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Fixed-Time Synchronization of Coupled Oscillator Networks with a Pacemaker

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Abstract: This paper investigates the fixed-time synchronization problem of a Kuramoto–oscillator network in the presence of a pacemaker. Based on the framework of the cyber-physical system (CPS), fixed-time synchronization criteria of such network are presented respectively for identical and non-identical oscillators. In virtue of Lyapunov stability analyses, sufficient conditions are deduced for achieving phase agreement and frequency synchronization for arbitrary initial phases and/or frequencies under distributed control strategies. Theoretical analysis shows that synchronization can be achieved in a fixed time, which is unrelated to initial phases/frequencies. Furthermore, the upper bounds of synchronization time are also obtained. Finally, the numerical simulations also verify the effectiveness of the derived results.

Keywords: fixed-time; Kuramoto oscillator; pacemaker; cyber-physical system (CPS); synchronization

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1. Introduction

In recent years, synchronous emergence of complex dynamical networks [1–8], as a representative class of self-organized synergistic behavior, has aroused numerous attention from many researchers in various fields, such as biology [1,2], engineering [3–6], physics [7,8], and so on. People have witnessed such phenomenon in biology and physics including flashing fireflies [1], chorus of crickets [2], Josephson junctions [7], and the rhythmic applause in the theater [8]. In the past several decades, the Kuramoto model [9,10] has become a hot topic in the control community. By now, many researchers [11,12] have devoted to derive synchronization criteria subject to coupling strength and initial phases for Kuramoto–oscillator networks. In [11], Jadbabaie considered a connected, undirected, and identical Kuramoto–oscillator network and presented that the phase agreement is achieved when the initial phases lie within the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Chopra and Song [12] pointed out that frequency synchronization is achieved if the initial phase differences of each oscillator are within the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Considering the impact of the pacemaker (the so-called leader) on Kuramoto oscillators is of significance from both theoretical and practical points of views. For example, in the clock synchronization of wireless networks [13], the time references in individual node are synchronized through cell-to-cell interactions and external coordination from a time base. Therefore, Kuramoto oscillators with a pacemaker have attracted considerable attention. In [14–17], researchers derived fruitful results by taking the pacemaker into consideration. In addition, based on the framework of cyber-physical system [18,19], the researchers in [20,21] used distributed linear controllers to synchronize the Kuramoto–oscillator network. The derived stability conditions were independent on the initial phases, which indicates that global synchronization was achieved. In [21], on the basis of such distributed linear control strategy, a sufficient criterion for the Kuramoto–oscillator network with a pacemaker was established. However, the above-mentioned synchronziation belongs to asymptotically synchronization, which means that synchronization is achieved when time approaches infinity.

Additionally, many researchers have focused on synchronization time of complex networks [22–29]. In [22], Shi and Ma presented a unified theoretical method to obtain the finite/fixed-time synchronization for complex networks with uncertain inner coupling by adopting the quantized controller and the quantized adaptive controller. Aouiti and Assali [23] obtained novel and effective criteria for achieving the finite-time synchronization. In [24], sufficient conditions were obtained to synchronize the nodes in each cluster to desired states in a settling time. Several fixed-time synchronization criteria expressed by some linear matrix inequalities are presented [25]. Zheng et al. [26] investigated the fixed-time synchronization of discontinuous competitive neural networks with time-varying delays and made the upper bound of the settling time less conservativeness. In [27], Wu and Li studied the finite-time and fixed-time synchronization of Kuramoto–oscillator network by adopting a novel multiplex controller. Based on a cyber-physical system, the finite-time synchronization of a Kuramoto–oscillator network with a pacemaker was studied [29]. Compared with the finite-time synchronization, the settling time in fixed-time synchronization can be adjusted to the desired value independent on initial states, which is of great practical importance. However, up to now, there are few results on fixed-time synchronization of a Kuramoto–oscillator network in the presence of a pacemaker.

Motivated by the above discussions, it is significant to consider the fixed-time synchronization of Kuramoto–oscillator network with a pacemaker. In this paper, based on CPS, we explore fixed-time synchronization criteria of such network by adopting distributed control strategies.

Our contributions are summarized as follows:

1. Effective criteria are established to deal with fixed-time phase agreement and frequency synchronization for the Kuramoto–oscillator network with a pacemaker, and upper bounds of synchronization time are also provided;
2. Compared with our previous results [29], synchronization can be achieved for arbitrary initial phases/ frequencies and the upper bounds of synchronization time are not affected by initial phases/ frequencies;
3. The requirement on the connectivity of physical system is relaxed.

The remainder of this paper is displayed as follows: Section 2 presents the detailed model description as well as preliminaries. In Section 3, sufficient conditions for fixed-time phase agreement and fixed-time frequency synchronization are derived for identical and non-identical oscillators, respectively. Section 4 provides numerical examples to verify the theoretical results, and Section 5 concludes the whole paper.

2. Model and Preliminaries

A Kuramoto–oscillator network consisting of N oscillators with control input u_i is described as

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} a_{ij} \sin\left(\theta_j - \theta_i\right) + u_i, \ i \in I, \tag{1}$$

where $I = \{1, \cdots, N\}$, θ_i and ω_i are the phase and natural frequency of oscillator i, respectively. $A = [a_{ij}] \in R^{N\times N}$ denotes the adjacency matrix of an undirected network, where $a_{ij} = a_{ji} = 1$ $(i \neq j)$ if and only if there is an edge between oscillators i and j; otherwise, $a_{ij} = 0$. Let $\mathcal{L}_A = D^A - A$ be the Laplacian matrix associated with the adjacency matrix A, where $D^A \in R^{N\times N}$ is a diagonal matrix with $D^A_{ii} = \sum_{j=1}^{N} a_{ij}$ $(\forall i \in I)$. The network associated with the adjacency matrix A is called the physical network.

Assume that there is a pacemaker with dynamics

$$\dot{\theta}_0 = \omega_0,$$

where θ_0 and ω_0 are the phase and natural frequency of the pacemaker, respectively.

In this paper, we focus on phase agreement and frequency synchronization with respect to the pacemaker in a fixed time.

Definition 1. *Network (1) with control input u_i achieves (pacemaker-based) fixed-time phase agreement, if there exists a settling time $T > 0$ independent on the initial states $\theta_i(0)$ $(i \in \{0\} \bigcup I)$, such that*

$$\lim_{t \to T}(\theta_i - \theta_0) = 0,\ i \in I, \tag{2}$$

and $\theta_i - \theta_0 \equiv 0$ for $t \geq T$.

Definition 2. *Network (1) with control input u_i achieves (pacemaker-based) fixed-time frequency synchronization, if there exists a settling time $T > 0$ independent on the initial states $\dot{\theta}_i(0)$ $(i \in \{0\} \bigcup I)$, such that*

$$\lim_{t \to T}(\dot{\theta}_i - \dot{\theta}_0) = 0,\ i \in I, \tag{3}$$

and $\dot{\theta}_i - \dot{\theta}_0 \equiv 0$ for $t \geq T$.

In order to obtain sufficient conditions for the above-mentioned synchronization, the following Lemmas are needed.

Lemma 1. *[30] Let $A = [a_{ij}] \in R^{N \times N}$ and $\mathcal{L}_A$ be the adjacency matrix and Laplacian matrix of an undirected graph $\mathcal{G}$ with N nodes, respectively. $x^T \mathcal{L}_A x = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(x_i - x_j)^2$ holds, where $x = (x_1, x_2, \cdots, x_N)^T$.*

Lemma 2. *[31] Consider the dynamic system defined by the following differential equation:*

$$\dot{x}(t) = f(t, x(t)),\ x(0) = x_0,\ t \in R_+, \tag{4}$$

where $x(t) \in R^{n \times 1}$ is the state vector and $f : R_+ \times R^{n \times 1} \to R^{n \times 1}$ is a nonlinear function.

If there exists a continuous radially unbounded function V: $R^{n \times 1} \to R_+ \cup \{0\}$ such that

(1) $V(x(t)) = 0 \Leftrightarrow x(t) = 0$;

(2) Any solution $x(t)$ of (4) satisfies the inequality

$$D^+ V(x(t)) \leq -(\alpha V^p(x(t)) + \beta V^q(x(t)))^k,$$

for some parameters α, β, p, q, $k > 0$ with $pk < 1$ and $qk > 1$, where $D^+ V(x(t))$ denotes the upper right-hand derivative of the function $V(x(t))$. Then, the origin is globally fixed time stable for system (4) and the settling time bounded by

$$T \leq \frac{1}{\alpha^k(1 - pk)} + \frac{1}{\beta^k(qk - 1)}.$$

For a real symmetric matrix $\mathcal{L}$, let $\lambda_{\min}(\mathcal{L})$ be the minimum eigenvalue of matrix $\mathcal{L}$. Denote $sig(x)^\alpha = sign(x)|x|^\alpha$, where the signum function $sign(x)$ is defined as

$$sign(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

In addition, denote the phase differences with respect to the pacemaker as

$$\xi_i = \theta_i - \theta_0.$$

3. Fixed-Time Phase Agreement and Frequency Synchronization

In this section, based on CPS, we move to seek sufficient criteria for fixed-time phase agreement and fixed-time frequency synchronization by designing distributed controllers.

3.1. Fixed-Time Phase Agreement

In this subsection, we consider the case of all oscillators with identical natural frequency, i.e., $\omega_i = \omega_0,\ \forall i \in I$. Thus, network (1) becomes

$$\dot{\theta}_i = \omega_0 + \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i) + u_i,\ i \in I. \tag{5}$$

For achieving fixed-time phase agreement, based on CPS, we design a distributed control strategy u_i as follows:

$$u_i = \sum_{j=1}^{N} b_{ij}(\theta_j - \theta_i) + f_i sig(\theta_0 - \theta_i)^{\alpha_1} + g_i sig(\theta_0 - \theta_i)^{\alpha_2},\ i \in I, \tag{6}$$

where $f_i > 0$, $g_i > 0$, $0 < \alpha_1 < 1$, and $\alpha_2 > 1$. $B = [b_{ij}] \in R^{N\times N}$ denotes the adjacency matrix of an undirected network, where $b_{ij} = b_{ji} = 1\ (i \neq j)$ if and only if there exists information exchange between oscillators i and j; otherwise, $b_{ij} = 0$. The network associated with the connections between the oscillators in the controller (6) is called a cyber network. Let $\mathcal{L}_B$ be the Laplacian matrix associated with the adjacency matrix B, where its elements are defined similarly to those of $\mathcal{L}_A$.

Theorem 1. *Network (5) under the distributed control strategy (6) achieves fixed-time phase agreement with the settling time bounded by*

$$T_1 \leq T_0 \triangleq \frac{1}{2^{\frac{\alpha_1-1}{2}}(1-\alpha_1)f_{\min}} + \frac{1}{2^{\frac{\alpha_2-1}{2}}(\alpha_2-1)g_{\min}}, \tag{7}$$

if

$$\lambda_{\min}(\cos\gamma \cdot \mathcal{L}_A + \mathcal{L}_B) \geq 0,$$

where $\gamma \in (\pi, 2\pi)$ *satisies* $\tan\gamma = \gamma$, $f_{\min} = \min\{f_1, \cdots, f_N\}$, *and* $g_{\min} = \min\{g_1, \cdots, g_N\}$.

Proof of Theorem 1. See Appendix A. □

According to (7), we find that the upper bound of synchronization time is independent on initial states. By Theorem 1, it is sufficient to achieve fixed-time phase agreement if $\lambda_{\min}(\cos\gamma \cdot \mathcal{L}_A + \mathcal{L}_B) \geq 0$. Therefore, even if the physical network is not connected, phase agreement could be also achieved with the help of the cyber network, which relaxes the requirement on the connectivity of the physical network.

3.2. Fixed-Time Frequency Synchronization

Now, we further consider the case of all oscillators with non-identical natural frequencies, i.e., there exists some $i \in I$ such that $\omega_i \neq \omega_0$.

For achieving fixed-time frequency synchronization, based on CPS, we design a distributed control strategy u_i as follows:

$$u_i = \sum_{j=1}^{N} b_{ij}(\theta_j - \theta_i) + U_i,\ i \in I, \tag{8}$$

where $\dot{U}_i = f_i sig(\dot{\theta}_0 - \dot{\theta}_i)^{\alpha_1} + g_i sig(\dot{\theta}_0 - \dot{\theta}_i)^{\alpha_2}$, $f_i > 0$, $g_i > 0$, $0 < \alpha_1 < 1$, $\alpha_2 > 1$, and b_{ij} denotes the same as that in (6).

Theorem 2. *Network (1) with non-identical oscillators under the distributed control strategy (8) achieves fixed-time frequency synchronization with the settling time bounded by*

$$T_2 \leq T_0 \triangleq \frac{1}{2^{\frac{\alpha_1-1}{2}}(1-\alpha_1)f_{\min}} + \frac{1}{2^{\frac{\alpha_2-1}{2}}(\alpha_2-1)g_{\min}}, \tag{9}$$

if

$$\lambda_{\min}(\mathcal{L}_B - \mathcal{L}_A) \geq 0,$$

where $\mathcal{L}_A$ and $\mathcal{L}_B$ represent the same matrices as those in Section 3.1.

Proof of Theorem 2. See Appendix B. □

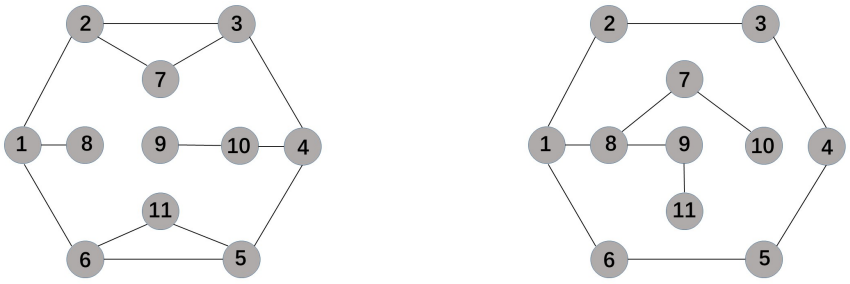
According to (9), we find that the upper bound of synchronization time is independent on initial states. By Theorem 2, it is sufficient to achieve fixed-time frequency synchronization if $\lambda_{\min}(\mathcal{L}_B - \mathcal{L}_A) \geq 0$. Therefore, similar to Theorem 1, we also relax the requirement on the connectivity of the physical network.

Remark 1. *By equations (7) and (9), the upper bounds of synchronization time T_0 increase as parameter α_1 grows and decrease as parameter α_2 grows both for identical and non-identical oscillators in the light of the monotonicity of T_0, which is also reflected in Figure 6 of Section 4. Additionally, T_0 decreases as $f_{\min}$ or $g_{\min}$ grows.*

Remark 2. *In this paper, it is assumed that both physical and cyber networks are undirected. However, for directed networks, it is difficult to guarantee that the Laplacian matrices associated with physical and cyber networks are semi-definite ones. Consequently, it is a challenging work to extend the presented theoretical results to directed networks, which is one of our future directions.*

4. Numerical Simulation

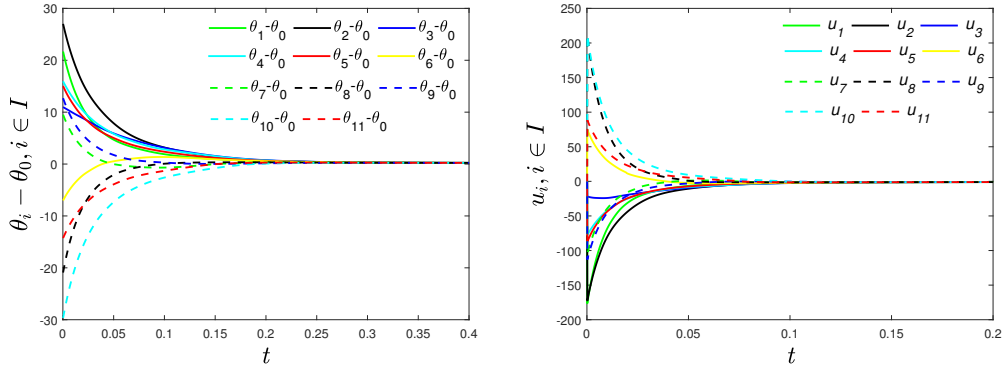
In this section, we assume networks associated with adjacency matrices A and B as shown in Figure 1a,b, respectively. In addition, $I = \{1, 2, \cdots, 11\}$.



(**a**) (**b**)

Figure 1. (**a**) Network associated with the adjacency matrix A; (**b**) network associated with the adjacency matrix B.

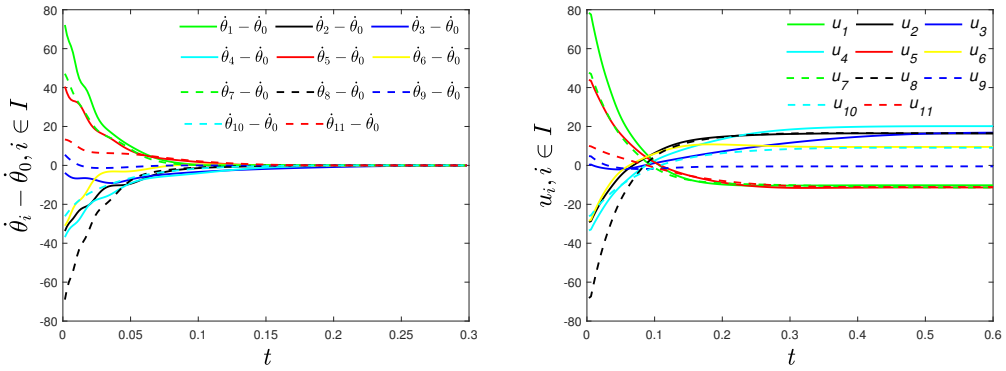
We first verify Theorem 1. Set ω_i= 1 ($i \in \{0\} \bigcup I$). Choose $\alpha_1 = 0.5$, $\alpha_2 = 1.5$ and $f_i = g_i = 2$ ($i \in I$). Initial phases $\theta_i(0)$ ($i \in \{0\} \bigcup I$) are randomly chosen from the continuous distribution over $[-10\pi, 10\pi]$. Obviously, $\lambda_{\min}(\cos\gamma \cdot \mathcal{L}_A + \mathcal{L}_B) = 0$. By (7), we can calculate the upper bound of settling time T_0 = 2.8710 s. In Figure 2a, we can see phase difference $\theta_i - \theta_0$ ($i \in I$) converge to zero, and phase agreement is achieved about 0.2840 s < T_0, which means fixed-time phase agreement is achieved. Time evolutions of the distributed control strategy (6) of each oscillator are shown in Figure 2b.



(**a**) (**b**)

Figure 2. (**a**) Time evolutions of phase differences θ_i-θ_0 $(i \in I)$ under the distributed control strategy (6); (**b**) time evolutions of the distributed control strategy (6).

Secondly, we verify Theorem 2. Set $(\omega_0, \omega_1, \omega_2, \cdots, \omega_{11}) = (2, -4, -3.2, -2.4, -1.6, -0.8, 0, 0.8, 1.6, 2.4, 3.2, 4)^T$. Choose $\alpha_1 = 0.5$, $\alpha_2 = 1.5$ and $f_i = g_i = 2$ $(i \in I)$. Obviously, $\lambda_{\min}(\mathcal{L}_B - \mathcal{L}_A) = 0$. By (9), we can calculate the upper bound of settling time $T_0 = 2.8710$ s. In Figure 3a, we can find frequency differences $\dot{\theta}_i - \dot{\theta}_0$ $(i \in I)$ that converge to zero, and frequency synchronization is achieved about 0.2840 s < T_0, which means that fixed-time frequency synchronization is achieved. Time evolutions of the distributed control strategy (8) of each oscillator are shown in Figure 3b.

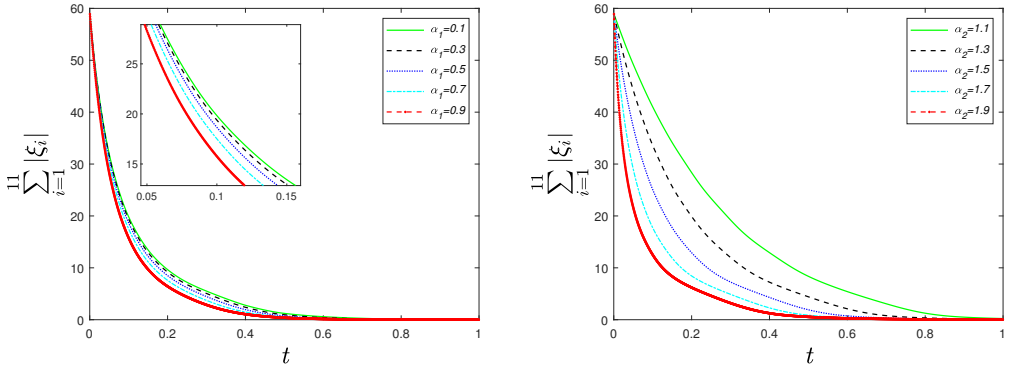


(**a**) (**b**)

Figure 3. (**a**) Timeevolutions of frequencies differences $\dot{\theta}_i$-$\dot{\theta}_0$ $(i \in I)$ under the distributed control strategy (8); (**b**) time evolutions of the distributed control strategy (8).

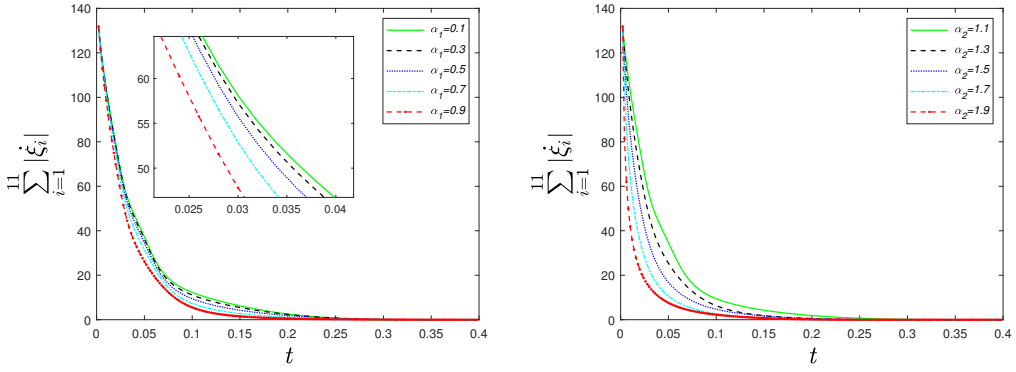
Thirdly, we move to see the influence of parameters α_1 and α_2 on $\sum_{i=1}^{11} |\xi_i|$. We choose $\alpha_1 = 0.1, 0.3, 0.5, 0.7, 0.9$ and $\alpha_2 = 1.1, 1.3, 1.5, 1.7, 1.9$. Figure 4 shows the convergence rate of $\sum_{i=1}^{11} |\xi_i|$ increases as parameter α_1 or α_2 grows for identical oscillators. Similarly, the influence of parameters α_1 and α_2 on $\sum_{i=1}^{11} |\dot{\xi}_i|$ is reflected in Figure 5. We can see that the convergence rate of $\sum_{i=1}^{11} |\dot{\xi}_i|$ increases as parameter α_1 or α_2 grows for non-identical oscillators.

Finally, we discuss the upper bounds of synchronization time T_0 for different parameters α_1 and α_2. Figure 6 shows that the upper bounds of synchronization time T_0 increase as α_1 grows and decrease as α_2 grows.



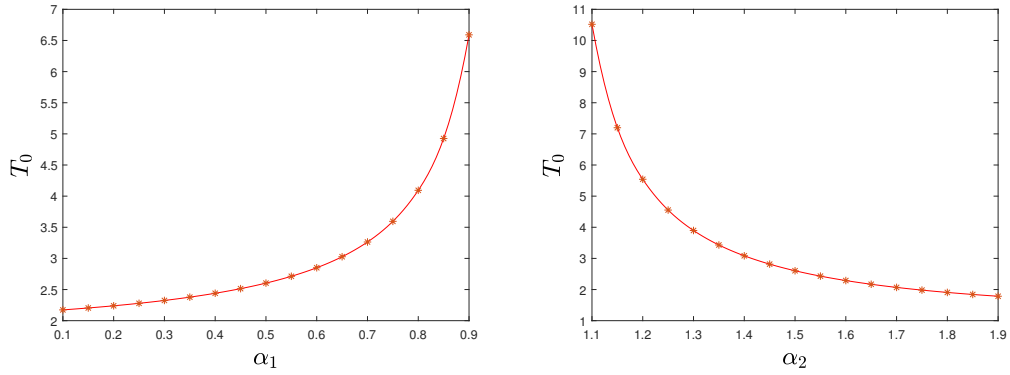
(**a**) (**b**)

Figure 4. (**a**) Time evolutions of $\sum_{i=1}^{11}|\xi_i|$ under the distributed control strategy (6) for different parameters $\alpha_1 = 0.1, 0.3, 0.5, 0.7, 0.9$; (**b**) time evolutions of $\sum_{i=1}^{11}|\xi_i|$ under the distributed control strategy (6) for different parameters $\alpha_2 = 1.1, 1.3, 1.5, 1.7, 1.9$.



(**a**) (**b**)

Figure 5. (**a**) Time evolutions of $\sum_{i=1}^{11}|\dot{\xi}_i|$ under the distributed control strategy (8) for different parameters $\alpha_1 = 0.1, 0.3, 0.5, 0.7, 0.9$; (**b**) time evolutions of $\sum_{i=1}^{11}|\dot{\xi}_i|$ under the distributed control strategy (8) for different parameters $\alpha_2 = 1.1, 1.3, 1.5, 1.7, 1.9$.



(**a**) (**b**)

Figure 6. (**a**) The upper bounds of synchronization time T_0 under the distributed control strategy (6) for different parameters α_1; (**b**) the upper bounds of synchronization time T_0 under the distributed control strategy (8) for different parameters α_2.

5. Conclusions

In this paper, the fixed-time phase agreement and frequency synchronization of Kuramoto–oscillator networks with a pacemaker have been investigated. Two distributed control strategies are designed to explore fixed-time synchronization for identical and non-identical oscillators. Furthermore, the upper bounds of synchronization time, which is not related to initial states, have been estimated. Numerical examples have validated the effectiveness of the derived theoretical results.

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Appendix A. Proof of Theorem 1

Proof. As $\xi_i = \theta_i - \theta_0$, then $\dot{\xi}_i = \dot{\theta}_i - \dot{\theta}_0 = \dot{\theta}_i - \omega_0$. Thus, network (5) with distributed control strategy (6) becomes

$$\dot{\xi}_i = \sum_{j=1}^{N} a_{ij} \sin(\xi_j - \xi_i) + \sum_{j=1}^{N} b_{ij}(\xi_j - \xi_i) - f_i sig(\xi_i)^{\alpha_1} - g_i sig(\xi_i)^{\alpha_2}. \tag{A1}$$

Consider the following Lyapunov functional candidate:

$$V_1 = \frac{1}{2}\xi^T\xi = \frac{1}{2}\sum_{i=1}^{N}\xi_i^2.$$

The derivation of V_1 along trajectories (A1) gives

$$\begin{aligned}\dot{V}_1 &= \sum_{i=1}^{N}\xi_i\dot{\xi}_i \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\xi_i \sin(\xi_j - \xi_i) + \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\xi_i \sin(\xi_j - \xi_i) - \sum_{i=1}^{N} f_i\xi_i sig(\xi_i)^{\alpha_1} \\ &\quad - \sum_{i=1}^{N} g_i\xi_i sig(\xi_i)^{\alpha_2}.\end{aligned} \tag{A2}$$

According to Lemma 1 and the fact $\frac{\sin(\theta_j - \theta_i)}{(\theta_j - \theta_i)} \in [\cos\gamma, 1]$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\xi_i \sin(\xi_j - \xi_i) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j)\sin(\xi_j - \xi_i) \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\frac{\sin(\xi_j - \xi_i)}{(\xi_j - \xi_i)}(\xi_j - \xi_i)^2 \\ &\leq -\frac{\cos\gamma}{2}\cdot\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\xi_j - \xi_i)^2 \\ &= -\cos\gamma\cdot\xi^T\mathcal{L}_A\xi.\end{aligned} \tag{A3}$$

In addition,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\xi_i(\xi_j - \xi_i) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}(\xi_j - \xi_i)^2 = -\xi^T\mathcal{L}_B\xi, \tag{A4}$$

$$\sum_{i=1}^{N} f_i\xi_i sig(\xi_i)^{\alpha_1} = \sum_{i=1}^{N} f_i\xi_i sign(\xi_i)|\xi_i|^{\alpha_1} \leq \sum_{i=1}^{N} f_i|\xi_i|^{1+\alpha_1}, \tag{A5}$$

$$\sum_{i=1}^{N} g_i\xi_i sig(\xi_i)^{\alpha_2} = \sum_{i=1}^{N} g_i\xi_i sign(\xi_i)|\xi_i|^{\alpha_2} \leq \sum_{i=1}^{N} g_i|\xi_i|^{1+\alpha_2}. \tag{A6}$$

Combining (A3)–(A6), Equation (A2) becomes

$$\dot{V}_1 \leq -\xi^T(\cos\gamma\cdot\mathcal{L}_A + \mathcal{L}_B)\xi - \sum_{i=1}^{N} f_i|\xi_i|^{1+\alpha_1} - \sum_{i=1}^{N} g_i|\xi_i|^{1+\alpha_2}.$$

If $\lambda_{\min}(\cos\gamma\cdot\mathcal{L}_A + \mathcal{L}_B) \geq 0$, we obtain

$$\begin{aligned}\dot{V}_1 &\leq -\sum_{i=1}^{N} f_i|\xi_i|^{1+\alpha_1} - \sum_{i=1}^{N} g_i|\xi_i|^{1+\alpha_2} \\ &\leq -f_{\min}2^{\frac{1+\alpha_1}{2}}[\frac{1}{2}\sum_{i=1}^{N}|\xi_i|^2]^{\frac{1+\alpha_1}{2}} - g_{\min}2^{\frac{1+\alpha_2}{2}}[\frac{1}{2}\sum_{i=1}^{N}|\xi_i|^2]^{\frac{1+\alpha_2}{2}} \\ &= -2^{\frac{1+\alpha_1}{2}}f_{\min}V_1^{\frac{1+\alpha_1}{2}} - 2^{\frac{1+\alpha_2}{2}}g_{\min}V_1^{\frac{1+\alpha_2}{2}}.\end{aligned}$$

By Lemma 2 and Definition 1, the fixed-time phase agreement can be achieved with the settling time bounded by

$$\begin{aligned}T_1 &\leq \frac{1}{2^{\frac{1+\alpha_1}{2}}(1-\frac{1+\alpha_1}{2})f_{\min}} + \frac{1}{2^{\frac{1+\alpha_2}{2}}(\frac{1+\alpha_2}{2}-1)g_{\min}} \\ &= \frac{1}{2^{\frac{\alpha_1-1}{2}}(1-\alpha_1)f_{\min}} + \frac{1}{2^{\frac{\alpha_2-1}{2}}(\alpha_2-1)g_{\min}}.\end{aligned}$$

□

Appendix B. Proof of Theorem 2

Proof. Under the control strategy (8), the derivation of (1) becomes

$$\begin{aligned}\ddot{\xi}_i = \ddot{\theta}_i &= \sum_{j=1}^{N} a_{ij}\cos(\xi_j - \xi_i)(\dot{\xi}_j - \dot{\xi}_i) + \dot{u}_i \\ &= \sum_{j=1}^{N} a_{ij}\cos(\xi_j - \xi_i)(\dot{\xi}_j - \dot{\xi}_i) + \sum_{j=1}^{N} b_{ij}(\dot{\xi}_j - \dot{\xi}_i) - f_i sig(\dot{\xi}_i)^{\alpha_1} - g_i sig(\dot{\xi}_i)^{\alpha_2}.\end{aligned} \tag{A7}$$

Consider the following Lyapunov functional candidate:

$$V_2 = \frac{1}{2}\dot{\xi}^T\dot{\xi} = \frac{1}{2}\sum_{i=1}^{N}\dot{\xi}_i^{\,2}.$$

The derivation of V_2 along trajectories (A7) gives

$$\begin{aligned}\dot{V}_2 &= \sum_{i=1}^{N}\dot{\xi}_i\ddot{\xi}_i \\ &= \sum_{i=1}^{N}\dot{\xi}_i[\sum_{j=1}^{N} a_{ij}\cos(\xi_j - \xi_i)(\dot{\xi}_j - \dot{\xi}_i) + \sum_{j=1}^{N} b_{ij}(\dot{\xi}_j - \dot{\xi}_i) - f_i sig(\dot{\xi}_i)^{\alpha_1} - g_i sig(\dot{\xi}_i)^{\alpha_2}] \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\dot{\xi}_i a_{ij}\cos(\xi_j - \xi_i)(\dot{\xi}_j - \dot{\xi}_i) + \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\dot{\xi}_i(\dot{\xi}_j - \dot{\xi}_i) - \sum_{i=1}^{N} f_i\dot{\xi}_i sig(\dot{\xi}_i)^{\alpha_1} \\ &\quad - \sum_{i=1}^{N} g_i\dot{\xi}_i sig(\dot{\xi}_i)^{\alpha_2}.\end{aligned} \tag{A8}$$

According to Lemma 1 and the fact $\left|\cos(\theta_j - \theta_i)\right| \le 1$, we can obtain

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N}\dot{\xi}_i a_{ij}\cos(\xi_j - \xi_i)(\dot{\xi}_j - \dot{\xi}_i) &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\dot{\xi}_j - \dot{\xi}_i)^2\cos(\xi_j - \xi_i) \\ &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\dot{\xi}_j - \dot{\xi}_i)^2 \\ &= \dot{\xi}^T\mathcal{L}_A\dot{\xi}.\end{aligned} \tag{A9}$$

In addition,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\dot{\xi}_i(\dot{\xi}_j - \dot{\xi}_i) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}(\dot{\xi}_j - \dot{\xi}_i)^2 = -\dot{\xi}^T\mathcal{L}_B\dot{\xi}, \tag{A10}$$

$$\sum_{i=1}^{N} f_i\dot{\xi}_i sig(\dot{\xi}_i)^{\alpha_1} = \sum_{i=1}^{N} f_i\dot{\xi}_i sign(\dot{\xi}_i)\left|\dot{\xi}_i\right|^{\alpha_1} \le \sum_{i=1}^{N} f_i\left|\dot{\xi}_i\right|^{1+\alpha_1}, \tag{A11}$$

$$\sum_{i=1}^{N} g_i\dot{\xi}_i sig(\dot{\xi}_i)^{\alpha_2} = \sum_{i=1}^{N} g_i\dot{\xi}_i sign(\dot{\xi}_i)\left|\dot{\xi}_i\right|^{\alpha_2} \le \sum_{i=1}^{N} g_i\left|\dot{\xi}_i\right|^{1+\alpha_2}. \tag{A12}$$

Combining (A9)–(A12), Equation (A8) becomes

$$\dot{V}_2 \le -\dot{\xi}^T(\mathcal{L}_B - \mathcal{L}_A)\dot{\xi} - \sum_{i=1}^{N} f_i\left|\dot{\xi}_i\right|^{1+\alpha_1} - \sum_{i=1}^{N} g_i\left|\dot{\xi}_i\right|^{1+\alpha_2}.$$

If $\lambda_{\min}(\mathcal{L}_B - \mathcal{L}_A) \geq 0$, we obtain

$$\begin{aligned}\dot{V}_2 &\leq -\sum_{i=1}^{N} f_i\left|\dot{\xi}_i\right|^{1+\alpha_1} - \sum_{i=1}^{N} g_i\left|\dot{\xi}_i\right|^{1+\alpha_2} \\ &= -\sum_{i=1}^{N} f_i\left[\left|\dot{\xi}_i\right|^2\right]^{\frac{1+\alpha_1}{2}} - \sum_{i=1}^{N} g_i\left[\left|\dot{\xi}_i\right|^2\right]^{\frac{1+\alpha_2}{2}} \\ &= -f_{\min} 2^{\frac{1+\alpha_1}{2}}\left[\frac{1}{2}\sum_{i=1}^{N}\left|\dot{\xi}_i\right|^2\right]^{\frac{1+\alpha_1}{2}} - g_{\min} 2^{\frac{1+\alpha_2}{2}}\left[\frac{1}{2}\sum_{i=1}^{N}\left|\dot{\xi}_i\right|^2\right]^{\frac{1+\alpha_2}{2}} \\ &= -2^{\frac{1+\alpha_1}{2}} f_{\min} V_2^{\frac{1+\alpha_1}{2}} - 2^{\frac{1+\alpha_2}{2}} g_{\min} V_2^{\frac{1+\alpha_2}{2}}.\end{aligned}$$

By Lemma 2 and Definition 2, network (1) with non-identical oscillators under the distributed control strategy (8) achieves fixed-time frequency synchronization with the settling time bounded by

$$\begin{aligned}T_2 &\leq \frac{1}{2^{\frac{1+\alpha_1}{2}}(1-\frac{1+\alpha_1}{2})f_{\min}} + \frac{1}{2^{\frac{1+\alpha_2}{2}}(\frac{1+\alpha_2}{2}-1)g_{\min}} \\ &= \frac{1}{2^{\frac{\alpha_1-1}{2}}(1-\alpha_1)f_{\min}} + \frac{1}{2^{\frac{\alpha_2-1}{2}}(\alpha_2-1)g_{\min}}.\end{aligned}$$

□

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