(t) &= -\int_0^{F_2(0)}\int_1^{F_1\left(tF_2^{-1}(v)\right)} \frac{\partial^2}{\partial u \partial v} C(u,v)\, du dv + \int_{F_2(0)}^1 \int_0^{F_2\left(tF_1^{-1}(v)\right)} \frac{\partial^2}{\partial u \partial v} C(u,v)\, dv du \\
&= -\int_0^{F_2(0)} \left[\frac{\partial}{\partial v} C\left(F_1\left(tF_2^{-1}(v)\right), v\right) - \frac{\partial}{\partial v} C(1,v)\right] dv + \int_{F_2(0)}^1 \frac{\partial}{\partial v} C\left(F_1\left(tF_2^{-1}(v)\right), v\right) dv \qquad (13) \\
&= F_2(0) + \int_0^1 \operatorname{sgn}\left(F_2^{-1}(v)\right) \frac{\partial}{\partial v} C\left(F_1\left(tF_2^{-1}(v)\right), v\right) dv.
\end{aligned}
$$

Thus, the assertions of Proposition 1 hold. □

From Proposition 1 and applying Equation (8), we obtain the following corollary on both density and distribution functions for the quotient $Y = \frac{X_1}{X_2}$ of two independent random variables X_1 and X_2 by using copulas:

Corollary 1. *When X_1 and X_2 are independent, then its copula $C(u,v) = uv$ has the density $c(u,v) = 1$, $\forall u, v \in \mathbb{I}$ and the density $f_Y(y)$ of the ratio $Y := \frac{X_1}{X_2}$ of two independent random variables becomes*

$$
f_Y(y) = \int_{-\infty}^{\infty} |x| f_1(xy) f_2(x) dx.
$$

This result is well known in the literature.

Next, we apply Equation (9) to derive both density and distribution function for $Y = \frac{X_1}{X_2}$ in case X_1 and X_2 are normal random variables and their dependence structure is captured by Gaussian Copulas. We first obtain the following corollary:

Corollary 2. *Assume that $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ and (X_1, X_2) follows Gaussian Copulas $C_r(u,v)$, $|r| < 1$, given in (46). Then, the density $f_Y(y)$ and distribution function $F_Y(y)$ of $Y = \frac{X_1}{X_2}$ have the forms*

$$
\begin{aligned}
f_Y(y) &= \int_0^1 sgn\left(\sigma_2 \Phi^{-1}(v) + \mu_2\right) \varphi\left(\frac{(y\sigma_2 - r\sigma_1)\Phi^{-1}(v) + y\mu_2 - \mu_1}{\sigma_1\sqrt{1-r^2}}\right) \frac{\sigma_2 \Phi^{-1}(v) + \mu_2}{\sigma_1 \sqrt{1-r^2}} dv, \qquad (14) \\
F_Y(y) &= \Phi\left(-\frac{\mu_2}{\sigma_2}\right) + \int_0^1 sgn\left(\sigma_2 \Phi^{-1}(v) + \mu_2\right) \Phi\left(\frac{(y\sigma_2 - r\sigma_1)\Phi^{-1}(v) + y\mu_2 - \mu_1}{\sigma_1\sqrt{1-r^2}}\right) dv, \qquad (15)
\end{aligned}
$$

respectively, where $\varphi(x)$ and $\Phi(x)$ are PDF and CDF of the standard normal distribution, respectively, and $\Phi^{-1}(x)$ denotes for the inverse function of $\Phi(x)$.

Proof. Let $X_1 \sim N(\mu_1, \sigma_1^2)$, and $X_2 \sim N(\mu_2, \sigma_2^2)$; then, their CDFs and inverse functions can be expressed in the following form:

$$
F_i(x) = \Phi\left(\frac{x - \mu_i}{\sigma_i}\right), \quad F_i^{-1}(v) = \sigma_i \Phi^{-1}(v) + \mu_i, \quad i = 1, 2.
$$

Given Gaussian Copulas $C_r(u,v)$ with $|r| < 1$, one can obtain its derivative $\frac{\partial C_r}{\partial v}(u,v)$, see Meyer (2013), as shown in the following:

$$
\frac{\partial C_r}{\partial v}(u,v) = \Phi\left(\frac{\Phi^{-1}(u) - r\Phi^{-1}(v)}{\sqrt{1-r^2}}\right).
$$

Now, applying Equation (9), we can simplify it to be

$$
\begin{aligned}
F_Y(y) &= \Phi\Big(-\frac{\mu_2}{\sigma_2}\Big) + \int_0^1 sgn\Big(\sigma_2\Phi^{-1}(v)+\mu_2\Big)\Phi\Big(\frac{\Phi^{-1}\Big(\Phi\Big(\frac{y\sigma_2\Phi^{-1}(v)+y\mu_2-\mu_1}{\sigma_1}\Big)\Big) - r\Phi^{-1}(v)}{\sqrt{1-r^2}}\Big) \\
&= \Phi\Big(-\frac{\mu_2}{\sigma_2}\Big) + \int_0^1 sgn\Big(\sigma_2\Phi^{-1}(v)+\mu_2\Big)\Phi\Big(\frac{(y\sigma_2 - r\sigma_1)\Phi^{-1}(v)+y\mu_2-\mu_1}{\sigma_1\sqrt{1-r^2}}\Big)dv.
\end{aligned}
$$

Taking derivative of $F_y(y)$ with respect to y, one gets the density $f_Y(y)$ defined as in (14). The assertions of Corollary 2 hold. □

We note that the probability exhibited in (15) can be easily computed by using the following Monte Carlo algorithm: For each $y \in \mathbb{R}$, we generate V from the uniform distribution on the unit interval $[0,1]$ with sample size N, say N = 10,000, and then the estimated probability is given by

$$
\widehat{F}_y(y) \approx \Phi\Big(-\frac{\mu_2}{\sigma_2}\Big) + \frac{1}{N}\sum_{i=1}^{N} sgn\Big(\sigma_2\Phi^{-1}(v_i)+\mu_2\Big)\Phi\Big(\frac{(y\sigma_2 - r\sigma_1)\Phi^{-1}(v_i)+y\mu_2-\mu_1}{\sigma_1\sqrt{1-r^2}}\Big). \tag{16}
$$

Using the result, we obtain the following corollary:

Corollary 3. *If $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(0, \sigma_2^2)$, and (X_1, X_2) follows Gaussian Copulas $C_r(u,v)$, $|r| < 1$, given in (46), then the median of $Y = \frac{X_1}{X_2}$ satisfies*

$$
median(Y) = r\frac{\sigma_1}{\sigma_2}, \quad \text{for all } \mu_1 \in \mathbb{R}. \tag{17}
$$

Proof. Since $X_2 \sim N(0, \sigma_2^2)$, $\Phi(-\frac{\mu_2}{\sigma_2}) = \Phi(0) = 0.5$. Hence, it is sufficient to prove that the integral term given in (15) is equal to zero. In fact, we find that

$$
\begin{aligned}
F_Y\Big(r\frac{\sigma_1}{\sigma_2}\Big) &= 0.5 + \int_0^1 sgn(\sigma_2\Phi^{-1}(v))\Phi\Big(\frac{-\mu_1}{\sigma_1\sqrt{1-r^2}}\Big)dv \\
&= 0.5 + \Phi\Big(\frac{-\mu_1}{\sigma_1\sqrt{1-r^2}}\Big)\Big[-\int_0^{1/2} dv + \int_{1/2}^1 dv\Big] \\
&= 0.5.
\end{aligned}
$$

Hence, the quantity $r\frac{\sigma_1}{\sigma_2}$ is the median of Y. The proof is complete. □

We turn to develop the proposition on density and distribution functions for the ratio of one variable over the sum of two variables $Z := \frac{X_1}{X_1+X_2}$ of two dependent random variables X_1 and X_2 by using copulas as stated in the following:

Proposition 2. *Suppose that (X_1, X_2) is a vector of two absolutely continuous random variables X_1 and X_2 with the marginal distributions F_1 and F_2, respectively, and let C be an absolutely continuous copula modeling dependence structure of the random vector (X_1, X_2), and define Z as*

$$
Z := \frac{X_1}{X_1+X_2}. \tag{18}
$$

Then, the density $f_Z(z)$ and distribution function $F_Z(z)$ of Z are

$$
f_Z(z) = \begin{cases} \int_0^1 \frac{|F_1^{-1}(u)|}{z^2} c\Big(u, F_2\Big(\frac{1-z}{z}F_1^{-1}(u)\Big)\Big) f_2\Big(\frac{1-z}{z}F_1^{-1}(u)\Big)\,du, & \text{if } z \neq 0, \\ f_1(0)\int_0^1 |F_2^{-1}(v)| c\Big(F_1(0), v\Big)dv, & \text{if } z = 0, \end{cases} \tag{19}
$$

$$F_Z(z) = \mathbf{1}_{\{z\geq 0\}} + \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right)\left[\frac{\partial}{\partial u}C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big) - \frac{\partial}{\partial u}C\left(u, F_2\left(\frac{1-z}{z}F_1^{-1}(u)\right)\right)\right] du, \tag{20}$$

respectively, where $\mathbf{1}_{\{\cdot\}}$ denotes an indicator function, F_i^{-1} denotes the inverse function of F_i for $i = 1, 2$, c is the density of copula C, and $\operatorname{sgn}(\cdot)$ is the sign function such that

$$\mathbf{1}_{\{z\geq 0\}} = \begin{cases} 1, & \text{if} \quad z \geq 0, \\ 0, & \text{if} \quad z < 0, \end{cases} \quad \text{and} \quad \operatorname{sgn}(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ -1, & \text{if} \quad x < 0. \end{cases}$$

Proof. By defining

$$\begin{cases} Z_1 = \dfrac{X_1}{X_1 + X_2}, \\ Z_2 = X_1 + X_2. \end{cases}$$

Here, we note that, since X_1 and X_2 are absolutely continuous, $P(X_1 + X_2 = 0) = 0$; that is, $X_1 + X_2 \neq 0$ almost surely. Hence, the transformation $Z_1 = \frac{X_1}{X_1+X_2}$ always exists with probability 1 and we obtain the following inverse transformation:

$$\begin{cases} X_1 = Z_1 Z_2, \\ X_2 = Z_2 - Z_1 Z_2, \end{cases}$$

and the Jacobian

$$J = \begin{vmatrix} Z_2 & Z_1 \\ -Z_2 & 1 - Z_1 \end{vmatrix} = Z_2.$$

Thus, the joint density of Z_1 and Z_2 becomes

$$\begin{aligned} h(z_1, z_2) &= f(z_1 z_2, z_2 - z_1 z_2)\,|z_2| \\ &= |z_2|\, c\left(F_1(z_1 z_2), F_2\left(z_2 - z_1 z_2\right)\right) f_1(z_1 z_2) f_2\left(z_2 - z_1 z_2\right), \end{aligned}$$

which leads us to get the density of Z_1 such that

$$f_{Z_1}(z_1) = \int_{-\infty}^{\infty} |z_2|\, c\left(F_1(z_1 z_2), F_2\left(z_2 - z_1 z_2\right)\right) f_1(z_1 z_2) f_2\left(z_2 - z_1 z_2\right) dz_2. \tag{21}$$

If $z_1 = 0$, then by taking $v := F_2(z_2)$, we get

$$f_{Z_1}(0) = f_1(0) \int_0^1 \left|F_2^{-1}(v)\right| c\Big(F_1(0), v\Big) dv.$$

If $z_1 > 0$, then by taking $u := F_1(z_1 z_2)$, we obtain

$$f_{Z_1}(z_1) = \int_0^1 \frac{\left|F_1^{-1}(u)\right|}{z_1^2} c\Big(u, F_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right)\Big) f_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) du.$$

If $z_1 < 0$, then also by taking $u := F_1(z_1 z_2)$, we yield

$$f_{Z_1}(z_1) = \int_1^0 \frac{\left|F_1^{-1}(u)\right|}{-z_1^2} c\Big(u, F_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right)\Big) f_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) du. \tag{22}$$

Hence, for $z_1 \neq 0$, we obtain

$$f_{Z_1}(z_1) = \int_0^1 \frac{\left|F_1^{-1}(u)\right|}{z_1^2} c\Big(u, F_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right)\Big) f_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) du \quad \text{for} \quad z_1 \neq 0.$$

As a consequence, the distribution of Z_1 becomes

$$F_{Y_1}(t) = \int_0^1 \int_{-\infty}^t \frac{\left|F_1^{-1}(u)\right|}{z_1^2} c\Big(u, F_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right)\Big) f_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) dz_1 du. \tag{23}$$

Setting $v = F_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) \quad \Longrightarrow dv = -\frac{F_1^{-1}(u)}{z_1^2} f_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) dz_1$, and note that

$$F_1^{-1}(u) \geq 0 \Longleftrightarrow u \geq F_1(0), \text{ and } F_1^{-1}(u) \leq 0 \Longleftrightarrow u \leq F_1(0),$$

we consider two cases as follows:

(i) **Case 1:** For $t < 0$, we have

$$\begin{aligned} F_{Z_1}(t) &= \int_0^{F_1(0)} \int_{F_2(-F_1^{-1}(u))}^{F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)} \frac{\partial^2}{\partial u \partial v} C(u,v)\, dv du - \int_{F_1(0)}^1 \int_{F_2(-F_1^{-1}(u))}^{F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)} \frac{\partial^2}{\partial u \partial v} C(u,v)\, dv du \\ &= \int_0^{F_1(0)} \left[\frac{\partial}{\partial u} C\left(u, F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)\right) - \frac{\partial}{\partial u} C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big)\right] du \\ &\quad - \int_{F_1(0)}^1 \left[\frac{\partial}{\partial u} C\left(u, F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)\right) - \frac{\partial}{\partial u} C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big)\right] du \\ &= \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right) \left[\frac{\partial}{\partial u} C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big) - \frac{\partial}{\partial u} C\left(u, F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)\right)\right] du. \end{aligned} \tag{24}$$

(ii) **Case 2:** For $t \geq 0$, we first split the integrals

$$\begin{aligned} F_{Z_1}(t) &= \int_0^1 \int_{-\infty}^0 \frac{\left|F_1^{-1}(u)\right|}{z_1^2} c\Big(u, F_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right)\Big) f_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) dz_1 du \\ &\quad + \int_0^1 \int_0^t \frac{\left|F_1^{-1}(u)\right|}{z_1^2} c\Big(u, F_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right)\Big) f_2\left(\frac{1-z_1}{z_1}F_1^{-1}(u)\right) dz_1 du \\ &=: I_1 + I_2. \end{aligned} \tag{25}$$

We then apply (24) to obtain the following expression for the integral I_1:

$$\begin{aligned} I_1 = F_{Z_1}(0) &= \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right) \left[\frac{\partial}{\partial u} C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big) - \frac{\partial}{\partial u} C\left(u, \lim_{t \to 0^-} F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)\right)\right] du \\ &= \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right) \frac{\partial}{\partial u} C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big) du \\ &\quad + \int_0^{F_1(0)} \frac{\partial}{\partial u} C(u,1)\, du - \int_{F_1(0)}^1 \frac{\partial}{\partial u} C(u,0)\, du \\ &= \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right) \frac{\partial}{\partial u} C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big) du + F_1(0), \end{aligned} \tag{26}$$

and obtain the following expression for the integral I_2:

$$\begin{aligned} I_2 &= \int_0^{F_1(0)} \int_0^{F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)} \frac{\partial^2}{\partial u \partial v} C(u,v)\, dv du - \int_{F_1(0)}^1 \int_1^{F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)} \frac{\partial^2}{\partial u \partial v} C(u,v)\, dv du \\ &= \int_0^{F_1(0)} \left[\frac{\partial}{\partial u} C\left(u, F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)\right) - \frac{\partial}{\partial u} C\Big(u, 0\Big)\right] du \\ &\quad - \int_{F_1(0)}^1 \left[\frac{\partial}{\partial u} C\left(u, F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)\right) - \frac{\partial}{\partial u} C\Big(u, 1\Big)\right] du \\ &= 1 - F_1(0) - \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right) \frac{\partial}{\partial u} C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big) du. \end{aligned} \tag{27}$$

From (26) and (27), we get the following for $t \geq 0$,

$$F_{Z_1}(t) = 1 + \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right)\left[\frac{\partial}{\partial u}C\Big(u, F_2\Big(-F_1^{-1}(u)\Big)\Big) - \frac{\partial}{\partial u}C\left(u, F_2\left(\frac{1-t}{t}F_1^{-1}(u)\right)\right)\right] du. \quad (28)$$

Combining (24) and (28) imply (20), we complete the proof. □

In the situation X_1 and X_2 are independent, applying Proposition 2, we obtain the following corollary:

Corollary 4. *When X_1 and X_2 are independent, then its copula $C(u,v) = uv$ has the density $c(u,v) = 1$, $\forall u, v \in \mathbb{I}$ and the density $f_Z(z)$ and distribution function $F_Z(z)$ for the ratio of one variable over the sum of two variables $Z := \frac{X_1}{X_1+X_2}$ of two independent random variables X_1 and X_2 become*

$$f_Z(z) = \int_{-\infty}^{\infty} |x|\, f_1(xz) f_2\left((1-x)z\right) dx$$

and

$$\begin{aligned} F_Z(z) &= \mathbf{1}_{\{z\geq 0\}} + \int_0^1 \operatorname{sgn}\left(F_1^{-1}(u)\right)\left[F_2\Big(-F_1^{-1}(u)\Big) - F_2\Big(\frac{1-z}{z}F_1^{-1}(u)\Big)\right] du, \\ &= \mathbf{1}_{\{z\geq 0\}} + \int_{-\infty}^{\infty} \operatorname{sgn}(x)\left[F_2(-x) - F_2\Big(\frac{1-z}{z}x\Big)\right] f_1(x)dx, \end{aligned}$$

respectively.

Next, we apply Equation (20) to derive the distribution function of $Z := \frac{X_1}{X_1+X_2}$ in the situation that both X_1 and X_2 are normal distributed such that their dependence structure can be captured by Gaussian Copulas as shown in the following corollary:

Corollary 5. *Assume that $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$, and (X_1, X_2) follows Gaussian Copulas $C_r(u,v)$, $|r| < 1$, given in (46). Then, distribution function $F_Z(z)$ of $Z := \frac{X_1}{X_1+X_2}$ has the form*

$$\begin{aligned} F_Z(z) = &= \mathbf{1}_{\{z\geq 0\}} + 2\Phi\Big(\frac{\mu_1}{\sigma_1}\Big) - 1 - \int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u) + \mu_1\Big)\Phi\Big(\frac{(\sigma_1 + r\sigma_2)\Phi^{-1}(u) + \mu_1 - \mu_2}{\sigma_2\sqrt{1-r^2}}\Big)du \\ &\quad - \int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u) + \mu_1\Big)\Phi\Big(\frac{[(1-z)\sigma_1 - zr\sigma_2]\Phi^{-1}(u) - z(\mu_1+\mu_2) + \mu_1}{z\sigma_2\sqrt{1-r^2}}\Big)du, \quad (29) \end{aligned}$$

where $\Phi(x)$ and $\Phi^{-1}(x)$ are CDF and its inverse of the standard normal random variable, respectively.

Proof. Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, the CDFs and their inverse functions can be written in the form

$$F_i(x) = \Phi\Big(\frac{x-\mu_i}{\sigma_i}\Big), \quad F_i^{-1}(v) = \sigma_i\Phi^{-1}(v) + \mu_i, \quad i = 1,2.$$

Given Gaussian Copulas $C_r(u,v)$, $|r| < 1$, we apply the results from Meyer (2013) to obtain its derivative $\frac{\partial C_r}{\partial u}(u,v)$ as shown in the following:

$$\frac{\partial C_r}{\partial u}(u,v) = \Phi\Big(\frac{\Phi^{-1}(v) - r\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big).$$

Applying Equation (20), one can simplify it to be

$$\begin{aligned}
F_Z(z) &= \mathbf{1}_{\{z\geq 0\}} + \int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u)+\mu_1\Big)\Phi\Big(\frac{\Phi^{-1}\Big(\Phi\Big(-\frac{\sigma_1\Phi^{-1}(u)+\mu_1-\mu_2}{\sigma_2}\Big)\Big)-r\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big)du \\
&\quad - \int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u)+\mu_1\Big)\Phi\Big(\frac{\Phi^{-1}\Big(\Phi\Big(\frac{\frac{1-z}{z}\big[\sigma_1\Phi^{-1}(u)+\mu_1\big]-\mu_2}{\sigma_2}\Big)\Big)-r\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big)du \\
&= \mathbf{1}_{\{z\geq 0\}} + \int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u)+\mu_1\Big)\Phi\Big(-\frac{(\sigma_1+r\sigma_2)\Phi^{-1}(u)+\mu_1-\mu_2}{\sigma_2\sqrt{1-r^2}}\Big)du \\
&\quad - \int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u)+\mu_1\Big)\Phi\Big(\frac{[(1-z)\sigma_1-zr\sigma_2]\Phi^{-1}(u)-z(\mu_1+\mu_2)+\mu_1}{z\sigma_2\sqrt{1-r^2}}\Big)du \\
&= \mathbf{1}_{\{z\geq 0\}} + 2\Phi\Big(\frac{\mu_1}{\sigma_1}\Big)-1-\int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u)+\mu_1\Big)\Phi\Big(\frac{(\sigma_1+r\sigma_2)\Phi^{-1}(u)+\mu_1-\mu_2}{\sigma_2\sqrt{1-r^2}}\Big)du \\
&\quad - \int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u)+\mu_1\Big)\Phi\Big(\frac{[(1-z)\sigma_1-zr\sigma_2]\Phi^{-1}(u)-z(\mu_1+\mu_2)+\mu_1}{z\sigma_2\sqrt{1-r^2}}\Big)du.
\end{aligned}$$

In the last step of the above, we use the property $\Phi(-x) = 1-\Phi(x)$ and

$$\int_0^1 sgn\Big(\sigma_1\Phi^{-1}(u)+\mu_1\Big)du = -\int_0^{\Phi\left(-\frac{\mu_1}{\sigma_1}\right)} du + \int_{\Phi\left(-\frac{\mu_1}{\sigma_1}\right)}^1 du = 1-2\Phi\Big(-\frac{\mu_1}{\sigma_1}\Big) = 2\Phi\Big(\frac{\mu_1}{\sigma_1}\Big)-1.$$

The proof is complete. □

We note that the probability given in (29) can also be easily computed by using the following Monte Carlo algorithm: For each $z \in \mathbb{R}$, we first generate U from the uniform distribution on the unit interval $[0,1]$ with sample size N, say $N = 10,000$. Then, we obtain the following estimated probability:

$$\begin{aligned}
\widehat{F}_Z(z) &\approx \mathbf{1}_{\{z\geq 0\}} + 2\Phi\Big(\frac{\mu_1}{\sigma_1}\Big)-1-\frac{1}{N}\sum_{i=1}^N sgn\Big(\sigma_1\Phi^{-1}(u_i)+\mu_1\Big)\Phi\Big(\frac{(\sigma_1+r\sigma_2)\Phi^{-1}(u_i)+\mu_1-\mu_2}{\sigma_2\sqrt{1-r^2}}\Big) \\
&\quad -\frac{1}{N}\sum_{i=1}^N sgn\Big(\sigma_1\Phi^{-1}(u_i)+\mu_1\Big)\Phi\Big(\frac{[(1-z)\sigma_1-zr\sigma_2]\Phi^{-1}(u_i)-z(\mu_1+\mu_2)+\mu_1}{z\sigma_2\sqrt{1-r^2}}\Big).
\end{aligned} \tag{30}$$

Using the above results, we obtain the following corollary:

Corollary 6. *Assume that $X_1 \sim N(0,\sigma^2)$, $X_2 \sim N(0,\sigma^2)$ and (X_1,X_2) follows Gaussian Copulas $C_r(u,v)$, $|r|<1$, given in (46). Then, the median of $Z := \frac{X_1}{X_1+X_2}$ is equal to $\frac{1}{2}$.*

Proof. Because $X_1 \sim N(0,\sigma^2)$, $X_2 \sim N(0,\sigma^2)$ and $sgn\Big(\sigma\Phi^{-1}(u)\Big) = sgn\Big(\Phi^{-1}(u)\Big)$, we obtain CDF of the ratio $Z := \frac{X_1}{X_1+X_2}$ from (29) as shown in the following:

$$F_Z\Big(z\Big) = \mathbf{1}_{\{z\geq 0\}} - \int_0^1 sgn\Big(\Phi^{-1}(u)\Big)\Big[\Phi\Big(\frac{(1+r)\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big)+\Phi\Big(\frac{[1-z-zr]\Phi^{-1}(u)}{z\sqrt{1-r^2}}\Big)\Big]du. \tag{31}$$

We get

$$F_Z\Big(\frac{1}{2}\Big) = 1 - \int_0^1 sgn\Big(\Phi^{-1}(u)\Big)\Big[\Phi\Big(\frac{(1+r)\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big)+\Phi\Big(\frac{[1-r]\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big)\Big]du. \tag{32}$$

Hence, it is sufficient to prove that the integral term given in (32) is equal to $\frac{1}{2}$. We let

$$I_1 := \int_0^1 sgn\Big(\Phi^{-1}(u)\Big)\Phi\Big(\frac{(1+r)\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big)du, \quad I_2 := \int_0^1 sgn\Big(\Phi^{-1}(u)\Big)\Phi\Big(\frac{(1-r)\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big)du,$$

and denote $\partial_i C_r(u,v), i = 1,2$ to be the partial derivative of $C_r(u,v)$ with respect to the ith variable, That is, $\partial_1 C_r(u,v) = \frac{\partial}{\partial u}C_r(u,v)$ and $\partial_2 C_r(u,v) = \frac{\partial}{\partial v}C_r(u,v)$. One can observe that

$$\partial_1 C_r(u,u) = \Phi\Big(\frac{(1-r)\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big).$$

Since Gaussian Copulas is symmetric; that is, $C_r(u,v) = C_r(v,u)$, we have $\partial_1 C_r(u,v) = \partial_2 C_r(v,u)$, and, thus, for $u = v$, we can derive $\partial_1 C_r(u,u) = \partial_2 C_r(u,u)$. Thereafter, the differentiation of $C_r(u,u)$ can be obtained:

$$dC_r(u,u) = \big[\partial_1 C_r(u,u) + \partial_2 C_r(u,u)\big]du = 2\partial_1 C_r(u,u),$$

and we get

$$\begin{aligned} I_2 = \frac{1}{2}\int_0^1 sgn\Big(\Phi^{-1}(u)\Big)dC_r(u,u) &= -\frac{1}{2}\int_0^{1/2} dC_r(u,u) + \frac{1}{2}\int_{1/2}^1 dC_r(u,u) \\ &= -\frac{1}{2}C_r\Big(\frac{1}{2},\frac{1}{2}\Big) + \frac{1}{2}\Big[1 - C_r\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] \\ &= \frac{1}{2} - C_r(\frac{1}{2},\frac{1}{2}). \end{aligned} \tag{33}$$

Similarly, for I_1, since $\Phi^{-1}(1-u) = -\Phi^{-1}(u)$, we obtain

$$\partial_1 C_r(u,1-u) = \Phi\Big(\frac{\Phi^{-1}(1-u) - r\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big) = \Phi\Big(-\frac{(1+r)\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big) = 1 - \Phi\Big(\frac{(1+r)\Phi^{-1}(u)}{\sqrt{1-r^2}}\Big),$$

and get

$$I_1 = \int_0^1 sgn\Big(\Phi^{-1}(u)\Big)du - \int_0^1 sgn\Big(\Phi^{-1}(u)\Big)\partial_1 C_r(u,1-u)du = -\int_0^1 sgn\Big(\Phi^{-1}(u)\Big)\partial_1 C_r(u,1-u)du,$$

in which we apply $\int_0^1 sgn\Big(\Phi^{-1}(u)\Big)du = 0$. From symmetry of the Gaussian Copulas, we also have $\partial_1 C_r(1-u,u) = \partial_2 C_r(u,1-u)$, obtain the differentiation of $C_r(u,1-u)$ given by

$$dC_r(u,1-u) = \big[\partial_1 C_r(u,1-u) - \partial_2 C_r(u,1-u)\big]du = \big[\partial_1 C_r(u,1-u) - \partial_1 C_r(1-u,u)\big]du,$$

and get

$$\begin{aligned} I_1 &= \int_0^{1/2} \partial_1 C_r(u,1-u)du - \int_{1/2}^1 \partial_1 C_r(u,1-u)du, \\ &= \int_0^{1/2} \partial_1 C_r(u,1-u)du + \int_{1/2}^0 \partial_1 C_r(1-u,u)du \\ &= \int_0^{1/2} \partial_1 C_r(u,1-u)du - \int_0^{1/2} \partial_1 C_r(1-u,u)du \\ &= \int_0^{1/2} dC_r(u,1-u) \\ &= C_r\Big(\frac{1}{2},\frac{1}{2}\Big). \end{aligned} \tag{34}$$

Combining (32)–(34), we find that $F_Z\left(\frac{1}{2}\right) = \frac{1}{2}$. Hence, the quantity $\frac{1}{2}$ is the median of Z. The proof is complete. □

Applying the proof of Corollary 6, we extend the result to obtain the following corollary for a larger family of symmetric distribution and symmetric copulas:

Corollary 7. *Assume that X_1 and X_2 are identically and symmetrically distributed with distribution F that has zero median and the dependence structure of (X_1, X_2) is modelled by a family of symmetric copulas $C(u,v)$, i.e., $C(u,v) = C(v,u)$ for all $u, v \in \mathbb{I}$. Then, the median of $Z := \frac{X_1}{X_1+X_2}$ is equal to $\frac{1}{2}$.*

Proof. Since X_1 and X_2 are identically and symmetrically distributed with distribution F and zero median, we have $F(-x) = 1 - F(x)$, for all $x \in \mathbb{R}$. By applying Equation (20), we obtain the CDF of the ratio Z, which is defined by

$$\begin{aligned} F_Z(z) &= \mathbf{1}_{\{z\geq 0\}} + \int_0^1 \operatorname{sgn}\left(F^{-1}(u)\right)\left[\frac{\partial}{\partial u}C\Big(u, F\Big(-F^{-1}(u)\Big)\Big) - \frac{\partial}{\partial u}C\left(u, F\left(\frac{1-z}{z}F^{-1}(u)\right)\right)\right]du, \\ &= \mathbf{1}_{\{z\geq 0\}} + \int_0^1 \operatorname{sgn}\left(F^{-1}(u)\right)\left[\frac{\partial}{\partial u}C\Big(u, 1-u\Big) - \frac{\partial}{\partial u}C\left(u, F\left(\frac{1-z}{z}F^{-1}(u)\right)\right)\right]du. \end{aligned} \tag{35}$$

Since copulas $C(u,v)$ are symmetric, i.e., $C(v,u) = C(u,v)$, we have $\partial_1 C(v,u) = \partial_2 C(u,v)$, and, thus, for $v = 1-u$, one can easily obtain $\partial_1 C(1-u,u) = \partial_2 C(u,1-u)$ and find that the differentiation of $C(u,1-u)$ with respect to u satisfies

$$dC(u,1-u) = \left[\partial_1 C(u,1-u) - \partial_2 C(u,1-u)\right]du = \left[\partial_1 C(u,1-u) - \partial_1 C(1-u,u)\right]du.$$

In addition, because the distribution F has zero median; that is, $F(0) = 0.5$, we have

$$\begin{aligned} \int_0^1 \operatorname{sgn}\left(F^{-1}(u)\right)\frac{\partial}{\partial u}C\Big(u,1-u\Big) &= -\int_0^{1/2}\partial_1 C(u,1-u)du + \int_{1/2}^1 \partial_1 C(u,1-u)du, \\ &= -\int_0^{1/2}\partial_1 C(u,1-u)du - \int_{1/2}^0 \partial_1 C(1-u,u)du \\ &= -\int_0^{1/2}\partial_1 C(u,1-u)du + \int_0^{1/2} \partial_1 C(1-u,u)du \\ &= -\int_0^{1/2} dC(u,1-u) = -C\Big(\frac{1}{2},\frac{1}{2}\Big). \end{aligned}$$

Therefore, we get

$$F_Z(z) = \mathbf{1}_{\{z\geq 0\}} - C\Big(\frac{1}{2},\frac{1}{2}\Big) - \int_0^1 \operatorname{sgn}\left(F^{-1}(u)\right)\frac{\partial}{\partial u}C\left(u, F\left(\frac{1-z}{z}F^{-1}(u)\right)\right)du, \tag{36}$$

and thus,

$$F_Z(0.5) = 1 - C\Big(\frac{1}{2},\frac{1}{2}\Big) - \int_0^1 \operatorname{sgn}\left(F^{-1}(u)\right)\frac{\partial}{\partial u}C\left(u,u\right)du. \tag{37}$$

Similarly, since copulas $C(u,v)$ are symmetric, i.e., $C(v,u) = C(u,v)$, we have $\partial_1 C(v,u) = \partial_2 C(u,v)$, and, thus, for $v = u$, we get $\partial_1 C(u,u) = \partial_2 C(u,u)$. Thus, the differentiation of $C(u,u)$ with respect to u satisfies

$$dC(u,u) = \left[\partial_1 C(u,u) + \partial_2 C(u,u)\right]du = 2\partial_1 C(u,u)du.$$

Applying this relation, we find

$$\int_0^1 \operatorname{sgn}\left(F^{-1}(u)\right) \frac{\partial}{\partial u} C(u,u)\, du = \frac{1}{2} \int_0^1 \operatorname{sgn}\left(F^{-1}(u)\right) dC(u,u) = \frac{1}{2} - C\Big(\frac{1}{2}, \frac{1}{2}\Big).$$

Hence, $F_Z(0.5) = 0.5$, i.e., $\frac{1}{2}$ is median of Z. The proof is complete. □

Remark: In the literature, they are many symmetric distributions with zero median, for example, normal $N(0,\sigma)$, Student-t t_ν, Cauchy distribution with location parameter $\alpha = 0$, uniform $U(-a,a), a \in \mathbb{R}_+$, and logistic distribution with zero location. In addition, Elliptical copulas (Gaussian, Student-t copulas) and Archimedian copulas (Clayton, Gumbel, Frank, Joe,...) are classes of symmetric copulas. Thus, if we apply (X_1, X_2) with these distributions and these copulas, Proposition 7 tells us that the random variable $Z := \frac{X_1}{X_1+X_2}$ always gets the median one-half. This theoretical result is consistent with our simulation results displayed in the next section.

We note that the formulas given in (8), (9), (19) and (20) may not have closed forms. However, they are easily computed by using the Monte Carlo (MC) simulation method or any techniques of numerical integration. We provide simulation studies in the next section.

5. A Simulation Study

Since the density and the CDF formula of the ratio $Y = \frac{X_1}{X_2}$ $[Z = \frac{X_1}{X_1+X_2}]$ expressed in (8) and (9) ((19) and (20)) are in terms of integrals and are very complicated, we cannot obtain the exact forms of the density and the CDF. To circumvent the difficulty, in this paper, we propose to use the Monte Carlo algorithm, numerical analysis and graphical approach to study behavior of density and distribution and the changes of their shapes when parameters are changing.

Suppose that X_1 and X_2 are normally distributed and denoted by $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1,2$ with PDF given by

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\Big(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\Big).$$

Without loss of generality, we consider $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. We note that, if X_1 and X_2 are independent and standard normal distributed, then it is well known that $Y = \dfrac{X_1}{X_2}$ follows standard Cauchy distribution. The Cauchy distribution is a type of distribution that has no mean and also does not exist any higher moments. To circumvent this problem, one may use median to measure the central tendency and use range or interquartile range to measure the spread of the distribution. The general Cauchy distribution has the following PDF:

$$f(x) := \frac{\beta}{\pi(\beta^2 + (x-\alpha)^2)}, \quad \alpha \in \mathbb{R}, \beta > 0.$$

Thus, location and scale parameter of $Y = \dfrac{X_1}{X_2}$ are $\alpha = 0$ and $\beta = 1$, respectively, if X_1 and X_2 are identically independent standard normal distributed.

We now investigate different dependence structures of X_1 and X_2 through several families of copulas and observe the shapes of the corresponding distributions for both Y and Z as well as estimate their percentiles at different levels including $0.05, 0.25, 0.5, 0.75, 0.95$ and denote the corresponding percentiles to be $Q_{0.05}, Q_{0.25}, Q_{0.50}, Q_{0.75}$, and $Q_{0.95}$, respectively. In risk analysis, the percentile $Q_{0.05}$ is often used as the Value-at-Risk (VaR) 5% while $Q_{0.25}$ and $Q_{0.75}$ are, respectively, called the first and third quartile of the random variable and the interquartile range (IQR) is defined by their difference; that is, $IQR = Q_{0.75} - Q_{0.25}$. The median measures the center of the distribution, which is equal to $Q_{0.5}$.

For each copula $C_\theta(u,v)$, the PDF and CDF of Y and Z can be plotted on the interval $[-4,4]$ by using the following steps:

(i) For each pair of y and $z \in \{-4, -3.9, -3.8, \ldots, 4\}$, generate the uniform random variables U and V on the unit interval; that is, $U, V \sim Uniform[0, 1]$ with the sample size N, say N = 10,000;

(ii) estimate the values for $f_Y(y)$, $F_Y(y)$, $f_Z(z)$, and $F_Z(z)$ by using

$$\widehat{f}_Y(y) \approx \frac{1}{N}\sum_{i=1}^{N}|F_2^{-1}(v_i)|c\Big(F_1\left(yF_2^{-1}(v_i)\right), v_i\Big)f_1\left(yF_2^{-1}(v_i)\right), \tag{38}$$

$$\widehat{F}_Y(y) \approx F_2(0) + \frac{1}{N}\sum_{i=1}^{N}\operatorname{sgn}\left(F_2^{-1}(v_i)\right)\frac{\partial}{\partial v}C\left(F_1\left(yF_2^{-1}(v_i)\right), v_i\right), \tag{39}$$

$$\widehat{f}_Z(z) \approx \begin{cases} \frac{1}{N}\sum\limits_{i=1}^{N}\frac{|F_1^{-1}(u_i)|}{z^2}c\Big(u_i, F_2\left(\frac{1-z}{z}F_1^{-1}(u_i)\right)\Big)f_2\left(\frac{1-z}{z}F_1^{-1}(u_i)\right), & \text{if } z \neq 0, \\ \frac{f_1(0)}{N}\sum\limits_{i=1}^{N}\left|F_2^{-1}(v_i)\right|c\Big(F_1(0), v_i\Big), & \text{if } z = 0, \end{cases} \tag{40}$$

$$\widehat{F}_Z(z) \approx \mathbf{1}_{\{z\geq 0\}} + \frac{1}{N}\sum_{i=1}^{N}\operatorname{sgn}\left(F_1^{-1}(u_i)\right)\Big[\frac{\partial C}{\partial u}(u_i, F_2(-F_1^{-1}(u_i))) - \frac{\partial C}{\partial u}(u_i, F_2(\frac{1-z}{z}F_1^{-1}(u_i)))\Big], \tag{41}$$

in which the density copula $c_\theta(u_i, v_i)$, the derivatives $\frac{\partial}{\partial u}C_\theta(u_i, v_i)$ and $\frac{\partial}{\partial v}C_\theta(u_i, v_i)$ can be obtained by using the packages of *VineCopula* in *R* language; and

(iii) plot $\widehat{f}_Y(y)$, $\widehat{F}_Y(y)$, $\widehat{f}_Z(z)$, and $\widehat{F}_Z(z)$ with $y, z \in \{-4, -3.9, -3.8, \cdots, 3.9, 4\}$.

To estimate percentiles Q_α's of Y and Z, we first construct the joint distribution of (X_1, X_2) by using Sklar's Theorem as shown in the following: For each copula $C_\theta(u, v)$, we first obtain the joint CDF of (X_1, X_2) such that

$$H_\theta(x_1, x_2) = C_\theta\Big(F_1(x_1), F_2(x_2)\Big).$$

We then repeat 5000 times, $k = 1, 2, \ldots, 5000$ for the following steps in the computation:

(1) For each repetition $k = 1, 2, \ldots, 5000$,

(i) generate (X_1, X_2) from $H_\theta(x_1, x_2)$ of sample size 10^4 by using the package *copula* in *R* language and define

$$y_i^{(k)} = \frac{x_{1i}^{(k)}}{x_{2i}^{(k)}}, \quad i = 1, 2, \cdots, 10^4; \tag{42}$$

$$z_i^{(k)} = \frac{x_{1i}^{(k)}}{x_{1i}^{(k)} + x_{2i}^{(k)}}, \quad i = 1, 2, \cdots, 10^4; \tag{43}$$

(ii) estimate the percentiles Q_α with $\alpha = 0.05, 0.25, 0.5, 0.75, 0.95$ for both Y and Z by using the following formula

$$\widehat{Q}_\alpha(Y^{(k)}) = y_{(\lfloor h\rfloor)}^{(k)} + (h - \lfloor h\rfloor)\left(y_{(\lfloor h\rfloor+1)}^{(k)} - y_{(\lfloor h\rfloor)}^{(k)}\right), \quad \textit{with } h = \left(10^4 - 1\right)\alpha + 1, \tag{44}$$

$$\widehat{Q}_\alpha(Z^{(k)}) = z_{(\lfloor h\rfloor)}^{(k)} + (h - \lfloor h\rfloor)\left(z_{(\lfloor h\rfloor+1)}^{(k)} - z_{(\lfloor h\rfloor)}^{(k)}\right), \tag{45}$$

where $y_{(1)}^{(k)} \leq y_{(2)}^{(k)} \leq \ldots \leq y_{(N)}^{(k)}$ and $z_{(1)}^{(k)} \leq z_{(2)}^{(k)} \leq \ldots \leq z_{(N)}^{(k)}$ denote the order statistics of both $y^{(k)}$ and $z^{(k)}$, respectively, and $\lfloor h\rfloor$ denotes integer part of h.

(2) Finally, we take average for each of the above quantities by using the following formula:

$$\widehat{Q}_\alpha(Y) = \frac{1}{5000}\sum_{k=1}^{5000} \widehat{Q}_\alpha(Y^{(k)}),$$
$$\widehat{Q}_\alpha(Z) = \frac{1}{5000}\sum_{k=1}^{5000} \widehat{Q}_\alpha(Z^{(k)}),$$

to obtain the estimates of the percentiles for Y and Z.

We note that the algorithms discussed in the above can be applied to any non-symmetric marginal distribution and skewed copulas family with absolutely continuous random variable that could contain negative range, for example, generalized skewed-*t* distribution and skewed-*t* Copulas. [2]

5.1. Gaussian Copulas

We first investigate dependence structures of X_1 and X_2 through Gaussian Copulas $C_r(u,v)$ and observe the shapes of the corresponding distributions for both Y and Z.

$$C_r(u,v) = \frac{1}{2\pi\sqrt{1-r^2}}\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \exp\left(-\frac{s^2-2rst+t^2}{2(1-r^2)}\right) dsdt, \tag{46}$$

where $\Phi^{-1}(x)$ is the inverse of standard normal CDF and r is Pearson's correlation coefficient between X_1 and X_2, $|r| < 1$. We now consider the cases with $r = -0.9, -0.5, 0, 0.5, 0.9$. When $r = 0$, it is corresponding to the independence situation, and we get PDFs and CDFs of Y and Z shown in Figures 1 and 2, respectively. As can be seen from the Figures and Tables 1 and 2, when the parameter r varies from negative to positive, the median is totally equal to the Pearson's correlation coefficient r. The more correlated, that is, the higher $|r|$ between X_1 and X_2 is, the smaller the spread, that is, IQR of Y, becomes. In contrast to Y, the center of Z is definitely unchanged (0.5), but the scale parameter of Z is smaller indicated by the higher height of the density. The shapes of both Y and Z are symmetric.

Table 1. Some percentiles of $Y = X_1/X_2$, where (X_1, X_2) follows Gaussian Copulas.

r	$\tau(C_r)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
−0.9	−0.71	−3.65	−1.34	−0.9	−0.46	1.85
−0.5	−0.33	−5.97	−1.37	−0.5	0.37	4.97
0	0	−6.31	−1.00	0.0	1.00	6.31
0.5	0.33	−4.97	−0.37	0.5	1.37	5.97
0.9	0.71	−1.85	0.46	0.9	1.34	3.65

Table 2. Some percentiles of $Z = X_1/(X_1 + X_2)$, where (X_1, X_2) follows Gaussian Copulas.

r	$\tau(C_r)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
−0.9	−0.71	−13.27	−1.68	0.5	2.68	14.26
−0.5	−0.33	−4.97	−0.37	0.5	1.37	5.97
0	0	−2.66	0.00	0.5	1.00	3.65
0.5	0.33	−1.33	0.21	0.5	0.79	2.32
0.9	0.71	−0.23	0.39	0.5	0.61	1.22

[2] We would like to show our appreciation to the anonymous reviewer to giving us helpful comments so that we could draw this conclusion.

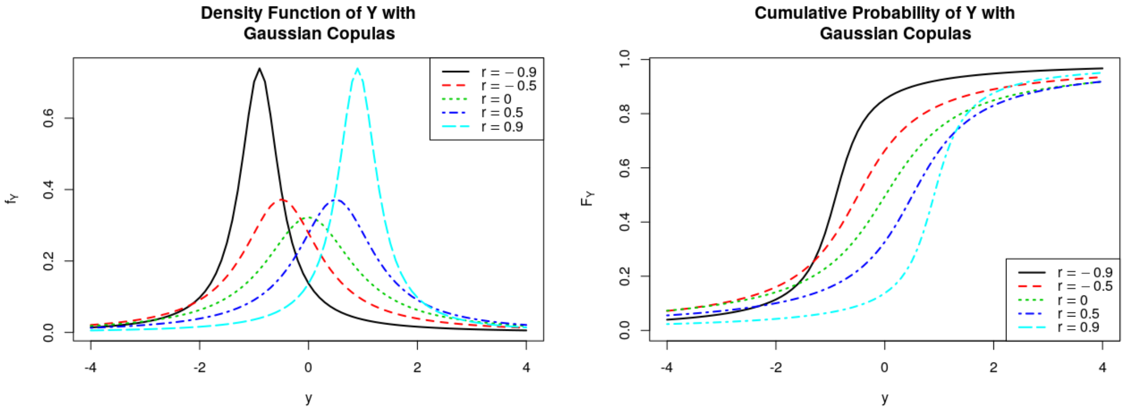

Figure 1. PDFs and CDFs of the ratio $Y = \frac{X_1}{X_2}$, where (X_1, X_2) follows Gaussian Copulas.

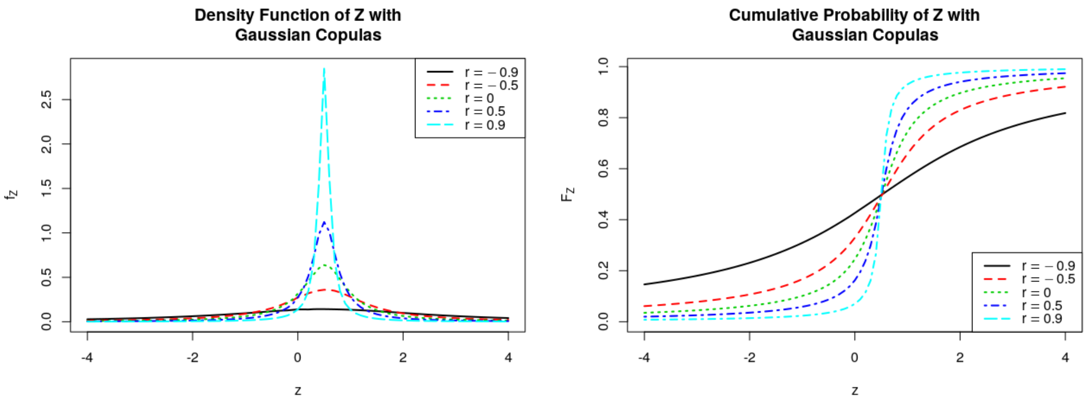

Figure 2. PDFs and CDFs of the ratio $Z = \frac{X_1}{X_1+X_2}$, where (X_1, X_2) follows Gaussian Copulas.

5.2. Student-t Copulas

We then investigate dependence structures of X_1 and X_2 through Student-t Copulas $C_{r,\nu}(u,v)$ and observe the shapes of the corresponding distributions of both Y and Z:

$$C_{r,\nu}(u,v) = \frac{1}{2\pi\sqrt{1-r^2}} \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \left(1 + \frac{s^2 - 2rst + t^2}{v(1-r^2)}\right)^{(v+2)/2} dsdt,$$

where $t_\nu^{-1}(x)$ is the inverse of Student CDF with degrees of freedom ν, r denotes Pearson's correlation coefficient between X_1 and X_2, and $|r| < 1$, and $\nu > 2$ is the degrees of freedom. We also consider $r = -0.9, -0.5, 0, 0.5, 0.9$ with three degrees of freedom ($\nu = 3$), where $r = 0$ is corresponding to no linear correlation. The PDFs and CDFs of Y of Z are, respectively, represented in Figures 3 and 4. Some percentiles are estimated and displayed in Tables 3 and 4. Similarly to Gaussian copula, in this case, the center and spread of Y and Z are also varying in the same way. However, one can see a representation of skewness and tailedness, that is, right skewed if $r < 0$ and left skewed if $r > 0$, since the fact that Student-t Copulas can capture tail dependence between X_1 and X_2.

Table 3. Some percentiles of $Y = X_1/X_2$, where (X_1, X_2) follows Student-t Copulas, $\nu = 3$.

r	$\tau(C_r)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
−0.9	−0.71	−3.42	−1.26	−0.92	−0.50	1.82
−0.5	−0.33	−5.18	−1.28	−0.56	0.40	4.42
0	0	−5.40	−1.00	0.00	1.00	5.41
0.5	0.33	−4.42	−0.40	0.56	1.28	5.18
0.9	0.71	−1.81	0.50	0.92	1.26	3.42

Table 4. Some percentiles of $Y = X_1/(X_1 + X_2)$, where (X_1, X_2) follows Student-t Copulas, $\nu = 3$.

r	$\tau(C_r)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
−0.9	−0.71	−18.20	−2.05	0.5	3.05	19.21
−0.5	−0.33	−6.62	−0.45	0.5	1.45	7.61
0	0	−3.35	0.00	0.5	1.00	4.35
0.5	0.33	−1.54	0.24	0.5	0.76	2.54
0.9	0.71	−0.24	0.40	0.5	0.60	1.24

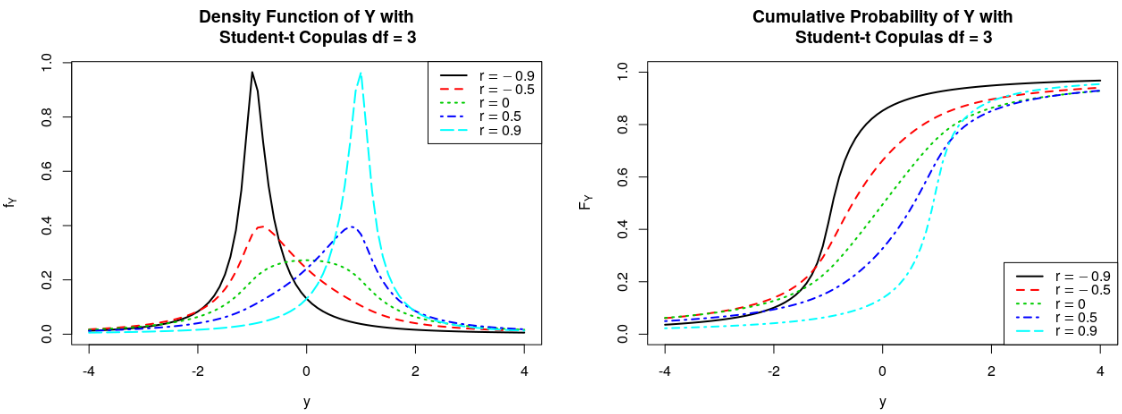

Figure 3. PDFs and CDFs of the ratio $Y = \frac{X_1}{X_2}$, where (X_1, X_2) follows Student-t Copulas, $\nu = 3$.

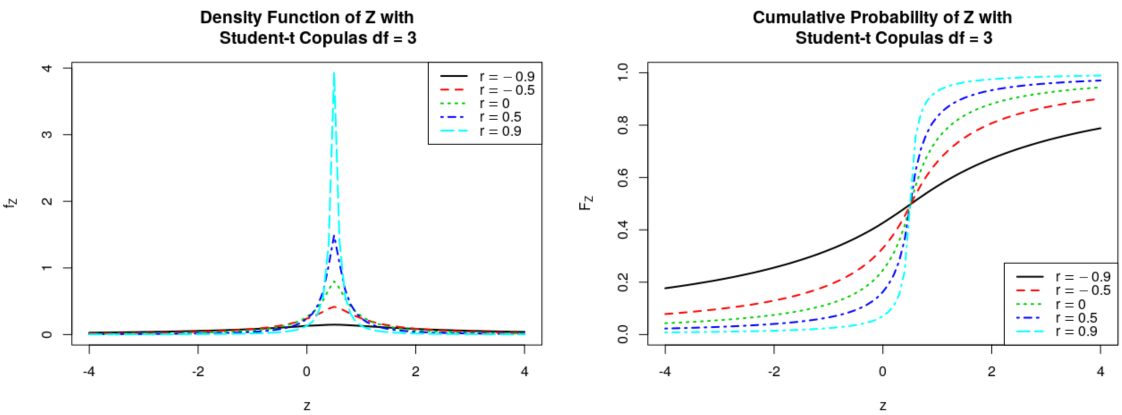

Figure 4. PDFs and CDFs of the ratio $Z = \frac{X_1}{X_1+X_2}$, where (X_1, X_2) follows Student-t Copulas, $\nu = 3$.

5.3. Clayton Copulas

We turn to investigate dependence structures of X_1 and X_2 through the following Clayton Copulas $C_\theta(u, v)$ and observe the shapes of the corresponding distributions of both Y and Z:

$$C_\theta(u,v) = \max\left\{u^{-\theta} + v^{-\theta} - 1\right\}^{-\frac{1}{\theta}}, \ \ \theta \in [-1;+\infty)\setminus 0.$$

In practice, we use $\theta > 0$ that leads to

$$C_\theta(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \ \ \theta > 0.$$

For $\theta = 1,2,3,4$, we obtain CDFs and PDFs of Y and Z and exhibit them in Figures 5 and 6, respectively, and their percentiles as shown in Tables 5 and 6. Clearly, Clayton Copulas affect Y to get heavier left tail and the more positive dependence is; that is, $\theta \to \infty$, the greater the median and the smaller the IQR of Y tend to be. On the other hand, the shape of distribution of Z is still symmetric with unchanged median, less spread, and more spike.

Table 5. Some percentiles of $Y = X_1/X_2$, where (X_1, X_2) follows Clayton Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0.33	−4.97	−0.36	0.53	1.31	5.83
2	0.5	−3.83	0.03	0.76	1.34	5.24
3	0.60	−2.87	0.25	0.86	1.34	4.71
4	0.67	−2.12	0.39	0.91	1.32	4.25

Table 6. Some percentiles of $Y = X_1/(X_1 + X_2)$, where (X_1, X_2) follows Clayton Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0.33	−1.31	0.22	0.5	0.78	2.31
2	0.5	−0.73	0.30	0.5	0.70	1.73
3	0.60	−0.42	0.35	0.5	0.65	1.42
4	0.67	−0.24	0.37	0.5	0.63	1.24

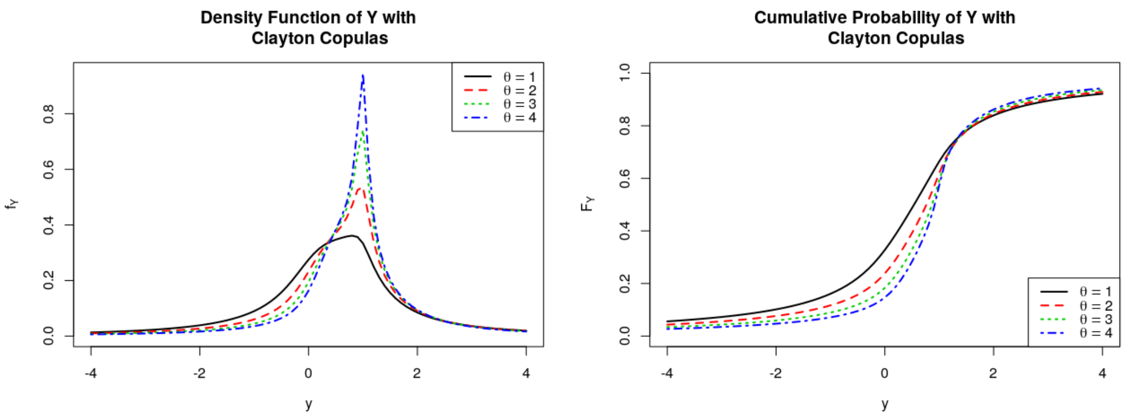

Figure 5. PDFs and CDFs of the ratio $Y = \frac{X_1}{X_2}$, where (X_1, X_2) follows Clayton Copulas.

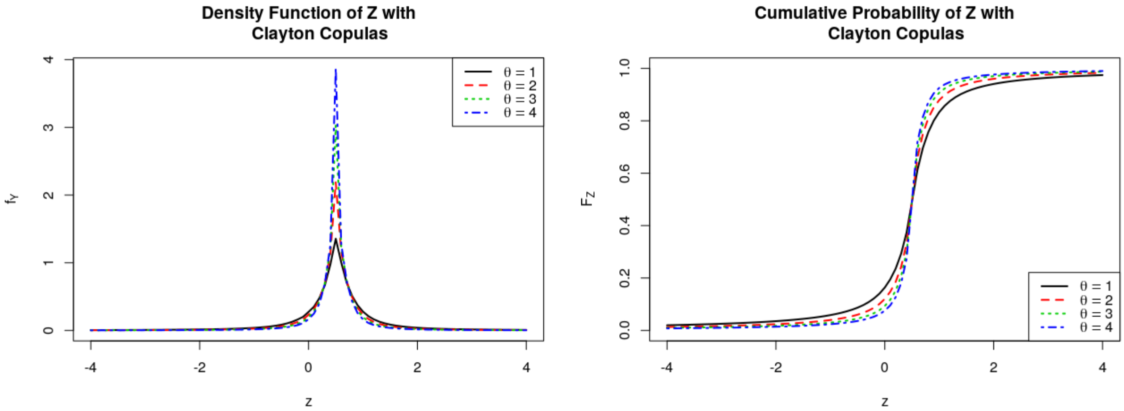

Figure 6. PDFs and CDFs of the ratio $Z = \frac{X_1}{X_1+X_2}$, where (X_1, X_2) follows Clayton Copulas.

5.4. Gumbel Copulas

We now investigate dependence structures of X_1 and X_2 through the following Gumbel Copulas $C_\theta(u,v)$ and observe the shapes of the corresponding distributions of both Y and Z:

$$C_\theta(u,v) = \exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{\frac{1}{\theta}}\right), \quad \theta > 0.$$

The parameter $\theta = 1$ implies uncorrelated (X_1, X_2). Figures 7 and 8 show the behavior of Y and Z via the copulas. Tables 7 and 8 represent some estimated percentiles for both Y and Z. In the comparison with Clayton, Gumbel Copula gets left skewness, higher median, and less spread for Y, but gets a symmetric shape, unchanged median, less spread, and smaller scale (higher spike) for Z. However, the shape of Y tends to be more symmetric when one increases the parameter θ.

Table 7. Some percentiles of $Y = X_1/X_2$, where (X_1, X_2) follows Gumbel Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0	−6.31	−1.00	0.00	1.00	6.32
2	0.5	−3.70	0.00	0.74	1.36	5.01
3	0.67	−2.26	0.38	0.89	1.32	3.95
4	0.75	−1.45	0.56	0.94	1.27	3.30

Table 8. Some percentiles of $Y = X_1/(X_1 + X_2)$, where (X_1, X_2) follows Gumbel Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0	−2.66	0.00	0.5	1.00	3.66
2	0.5	−0.82	0.30	0.5	0.70	1.82
3	0.67	−0.34	0.37	0.5	0.63	1.34
4	0.75	−0.12	0.41	0.5	0.59	1.12

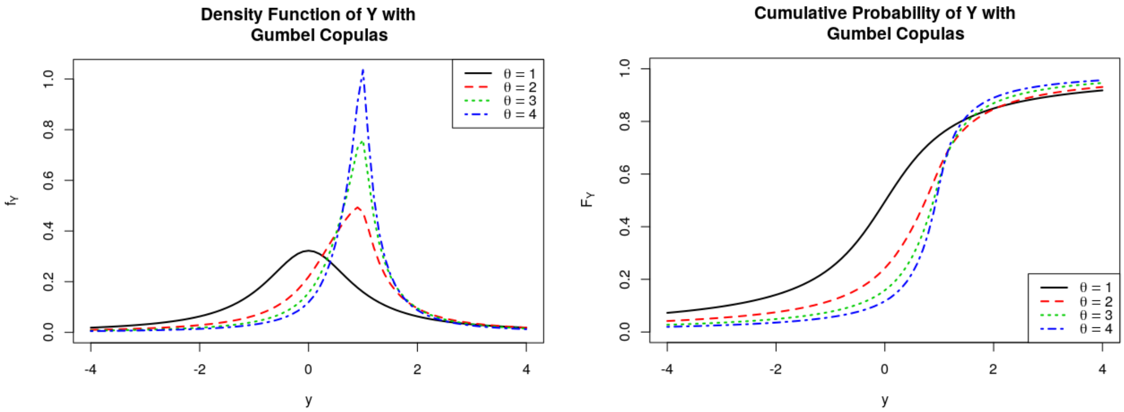

Figure 7. PDFs and CDFs of the ratio $Y = \frac{X_1}{X_2}$, where (X_1, X_2) follows Gumbel Copulas.

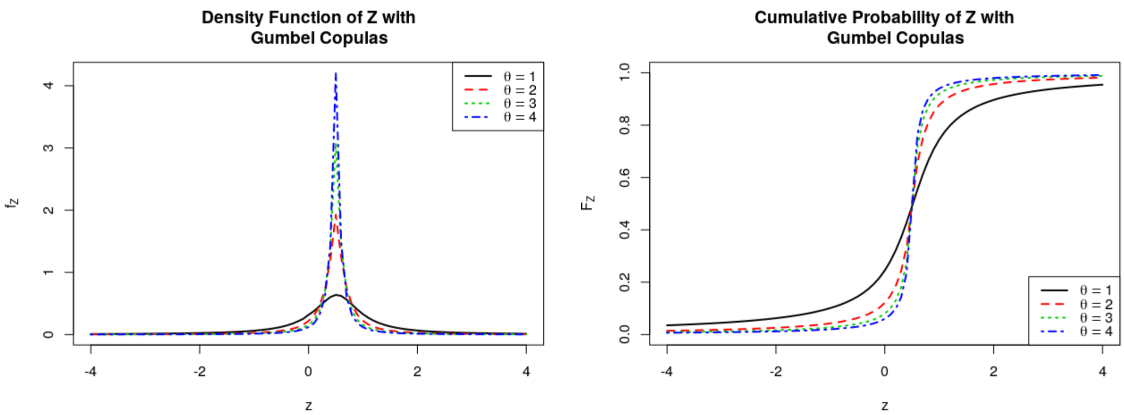

Figure 8. PDFs and CDFs of the ratio $Z = \frac{X_1}{X_1+X_2}$, where (X_1, X_2) follows Gumbel Copulas.

5.5. Frank Copulas

In addition, we investigate dependence structures of X_1 and X_2 through the following Frank Copulas $C_\theta(u, v)$ and observe the shapes of the corresponding distributions of both Y and Z:

$$C_\theta(u, v) = -\frac{1}{\theta} \ln \left(1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{e^{-\theta} - 1} \right), \quad \theta \in \mathbb{R} \backslash \{0\}.$$

For Frank Copulas, the parameter θ represents two independent random variables when it tends to zero, becomes more monotonic structure when $\theta \to \infty$, and becomes more counter-monotonic when $\theta \to -\infty$. For $\theta = 1, 2, 3, 4$, we have Tables 9 and 10, and Figures 9 and 10. In contrast to both Clayton and Gumbel Copulas, the density of Y via Frank Copulas is more symmetric and the density of Z is not scaling too much, but for the median and spread of Y, it behaves like the Gumbel Copulas and Clayton Copulas; that is, if we increase the parameter θ, the median also increases, whereas the spread (via IQR) decreases.

Table 9. Some percentiles of $Y = X_1/X_2$, where (X_1, X_2) follows Frank Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0.11	−6.03	−0.79	0.19	1.18	6.42
2	0.21	−5.59	−0.56	0.36	1.31	6.37
3	0.31	−5.05	−0.33	0.50	1.40	6.19
4	0.39	−4.46	−0.14	0.60	1.46	5.90

Table 10. Some percentiles of $Y = X_1/(X_1 + X_2)$, where (X_1, X_2) follows Frank Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0.11	−2.09	0.09	0.5	0.91	3.09
2	0.21	−1.62	0.16	0.5	0.84	2.62
3	0.31	−1.25	0.21	0.5	0.79	2.25
4	0.39	−0.96	0.25	0.5	0.75	1.96

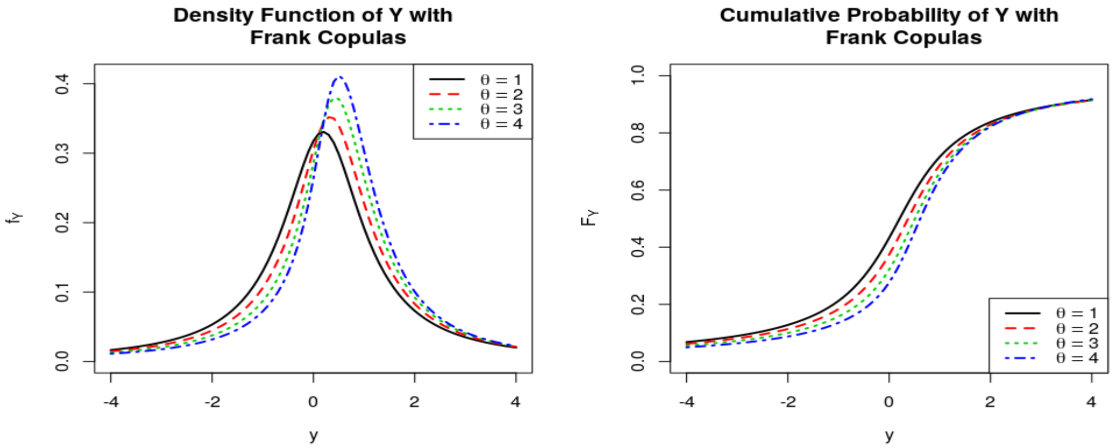

Figure 9. PDF and CDF of quotient of the ratio $Y = \frac{X_1}{X_2}$, where (X_1, X_2) follows Frank Copulas.

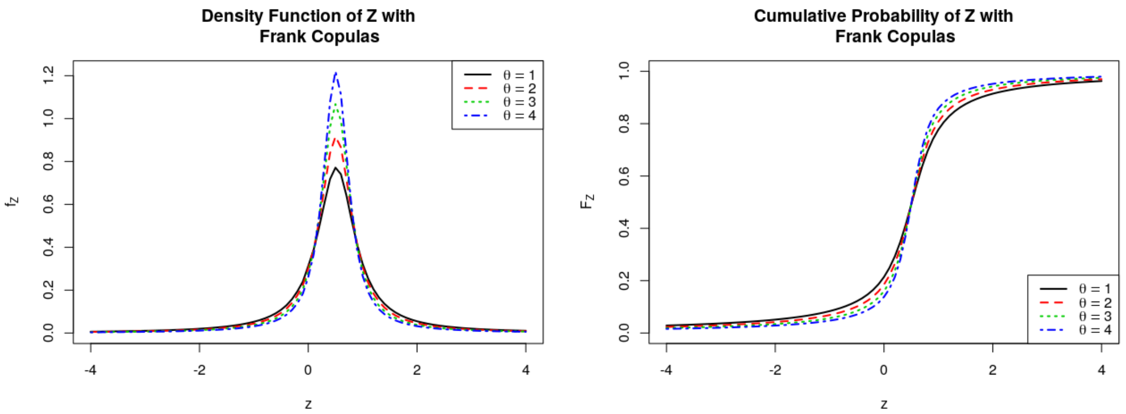

Figure 10. PDF and CDF of the ratio $Z = \frac{X_1}{X_1+X_2}$, where (X_1, X_2) follows Frank Copulas.

5.6. Joe Copulas

Finally, we investigate dependence structures of X_1 and X_2 through the following Joe Copulas $C_\theta(u, v)$ and observe the shapes of the corresponding distributions of both Y and Z:

$$C_\theta(u, v) = 1 - [(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta]^{1/\theta}, \quad \theta \in [1, \infty).$$

Similar to Gumbel Copulas, Joe Copula shows independence when $\theta = 1$ and becomes more monotonicity if $\theta \to \infty$. With assistance of the tables and graphs with $\theta = 1, 2, 3, 4$, Tables 11 and 12 and Figures 11 and 12 tell us that the distribution behaviors of Y are also affected with higher median, less IQR, smaller scale (higher spike), and the shape is more asymmetric if one increases the parameter θ. On the other hand, Z still gets median unchanged with *median* $= 0.5$ and less spread with the sum of the first quartile and third quartile is always equal to 1, i.e., $(Q_{0.25} + Q_{0.75} = 1)$.

Table 11. Some percentiles of $Y = X_1/X_2$, where (X_1, X_2) follows Joe Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0	−6.31	−1.00	0.00	1.00	6.32
2	0.36	−4.82	−0.32	0.57	1.30	5.67
3	0.52	−3.66	0.07	0.79	1.33	5.13
4	0.61	−2.71	0.29	0.88	1.33	4.62

Table 12. Some percentiles of $Y = X_1/(X_1 + X_2)$, where (X_1, X_2) follows Joe Copulas.

θ	$\tau(C_\theta)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
1	0	−2.66	0.00	0.5	1.00	3.66
2	0.36	−1.25	0.23	0.5	0.77	2.25
3	0.52	−0.66	0.31	0.5	0.69	1.66
4	0.61	−0.37	0.35	0.5	0.65	1.37

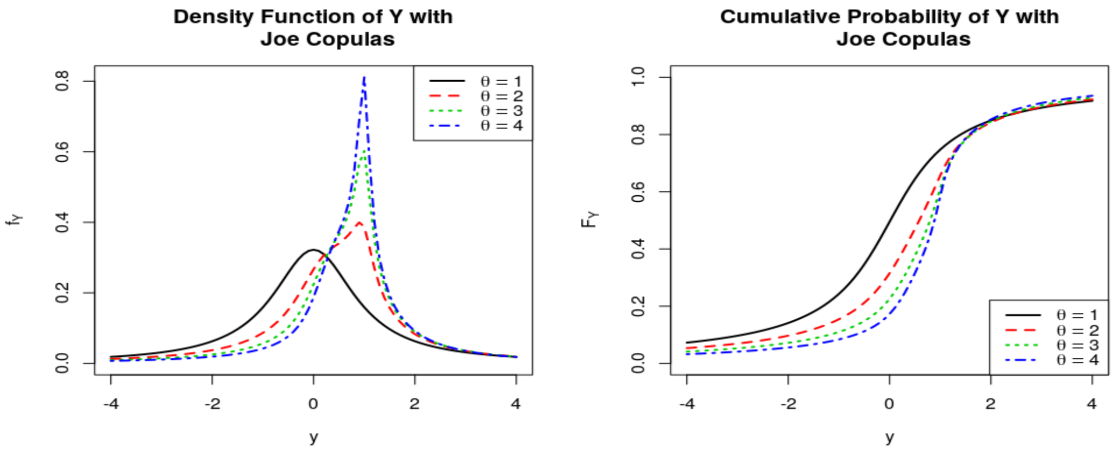

Figure 11. PDFs and CDFs of the ratio $Y = \frac{X_1}{X_2}$, where (X_1, X_2) follows Joe Copulas.

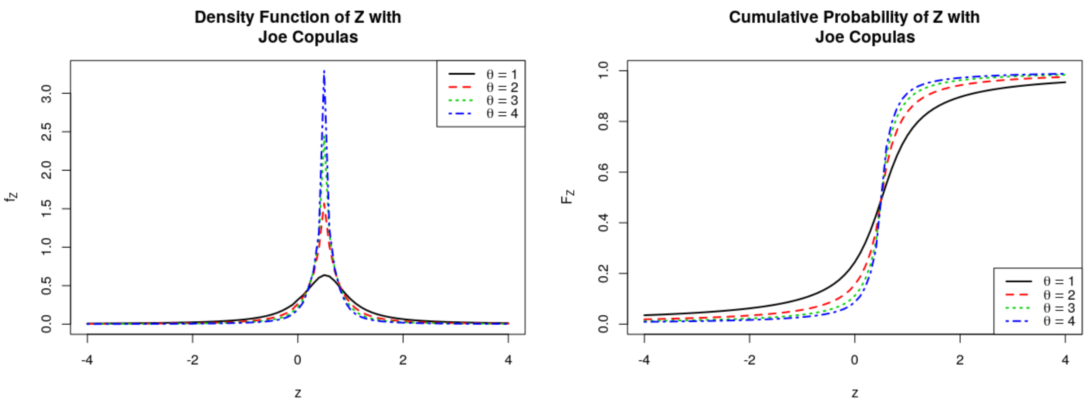

Figure 12. PDFs and CDFs of the ratio $Z = \frac{X_1}{X_1+X_2}$, where (X_1, X_2) follows Joe Copulas.

5.7. Comparison of Copulas with the Same Measure of Dependence

In this section, we investigate the effects of the six copulas families as discussed above on the shapes of different distributions for the random variables $Y := X_1/X_2$ and $Z := X_1/(X_1 + X_2)$ when they have the same measure of dependence—the Kendall's coefficient τ. Here, the parameters are chosen to each copula to correspond to Kendall $\tau = 0.49$. We exhibit the corresponding CDFs and PDFs of both Y and Z in Figures 13 and 14, estimate the percentiles Q_α for some $\alpha = 0.05, 0.25, 0.5, 0.75, 0.95$, and display the values in Tables 13 and 14. As can be seen from the tables and the figures, Y attains the greatest median (0.76) and the smallest IQR (1.32) with left skewed shape when both X_1 and X_2 follow Joe Copula and Student-t Copula. In contrast, Gaussian Copula produces the smallest median (0.70) and the largest IQR (1.42) with symmetric shape. Using Clayton Copula, Y gets the second biggest median (0.74). The ratio random variable Y has the same median (0.72) for both Gumbel and Frank Copula, but the Frank makes Y get higher IQR than the Gumbel (1.41 $>$ 1.33). On the other hand, the random variable Z has symmetric shape with unchanged median (0.5) among all investigated copulas. It only changes the scale, where the Joe Copula affects Z with the smallest scale (i.e., the tallest height of density) whilst the Frank Copula causes Z to get the greatest scale (i.e., the shortest height of density).

Table 13. Some percentiles of $Y = X_1/X_2$, where (X_1, X_2) modelled with six copulas having the same Kendall coefficient $\tau = 0.49$.

Copulas	Parameters	$\tau(C)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
Gaussian	0.7	0.49	−3.81	−0.01	0.70	1.41	5.21
Student-t	0.7, $\nu = 3$	0.49	−3.52	−0.02	0.76	1.31	4.63
Clayton	1.90	0.49	−3.93	0.00	0.74	1.34	5.30
Gumbel	1.95	0.49	−3.80	−0.03	0.72	1.36	5.07
Frank	5.5	0.49	−3.60	0.08	0.72	1.49	5.41
Joe	2.8	0.49	−3.88	0.01	0.76	1.33	5.24

Table 14. Some percentiles of $Y = X_1/(X_1 + X_2)$, where (X_1, X_2) modelled with six copulas having the same Kendall coefficient $\tau = 0.49$.

Copulas	Parameters	$\tau(C)$	$Q_{0.05}$	$Q_{0.25}$	Median	$Q_{0.75}$	$Q_{0.95}$
Gaussian	0.7	0.49	−0.83	0.29	0.5	0.71	1.83
Student-t	0.7, $\nu = 3$	0.49	−0.92	0.31	0.5	0.69	1.92
Clayton	1.90	0.49	−0.77	0.30	0.5	0.70	1.77
Gumbel	1.95	0.49	−0.86	0.30	0.5	0.70	1.86
Frank	5.5	0.49	−0.65	0.29	0.5	0.71	1.65
Joe	2.8	0.49	−0.74	0.30	0.5	0.70	1.74

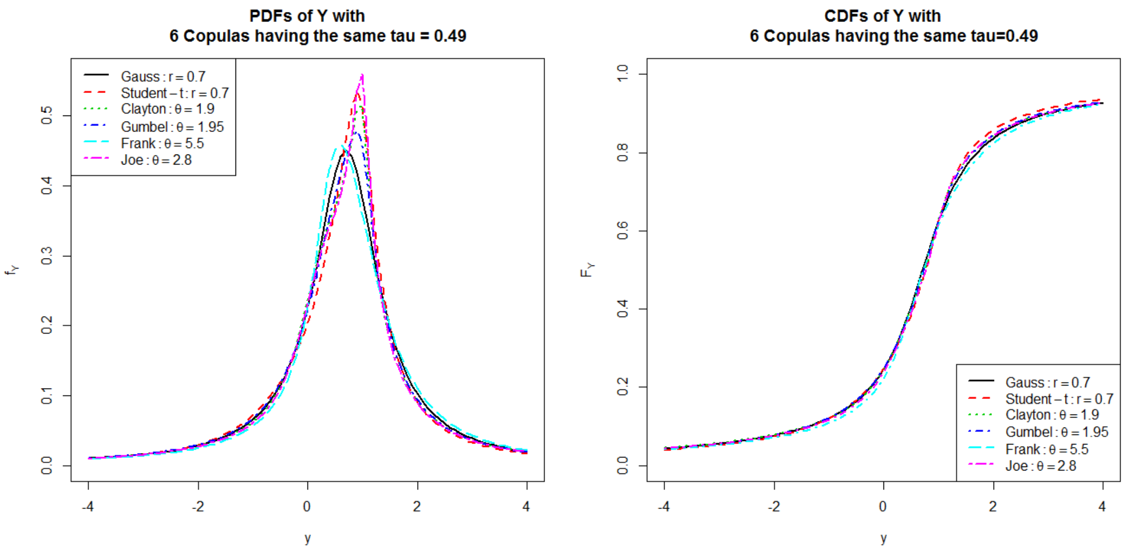

Figure 13. PDFs and CDFs of the ratio $Y = \frac{X_1}{X_2}$, where (X_1, X_2) modeled with six Copulas having the same $\tau = 0.49$.

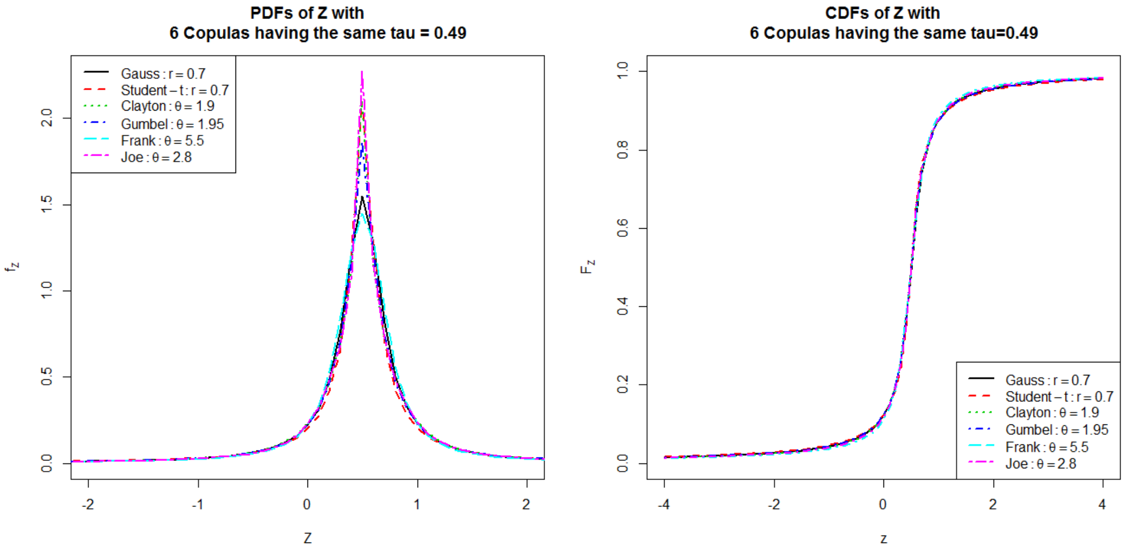

Figure 14. PDF and CDF of the ratio $Z = \frac{X_1}{X_1+X_2}$, where (X_1, X_2) modeled with six Copulas having the same $\tau = 0.49$.

6. Conclusions

Determining distributions of the functions of random variables is a very crucial task and this problem has attracted a number of researchers because there are numerous applications in Economics, Science, and many other areas, especially in the areas of finance including risk management and option pricing. However, to the best of our knowledge, the problem of determining distribution functions of quotient of dependent random variables using copulas has not been widely studied and, as far as we know, no published paper or working paper has done the work we are doing in this paper. Thus, to bridge the gap in the literature, in this paper, we first develop two general propositions on both density and distribution functions for the quotient $Y = \frac{X_1}{X_2}$ and the ratio of one variable over the sum of two variables $Z := \frac{X_1}{X_1+X_2}$ of two dependent random variables X_1 and X_2 by using copulas. We then derive two corollaries on both density and distribution functions for the two quotients $Y = \frac{X_1}{X_2}$ and $Z = \frac{X_1}{X_1+X_2}$ of two dependent normal random variables X_1 and X_2 in case of Gaussian Copulas by applying the two main general propositions developed in our paper. From the results, we derive the corollaries on the median for the ratios of both Y and Z of two normal random variables X_1 and X_2. Furthermore, the result of median for Z is also extended to a larger family of symmetric distributions and symmetric copulas of X_1 and X_2.

Since the density and the CDF formula of the ratios of both Y and Z are in terms of integrals and are very complicated, we cannot obtain the exact forms of the density and the CDF. To circumvent the difficulty, in this paper, we propose to use the Monte Carlo algorithm, numerical analysis, and graphical approach that can efficiently compute complicated integrals and study the behaviors of both density and distribution and the changes of their shapes when parameters are changing. We illustrate our proposed approaches by using a simulation study with ratios of normal random variables on several different copulas, including Gaussian, Student-*t*, Clayton, Gumbel, Frank, and Joe Copulas. We find that copulas make big impacts on behavior of distributions, and since Gaussian and Student-*t* Copulas belong to an elliptical family, they similarly act on shapes of Y and Z in the same fashion. We also document the effects when using Archimedean copulas including Clayton, Gumbel, Frank, and Joe Copulas. However, there are also some differences, especially on location and scale effects. For example, distribution of Z does not change the median and its shape is always symmetric for all investigated copulas while the random variable Y is affected in skewness, median and spread. Our findings are useful for academics in their study of the shapes, center and spread of both density and distribution functions for the ratios by using different copulas. Since the ratios in different copulas are widely used in many important empirical studies in Finance, Economics, and many other areas, our findings are useful to practitioners in Finance, Economics, and many other areas who need to study the shapes, center and spread of the ratios by using different copulas in their analysis and useful to policy makers if they need to consider the shapes, center and spread of both density and distribution functions for the ratios by using different copulas in their policy decision-making. Finally, we note that, although all of the propositions and corollaries developed in our paper are relatively easy to derive, all the results developed in our paper are useful to both academics and practitioners because there is a wide range of applications with variables that have a negative range. Readers may refer to (Chang et al. 2016, 2018a, 2018b, 2018c; Chang et al. 2015; Wong 2016) for more information on the applications in different areas.

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