

Article

The Spectral Representation of a Discrete Version of Blackwell's Markov Chain

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Abstract

We consider a Markov chain that can be termed a discrete version of Blackwell's example from 1958. It is constructed with the aid of a sequence of independent Markov chains with two states. It turns out its stationary distribution π and transition matrix P are in detailed balance. As a result, the transition operator associated with P is self-adjoint in $\ell^2(\pi)$, the Hilbert space of all square summable sequences with respect to π . All eigenvalues of P are therefore real, and we give explicit formulae for them. Their corresponding eigenvectors form an orthogonal family in $\ell^2(\pi)$. Consequently, P can be diagonalized, and we find manageable formulae for P^n , where $n \geq 2$.

Keywords: Markov chain; infinite state space; spectral representation; stationary distribution

1. Introduction

In 1958, D. Blackwell provided an example of a continuous-time Markov chain with all states instantaneous (see [1] or more recent books [2,3]). The fact that all states of this process are instantaneous means its Q -matrix has $-\infty$ in all its diagonal entries. This is one of the few such known examples; see [4] for another one. Some results related to this chain have been obtained recently (see [5,6]). As noted in these papers, the chain constructed by Blackwell is interesting because some explicit calculations can be carried out that are related to this process. For example, its stationary distribution was found in [6], where a stable case of this chain was also investigated.

In this paper we consider a discrete version of Blackwell's example, which is constructed as follows. Let $X_1(n), X_2(n), \dots$ be a sequence of independent Markov chains, each with two states 0 and 1. Our chain $(X(n))_{n \geq 0}$ will be the joint process $(X_1(n), X_2(n), \dots)$. Such a combination of independent Markov processes defines a new Markov process, whose state space is the set $\{0, 1\}^\infty$ in our case. Details can be found in [7] (p. 45) and [3] (p. 65). Notice, however, that the set $\{0, 1\}^\infty$ is uncountable. Therefore, in Lemma 2, we demonstrate that under certain conditions imposed on its components, $(X(n))_{n \geq 0}$ takes values in a countable subset of $\{0, 1\}^\infty$, which can be identified with the set of natural numbers $\mathbb{N}$. In other words, the process formally defined by (28) will be a Markov chain on $\mathbb{N}$. Its transition matrix P is determined by (30).

In Lemma 4, which is a generalization of [6] (Theorem 4), we find the stationary distribution of $(X(n))_{n \geq 0}$. This distribution, denoted by π , is in detailed balance with P , which means Equation (39) hold true. As a result, the transition operator T associated with P (see (56) for definition) is a self-adjoint operator in $\ell^2(\pi)$, the Hilbert space of all sequences that are square summable with respect to π . This, in turn, implies that all



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eigenvalues of T , and therefore of P , are real. In Theorem 3, we give explicit formulae for these eigenvalues. Their corresponding eigenvectors form an orthogonal basis in $\ell^2(\pi)$ (see Theorem 4). Then, we can express P in the form $W^{-1}DW$, where D is a diagonal matrix with eigenvalues of P on its diagonal. The columns of W are eigenvectors corresponding to those eigenvalues. In Lemma 5, we show that W^{-1} equals $c^{-1}W$, for a certain constant $c > 0$. We finally obtain explicit formulae for $P^n = c^{-1}WD^nW$ for any $n \geq 2$ (see Examples 1 and 3). Such formulae, for a single two-state Markov chain, are available in many textbooks (see, e.g., [8,9]).

It is well known that transition matrices related to Markov chains can also be viewed as Markov operators in $l^1(\mathbb{N})$, the Banach space of all absolutely summable sequences (see [2] (p. 42)). Theorem 3 is formulated from this perspective. In consequence, the rows of W are vectors from $l^1(\mathbb{N})$.

It should also be added that there is no notion of an instantaneous state for a discrete-time Markov chain. Nevertheless, the Markov chain investigated here can be called a discrete version of Blackwell's example because their construction is similar.

Probabilistic models using sequences of independent Markov chains with two or more states have some applications, especially in biology and medicine (see [10,11] and the references therein). For example, in [11], a model of m independent binary chains was studied. It has been used to detect genomic deletions in cancer patients. Statistical physics is another field where such models appear (see [3,12]). They are typically used to describe the behavior of independent particle systems.

The Markov chains studied in Sections 2 and 3 may seem related to the Ising model, described in [13] (Chapter 7), as well as to birth and death processes (see [8] (p. 155)), whether with bounded or unbounded population. However, there are essential differences between them. The reason for this is the following. All entries of the transition matrix given by (5) or (30) are non-zero. The conclusion is that in the chains under study, each state is accessible from every other state in one step. Although a Markov chain used in the one-dimensional Ising model is defined on the state space $\{-1, +1\}^m$, which would be a case similar to the one considered in Section 2, not all transitions in one step are possible (see [13] (pp. 195–197)). Similarly, birth and death processes usually allow only for some, not all, transitions in one step (see [9] (pp. 185–187)).

The structure of this article is as follows. In Section 2, we begin with a Markov chain composed of a finite number of independent, two-state Markov chains. The notation introduced in this section, especially the binary representation of integers, will be used throughout the article. In Section 3, we consider a Markov chain with infinite state space. We find its stationary distribution and show that its state space can be identified with the set of natural numbers. Theorems 3 and 4 are the main results of this article. Theorem 1, being a special case of Theorem 3, is formulated in Section 2. Theorem 2 is equivalent to Theorem 1. Some additional results are included in Section 4.

2. The Spectral Representation of a 2^m -State Markov Chain

The set of natural numbers $\{1, 2, 3, \dots\}$ will be denoted by $\mathbb{N}$. The set of all integers is denoted by $\mathbb{Z}$. Additionally, in this section, $\mathbb{S}_m$ stands for the set of integers from 1 to 2^m , $\mathbb{S}_m = \{1, 2, 3, \dots, 2^m\}$.

Let an integer $m \geq 2$ be fixed and consider a Markov chain $(X(n))_{n \geq 0}$ defined as

$$X(n) := (X_1(n), X_2(n), \dots, X_m(n)) \in \{0, 1\}^m, \quad n \geq 0, \quad (1)$$

where $X_1(\cdot), \dots, X_m(\cdot)$ are mutually independent, homogeneous Markov chains, each with two states 0 and 1. As mentioned earlier, such a combination of independent chains defines

another Markov chain (see [7] (p. 45)). The transition matrix of the component $X_k(\cdot)$ has then the form

$$P_k = \begin{bmatrix} 1 - b_k & b_k \\ a_k & 1 - a_k \end{bmatrix} = \begin{bmatrix} b'_k & b_k \\ a_k & a'_k \end{bmatrix}, \quad (2)$$

where, in general, $a_k, b_k \in [0, 1]$. In this paper we assume $a_k, b_k \in (0, 1)$ for $k = 1, \dots, m$ and denote also $a'_k := 1 - a_k$, $b'_k := 1 - b_k$. A homogeneous Markov chain is fully characterized by its transition matrix and initial distribution (see [8,9]). Thus, for the chain $X_k(\cdot)$ it would be P_k and $X_k(0)$. The meaning of a_k and b_k in (2) is as follows:

$$\begin{cases} \mathbb{P}(X_k(n+1) = 1 | X_k(n) = 0) = b_k \\ \mathbb{P}(X_k(n+1) = 0 | X_k(n) = 0) = 1 - b_k = b'_k \\ \mathbb{P}(X_k(n+1) = 0 | X_k(n) = 1) = a_k \\ \mathbb{P}(X_k(n+1) = 1 | X_k(n) = 1) = 1 - a_k = a'_k \end{cases} \quad (3)$$

for any $n \geq 0$, so these are the transition probabilities for $X_k(\cdot)$ in one step.

It is clear from the independence of its components that $(X(n))_{n \geq 0}$ is also a homogeneous Markov chain, and in order to write its transition matrix, we will use the binary representation of integers introduced in [5,6]. This turned out to be a useful way of encoding natural numbers. We explain this representation here as well.

Basically, it all comes down to the fact that for every $i \in \mathbb{N}$ there exists a unique finite set $I_i \subset \mathbb{N}$, such that

$$i = 1 + \sum_{k \in I_i} 2^{k-1}, \quad (4)$$

where for $i = 1$ we assume $I_1 := \emptyset$ and the sum over I_1 equals zero. For example, $I_{19} = \{2, 5\}$ because $19 = 1 + 2^{2-1} + 2^{5-1}$. The set I_i in (4) we call the representation set of i . In this section, by $\bar{I}_i$ we denote the complement of I_i with respect to $\{1, 2, \dots, m\}$, i.e., $\bar{I}_i := \{1, 2, \dots, m\} \setminus I_i$. To summarize, we list representation sets of integers from 1 to 8: $I_1 = \emptyset$, $I_2 = \{1\}$, $I_3 = \{2\}$, $I_4 = \{1, 2\}$, $I_5 = \{3\}$, $I_6 = \{1, 3\}$, $I_7 = \{2, 3\}$, $I_8 = \{1, 2, 3\}$.

Lemma 1. *The set of states of $(X(n))_{n \geq 0}$, defined by (1), can be identified with $\mathbb{S}_m = \{1, 2, \dots, 2^m\}$. Moreover, let $P = (p_{ij})_{i,j \in \mathbb{S}_m}$ be its transition matrix. Then*

$$p_{ij} = \prod_{k \in I_i \cap I_j} a'_k \cdot \prod_{k \in I_i \cap \bar{I}_j} a_k \cdot \prod_{k \in \bar{I}_i \cap \bar{I}_j} b'_k \cdot \prod_{k \in \bar{I}_i \cap I_j} b_k, \quad (5)$$

where I_i, I_j are representation sets of i and j , respectively. We assume that if in (5) one of the sets $I_i \cap I_j$, $I_i \cap \bar{I}_j$, $\bar{I}_i \cap \bar{I}_j$, $\bar{I}_i \cap I_j$ is empty then the product over this set equals 1.

Proof. Let $\mathbb{S}$ denote the state space of $(X(n))_{n \geq 0}$. Thus, it consists of all sequences $(\delta_1, \delta_2, \dots, \delta_m)$, where $\delta_k \in \{0, 1\}$, for $k = 1, 2, \dots, m$. As explained in [6], a sequence $\delta = (\delta_1, \delta_2, \dots, \delta_m)$ can be identified with the binary representation of an integer $i = 1 + \sum_{k=1}^m \delta_k 2^{k-1}$. The smallest such integer equals 1, for the sequence $(0, 0, \dots, 0)$, and the biggest equals 2^m , for $(1, 1, \dots, 1)$. In other words, we can define a map $f : \mathbb{S} \rightarrow \mathbb{S}_m$ in the following way

$$f((\delta_1, \delta_2, \dots, \delta_m)) = 1 + \sum_{k=1}^m \delta_k 2^{k-1}. \quad (6)$$

Notice that f is a bijection because every natural number has a unique binary representation. We can rewrite (6) in an equivalent form $f(\delta) = i$, where

$$\mathbb{S} \ni \delta = (\delta_1, \delta_2, \dots, \delta_m) \longleftrightarrow i = 1 + \sum_{k=1}^m \delta_k 2^{k-1} \in \mathbb{S}_m.$$

For example, for $m = 3$ we have $(0, 0, 0) \rightarrow 1$, $(1, 0, 0) \rightarrow 2$, $(1, 1, 0) \rightarrow 4$, etc. The above observations justify that the set of states of $(X(n))_{n \geq 0}$ can be regarded as $\mathbb{S}_m$ and I assume we do so. Also, it can be seen from (4) and (6) that I_i contains the indices of those components of δ which are equal to 1. Thus, $\bar{I}_i$ is the set of indices of δ which are equal to 0.

Fix $i, j \in \mathbb{S}_m$ and let I_i, I_j be their representation sets determined by (4). Then, denote $\delta := f^{-1}(i)$ and $\delta' := f^{-1}(j)$, where f is defined by (6). Because I_i and I_j are subsets of $\{1, 2, \dots, m\}$, with possibility that $I_i = \emptyset$ and/or $I_j = \emptyset$, therefore

$$\{1, 2, \dots, m\} = \underbrace{(I_i \cap I_j)}_{1's \rightarrow 1's} \cup \underbrace{(I_i \cap \bar{I}_j)}_{1's \rightarrow 0's} \cup \underbrace{(\bar{I}_i \cap \bar{I}_j)}_{0's \rightarrow 0's} \cup \underbrace{(\bar{I}_i \cap I_j)}_{0's \rightarrow 1's}. \quad (7)$$

The meaning of the four pairwise disjoint sets in (7) is clear. They contain the indices of those components of δ which change to their corresponding components of δ' : from 1 to 1, from 1 to 0, from 0 to 0 and from 0 to 1. Hence, (5) follows from the independence of $X_1(\cdot), \dots, X_m(\cdot)$ and (3). $\square$

As can be seen from (5), every $p_{ij} \in (0, 1)$, and it is a product of exactly m numbers. This, in turn, implies that $(X(n))_{n \geq 0}$ is irreducible (all states communicate) and aperiodic, which means that each state is accessible from every other state in one step.

Since the state space of $(X(n))_{n \geq 0}$ is finite, $|\mathbb{S}_m| = 2^m$, there exists a unique stationary distribution $\pi = (\pi_i)_{i \in \mathbb{S}_m}$ for this chain, which is determined by the equation $\pi = \pi P$; see, e.g., [9] (p. 133). In coordinates, it means

$$\pi_j = \sum_{i \in \mathbb{S}_m} \pi_i p_{ij}, \quad j \in \mathbb{S}_m.$$

In [6] (Theorem 4), it was proved that this distribution is given by

$$\pi_i = \frac{1}{\prod_{k=1}^m \left(1 + \frac{b_k}{a_k}\right)} \cdot \prod_{k \in I_i} \frac{b_k}{a_k}, \quad i \in \mathbb{S}_m. \quad (8)$$

For example, for $m = 3$ it takes the form

$$\pi = \frac{1}{\prod_{k=1}^3 \left(1 + \frac{b_k}{a_k}\right)} \cdot \left(1, \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_1 b_2}{a_1 a_2}, \frac{b_3}{a_3}, \frac{b_1 b_3}{a_1 a_3}, \frac{b_2 b_3}{a_2 a_3}, \frac{b_1 b_2 b_3}{a_1 a_2 a_3}\right).$$

Let $l^1(\mathbb{R}^{2^m})$ denote $\mathbb{R}^{2^m}$ with the norm defined by

$$\|x\|_{l^1(\mathbb{R}^{2^m})} = \sum_{i=1}^{2^m} |\xi_i|, \quad x = (\xi_i)_{i \leq 2^m} = (\xi_1, \xi_2, \dots, \xi_{2^m}).$$

Then, the map $x \rightarrow x \cdot P$, where P is the transition matrix (5), is a Markov operator in $l^1(\mathbb{R}^{2^m})$ (see [2] (p. 42)). It means that $\|x\|_{l^1(\mathbb{R}^{2^m})} = \|x \cdot P\|_{l^1(\mathbb{R}^{2^m})}$ for any $x \geq 0$. By $x \geq 0$ we understand all components of the x are non-negative, i.e., $\xi_i \geq 0$.

Recall that a number λ is called an eigenvalue of P if there exists $x \in \mathbb{R}^{2^m}$ such that $x \neq 0$ and $xP = \lambda x$ (see [14] (p. 134)). In such a case, x is termed an eigenvector of P

corresponding to λ . In view of (8), it is clear that $\lambda_1 = 1$ is an eigenvalue of P and $x_{\lambda_1} = \pi$ is its corresponding eigenvector.

This theorem can be considered a generalization of the results from [8,9].

Theorem 1. The matrix $P = (p_{ij})_{i,j \in \mathbb{S}_m}$, determined by (5), has 2^m eigenvalues. These eigenvalues, denoted as $\lambda_1, \lambda_2, \dots, \lambda_{2^m}$, are given by

$$\lambda_i = \prod_{k \in I_i} (1 - a_k - b_k) = \prod_{k \in I_i} (a'_k - b_k), \quad i \in \mathbb{S}_m. \quad (9)$$

The eigenvector $x_{\lambda_i} = (\xi_{i,j})$, corresponding to λ_i , equals

$$\xi_{i,j} = (-1)^{|I_j \cap I_i|} \prod_{k \in I_j \cap \bar{I}_i} \frac{b_k}{a_k}, \quad j = 1, 2, \dots, 2^m. \quad (10)$$

Proof. See the proof of Theorem 3, where we consider a more general case. Those calculations also apply to this theorem. Another approach to the eigenvalues of P is presented in Theorem 2. $\square$

Remark 1. Notice that $\lambda_1 = 1$ and $|\lambda_i| < 1$, for $i = 2, 3, \dots, 2^m$. This is because the Markov chain under study is irreducible and aperiodic (see [8] (p. 137)). We assume all $a_k, b_k \in (0, 1)$. However, not all these eigenvalues have to be distinct. For example, we may take $a_k = b_k = \frac{1}{2}$, for $k = 1, 2, \dots, m$. In this case, there are only two: $\lambda_1 = 1$ and $\lambda_2 = 0$.

Nevertheless, all vectors defined by (10) are linearly independent, and Theorem 1 allows us to diagonalize P . It can be written as

$$P = W^{-1}DW, \quad (11)$$

where D and W are given by

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{2^m} \end{bmatrix} \quad W = \begin{bmatrix} x_{\lambda_1} \\ x_{\lambda_2} \\ \dots \\ x_{\lambda_{2^m}} \end{bmatrix} = \begin{bmatrix} \xi_{1,1} & \xi_{1,2} & \dots & \xi_{1,2^m} \\ \xi_{2,1} & \xi_{2,2} & \dots & \xi_{2,2^m} \\ \dots & \dots & \dots & \dots \\ \xi_{2^m,1} & \xi_{2^m,2} & \dots & \xi_{2^m,2^m} \end{bmatrix}$$

In Lemma 5, we prove that $W^{-1} = c^{-1} \cdot W$, where $c = \prod_{k=1}^m \left(1 + \frac{b_k}{a_k}\right)$. Hence, from (11), we would obtain

$$P^n = \frac{1}{\prod_{k=1}^m \left(1 + \frac{b_k}{a_k}\right)} WD^nW, \quad n \geq 1. \quad (12)$$

Both (11) and (12) hold true also when D and W are infinite (see Theorem 3).

Example 1. Consider $m = 2$. The transition matrix of $X(n) = (X_1(n), X_2(n))$ has the form

$$P = \begin{bmatrix} b'_1 b'_2 & b_1 b'_2 & b'_1 b_2 & b_1 b_2 \\ a'_1 b'_2 & a'_1 b'_2 & a'_1 b_2 & a'_1 b_2 \\ b'_1 a_2 & b_1 a_2 & b'_1 a'_2 & b_1 a'_2 \\ a_1 a_2 & a'_1 a_2 & a_1 a'_2 & a'_1 a'_2 \end{bmatrix} \quad (13)$$

The four eigenvalues of P (see (9)) are as follows:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a'_1 - b_1 & 0 & 0 \\ 0 & 0 & a'_2 - b_2 & 0 \\ 0 & 0 & 0 & (a'_1 - b_1)(a'_2 - b_2) \end{bmatrix}$$

and their corresponding eigenvectors are given by

$$W = \begin{bmatrix} x_{\lambda_1} \\ x_{\lambda_2} \\ x_{\lambda_3} \\ x_{\lambda_4} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b_1}{a_1} & \frac{b_2}{a_2} & \frac{b_1 b_2}{a_1 a_2} \\ 1 & -1 & \frac{b_2}{a_2} & -\frac{b_2}{a_2} \\ 1 & \frac{b_1}{a_1} & -1 & -\frac{b_1}{a_1} \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

In particular, for any $n \geq 2$, we have

$$\begin{cases} p_{11}^{(n)} = \frac{1}{\prod_{k=1}^2 (1 + \frac{b_k}{a_k})} \left(1 + \frac{b_1}{a_1} \lambda_2^n + \frac{b_2}{a_2} \lambda_3^n + \frac{b_1 b_2}{a_1 a_2} \lambda_4^n \right) \\ p_{12}^{(n)} = \frac{1}{\prod_{k=1}^2 (1 + \frac{b_k}{a_k})} \left(\frac{b_1}{a_1} (1 - \lambda_2^n) + \frac{b_1 b_2}{a_1 a_2} (\lambda_3^n - \lambda_4^n) \right) \\ p_{13}^{(n)} = \frac{1}{\prod_{k=1}^2 (1 + \frac{b_k}{a_k})} \left(\frac{b_2}{a_2} (1 - \lambda_3^n) + \frac{b_1 b_2}{a_1 a_2} (\lambda_2^n - \lambda_4^n) \right) \\ p_{14}^{(n)} = \frac{1}{\prod_{k=1}^2 (1 + \frac{b_k}{a_k})} \left(\frac{b_1 b_2}{a_1 a_2} (1 - \lambda_2^n - \lambda_3^n + \lambda_4^n) \right). \end{cases}$$

Taking the limit, we obtain

$$\lim_{n \rightarrow +\infty} (p_{11}^{(n)}, p_{12}^{(n)}, p_{13}^{(n)}, p_{14}^{(n)}) = \frac{1}{\prod_{k=1}^2 (1 + \frac{b_k}{a_k})} \cdot \left(1, \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_1 b_2}{a_1 a_2} \right),$$

i.e., the stationary distribution for $X(n)$ (see (8)).

The transition matrix P_k , given by (2), has the following characteristic polynomial:

$$p(\mu) = \det(P_k - \mu I) = (\mu - 1)(\mu - (1 - a_k - b_k)), \quad \mu \in \mathbb{C}.$$

The eigenvalues of P_k , together with their corresponding left eigenvectors, are therefore equal

$$\begin{cases} \mu_1^k = 1, & v_1^k = (1, \frac{b_k}{a_k}), \\ \mu_2^k = 1 - a_k - b_k, & v_2^k = (1, -1). \end{cases} \quad (14)$$

In other words, $v_i^k P_k = \mu_i^k v_i^k$, for $i = 1, 2$. Notice that each λ_i (see (9)) is a product of μ_i^k , for a suitable choice of i . This suggests that the transition matrix of the 2^m -state Markov chain is the Kronecker product of the matrices $P_1, \dots, P_m$. In Theorem 2, we prove that this is indeed the case, and we begin with the definition of the tensor product of matrices.

The tensor product or the Kronecker product of two matrices, $A = (a_{ij})$ and $B = (b_{ij})$, is defined to be the block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

See [15] (p. 243). So, if A is an $m \times n$ matrix and B is a $p \times q$ matrix, then $A \otimes B$ has mp rows and nq columns. Its (i, j) entry equals

$$(A \otimes B)_{ij} = a_{f_1(i)f_2(j)} b_{g_1(i)g_2(j)}, \quad (15)$$

where

$$\begin{cases} f_1(i) = (i-1) \operatorname{div} p + 1, & f_2(j) = (j-1) \operatorname{div} q + 1, \\ g_1(i) = (i-1) \bmod p + 1, & g_2(j) = (j-1) \bmod q + 1, \end{cases}$$

for $i = 1, 2, \dots, mp$ and $j = 1, 2, \dots, nq$. Here, the div notation stands for the usual integer division function: $i \operatorname{div} p := \lfloor \frac{i}{p} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function (see [16] (p. 120)).

In general, $A \otimes B \neq B \otimes A$; however, the tensor product is associative, i.e., $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, for any matrices. For example, $v_1^1 \otimes v_1^2$ and $v_1^2 \otimes v_1^1$ are

$$\left(1, \frac{b_1}{a_1}\right) \otimes \left(1, \frac{b_2}{a_2}\right) = \left(1, \frac{b_2}{a_2}, \frac{b_1}{a_1}, \frac{b_1 b_2}{a_1 a_2}\right), \quad \left(1, \frac{b_2}{a_2}\right) \otimes \left(1, \frac{b_1}{a_1}\right) = \left(1, \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_1 b_2}{a_1 a_2}\right).$$

The following theorem is equivalent to Theorem 1. It explains why λ_i s are products of μ_i^k s, and also unveils the structure of x_{λ_i} s, their corresponding eigenvectors.

Theorem 2. Let $P_1, P_2, \dots, P_m$ and P be given by (2) and (5), respectively. Then,

$$P = P_m \otimes P_{m-1} \otimes \dots \otimes P_2 \otimes P_1. \quad (16)$$

Moreover, any eigenvalue of P has the form

$$\mu_{i_m}^m \cdot \mu_{i_{m-1}}^{m-1} \dots \mu_{i_2}^2 \cdot \mu_{i_1}^1 \quad \left(\lambda_i = \prod_{k \in I_i} \mu_k^k \right), \quad (17)$$

for some $i_1, i_2, \dots, i_m \in \{1, 2\}$, and its corresponding eigenvector equals

$$v_{i_m}^m \otimes v_{i_{m-1}}^{m-1} \otimes \dots \otimes v_{i_2}^2 \otimes v_{i_1}^1. \quad (18)$$

Proof. In view of [15] (Theorem 4.2.12), the only thing that needs proof is (16). (17) and (18) follow from it. So we show (16) and the proof is by induction on m . We begin with $m = 2$. This case does not require additional notation. The two matrices are

$$P_2 = \begin{bmatrix} b'_2 & b_2 \\ a_2 & a'_2 \end{bmatrix}, \quad P_1 = \begin{bmatrix} b'_1 & b_1 \\ a_1 & a'_1 \end{bmatrix},$$

so their Kronecker product equals

$$P_2 \otimes P_1 = \begin{bmatrix} b'_2 \cdot \begin{bmatrix} b'_1 & b_1 \\ a_1 & a'_1 \end{bmatrix} & b_2 \cdot \begin{bmatrix} b'_1 & b_1 \\ a_1 & a'_1 \end{bmatrix} \\ a_2 \cdot \begin{bmatrix} b'_1 & b_1 \\ a_1 & a'_1 \end{bmatrix} & a'_2 \cdot \begin{bmatrix} b'_1 & b_1 \\ a_1 & a'_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} b'_1 b'_2 & b_1 b'_2 & b'_1 b_2 & b_1 b_2 \\ a_1 b'_2 & a'_1 b'_2 & a_1 b_2 & a'_1 b_2 \\ b'_1 a_2 & b_1 a_2 & b'_1 a'_2 & b_1 a'_2 \\ a_1 a_2 & a'_1 a_2 & a_1 a'_2 & a'_1 a'_2 \end{bmatrix},$$

which coincides with the matrix given by (13). The first step of induction is completed.

To continue the proof, let us introduce the following notation. The matrix $P = (p_{ij})_{i,j \in \mathbb{S}_m}$ is now denoted $P(m) = (p_{ij}(m))_{i,j \in \mathbb{S}_m}$. The matrix P_k is now $P_k = (\alpha_{ij}(k))_{i,j \in \{1,2\}}$, i.e.,

$$P_k = \begin{bmatrix} \alpha_{11}(k) & \alpha_{12}(k) \\ \alpha_{21}(k) & \alpha_{22}(k) \end{bmatrix} = \begin{bmatrix} b'_k & b_k \\ a_k & a'_k \end{bmatrix}, \quad k \geq 1. \quad (19)$$

Also, let $\mathbb{N}_m := \{1, 2, \dots, m\}$, for $m \geq 2$. The second step of induction requires us to prove

$$P(m+1) = P_{m+1} \otimes P(m), \quad (20)$$

for $m \geq 2$. At first, we write explicit expressions for the entries of the two $2^{m+1} \times 2^{m+1}$ matrices in (20). Fix $i, j \in \mathbb{S}_{m+1} = \{1, 2, \dots, 2^{m+1}\}$. Based on (15), we have

$$(P_{m+1} \otimes P(m))_{ij} = \alpha_{f_1(i)f_2(j)}(m+1)p_{g_1(i)g_2(j)}(m), \quad (21)$$

where

$$\begin{cases} f_1(i) = (i-1) \operatorname{div} 2^m + 1, & f_2(j) = (j-1) \operatorname{div} 2^m + 1, \\ g_1(i) = (i-1) \bmod 2^m + 1, & g_2(j) = (j-1) \bmod 2^m + 1. \end{cases}$$

Notice that $g_1(i), g_2(j) \in \mathbb{S}_m$. This is because $(-1) \bmod 2^m = 2^m - 1$, which implies

$$(2^{m+1} - 1) \bmod 2^m + 1 = 2^m - 1 + 1 = 2^m \in \mathbb{S}_m.$$

The (i, j) entry of $P(m+1)$ is

$$p_{ij}(m+1) = \prod_{k \in I_i \cap I_j} a'_k \cdot \prod_{k \in I_i \cap \bar{I}_j} a_k \cdot \prod_{k \in \bar{I}_i \cap \bar{I}_j} b'_k \cdot \prod_{k \in \bar{I}_i \cap I_j} b_k, \quad (22)$$

where $I_i, I_j \subset \mathbb{N}_{m+1}$. We will write $p_{ij}(m+1)$ in such a way that it can be directly compared with the expression on the right side of (21). To this end, let us define

$$I_i^1 := I_i \cap \mathbb{N}_m, \quad I_i^2 := I_i \cap \{m+1\} \Rightarrow I_i = I_i^1 \cup I_i^2, \quad I_i^1 \cap I_i^2 = \emptyset.$$

In particular, $I_i^1 \subset \mathbb{N}_m$ and $I_i^2 \subset \{m+1\}$. In the latter case, $I_i^2 = \emptyset$ or $I_i^2 = \{m+1\}$. By direct calculation, we obtain the equalities

$$\begin{cases} I_i \cap I_j = (I_i^1 \cap I_j^1) \cup (I_i^2 \cap I_j^2), \\ I_i \cap \bar{I}_j = (I_i^1 \cap \bar{I}_j^1) \cup (I_i^2 \cap \bar{I}_j^2), \\ \bar{I}_i \cap \bar{I}_j = (\bar{I}_i^1 \cap \bar{I}_j^1 \cap \mathbb{N}_m) \cup (\bar{I}_i^2 \cap \bar{I}_j^2 \cap \{m+1\}), \\ \bar{I}_i \cap I_j = (\bar{I}_i^1 \cap I_j^1) \cup (\bar{I}_i^2 \cap I_j^2). \end{cases} \quad (23)$$

All complements in (22) and (23) are with respect to $\mathbb{N}_{m+1}$. For example, the proof of the first equality is this

$$I_i \cap I_j = (I_i^1 \cup I_i^2) \cap (I_j^1 \cup I_j^2) = (I_i^1 \cap (I_j^1 \cup I_j^2)) \cup (I_i^2 \cap (I_j^1 \cup I_j^2)) = (I_i^1 \cap I_j^1) \cup (I_i^2 \cap I_j^2),$$

due to $I_i^1 \cap I_j^2 = \emptyset$ and $I_i^2 \cap I_j^1 = \emptyset$. Taking into account (23), the right hand side of (22) can be factorized as follows

$$p_{ij}(m+1) = p_{ij}^1(m+1)p_{ij}^2(m+1), \quad (24)$$

where

$$\begin{cases} p_{ij}^1(m+1) := \prod_{k \in I_i^1 \cap I_j^1} a'_k \cdot \prod_{k \in I_i^1 \cap \bar{I}_j^1} a_k \cdot \prod_{k \in \bar{I}_i^1 \cap \bar{I}_j^1 \cap \mathbb{N}_m} b'_k \cdot \prod_{k \in \bar{I}_i^1 \cap I_j^1} b_k, \\ p_{ij}^2(m+1) := \prod_{k \in I_i^2 \cap I_j^2} a'_k \cdot \prod_{k \in I_i^2 \cap \bar{I}_j^2} a_k \cdot \prod_{k \in \bar{I}_i^2 \cap \bar{I}_j^2 \cap \{m+1\}} b'_k \cdot \prod_{k \in \bar{I}_i^2 \cap I_j^2} b_k. \end{cases}$$

Comparing (21) with (24), we conclude that to prove (20), we have to show

$$p_{ij}^1(m+1) = p_{g_1(i)g_2(j)}(m), \quad p_{ij}^2(m+1) = \alpha_{f_1(i)f_2(j)}(m+1). \quad (25)$$

I_i^1 and I_j^1 are subsets of $\mathbb{N}_m$, as are $I_i^1 \cap I_j^1$, $I_i^1 \cap \overline{I_j^1}$, $\overline{I_i^1} \cap I_j^1$ and $\overline{I_i^1} \cap \overline{I_j^1} \cap \mathbb{N}_m$. So the proof of the first equality in (25) will follow from

$$I_{g_1(i)} = I_i^1 = I_i \cap \mathbb{N}_m, \quad i \in \mathbb{S}_{m+1}. \quad (26)$$

To substantiate it, we consider two cases. In the first one, let $i \leq 2^m$, i.e., $i \in \mathbb{S}_m$. Then, $I_i \subset \mathbb{N}_m$, so $I_i^1 = I_i$, and

$$g_1(i) = (i - 1) \bmod 2^m + 1 = i \Rightarrow I_{g_1(i)} = I_i.$$

In the second case, $i \in \mathbb{S}_{m+1} \setminus \mathbb{S}_m$, i.e., $i \in \{2^m + 1, \dots, 2^{m+1}\}$. Then, certainly $\{m + 1\} \subset I_i$, which implies $I_i^1 = I_i \cap \mathbb{N}_m = I_i \setminus \{m + 1\}$. On the other hand, $2^m \bmod 2^m = 0$ results in

$$g_1(i) = i - 2^m = i - 2^{(m+1)-1} \Rightarrow I_{i-2^m} = I_i \setminus \{m + 1\}.$$

Thus, (26) is proved. The last thing to demonstrate is

$$\alpha_{f_1(i)f_2(j)}(m + 1) = \prod_{k \in I_i^2 \cap I_j^2} a'_k \cdot \prod_{k \in I_i^2 \cap \overline{I_j^2}} a_k \cdot \prod_{k \in \overline{I_i^2} \cap I_j^2} b'_k \cdot \prod_{k \in \overline{I_i^2} \cap \overline{I_j^2}} b_k, \quad (27)$$

where $\alpha_{ij}(m + 1)$ s are given by (19). This time, there are four cases to consider.

Suppose $i \leq 2^m$ and $j \leq 2^m$. Then, $I_i^2 = \emptyset$, $I_j^2 = \emptyset$, $\overline{I_i^2} = \mathbb{N}_{m+1}$, $\overline{I_j^2} = \mathbb{N}_{m+1}$ and

$$f_1(i) = (i - 1) \bmod 2^m + 1 = 1, \quad f_2(j) = (j - 1) \bmod 2^m + 1 = 1,$$

so (27) takes the form

$$\alpha_{11}(m + 1) = b'_{m+1} = \prod_{k \in \{m+1\}} b'_k.$$

If $i \leq 2^m$ and $j \in \{2^m + 1, \dots, 2^{m+1}\}$, then $I_i^2 = \emptyset$, $I_j^2 = \{m + 1\}$, $\overline{I_i^2} = \mathbb{N}_{m+1}$, $\overline{I_j^2} = \mathbb{N}_m$, with $f_1(i) = 1$ and $f_2(j) = 2$. Thus, we have

$$\alpha_{12}(m + 1) = b_{m+1} = \prod_{k \in \{m+1\}} b_k.$$

In a similar way, if $i \in \{2^m + 1, \dots, 2^{m+1}\}$ and $j \leq 2^m$, then $I_i^2 = \{m + 1\}$, $I_j^2 = \emptyset$, $\overline{I_i^2} = \mathbb{N}_m$, $\overline{I_j^2} = \mathbb{N}_{m+1}$, with $f_1(i) = 2$ and $f_2(j) = 1$. Therefore, in this case

$$\alpha_{21}(m + 1) = a_{m+1} = \prod_{k \in \{m+1\}} a_k.$$

Finally, for $i, j \in \{2^m + 1, \dots, 2^{m+1}\}$ we have $I_i^2 = I_j^2 = \{m + 1\}$, $\overline{I_i^2} = \overline{I_j^2} = \mathbb{N}_m$ and $f_1(i) = 2$, $f_2(j) = 2$. Hence, we obtain

$$\alpha_{22}(m + 1) = a'_{m+1} = \prod_{k \in \{m+1\}} a'_k.$$

This proves (27), which implies (25). In consequence, (20) holds true and the proof of the theorem is completed. $\square$

Example 2. Given Theorem 2 and (14), the four eigenvalues from Example 1 can be expressed in the form

$$\begin{cases} \lambda_1 = \mu_1^2 \mu_1^1 = 1 \cdot 1, \\ \lambda_2 = \mu_1^2 \mu_2^1 = 1 \cdot (a'_1 - b_1), \\ \lambda_3 = \mu_2^2 \mu_1^1 = (a'_2 - b_2) \cdot 1, \\ \lambda_4 = \mu_2^2 \mu_2^1 = (a'_2 - b_2) \cdot (a'_1 - b_1), \end{cases}$$

and their corresponding eigenvectors as tensor products

$$\begin{cases} x_{\lambda_1} = v_1^2 \otimes v_1^1 = (1, \frac{b_2}{a_2}) \otimes (1, \frac{b_1}{a_1}) = (1, \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_1 b_2}{a_1 a_2}), \\ x_{\lambda_2} = v_1^2 \otimes v_2^1 = (1, \frac{b_2}{a_2}) \otimes (1, -1) = (1, -1, \frac{b_2}{a_2}, -\frac{b_2}{a_2}), \\ x_{\lambda_3} = v_2^2 \otimes v_1^1 = (1, -1) \otimes (1, \frac{b_1}{a_1}) = (1, \frac{b_1}{a_1}, -1, -\frac{b_1}{a_1}), \\ x_{\lambda_4} = v_2^2 \otimes v_2^1 = (1, -1) \otimes (1, -1) = (1, -1, -1, 1). \end{cases}$$

Using the tensor product notation, the stationary distribution π (see (8)) may be written as

$$\pi = \frac{1}{\prod_{k=1}^m (1 + \frac{b_k}{a_k})} \cdot \left(1, \frac{b_m}{a_m}\right) \otimes \left(1, \frac{b_{m-1}}{a_{m-1}}\right) \otimes \dots \otimes \left(1, \frac{b_1}{a_1}\right).$$

3. A Discrete Version of Blackwell's Markov Chain

Let $X_1(\cdot), X_2(\cdot), X_3(\cdot), \dots$ be an infinite sequence of mutually independent, two-state chains as described in Section 2. A discrete analogue of the Markov chain introduced in [1] can be defined as follows:

$$X(n) := (X_1(n), X_2(n), X_3(n), \dots) \in \{0, 1\}^\infty, \quad n \geq 0. \quad (28)$$

This joint process is well-defined by the Kolmogorov extension theorem (see [3] (p. 65)), and it is also a Markov process. At this point, however, it is not clear whether it is a Markov chain, i.e., that its state space is countable. We deal with this issue in Lemma 2.

Let us first notice that $(X(n))_{n \geq 0}$ is characterized by two sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$, where $a_k, b_k \in (0, 1)$ for $k \geq 1$ (see (3)). As noted in the introduction, there is no notion of an instantaneous state for a discrete-time Markov chain.

In a similar way as in [6] (Lemma 5), we show that under certain conditions imposed on $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$, the set of states of $(X(n))_{n \geq 0}$ can be regarded as $\mathbb{N}$.

Lemma 2. Let $\mathbb{S}$ denote the state space of the Markov process defined by (28). If the sequence $(b_k)_{k \in \mathbb{N}}$ satisfies

$$\sum_{k=1}^{\infty} b_k < +\infty, \quad (29)$$

then $\mathbb{S}$ is countable and it can be identified with $\mathbb{N}$. The transition matrix $P = (p_{ij})_{i,j \in \mathbb{N}}$ of this chain equals

$$p_{ij} = \prod_{k \in I_i \cap I_j} a'_k \cdot \prod_{k \in I_i \cap \bar{I}_j} a_k \cdot \prod_{k \in \bar{I}_i \cap I_j} b'_k \cdot \prod_{k \in \bar{I}_i \cap \bar{I}_j} b_k. \quad (30)$$

Moreover, every $p_{ij} \in (0, 1)$ and it is a product of infinitely many numbers.

Proof. For now, we have $\mathbb{S} \subset \{0, 1\}^\infty$. The set $\{0, 1\}^\infty$, consisting of all sequences of 0 s and 1 s, including those with infinitely many 1 s, is uncountable by the known Cantor's diagonal argument. So, in general, $\mathbb{S}$ could be uncountable.

However, if the condition (29) holds true, then, with probability one, elements of $\mathbb{S}$ are sequences with a finite number of 1 s. This follows from the Borel–Cantelli lemma (see, e.g., [3] (p. 66)). Namely, we denote $A_k := \{X_k(n+1) = 1 | X_k(n) = 0\}$, where $k \geq 1$. Then, $\mathbb{P}(A_k) = b_k$ (see (3)) and $\mathbb{P}(A_k \text{ i.o.}) = 0$, provided that

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \sum_{k=1}^{\infty} b_k < +\infty.$$

In other words, the condition (29) guarantees that only a finite number of 0 s change to 1 s. Formally, we can write $\mathbb{S}$ as

$$\mathbb{S} = \left\{ (\delta_k)_{k \in \mathbb{N}} : \delta_k = 0 \text{ or } 1 \text{ and } \sum_{k=1}^{\infty} \delta_k < +\infty \right\}$$

and the set thus defined is countable, as a countable sum of finite sets. Because $\mathbb{S}$ is infinite and countable, there exists a bijection between $\mathbb{S}$ and $\mathbb{N}$. We define $f : \mathbb{S} \rightarrow \mathbb{N}$ by

$$f((\delta_1, \delta_2, \delta_3, \dots)) = 1 + \sum_{k=1}^{\infty} \delta_k 2^{k-1} < +\infty \quad (31)$$

(see also (6)). That f is a bijection is a consequence of the fact that every natural number has a unique binary representation, which is encoded in $(\delta_k)_{k \in \mathbb{N}}$. So we may write (31) in an equivalent way $f(\delta) = i$, where

$$\mathbb{S} \ni \delta = (\delta_1, \delta_2, \dots, \delta_m, 0, 0, 0, \dots) \longleftrightarrow i = 1 + \sum_{k=1}^m \delta_k 2^{k-1} \in \mathbb{N}, \quad (32)$$

In contrast to (6), where m is fixed, an integer m in (32) is not fixed, but depends on δ . For every $\delta \in \mathbb{S}$ such an m exists, because it has a finite number of 1 s. Therefore, $\mathbb{S}$ can be identified with $\mathbb{N}$, which means

$$\mathbb{P}(X(n) \in \mathbb{N} | X(0) \in \mathbb{N}) = 1, \quad \text{for all } n \geq 0.$$

From now on we assume $\mathbb{S} = \mathbb{N}$. Fix $i, j \in \mathbb{N}$ and let I_i, I_j be their representation sets determined by (4). Let $\delta := f^{-1}(i)$ and $\delta' := f^{-1}(j)$, where f is defined by (31). This time, I_i and I_j , are subsets of $\mathbb{N}$ with $\bar{I}_i = \mathbb{N} \setminus I_i$ and $\bar{I}_j = \mathbb{N} \setminus I_j$. Therefore, we have

$$\mathbb{N} = \underbrace{(I_i \cap I_j)}_{1's \rightarrow 1's} \cup \underbrace{(I_i \cap \bar{I}_j)}_{1's \rightarrow 0's} \cup \underbrace{(\bar{I}_i \cap \bar{I}_j)}_{0's \rightarrow 0's} \cup \underbrace{(\bar{I}_i \cap I_j)}_{0's \rightarrow 1's}. \quad (33)$$

This is an analogous situation to that of (7) and the meaning of the four pairwise disjoint sets in (33) is similar. What is different here is that the set $\bar{I}_i \cap \bar{I}_j$ is infinite. In other words, infinitely many 0 s change to 0 s. Notice that the sets $I_i \cap I_j$, $I_i \cap \bar{I}_j$ and $\bar{I}_i \cap I_j$ are finite. Therefore, to justify (30) we need to prove that the infinite product $\prod_{k \in \bar{I}_i \cap \bar{I}_j} b'_k$ converges. It is enough to consider the case of p_{11} and to prove

$$p_{11} = \prod_{k=1}^{\infty} b'_k = \prod_{k=1}^{\infty} (1 - b_k) \in (0, 1). \quad (34)$$

Because we assume $b_k \in (0, 1)$, for every $k \geq 1$, it turns out the conditions (29) and (34) are equivalent (see [17] (Problem 3.8.4)). Thus, (29) implies (34). Moreover, (34) implies that $p_{ij} \in (0, 1)$, for any $i, j \in \mathbb{N}$, since the product defining p_{ij} differs from p_{11} by a finite number of terms. This completes the proof of (30). $\square$

Lemma 3. The matrix given by (30) is stochastic; i.e., it satisfies

$$\sum_{j=1}^{\infty} p_{ij} = 1, \quad \text{for every } i \in \mathbb{N}. \quad (35)$$

Proof. For fixed $i \in \mathbb{N}$ we have

$$\sum_{j=1}^{\infty} p_{ij} = \sum_{I_j \subset \mathbb{N}} \left(\prod_{k \in I_j \cap I_i} a'_k \cdot \prod_{k \in I_j \cap \bar{I}_i} a_k \cdot \prod_{k \in \bar{I}_i \cap \bar{I}_j} b'_k \cdot \prod_{k \in \bar{I}_i \cap I_j} b_k \right), \quad (36)$$

where the summation runs over all finite subsets of $\mathbb{N}$. Denote $I := I_j \cap I_i$ and $I' := I_j \cap \bar{I}_i$. The sets I, I' are disjoint sets, $I \cap I' = \emptyset$, with $I \subset I_i$ and $I' \subset \bar{I}_i$. Therefore, we write (36) as

$$\sum_{j=1}^{\infty} p_{ij} = \sum_{I \subset I_i} \left[\sum_{I' \subset \bar{I}_i} \left(\prod_{k \in I} a'_k \cdot \prod_{k \in I_i \cap \bar{I}} a_k \cdot \prod_{k \in \bar{I}_i \cap \bar{I}'} b'_k \cdot \prod_{k \in I'} b_k \right) \right],$$

where the outer sum is over all finite subsets of I_i and the interior sum is over all finite subsets of $\bar{I}_i$ (see Section 4.2). Moreover, notice that in the interior sum, we can factor out the products that do not depend on I , namely,

$$\sum_{j=1}^{\infty} p_{ij} = \sum_{I \subset I_i} \left[\left(\prod_{k \in I} a'_k \cdot \prod_{k \in I_i \cap \bar{I}} a_k \right) \cdot \sum_{I' \subset \bar{I}_i} \left(\prod_{k \in \bar{I}_i \cap \bar{I}'} b'_k \cdot \prod_{k \in I'} b_k \right) \right].$$

Based on Lemma 6, we have

$$\sum_{I' \subset \bar{I}_i} \left(\prod_{k \in \bar{I}_i \cap \bar{I}'} b'_k \cdot \prod_{k \in I'} b_k \right) = \prod_{k \in \bar{I}_i} (b_k + b'_k) = 1,$$

and using (83), we obtain

$$\sum_{j=1}^{\infty} p_{ij} = \sum_{I \subset I_i} \left(\prod_{k \in I} a'_k \cdot \prod_{k \in I_i \cap \bar{I}} a_k \right) = \prod_{k \in I_i} (a_k + a'_k) = 1,$$

which proves (35). $\square$

The (infinite) matrix $P = (p_{ij})_{i,j \in \mathbb{N}}$, as a Markov operator, acts in $l^1(\mathbb{N})$. Recall this is the Banach space of all absolutely summable sequences, which means that $x = (\xi_i)_{i \in \mathbb{N}}$ belongs to $l^1(\mathbb{N})$ if and only if $\sum_{i \in \mathbb{N}} |\xi_i| < +\infty$, and then $\|x\|_{l^1(\mathbb{N})} = \sum_{i \in \mathbb{N}} |\xi_i|$. It is a separable space in which every $x \in l^1(\mathbb{N})$ can be written as $\sum_{i \in \mathbb{N}} \xi_i e_i$, where $\{e_i\}_{i \in \mathbb{N}}$ is the standard Schauder basis in this space, i.e., $e_i = (\dots, 0, 1, 0, \dots)$ with 1 in the i -th coordinate (see, e.g., [18] (Chapter 7)). In particular, $\|x\|_{l^1(\mathbb{N})} = \|x \cdot P\|_{l^1(\mathbb{N})}$ for any $x \in l^1(\mathbb{N})$ such that $x \geq 0$. If x is arbitrary, then $\|x \cdot P\|_{l^1(\mathbb{N})} \leq \|x\|_{l^1(\mathbb{N})}$, which means P is a linear contraction (see [2] (p. 15)). This lemma is an analogue of [6] (Theorem 2).

Lemma 4. Suppose the sequences $(a_k)_{k \in \mathbb{N}}$, $(b_k)_{k \in \mathbb{N}}$ satisfy (29), and additionally

$$\sum_{k=1}^{\infty} \frac{b_k}{a_k} < +\infty. \quad (37)$$

Then the stationary distribution $\pi = (\pi_i)_{i \in \mathbb{N}}$ for the Markov chain defined by (28) equals

$$\pi_i = \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right)} \cdot \prod_{k \in I_i} \frac{b_k}{a_k}, \quad (38)$$

and it satisfies the following detailed balance condition:

$$\pi_i p_{ij} = \pi_j p_{ji}, \quad i, j \in \mathbb{N}. \quad (39)$$

Proof. First, we prove $\sum_{i=1}^{\infty} \pi_i = 1$, which means $\pi \in l^1(\mathbb{N})$. The condition (37) implies that the product $\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right)$ converges, and it can be written as

$$\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right) = 1 + \sum_{k=1}^{\infty} \frac{b_k}{a_k} + \sum_{k_1 < k_2} \frac{b_{k_1} b_{k_2}}{a_{k_1} a_{k_2}} + \sum_{k_1 < k_2 < k_3} \frac{b_{k_1} b_{k_2} b_{k_3}}{a_{k_1} a_{k_2} a_{k_3}} + \dots,$$

or, equivalently using our notation, in the form

$$\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right) = \sum_{i=1}^{+\infty} \left(\prod_{k \in I_i} \frac{b_k}{a_k} \right)$$

See [17] (Problems 3.8.3, 3.8.26) and Section 4.1. This shows π is a probability distribution. Now we prove π is stationary. For this purpose, define $\sigma = (\sigma_i)_{i \in \mathbb{N}}$, a multiple of π , as

$$\sigma_i := \pi_i \cdot \prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right) = \prod_{k \in I_i} \frac{b_k}{a_k}. \quad (40)$$

We will prove that $\sigma = \sigma P$ and $\sigma_i p_{ij} = \sigma_j p_{ji}$, which is equivalent to $\pi = \pi P$ and (39), respectively. We begin with $\sigma_i p_{ij}$. Based on (30) and (40), it equals

$$\sigma_i p_{ij} = \prod_{k \in I_i} \frac{b_k}{a_k} \cdot \prod_{k \in I_i \cap I_j} a'_k \cdot \prod_{k \in I_i \cap \bar{I}_j} a_k \cdot \prod_{k \in \bar{I}_i \cap \bar{I}_j} b'_k \cdot \prod_{k \in \bar{I}_i \cap I_j} b_k. \quad (41)$$

Notice that $I_i \cup (\bar{I}_i \cap I_j) = I_i \cup I_j$; hence, we have

$$\prod_{k \in I_i} b_k \cdot \prod_{k \in \bar{I}_i \cap I_j} b_k = \prod_{k \in I_i \cup I_j} b_k = \prod_{k \in I_j} b_k \cdot \prod_{k \in \bar{I}_j \cap I_i} b_k$$

and

$$\prod_{k \in I_i} \frac{1}{a_k} \cdot \prod_{k \in I_i \cap \bar{I}_j} a_k = \prod_{k \in I_i \cap I_j} \frac{1}{a_k} = \prod_{k \in I_j} \frac{1}{a_k} \cdot \prod_{k \in I_j \cap \bar{I}_i} a_k.$$

Thus, we can write (41) as follows:

$$\sigma_i p_{ij} = \prod_{k \in I_j} \frac{b_k}{a_k} \cdot \prod_{k \in I_i \cap I_j} a'_k \cdot \prod_{k \in I_j \cap \bar{I}_i} a_k \cdot \prod_{k \in \bar{I}_i \cap \bar{I}_j} b'_k \cdot \prod_{k \in \bar{I}_j \cap I_i} b_k = \sigma_j p_{ji}. \quad (42)$$

This proves (39). Now we prove $\sigma = \sigma P$. Taking into account (42) and (35), we obtain

$$(\sigma P)_j = \sum_{i \in \mathbb{N}} \sigma_i p_{ij} = \sum_{i \in \mathbb{N}} \sigma_j p_{ji} = \sigma_j \sum_{i \in \mathbb{N}} p_{ji} = \sigma_j, \quad j \in \mathbb{N},$$

which proves $\pi = \pi P$. The above calculation shows the detailed balance condition is stronger than stationarity. In other words, not every stationary distribution must satisfy (39). This completes the proof. $\square$

Theorem 3. Suppose the sequences $(a_k)_{k \in \mathbb{N}}$, $(b_k)_{k \in \mathbb{N}}$ satisfy (29) and (37). Then the transition matrix $P = (p_{ij})_{i,j \in \mathbb{N}}$, determined by (30), has infinitely many eigenvalues given by

$$\lambda_i = \prod_{k \in I_i} (1 - a_k - b_k) = \prod_{k \in I_i} (a'_k - b_k), \quad i \in \mathbb{N}. \quad (43)$$

The eigenvector $x_{\lambda_i} = (\xi_{i,j})$, corresponding to λ_i , has the form

$$\xi_{i,j} = (-1)^{|I_j \cap I_i|} \prod_{k \in I_j \cap \bar{I}_i} \frac{b_k}{a_k}, \quad j = 1, 2, 3, \dots \quad (44)$$

and its l^1 -norm equals

$$\|x_{\lambda_i}\|_{l^1(\mathbb{N})} = 2^{|I_i|} \cdot \prod_{k \in \bar{I}_i} \left(1 + \frac{b_k}{a_k}\right). \quad (45)$$

Proof. Fix $i \in \mathbb{N}$ and let $y = (y_j)_{j \in \mathbb{N}}$ denote the product of x_{λ_i} and P , i.e., $y = x_{\lambda_i} \cdot P$. We will show that

$$y_j = \lambda_i \xi_{i,j}, \quad j \in \mathbb{N}. \quad (46)$$

Now also fix $j \in \mathbb{N}$. Then $y_j = (x_{\lambda_i} \cdot P)_j$ and

$$y_j = \sum_{l=1}^{\infty} \xi_{i,l} \cdot p_{l,j} = \sum_{l=1}^{\infty} \left((-1)^{|I_l \cap I_i|} \prod_{k \in I_l \cap \bar{I}_i} \frac{b_k}{a_k} \prod_{k \in I_l \cap I_j} a'_k \cdot \prod_{k \in I_l \cap \bar{I}_j} a_k \cdot \prod_{k \in \bar{I}_l \cap \bar{I}_j} b'_k \cdot \prod_{k \in \bar{I}_l \cap I_j} b_k \right).$$

Equivalently, we may write y_j as

$$y_j = \sum_{I_l \subset \mathbb{N}} \left((-1)^{|I_l \cap I_i|} \prod_{k \in I_l \cap \bar{I}_i} \frac{b_k}{a_k} \prod_{k \in I_l \cap I_j} a'_k \cdot \prod_{k \in I_l \cap \bar{I}_j} a_k \cdot \prod_{k \in \bar{I}_l \cap \bar{I}_j} b'_k \cdot \prod_{k \in \bar{I}_l \cap I_j} b_k \right), \quad (47)$$

where the sum is over all finite subsets of $\mathbb{N}$. Because I_i and I_j are fixed, I_l can be written as a sum of the following three pairwise disjoint sets

$$I_l = \underbrace{(I_l \cap I_i)}_{=I} \cup \underbrace{(I_l \cap \bar{I}_i \cap I_j)}_{=I'} \cup \underbrace{(I_l \cap \bar{I}_i \cap \bar{I}_j)}_{=I''}. \quad (48)$$

In particular, $I \subset I_i$, $I' \subset \bar{I}_i \cap I_j$ and $I'' \subset \bar{I}_i \cap \bar{I}_j$. Thanks to (48), equality (47) is equivalent to

$$y_j = \sum_{I \subset I_i} \left[\sum_{I' \subset I_j \cap \bar{I}_i} \left(\sum_{I'' \subset \bar{I}_i \cap \bar{I}_j} (-1)^{|I|} \prod_{k \in J_1} \frac{b_k}{a_k} \prod_{k \in J_2} a'_k \prod_{k \in J_3} a_k \prod_{k \in J_4} b'_k \prod_{k \in J_5} b_k \right) \right], \quad (49)$$

where the first sum is over all subsets of I_i , the second sum is over all subsets of $I_j \cap \bar{I}_i$ and the third sum is over all finite subsets of $\bar{I}_i \cap \bar{I}_j$. The sets $J_1, \dots, J_5$ in (49) are expressed in terms of I , I' and I'' , and are given by

$$J_1 = I' \cup I'', \quad J_2 = I' \cup (I \cap I_j), \quad J_3 = I'' \cup (I \cap \bar{I}_j), \quad J_4 = \bar{I}_j \cap \bar{I}'' \cap \bar{I}, \quad J_5 = I_j \cap \bar{I} \cap \bar{I}.$$

Of these five sets, J_4 is the only infinite one. To perform the summation over I'' in (49), first notice that J_4 can be written as a sum of two disjoint sets, namely,

$$J_4 = \bar{I}_j \cap \bar{I}'' \cap \bar{I} = (\bar{I}_j \cap \bar{I} \cap I_i) \cup (\bar{I}_j \cap \bar{I}'' \cap \bar{I}_i).$$

This holds true because $I_i \cup \bar{I}_i = \mathbb{N}$ and I'' is a subset of $\bar{I}_i \cap \bar{I}_j$. By Lemma 6, we have

$$\begin{aligned} & \sum_{I'' \subset \bar{I}_i \cap \bar{I}_j} \left(\prod_{k \in I''} \frac{b_k}{a_k} \prod_{k \in I''} a_k \prod_{k \in \bar{I}_j \cap I'' \cap \bar{I}_i} b'_k \right) \\ &= \sum_{I'' \subset \bar{I}_i \cap \bar{I}_j} \left(\prod_{k \in I''} b_k \prod_{k \in \bar{I}_j \cap I'' \cap \bar{I}_i} b'_k \right) = \prod_{k \in \bar{I}_i \cap \bar{I}_j} (b_k + b'_k) = 1, \end{aligned}$$

and (49) takes the form

$$\begin{aligned} y_j &= \sum_{I \subset I_i} \left[\sum_{I' \subset I_j \cap \bar{I}_i} \left((-1)^{|I|} \prod_{k \in I'} \frac{b_k}{a_k} \prod_{k \in I_2} a'_k \prod_{k \in I \cap \bar{I}_j} a_k \prod_{k \in I_i \cap \bar{I} \cap \bar{I}_j} b'_k \prod_{k \in I_5} b_k \right) \right] \\ &= \prod_{k \in I_j \cap \bar{I}_i} b_k \sum_{I \subset I_i} \left[\sum_{I' \subset I_j \cap \bar{I}_i} \left((-1)^{|I|} \prod_{k \in \bar{I} \cap I_i \cap I_j} b_k \prod_{k \in I \cap \bar{I}_j} a_k \prod_{k \in I_i \cap \bar{I} \cap \bar{I}_j} b'_k \prod_{k \in I \cap I_j} a'_k \prod_{k \in I'} \frac{a'_k}{a_k} \right) \right]. \end{aligned}$$

The application of (83), together with $a_k + a'_k = 1$, gives

$$\sum_{I' \subset I_j \cap \bar{I}_i} \left(\prod_{k \in I'} \frac{a'_k}{a_k} \right) = \prod_{k \in I_j \cap \bar{I}_i} \left(1 + \frac{a'_k}{a_k} \right) = \prod_{k \in I_j \cap \bar{I}_i} \frac{1}{a_k}.$$

This leads to

$$y_j = \left(\prod_{k \in I_j \cap \bar{I}_i} \frac{b_k}{a_k} \right) \sum_{I \subset I_i} \left((-1)^{|I|} \prod_{k \in \bar{I} \cap I_i \cap I_j} b_k \prod_{k \in I \cap \bar{I}_j} a_k \prod_{k \in I_i \cap \bar{I} \cap \bar{I}_j} b'_k \prod_{k \in I \cap I_j} a'_k \right). \quad (50)$$

In view of (43), (44) and (50), to prove (46), we need to show

$$\lambda_i = (-1)^{|I_j \cap I_i|} \sum_{I \subset I_i} \left((-1)^{|I|} \prod_{k \in \bar{I} \cap I_i \cap I_j} b_k \prod_{k \in I \cap \bar{I}_j} a_k \prod_{k \in I_i \cap \bar{I} \cap \bar{I}_j} b'_k \prod_{k \in I \cap I_j} a'_k \right). \quad (51)$$

So we prove that right hand sides of (43) and (51) are equal. We begin with (43). First, from $I_i = (I_i \cap I_j) \cup (I_i \cap \bar{I}_j)$ we have

$$\lambda_i = \prod_{k \in I_i} (a'_k - b_k) = \prod_{k \in I_i \cap I_j} (a'_k - b_k) \prod_{k \in I_i \cap \bar{I}_j} (a'_k - b_k).$$

The set $I_i \cap I_j$ is finite, so using (84), we can write

$$\prod_{k \in I_i \cap I_j} (a'_k - b_k) = \sum_{I' \subset I_i \cap I_j} \left((-1)^{|\bar{I}' \cap I_i \cap I_j|} \prod_{k \in I'} a'_k \prod_{k \in \bar{I}' \cap I_i \cap I_j} b_k \right). \quad (52)$$

Because $a'_k - b_k = -(a_k - b'_k)$, then

$$\prod_{k \in I_i \cap \bar{I}_j} (a'_k - b_k) = (-1)^{|I_i \cap \bar{I}_j|} \prod_{k \in I_i \cap \bar{I}_j} (a_k - b'_k)$$

and from this, again using (84), we have

$$\prod_{k \in I_i \cap \bar{I}_j} (a'_k - b_k) = (-1)^{|I_i \cap \bar{I}_j|} \sum_{I'' \subset I_i \cap \bar{I}_j} \left((-1)^{|\bar{I}'' \cap I_i \cap \bar{I}_j|} \prod_{k \in I''} a_k \prod_{k \in \bar{I}'' \cap I_i \cap \bar{I}_j} b'_k \right). \quad (53)$$

To combine (52) and (53), notice that the sets I' and I'' in (52) and (53), respectively, are disjoint sets and are subsets of I_i . Then, the set $I := I' \cup I''$ is also a subset of I_i with $I' = I \cap I_j$, $I'' = I \cap \bar{I}_j$. Moreover,

$$|\bar{I} \cap I_i \cap I_j| + |\bar{I}'' \cap I_i \cap \bar{I}_j| = |\bar{I} \cap I_i|$$

and

$$(-1)^{|\bar{I} \cap I_i|} = (-1)^{|I_i| - |I|} = (-1)^{|I_i|} \cdot (-1)^{|I|}.$$

Combining (52) and (53) results in

$$\lambda_i = (-1)^{|I_i \cap \bar{I}_j| + |I_i|} \sum_{I \subset I_i} \left((-1)^{|I|} \prod_{k \in \bar{I} \cap I_i \cap I_j} b_k \prod_{k \in I \cap \bar{I}_j} a_k \prod_{k \in \bar{I} \cap I_i \cap \bar{I}_j} b'_k \prod_{k \in I \cap I_j} a'_k \right). \quad (54)$$

The final step is to notice

$$|I_i \cap \bar{I}_j| + |I_i| = 2|I_i \cap \bar{I}_j| + |I_i \cap I_j|$$

and

$$(-1)^{|I_i \cap \bar{I}_j| + |I_i|} = (-1)^{|I_i \cap I_j|}.$$

This concludes the proof of (51); i.e., the right hand sides of (51) and (54) are equal.

Now we prove (45). Based on (44), we have

$$\|x_{\lambda_i}\|_{l^1(\mathbb{N})} = \sum_{j=1}^{\infty} \left(\prod_{k \in I_j \cap \bar{I}_i} \frac{b_k}{a_k} \right) = \sum_{I_j \subset \mathbb{N}} \left(\prod_{k \in I_j \cap \bar{I}_i} \frac{b_k}{a_k} \right), \quad (55)$$

where the sum is over all finite subsets of $\mathbb{N}$. Denote $I := I_j \cap I_i$ and $I' := I_j \cap \bar{I}_i$. Then, $I \cap I' = \emptyset$ and (55) becomes

$$\|x_{\lambda_i}\|_{l^1(\mathbb{N})} = \sum_{I \subset I_i} \left[\sum_{I' \subset \bar{I}_i} \left(\prod_{k \in I'} \frac{b_k}{a_k} \right) \right],$$

where the outer sum runs over all subsets of I_i and the interior sum is over all finite subsets of $\bar{I}_i = \mathbb{N} \setminus I_i$. From (37) and Lemma 6, we have

$$\sum_{I' \subset \bar{I}_i} \left(\prod_{k \in I'} \frac{b_k}{a_k} \right) = \prod_{k \in \bar{I}_i} \left(1 + \frac{b_k}{a_k} \right) < +\infty.$$

The set I_i is finite and the number of all its subsets equals $2^{|I_i|}$; therefore,

$$\|x_{\lambda_i}\|_{l^1(\mathbb{N})} = \prod_{k \in \bar{I}_i} \left(1 + \frac{b_k}{a_k} \right) \cdot \sum_{I \subset I_i} 1 = 2^{|I_i|} \cdot \prod_{k \in \bar{I}_i} \left(1 + \frac{b_k}{a_k} \right).$$

This completes the proof of (45) and the theorem. $\square$

Remark 2. As in the finite case, we have $\lambda_1 = 1$ and $|\lambda_i| < 1$, for $i \geq 2$. Although there are infinitely many eigenvalues, not necessarily distinct, each is expressed as a finite product. All vectors given by (44) are linearly independent (see Lemma 5). Moreover, it should be added the conditions (29) and (37) are not contradictory. For example, the sequences

$$b_k = \frac{1}{3^k}, \quad a_k = \frac{1}{2^k}, \quad k \geq 1,$$

satisfy both (29) and (37).

The results in Theorem 3 are formulated from the point of view of the Banach space $l^1(\mathbb{N})$ and the map $x \rightarrow x \cdot P$. These results can also be looked at from a slightly better and more natural perspective. Namely, that of the Hilbert space $\ell^2(\pi)$. This is shown in Theorem 4. To make everything precise, we start by recalling a few concepts.

The Banach space of all bounded sequences, usually denoted as $l^\infty(\mathbb{N})$, is a vector space equipped with the norm $\|x\|_{l^\infty(\mathbb{N})} := \sup_{i \in \mathbb{N}} |\xi_i|$. Thus, a sequence $x = (\xi_i)_{i \in \mathbb{N}}$ belongs to $l^\infty(\mathbb{N})$ if and only if $\sup_{i \in \mathbb{N}} |\xi_i| < +\infty$. Let P^T denote the transpose of P given by (30). The transition operator T , associated with P , is the map defined by

$$Tx := x \cdot P^T, \quad x \in l^\infty(\mathbb{N}) \quad (56)$$

(see [19] (p. 113)). For this definition to be correct, we assume (as before) that x is a row vector. We can also write T in coordinates. To this end, let $(\zeta_i)_{i \in \mathbb{N}} := Tx$. Then, $\zeta_j = \sum_{i \in \mathbb{N}} \xi_i p_{ji}$ for $j \in \mathbb{N}$. It is clear that T is a linear operator. Moreover, one can show that $\|Tx\|_{l^\infty(\mathbb{N})} \leq \|x\|_{l^\infty(\mathbb{N})}$, which means it is a contraction.

Let π be the probability distribution given by (38). We define $\ell^2(\pi)$ as the Hilbert space of all infinite sequences of complex numbers equipped with the inner product

$$\langle x, y \rangle_\pi := \sum_{i=1}^{\infty} \xi_i \overline{\eta_i} \pi_i, \quad x = (\xi_i)_{i \in \mathbb{N}}, \quad y = (\eta_i)_{i \in \mathbb{N}},$$

and the corresponding norm $\|x\|_\pi = (\sum_{i=1}^{\infty} |\xi_i|^2 \pi_i)^{1/2}$. In other words, $x \in \ell^2(\pi)$ if and only if $\sum_{i=1}^{\infty} |\xi_i|^2 \pi_i < +\infty$. For example, a constant vector $(1, 1, 1, \dots)$ is an element of $\ell^2(\pi)$, because

$$\|(1, 1, 1, \dots)\|_\pi^2 = \sum_{i=1}^{\infty} \pi_i = 1 < +\infty.$$

In a similar way, it can be shown that any $x \in l^\infty(\mathbb{N})$ belongs to $\ell^2(\pi)$, i.e., $l^\infty(\mathbb{N}) \subset \ell^2(\pi)$ (see [19] (p. 119)). So for now, the domain of T is $l^\infty(\mathbb{N})$.

Theorem 4. Let P and π be given by (30) and (38), respectively. Then, the transition operator T , defined by (56), can be extended to $\ell^2(\pi)$. Such an extended operator is contractive, $\|Tx\|_\pi \leq \|x\|_\pi$, and self-adjoint in $\ell^2(\pi)$; i.e., it satisfies

$$\langle Tx, y \rangle_\pi = \langle x, Ty \rangle_\pi, \quad x, y \in \ell^2(\pi).$$

As a result, the spectrum of T is real and its point spectrum $\sigma_p(T)$, which is defined as the set of eigenvalues of T , equals

$$\sigma_p(T) = \sigma_p(P^T) = \sigma_p(P) = \{\lambda_i : i \in \mathbb{N}\}, \quad (57)$$

where λ_i is given by (43). The eigenvector $y_{\lambda_i} = (\eta_{j,i})$, corresponding to λ_i , has the form

$$\eta_{j,i} = (-1)^{|I_j \cap I_i|} \prod_{k \in I_i \cap \bar{I}_j} \frac{b_k}{a_k}, \quad j = 1, 2, 3, \dots \quad (58)$$

Moreover, the eigenvectors $y_{\lambda_1}, y_{\lambda_2}, \dots$ form an orthogonal family in $\ell^2(\pi)$ and

$$\langle y_{\lambda_{i_1}}, y_{\lambda_{i_2}} \rangle_\pi = \begin{cases} \|y_{\lambda_{i_1}}\|_\pi^2 = \prod_{k \in I_{i_1}} \frac{b_k}{a_k}, & \text{if } i_1 = i_2, \\ 0, & \text{if } i_1 \neq i_2. \end{cases} \quad (59)$$

Proof. P and π satisfy (39), so T is a self-adjoint operator (see, e.g., [19] (pp. 119–120)). It is well known that the spectrum of a self-adjoint operator is real (see [18] (p. 301) and [18] (p. 309) for the proof of (57)). The argument that $y_{\lambda_i} \cdot P^T = \lambda_i y_{\lambda_i}$ is as follows. First, Theorem 3 says that $x_{\lambda_i} \cdot P = \lambda_i x_{\lambda_i}$. Equivalently, it means $WP = DW$, where the matrices D and W are given by (64) and (65), respectively. In Lemma 5, we prove that $W^{-1} = c^{-1}W$ for some $c > 0$. Thus, $W = cW^{-1}$. As a result, $P = W^{-1}DW$ and $P^T = W^T D (W^{-1})^T$. Finally, we obtain

$$P^T = (cW^{-1})^T D (c^{-1}W)^T = (W^T)^{-1} D W^T.$$

The above equality shows that the rows of W^T , or equivalently, the columns of W , are eigenvectors of P^T corresponding to its eigenvalues. But that is how y_{λ_i} 's are defined; i.e., y_{λ_i} determined by (58) is the i -th column of W . This proves $y_{\lambda_i} P^T = \lambda_i y_{\lambda_i}$.

What remains to be proven is (59). To carry out the proof, let us denote

$$c_k := \frac{b_k}{a_k}, \quad c := \prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right)$$

(see also Section 4). With this notation, we have

$$\|y_{\lambda_{i_1}}\|_\pi^2 = \sum_{j=1}^{\infty} \eta_{j,i_1}^2 \pi_j = \frac{1}{c} \sum_{j=1}^{\infty} \left(\prod_{k \in I_{i_1} \cap \bar{I}_j} c_k^2 \cdot \prod_{k \in I_j} c_k \right). \quad (60)$$

It is clear that the set $I_{i_1} \cup I_j$ equals $I_{i_1} \cup (I_j \cap \bar{I}_{i_1})$. Then, in (60), we can factor out the product $\prod_{k \in I_{i_1}} c_k$, which gives

$$\|y_{\lambda_{i_1}}\|_\pi^2 = \frac{1}{c} \prod_{k \in I_{i_1}} c_k \cdot \sum_{j=1}^{\infty} \left(\prod_{k \in I_{i_1} \cap \bar{I}_j} c_k \cdot \prod_{k \in I_j \cap \bar{I}_{i_1}} c_k \right) = \prod_{k \in I_{i_1}} c_k \quad (61)$$

(see (75)), where the key argument used to prove it is the set $(I_{i_1} \cap \bar{I}_j) \cup (I_j \cap \bar{I}_{i_1})$ equals the symmetric difference of I_{i_1} and I_j (see Section 4.1). Now let $i_1 \neq i_2$. Then

$$\langle y_{\lambda_{i_1}}, y_{\lambda_{i_2}} \rangle_\pi = \frac{1}{c} \sum_{j=1}^{\infty} \left[\left((-1)^{|I_{i_1} \cap I_j|} \prod_{k \in I_{i_1} \cap \bar{I}_j} c_k \right) \left((-1)^{|I_{i_2} \cap I_j|} \prod_{k \in I_{i_2} \cap \bar{I}_j} c_k \right) \left(\prod_{k \in I_j} c_k \right) \right]. \quad (62)$$

Notice that

$$(I_{i_1} \cap \bar{I}_j) \cap (I_{i_2} \cap \bar{I}_j) = (I_{i_1} \cap I_{i_2}) \cap \bar{I}_j,$$

so Equation (62) takes the form

$$\langle y_{\lambda_{i_1}}, y_{\lambda_{i_2}} \rangle_{\pi} = \frac{1}{c} \prod_{k \in I_{i_1}} c_k \cdot \sum_{j=1}^{\infty} \left[\left((-1)^{|I_j \cap I_{i_1}|} \prod_{k \in I_j \cap I_{i_1}} c_k \right) \left((-1)^{|I_j \cap I_{i_2}|} \prod_{k \in I_j \cap I_{i_2}} c_k \right) \right] = 0,$$

see (77) and (82). This concludes the proof of (59). $\square$

Example 3. As noted in Remark 1, the Formula (12) is also true when D and W are infinite-dimensional matrices. In Remark 3, we give a more detailed explanation of the right hand side of the formula

$$P^n = \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k} \right)} W D^n W, \quad n \geq 1, \quad (63)$$

where the diagonal matrix D equals

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & a'_1 - b_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & a'_2 - b_2 & 0 & 0 & \dots \\ 0 & 0 & 0 & (a'_1 - b_1)(a'_2 - b_2) & 0 & \dots \\ 0 & 0 & 0 & 0 & a'_3 - b_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (64)$$

and

$$W = [\xi_{i,j}] = \begin{bmatrix} 1 & \frac{b_1}{a_1} & \frac{b_2}{a_2} & \frac{b_1 b_2}{a_1 a_2} & \frac{b_3}{a_3} & \frac{b_1 b_3}{a_1 a_3} & \frac{b_2 b_3}{a_2 a_3} & \frac{b_1 b_2 b_3}{a_1 a_2 a_3} & \frac{b_4}{a_4} & \dots \\ 1 & -1 & \frac{b_2}{a_2} & -\frac{b_2}{a_2} & \frac{b_3}{a_3} & -\frac{b_3}{a_3} & \frac{b_2 b_3}{a_2 a_3} & -\frac{b_2 b_3}{a_2 a_3} & \frac{b_4}{a_4} & \dots \\ 1 & \frac{b_1}{a_1} & -1 & -\frac{b_1}{a_1} & \frac{b_3}{a_3} & \frac{b_1 b_3}{a_1 a_3} & -\frac{b_3}{a_3} & -\frac{b_1 b_3}{a_1 a_3} & \frac{b_4}{a_4} & \dots \\ 1 & -1 & -1 & 1 & \frac{b_3}{a_3} & -\frac{b_3}{a_3} & -\frac{b_3}{a_3} & \frac{b_3}{a_3} & \frac{b_4}{a_4} & \dots \\ 1 & \frac{b_1}{a_1} & \frac{b_2}{a_2} & \frac{b_1 b_2}{a_1 a_2} & -1 & -\frac{b_1}{a_1} & -\frac{b_2}{a_2} & -\frac{b_1 b_2}{a_1 a_2} & \frac{b_4}{a_4} & \dots \\ 1 & -1 & \frac{b_2}{a_2} & -\frac{b_2}{a_2} & -1 & 1 & -\frac{b_2}{a_2} & \frac{b_2}{a_2} & \frac{b_4}{a_4} & \dots \\ 1 & \frac{b_1}{a_1} & -1 & -\frac{b_1}{a_1} & -1 & -\frac{b_1}{a_1} & 1 & \frac{b_1}{a_1} & \frac{b_4}{a_4} & \dots \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & \frac{b_4}{a_4} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (65)$$

(see (44)). From (63) we obtain an explicit formula for $p_{ij}^{(n)}$, namely,

$$p_{ij}^{(n)} = \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k} \right)} \sum_{l=1}^{\infty} \left[\lambda_l^n (-1)^{k_l} \left(\prod_{k \in K_l} \frac{b_k}{a_k} \right) \right], \quad (66)$$

where $k_l = |I_l \cap I_i| + |I_j \cap I_l|$ and $K_l = (I_l \cap \bar{I}_i) \cup (I_j \cap \bar{I}_l)$. In particular, based on (66), we have

$$p_{11}^{(n)} = \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k} \right)} \sum_{i=1}^{\infty} \left[\lambda_i^n \left(\prod_{k \in I_i} \frac{b_k}{a_k} \right) \right], \quad n \geq 1,$$

taking the limit

$$\lim_{n \rightarrow +\infty} p_{11}^{(n)} = \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k} \right)} = \pi_1,$$

where π_1 is the first component of $\pi = (\pi_i)_{i \in \mathbb{N}}$ (see (38)). Recall that this stationary distribution is given by

$$\pi = \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right)} \cdot \left(1, \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_1 b_2}{a_1 a_2}, \frac{b_3}{a_3}, \frac{b_1 b_3}{a_1 a_3}, \frac{b_2 b_3}{a_2 a_3}, \frac{b_1 b_2 b_3}{a_1 a_2 a_3}, \dots\right)$$

and it is an eigenvector corresponding to $\lambda_1 = 1$.

Remark 3. We show that with the expression on the right side of Equation (63), we can associate a bounded operator in $\ell^2(\pi)$. For this purpose, define the map

$$y \rightarrow W \cdot y^T, \quad y \in \ell^2(\pi), \quad (67)$$

where W is given by (65) and y^T denotes the transpose of a row vector y . We prove that (67) defines a bounded operator. Based on (59), for $e_i \in \ell^2(\pi)$, we have

$$\|We_i^T\|_{\pi}^2 = \|y_{\lambda_i}\|_{\pi}^2 = \prod_{k \in I_i} \frac{b_k}{a_k} = \pi_i \cdot \prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right) \leq \prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right), \quad i \in \mathbb{N}.$$

Let $y = (\xi_i)_{i \in \mathbb{N}} \in \ell^2(\pi)$ be arbitrary. Then, $y = \sum_{i=1}^{\infty} \xi_i e_i$ with $\sum_{i=1}^{\infty} |\xi_i|^2 \pi_i < +\infty$ and

$$\|Wy^T\|_{\pi}^2 = \left\| \sum_{i=1}^{\infty} \xi_i We_i^T \right\|_{\pi}^2 \leq \sum_{i=1}^{\infty} |\xi_i|^2 \cdot \|We_i^T\|_{\pi}^2 = \|y\|_{\pi}^2 \cdot \prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right). \quad (68)$$

The diagonal matrix D^n represents a bounded operator. Its all-diagonal entries do not exceed 1 in absolute value. Thus, the product WD^nW is a composition of three bounded operators in $\ell^2(\pi)$. Finally, inequality (68) makes it clear that the operator

$$y \rightarrow \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{b_k}{a_k}\right)} WD^nW \cdot y^T, \quad y \in \ell^2(\pi),$$

is a contractive one.

It is worth to add that the operator $x \rightarrow x \cdot W$, acting in $l^1(\mathbb{N})$, is unbounded. To see it, take an integer $N \geq 2$ and $i := 1 + \sum_{k=1}^N 2^{k-1}$. Then, $I_i = \{1, 2, \dots, N\}$; therefore, $|I_i| = N$. Moreover, $\|e_i\|_{l^1(\mathbb{N})} = 1$, for any N , and from (45) we obtain

$$\|e_i \cdot W\|_{l^1(\mathbb{N})} = \|x_{\lambda_i}\|_{l^1(\mathbb{N})} = 2^N \cdot \prod_{k=N+1}^{\infty} \left(1 + \frac{b_k}{a_k}\right) \geq 2^N \rightarrow +\infty,$$

as $N \rightarrow +\infty$. Hence, W cannot be bounded, as an operator, in $l^1(\mathbb{N})$.

4. Some Additional Results

4.1. Computation of W^{-1}

The matrix W , determined by (65), as well as W in Section 2, is a special case of the matrix $\tilde{W}$ defined by (70). So if, in $\tilde{W}$, we substitute $c_k = b_k/a_k$, we will obtain W . Therefore, we consider this more general case here.

We assume that $(c_k)_{k \in \mathbb{N}}$ is a sequence of positive real numbers satisfying the summability condition $\sum_{k=1}^{+\infty} c_k < +\infty$. Then, the product $\prod_{k=1}^{+\infty} (1 + c_k)$ converges and

$$c := \prod_{k=1}^{+\infty} (1 + c_k) = \sum_{i=1}^{+\infty} \left(\prod_{k \in I_i} c_k \right) < +\infty \quad (69)$$

(see [17] (Problems 3.8.3, 3.8.26)). Equivalently, the right hand side of (69) may be written as $\sum_{I_i \subset \mathbb{N}} (\prod_{k \in I_i} c_k)$, where the sum is over all finite subsets of $\mathbb{N}$.

Based on the sequence $(c_k)_{k \in \mathbb{N}}$, we define an infinite matrix $\tilde{W}$ as

$$\tilde{W} = [\tilde{w}_{i,j}], \quad \tilde{w}_{i,j} = (-1)^{|I_i \cap I_j|} \cdot \prod_{k \in \overline{I_i} \cap I_j} c_k, \quad i, j \in \mathbb{N}, \quad (70)$$

where I_i, I_j are representation sets of i and j , respectively. In a more explicit form, its first eight rows are

$$\tilde{W} = \begin{bmatrix} 1 & c_1 & c_2 & c_1 c_2 & c_3 & c_1 c_3 & c_2 c_3 & c_1 c_2 c_3 & c_4 & \dots \\ 1 & -1 & c_2 & -c_2 & c_3 & -c_3 & c_2 c_3 & -c_1 c_2 c_3 & c_4 & \dots \\ 1 & c_1 & -1 & -c_1 & c_3 & c_1 c_3 & -c_3 & -c_1 c_3 & c_4 & \dots \\ 1 & -1 & -1 & 1 & c_3 & -c_3 & -c_3 & c_3 & c_4 & \dots \\ 1 & c_1 & c_2 & c_1 c_2 & -1 & -c_1 & -c_2 & -c_1 c_2 & c_4 & \dots \\ 1 & -1 & c_2 & -c_2 & -1 & 1 & -c_2 & c_2 & c_4 & \dots \\ 1 & c_1 & -1 & -c_1 & -1 & -c_1 & 1 & c_1 & c_4 & \dots \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & c_4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

In case the sequence $(c_k)_{k \in \mathbb{N}}$ is finite, e.g., it consists of c_1 and c_2 , then

$$\tilde{W} = \begin{bmatrix} 1 & c_1 & c_2 & c_1 c_2 \\ 1 & -1 & c_2 & -c_2 \\ 1 & c_1 & -1 & -c_1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

For a given set X , let $\mathcal{P}(X)$ denote its power set, i.e., the set consisting of all subsets of X . The symmetric difference of A and B is defined as

$$A \triangle B := (A \cap \overline{B}) \cup (\overline{A} \cap B), \quad A, B \in \mathcal{P}(X).$$

It is well known that $(\mathcal{P}(X), \triangle)$ is an (abelian) group (see [20] (p. 10)). The empty set is the neutral element of this group and every element of $\mathcal{P}(X)$ equals its own inverse. More precisely, it means that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$, $A \triangle \emptyset = A$ and $A \triangle A = \emptyset$ for any $A, B, C \in \mathcal{P}(X)$. Because $(\mathcal{P}(X), \triangle)$ is a group, for any fixed $A \in \mathcal{P}(X)$, the map $f : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, defined by

$$f(B) = B \triangle A \quad (71)$$

is a bijection (see [20] (p. 53)). This means $\mathcal{P}(X) = \{f(B) : B \in \mathcal{P}(X)\}$.

If X is finite and non-empty, say $|X| = n$, where $n \geq 1$, then

$$\sum_{I \in \mathcal{P}(X)} (-1)^{|I|} = 0 \quad (72)$$

(see, e.g., [16] (p. 201)). It follows from $\sum_{k=0}^n \binom{n}{k} (-1)^k = 0$, which is a consequence of the binomial expansion $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$, if we put $a = 1$ and $b = -1$. For example, take $X = \{1, 2\}$. Then, $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and

$$(-1)^{|\emptyset|} + (-1)^{|\{1\}|} + (-1)^{|\{2\}|} + (-1)^{|\{1,2\}|} = 1 - 1 - 1 + 1 = 0.$$

We will use (71) and (72) in the proof of the following lemma, where $I = [\delta_{i,j}]$ denotes the identity matrix; i.e., $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$.

Lemma 5. Let $\tilde{W}$ be the matrix defined by (70). Then $\tilde{W}^2 = c \cdot I$, which implies

$$\tilde{W}^{-1} = c^{-1} \cdot \tilde{W}, \quad (73)$$

where c is given by (69).

Proof. Denote $V = [v_{i,j}] = \tilde{W}^2$. First, we compute $v_{1,1}$. Recall that $I_1 = \emptyset$ and $\overline{I_1} = \mathbb{N}$; hence, $\tilde{w}_{1,j} = \prod_{k \in I_j} c_k$ and $\tilde{w}_{j,1} = 1$, for any $j \in \mathbb{N}$. Then, from (69) we get

$$v_{1,1} = \sum_{j=1}^{\infty} \tilde{w}_{1,j} \tilde{w}_{j,1} = \sum_{j=1}^{\infty} \left(\prod_{k \in I_j} c_k \right) = c.$$

For $i \geq 2$ we have

$$v_{i,i} = \sum_{j=1}^{\infty} \tilde{w}_{i,j} \tilde{w}_{j,i} = \sum_{j=1}^{\infty} \left[\left((-1)^{|I_i \cap I_j|} \prod_{k \in \overline{I_i} \cap I_j} c_k \right) \left((-1)^{|I_j \cap I_i|} \prod_{k \in \overline{I_j} \cap I_i} c_k \right) \right]. \quad (74)$$

Notice that in (74) the powers of -1 cancel each other out because

$$(-1)^{|I_i \cap I_j|} \cdot (-1)^{|I_j \cap I_i|} = (-1)^{2|I_i \cap I_j|} = 1.$$

Moreover, $(\overline{I_i} \cap I_j) \cup (\overline{I_j} \cap I_i) = I_i \triangle I_j$. In other words, it is the symmetric difference of I_i and I_j . The set I_i is fixed, so by (71) the map $I_j \rightarrow I_i \triangle I_j$ is a bijection. As a result, (74) takes the form

$$v_{i,i} = \sum_{j=1}^{\infty} \left(\prod_{k \in I_i \triangle I_j} c_k \right) = \sum_{j'=1}^{\infty} \left(\prod_{k \in I_{j'}} c_k \right) = c. \quad (75)$$

Now we will prove that $v_{i_1, i_2} = 0$, if $i_1 \neq i_2$. So let i_1, i_2 be fixed and assume $i_1 \neq i_2$. Then $I_{i_1} \neq I_{i_2}$, which implies $I_{i_1} \triangle I_{i_2} \neq \emptyset$, or equivalently

$$|I_{i_1} \triangle I_{i_2}| \geq 1. \quad (76)$$

Based on (70), we have

$$v_{i_1, i_2} = \sum_{j=1}^{\infty} \tilde{w}_{i_1, j} \tilde{w}_{j, i_2} = \sum_{j=1}^{\infty} \left[\left((-1)^{|I_{j_1} \cap I_{i_1}|} \prod_{k \in \overline{I_{j_1}} \cap I_{i_1}} c_k \right) \left((-1)^{|I_{j_2} \cap I_{i_2}|} \prod_{k \in \overline{I_{j_2}} \cap I_{i_2}} c_k \right) \right]. \quad (77)$$

Let $\mathcal{P}_1(\mathbb{N})$ denote the family of all finite subsets of $\mathbb{N}$. The set $\mathcal{P}_1(\mathbb{N})$ is countable, as a countable sum of finite sets. Then, define the map $g : \mathcal{P}_1(\mathbb{N}) \rightarrow \mathcal{P}_1(\mathbb{N})$ as follows:

$$g(I_j) := (I_j \cap \overline{I_{i_1}}) \cup (\overline{I_j} \cap I_{i_2}), \quad |I_j| < \infty. \quad (78)$$

Equivalently, the set $g(I_j)$ can be written in the form

$$g(I_j) = (I_j \cup I_{i_2}) \setminus (I_j \cap I_{i_1}) = (I_j \cap \overline{I_{i_1}} \cap \overline{I_{i_2}}) \cup (I_{i_2} \setminus (I_{i_1} \cap I_{i_2} \cap I_j)).$$

The assumption $i_1 \neq i_2$ means that g is not a bijection. To see it, define the relation

$$I_{j_1} \sim I_{j_2} \Leftrightarrow g(I_{j_1}) = g(I_{j_2}),$$

where g is given by (78). It is straightforward to verify that it is an equivalence relation on $\mathcal{P}_1(\mathbb{N})$. Moreover, the equivalence class of I_j equals

$$[I_j] = \left\{ S_I =: (I_j \cap \overline{(I_{i_1} \triangle I_{i_2})}) \cup I : I \subset I_{i_1} \triangle I_{i_2} \right\}. \quad (79)$$

Therefore, from (76), we obtain

$$|[I_j]| = 2^{|I_{i_1} \triangle I_{i_2}|} \geq 2,$$

which proves that g is not injective. Furthermore, the set S_I , in (79), can be written as a sum of three mutually disjoint sets, namely,

$$S_I = (I_j \cap \overline{I_{i_1}} \cap \overline{I_{i_1}}) \cup (I_j \cap I_{i_1} \cap I_{i_2}) \cup I.$$

We will show that $g(S_I) = g(I_j)$ for any $I \subset I_{i_1} \triangle I_{i_2}$. Based on (78), the set $g(S_I)$ can be written as $S'_I \cup S''_I$, where

$$S'_I = [(I_j \cap \overline{I_{i_1}} \cap \overline{I_{i_1}}) \cup (I_j \cap I_{i_1} \cap I_{i_2}) \cup I] \cap \overline{I_{i_1}} = (I_j \cap \overline{I_{i_1}} \cap \overline{I_{i_1}}) \cup (I \cap \overline{I_{i_1}})$$

and

$$S''_I = [(\overline{I_j} \cup I_{i_1} \cup I_{i_2}) \cap (\overline{I_{i_1}} \cup \overline{I_{i_2}} \cup \overline{I_j}) \cap \overline{I}] \cap I_{i_2} = \overline{I} \cap I_{i_2} \cap (\overline{I_{i_1}} \cup \overline{I_j}).$$

Because $I \subset I_{i_1} \triangle I_{i_2}$, we have $I \cap \overline{I_{i_1}} = I \cap I_{i_2}$ and

$$I_{i_2} \cap (\overline{I_{i_1}} \cup \overline{I_j}) = I_{i_2} \cap \overline{(I_{i_1} \cap I_j)} = I_{i_2} \setminus (I_{i_1} \cap I_{i_2} \cap I_j).$$

Hence, we get

$$(I \cap I_{i_2}) \cup [\overline{I} \cap (I_{i_2} \setminus (I_{i_1} \cap I_{i_2} \cap I_j))] = I_{i_2} \setminus (I_{i_1} \cap I_{i_2} \cap I_j),$$

and as a result,

$$g(S_I) = S'_I \cup S''_I = (I_j \cap \overline{I_{i_1}} \cap \overline{I_{i_2}}) \cup (I_{i_2} \setminus (I_{i_1} \cap I_{i_2} \cap I_j)) = g(I_j). \quad (80)$$

Thanks to (80), we can write (77) in the form

$$v_{i_1, i_2} = \sum_{g(I_j)} \left[\left(\prod_{k \in g(I_j)} c_k \right) \cdot \Gamma_{I_j} \right], \quad (81)$$

where g is given by (78) and

$$\Gamma_{I_j} = \sum_{I \subset I_{i_1} \triangle I_{i_2}} (-1)^{|[(I_j \cap \overline{I_{i_1}} \triangle \overline{I_{i_2}}) \cup I] \cap I_{i_1}| + |[(I_j \cap \overline{I_{i_1}} \triangle \overline{I_{i_2}}) \cup I] \cap I_{i_2}|}.$$

However, notice that

$$[(I_j \cap \overline{I_{i_1}} \triangle \overline{I_{i_2}}) \cup I] \cap I_{i_1} = (I \cap I_{i_1}) \cup (I_{i_1} \cap I_{i_2} \cap I_j),$$

and similarly

$$[(I_j \cap \overline{I_{i_1}} \triangle \overline{I_{i_2}}) \cup I] \cap I_{i_2} = (I \cap I_{i_2}) \cup (I_{i_1} \cap I_{i_2} \cap I_j).$$

Then, from (72), we have

$$\Gamma_{I_j} = \sum_{I \subset I_{i_1} \triangle I_{i_2}} (-1)^{|I \cap I_{i_1}| + |I \cap I_{i_2}|} = \sum_{I \subset I_{i_1} \triangle I_{i_2}} (-1)^{|I|} = 0.$$

Taking into account (81), we finally obtain

$$v_{i_1, i_2} = 0, \quad \text{if } i_1 \neq i_2. \quad (82)$$

This completes the proof. $\square$

4.2. Computation of $\prod_{k \in J} (c_k + d_k)$

Suppose J is a finite subset of $\mathbb{N}$. It is straightforward to check that the product $\prod_{k \in J} (c_k + d_k)$, where $c_k, d_k \in \mathbb{R}$, can be written in the following way:

$$\prod_{k \in J} (c_k + d_k) = \sum_{I \subset J} \left(\prod_{k \in I} c_k \cdot \prod_{k \in \bar{I} \cap J} d_k \right). \quad (83)$$

The above sum runs over all subsets I of J , so there are $2^{|J|}$ terms in the expression on the right hand side of (83) (see [20] (p. 12)). For example, for $J = \{1, 2\}$, we have

$$\prod_{k \in \{1, 2\}} (c_k + d_k) = c_1 c_2 + c_1 d_2 + c_2 d_1 + d_1 d_2,$$

and the corresponding subsets of J are $\{1, 2\}$, $\{1\}$, $\{2\}$ and $\emptyset$, respectively.

Similarly to (83), by putting $-d_k$ in place of d_k , we would obtain

$$\prod_{k \in J} (c_k - d_k) = \sum_{I \subset J} \left((-1)^{|\bar{I} \cap J|} \cdot \prod_{k \in I} c_k \cdot \prod_{k \in \bar{I} \cap J} d_k \right), \quad (84)$$

where, again, the sum is over all subsets of J . For example, consider $\prod_{k \in \{1, 2\}} (c_k - d_k)$. Then,

$$\prod_{k \in \{1, 2\}} (c_k - d_k) = c_1 c_2 - c_1 d_2 - c_2 d_1 + d_1 d_2,$$

which can be rewritten as

$$(-1)^{|\{1, 2\}|} c_1 c_2 + (-1)^{|\{1\}|} c_1 d_2 + (-1)^{|\{2\}|} c_2 d_1 + (-1)^{|\emptyset|} d_1 d_2.$$

It turns out that if J is infinite and the product $\prod_{k \in J} (c_k + d_k)$ converges, then (83) also holds true, so for $J = \mathbb{N}$, it would take the form

$$\prod_{k=1}^{\infty} (c_k + d_k) = \sum_{I \subset \mathbb{N}} \left(\prod_{k \in I} c_k \cdot \prod_{k \in \bar{I} \cap \mathbb{N}} d_k \right).$$

We will prove it in the following lemma.

Lemma 6. Assume J is an infinite subset of $\mathbb{N}$ and let $(c_k)_{k \in J}$, $(d_k)_{k \in J}$ be sequences of positive numbers such that the products $\prod_{k \in J} (c_k + d_k)$, $\prod_{k \in J} d_k$ converge. Then,

$$\prod_{k \in J} (c_k + d_k) = \sum_{I \subset J} \left(\prod_{k \in I} c_k \cdot \prod_{k \in \bar{I} \cap J} d_k \right), \quad (85)$$

where the sum is taken over all finite subsets of J .

Proof. Because the product $\prod_{k \in J} (c_k + d_k)$ converges, then

$$\prod_{k \in J} (c_k + d_k) = \lim_{n \rightarrow +\infty} \prod_{k \in J \cap \mathbb{S}_n} (c_k + d_k),$$

where $\mathbb{S}_n = \{1, 2, \dots, 2^n\}$. The set $J \cap \mathbb{S}_n$ is finite, so using (83) results in

$$\prod_{k \in J} (c_k + d_k) = \lim_{n \rightarrow +\infty} \left[\sum_{I \subset J \cap \mathbb{S}_n} \left(\prod_{k \in I} c_k \cdot \prod_{k \in I \cap J \cap \mathbb{S}_n} d_k \right) \right], \quad (86)$$

where the sum is over all subsets of $J \cap \mathbb{S}_n$ (see also [17] (Problem 3.8.26)). But the product $\prod_{k \in J} d_k$ converges, so

$$\lim_{n \rightarrow +\infty} \prod_{k \in I \cap J \cap \mathbb{S}_n} d_k = \prod_{k \in I \cap J} d_k.$$

All terms on the right side of (86) are non-negative, so taking the limit, we obtain (85). $\square$

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