

Article

Self-Normalized Moderate Deviations for Degenerate U -Statistics

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Abstract: In this paper, we study self-normalized moderate deviations for degenerate U -statistics of order 2. Let $\{X_i, i \geq 1\}$ be i.i.d. random variables and consider symmetric and degenerate kernel functions in the form $h(x, y) = \sum_{l=1}^{\infty} \lambda_l g_l(x) g_l(y)$, where $\lambda_l > 0$, $Eg_l(X_1) = 0$, and $g_l(X_1)$ is in the domain of attraction of a normal law for all $l \geq 1$. Under the condition $\sum_{l=1}^{\infty} \lambda_l < \infty$ and some truncated conditions for $\{g_l(X_1) : l \geq 1\}$, we show that $\log P\left(\frac{\sum_{1 \leq i \neq j \leq n} h(X_i, X_j)}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq x_n^2\right) \sim -\frac{x_n^2}{2}$ for $x_n \rightarrow \infty$ and $x_n = o(\sqrt{n})$, where $V_{n,l}^2 = \sum_{i=1}^n g_l^2(X_i)$. As application, a law of the iterated logarithm is also obtained.

Keywords: moderate deviation; degenerate U -statistics; law of the iterated logarithm; self-normalization

MSC: 60F15; 60F10; 62E20

1. Introduction and Main Results

The recent three decades have witnessed significant developments on self-normalized limit theory, especially on large deviations, Cramér-type moderate deviations, and the law of the iterated logarithm. Compared with the classical limit theorems, these self-normalized limit theorems usually require much less moment assumptions.

Let $X, X_1, X_2, \dots$ be independent identically distributed (i.i.d.) random variables. Set

$$S_n = \sum_{i=1}^n X_i \text{ and } V_n^2 = \sum_{i=1}^n X_i^2.$$

Griffin and Kuelbs [1] obtained a law of the iterated logarithm (LIL) for the self-normalized sum of i.i.d. random variables with distributions in the domain of attraction of a normal or stable law. They proved that

1. If $EX = 0$ and X is in the domain of attraction of a normal law, then

$$\limsup_{n \rightarrow \infty} \frac{S_n}{V_n(2 \log \log n)^{1/2}} = 1 \text{ a.s.} \quad (1)$$

2. If X is symmetric and is in the domain of attraction of a stable law, then there exists a positive constant C such that

$$\limsup_{n \rightarrow \infty} \frac{S_n}{V_n(2 \log \log n)^{1/2}} = C \text{ a.s.} \quad (2)$$



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Shao [2] obtained the following self-normalized moderate deviations and specified the constant C in (2). Let $\{x_n, n \geq 1\}$ be a sequence of positive numbers such that $x_n \rightarrow \infty$ and $x_n = o(\sqrt{n})$ as $n \rightarrow \infty$.

1. If $EX = 0$ and X is in the domain of attraction of a normal law, then

$$\lim_{n \rightarrow \infty} x_n^{-2} \log P\left(\frac{S_n}{V_n} \geq x_n\right) = -\frac{1}{2}. \quad (3)$$

2. If X is in the domain of attraction of a stable law such that $EX = 0$ with index $1 < \alpha < 2$, or X_1 is symmetric with index $\alpha = 1$, then

$$\lim_{n \rightarrow \infty} x_n^{-2} \log P\left(\frac{S_n}{V_n} \geq x_n\right) = -\beta(\alpha, c_1, c_2),$$

where $\beta(\alpha, c_1, c_2)$ is a constant depending on the tail distribution; see [2] for an explicit definition.

Shao [3] refined (3) and obtained the following Cramér-type moderate deviation theorem under a finite third moment: if $EX = 0$ and $E|X|^3 < \infty$, then

$$\frac{P(S_n/V_n \geq x_n)}{P(Z \geq x_n)} \rightarrow 1 \quad (4)$$

for any $x_n \in [0, o(n^{1/6})]$, where Z is the standard normal random variable.

Jing, Shao and Wang [4] further extended (4) to general independent random variables under a Lindeberg-type condition, while Shao and Zhou [5] established the result for self-normalized non-linear statistics, which include U -statistics as a special case.

The U -statistics were introduced by Halmos [6] and Hoeffding [7]. The LIL for nondegenerate U -statistics was obtained by Serfling [8]. The LIL for degenerate U -statistics was studied by Dehling, Denker and Philipp ([9,10]), Dehling [11], Arcones and Giné [12], Teicher [13], Giné and Zhang [14], and others. Giné, Kwapień, Latała and Zinn [15] provided necessary and sufficient conditions for the LIL of degenerate U -statistics of order 2, which was extended to any order by Adamczak and Latała [16].

The main purpose of this paper is to study the self-normalized moderate deviations and the LIL for degenerate U -statistics of order 2. Let

$$U_n = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} h(X_i, X_j),$$

where

$$h(x, y) = \sum_{l=1}^{\infty} \lambda_l g_l(x) g_l(y). \quad (5)$$

A motivation example for the LIL is the one with the kernel $h(x, y) = xy$. Obviously, $V_n^2 / (2V_n^2 \log \log n) \rightarrow 0$. Then, via (1) and (2), we have

1. If $EX = 0$ and X is in the domain of attraction of a normal law, then

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{2V_n^2 \log \log n} \left| \sum_{1 \leq i \neq j \leq n} X_i X_j \right| \\ &= \limsup_{n \rightarrow \infty} \left[\frac{S_n}{V_n (2 \log \log n)^{1/2}} \right]^2 = 1 \text{ a.s.} \end{aligned} \quad (6)$$

2. If X is symmetric and is in the domain of attraction of a stable law, then there exists a positive constant C such that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{2V_n^2 \log \log n} \left| \sum_{1 \leq i \neq j \leq n} X_i X_j \right| \\ &= \limsup_{n \rightarrow \infty} \left[\frac{S_n}{V_n (2 \log \log n)^{1/2}} \right]^2 = C^2 \text{ a.s.} \end{aligned}$$

For the general degenerate kernel h defined in (5), we have

$$\begin{aligned} n(n-1)U_n &= \sum_{l=1}^{\infty} \lambda_l \sum_{1 \leq i \neq j \leq n} g_l(X_i) g_l(X_j) \\ &= \sum_{l=1}^{\infty} \lambda_l \left(\left(\sum_{i=1}^n g_l(X_i) \right)^2 - \sum_{i=1}^n g_l^2(X_i) \right). \end{aligned}$$

Suppose that $g_l(X)$ is in the domain of attraction of a normal law for every $l \geq 1$. Then, $L_l(x) := E g_l^2(X_1) I(|g_l(X_1)| \leq x)$ is a slowly varying function for all $l \geq 1$ as $x \rightarrow \infty$. Let $\{x_n, n \geq 1\}$ be a sequence of positive numbers such that $x_n \rightarrow \infty$ and $x_n = o(\sqrt{n})$ as $n \rightarrow \infty$. For each $l \geq 1$, set

$$\begin{aligned} b_l &= \inf \left\{ x \geq 1 : L_l(x) > 0 \right\}, \\ z_{n,l} &= \inf \left\{ s : s \geq b_l + 1, \frac{L_l(s)}{s^2} \leq \frac{x_n^2}{n} \right\}. \end{aligned} \quad (7)$$

Write

$$W_n = \frac{n(n-1)U_n}{\max_{1 \leq l < \infty} \lambda_l \sum_{i=1}^n g_l^2(X_i)}.$$

We have the following self-normalized moderate deviation:

Theorem 1. Let $E g_l(X) = 0$ and $\lambda_l \geq 0$ for every $l \geq 1$.

$$\sup_{x \in \mathbb{R}} \frac{\sum_{l=m+1}^{\infty} \lambda_l g_l^2(x)}{\sum_{1 \leq l < \infty} \lambda_l g_l^2(x)} \rightarrow 0 \text{ as } m \rightarrow \infty \quad (8)$$

and

$$\lim_{n \rightarrow \infty} \frac{E g_l(X) I(|g_l(X)| \leq z_{n,l}) g_k(X) I(|g_k(X)| \leq z_{n,k})}{\sqrt{L_l(z_{n,l}) L_k(z_{n,k})}} \rightarrow 0 \quad (9)$$

for any $l \neq k$. Then, for $x_n \rightarrow \infty$ and $x_n = o(\sqrt{n})$,

$$\lim_{n \rightarrow \infty} x_n^{-2} \log P(W_n \geq x_n^2) = -\frac{1}{2}.$$

As an application, we have the following self-normalized LIL:

Theorem 2. Under the assumptions in Theorem 1, and instead of (8), we assume that for each $l \in [1, \infty)$, there is a constant $c_l > 0$ such that

$$\sup_{x \in \mathbb{R}} \frac{\lambda_l g_l^2(x)}{\sum_{l=1}^{\infty} \lambda_l g_l^2(x)} \leq c_l \text{ and } \sum_{l=1}^{\infty} c_l < \infty. \quad (10)$$

Then,

$$\limsup_{n \rightarrow \infty} \frac{W_n}{\log \log n} = 2 \text{ a.s.} \quad (11)$$

Remark 1. We use an example to show that (9) cannot be removed. Let $g_1(x) = x$, $g_2(x) = x^3$ and $\lambda_1 = \lambda_2 = 1$, $\lambda_l = 0$ for $l \geq 3$. Let X be a Rademacher random variable. Then, $n(n-1)U_n = \sum_{1 \leq i \neq j \leq n} (X_i X_j + X_i^3 X_j^3) = 2 \sum_{1 \leq i \neq j \leq n} X_i X_j$ and $W_n = 2 \sum_{1 \leq i \neq j \leq n} X_i X_j / \sum_{i=1}^n X_i^2$. By (6), $\limsup_{n \rightarrow \infty} W_n / \log \log n = 4$ a.s. which contradicts (11).

2. Proofs

In the proofs of theorems, we will use the following properties for the slowly varying functions g_l (e.g., Bingham et al. [17]). As $x \rightarrow \infty$,

$$P(|g_l(X)| \geq x) = o(L_l(x)/x^2), \quad (12)$$

$$E|g_l(X)|I(|g_l(X)| \geq x) = o(L_l(x)/x), \quad (13)$$

$$E|g_l(X)|^p I(|g_l(X)| \leq x) = o(x^{p-2}L_l(x)), \quad p > 2. \quad (14)$$

Since $L_l(s)/s^2 \rightarrow 0$ as $s \rightarrow \infty$ and $L_l(x)$ is right continuous, in (7), $z_{n,l} \rightarrow \infty$ and for all sufficiently large n values, we have

$$nL_l(z_{n,l}) = x_n^2 z_{n,l}^2. \quad (15)$$

2.1. The Upper Bound of Theorem 1

For each $l \geq 1$ and $i \geq 1$, denote the truncated function

$$\bar{g}_l(X_i) = g_l(X_i)I(|g_l(X_i)| \leq z_{n,l}).$$

Since $Eg_l(X_i) = 0$, we have

$$E\bar{g}_l(X_i) = o(L_l(z_{n,l})/z_{n,l}) = o(x_n \sqrt{L_l(z_{n,l})}/\sqrt{n}) \quad (16)$$

by (13) and (15). For each $l \geq 1$, write

$$Y_{n,l} = \sum_{i=1}^n g_l(X_i), \quad \bar{Y}_{n,l} = \sum_{i=1}^n \bar{g}_l(X_i) \quad \text{and} \quad V_{n,l}^2 = \sum_{i=1}^n g_l^2(X_i).$$

By Condition (8), for each $0 < \varepsilon < 1$, there exists $1 \leq m < \infty$ such that

$$m \max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \geq \sum_{l=1}^m \lambda_l V_{n,l}^2 \geq (1 - \varepsilon) \sum_{l=1}^{\infty} \lambda_l V_{n,l}^2. \quad (17)$$

Hence, for $x_n \rightarrow \infty$,

$$P(W_n \geq (1 + \varepsilon)x_n^2) \leq P\left(\frac{\sum_{l=1}^{\infty} \lambda_l Y_{n,l}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq x_n^2\right). \quad (18)$$

Observe that for any random variables $\{U_l\}_{l=1}^\infty$ and $\{Z_l\}_{l=1}^\infty$ and any constants $x > 0$ and $0 < a < 1/3$, we have the following via the Cauchy inequality:

$$\begin{aligned} & P\left(\sum_l (U_l + Z_l)^2 \geq x\right) \\ & \leq P\left(\sum_l U_l^2 \geq (1-a)^2 x\right) + P\left(\sum_l Z_l^2 \geq a^2 x\right) \\ & \leq P\left(\sum_l U_l^2 \geq (1-2a)x\right) + P\left(\sum_l Z_l^2 \geq a^2 x\right). \end{aligned} \quad (19)$$

For any $0 < \varepsilon < 1/3$, by (18) and (19), we have

$$\begin{aligned} & P\left(W_n \geq (1+\varepsilon)x_n^2\right) \\ & \leq P\left(\frac{\sum_{l=1}^\infty \lambda_l (\sum_{i=1}^n g_l(X_i) I(|g_l(X_i)| > z_{n,l}))^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq \varepsilon^2 x_n^2\right) \\ & \quad + P\left(\frac{\sum_{l=1}^\infty \lambda_l \tilde{Y}_{n,l}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq (1-2\varepsilon)x_n^2\right). \end{aligned} \quad (20)$$

For any integer $m \geq 1$ and any constant $C_1 > 0$ with $C_1 \varepsilon < 1$, we have

$$\begin{aligned} & P\left(\frac{\sum_{l=1}^\infty \lambda_l \tilde{Y}_{n,l}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq (1-2\varepsilon)x_n^2\right) \\ & \leq P\left(\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq \frac{n}{\varepsilon} \sum_{l=m+1}^\infty \lambda_l L_l(z_{n,l})\right) \\ & \quad + P\left(\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq (1-\varepsilon)n \max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l})\right) \\ & \quad + P\left(\frac{\sum_{l=m+1}^\infty \lambda_l \tilde{Y}_{n,l}^2}{(n/\varepsilon) \sum_{l=m+1}^\infty \lambda_l L_l(z_{n,l})} \geq C_1 \varepsilon (1-2\varepsilon)x_n^2\right) \\ & \quad + P\left(\frac{\sum_{l=1}^m \lambda_l \tilde{Y}_{n,l}^2}{(1-\varepsilon)n \max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l})} \geq (1-C_1 \varepsilon)(1-2\varepsilon)x_n^2\right). \end{aligned} \quad (21)$$

Applying (21) to (20), we have

$$\begin{aligned} & P\left(W_n \geq (1+\varepsilon)x_n^2\right) \\ & \leq P\left(\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq \frac{n}{\varepsilon} \sum_{l=m+1}^\infty \lambda_l L_l(z_{n,l})\right) \\ & \quad + P\left(\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq (1-\varepsilon)n \max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l})\right) \\ & \quad + P\left(\frac{\sum_{l=1}^\infty \lambda_l (\sum_{i=1}^n g_l(X_i) I(|g_l(X_i)| > z_{n,l}))^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq \varepsilon^2 x_n^2\right) \\ & \quad + P\left(\frac{\sum_{l=m+1}^\infty \lambda_l \tilde{Y}_{n,l}^2}{n \sum_{l=m+1}^\infty \lambda_l L_l(z_{n,l})} \geq C_1 (1-2\varepsilon)x_n^2\right) \\ & \quad + P\left(\frac{\sum_{l=1}^m \lambda_l \tilde{Y}_{n,l}^2}{n \max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l})} \geq (1-C_1 \varepsilon)(1-2\varepsilon)^2 x_n^2\right) \\ & := I_{1,1} + I_{1,2} + I_2 + I_3 + I_4. \end{aligned} \quad (22)$$

2.2. Estimation of $I_{1,1}$ and $I_{1,2}$

Proposition 1. For $m \geq 1$ that is sufficiently large,

$$I_{1,1} = P\left(\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq \frac{n}{\varepsilon} \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})\right) \leq \exp(-2x_n^2) \quad (23)$$

and for any constants $\delta > 0$ and $0 < \eta < 1$,

$$P\left(2m \max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq (1 - \eta)n \sum_{1 \leq l < \infty} \lambda_l L_l(z_{n,l})\right) \leq \exp(-2x_n^2), \quad (24)$$

$$\begin{aligned} & P\left(\max_{1 \leq l < \infty} \lambda_l \sum_{i=1}^n g_l^2(X_i) I(|g_l(X_i)| \leq \delta z_{n,l}) \leq (1 - \eta)n \max_{1 \leq l < \infty} \lambda_l L_l(z_{n,l})\right) \\ & \leq \exp(-2x_n^2). \end{aligned} \quad (25)$$

In particular,

$$I_{1,2} \leq P\left(\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq (1 - \varepsilon)n \max_{1 \leq l < \infty} \lambda_l L_l(z_{n,l})\right) \leq \exp(-2x_n^2).$$

Proof. We shall apply the following exponential inequality (see, e.g., Theorem 2.19 of de la Peña, Lai and Shao [18]). If $Y_1, \dots, Y_n$ are independent random variables with $Y_i \geq 0$, $\mu_n = \sum_{i=1}^n EY_i$ and $B_n^2 = \sum_{i=1}^n EY_i^2 < \infty$, then for $0 < x < \mu_n$,

$$P\left(\sum_{i=1}^n Y_i \leq x\right) \leq \exp\left(-\frac{(\mu_n - x)^2}{2B_n^2}\right).$$

By (8), $\sum_{l=m+1}^{\infty} \lambda_l V_{n,l}^2 / \sum_{l=1}^{\infty} \lambda_l V_{n,l}^2 \rightarrow 0$ as $m \rightarrow \infty$. Then, by (17),

$$\frac{\sum_{l=m+1}^{\infty} \lambda_l V_{n,l}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (26)$$

Hence,

$$\varepsilon \max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \geq 2 \sum_{l=m+1}^{\infty} \lambda_l V_{n,l}^2 \geq 2 \sum_{l=m+1}^{\infty} \lambda_l \sum_{i=1}^n g_l^2(X_i).$$

Then,

$$\begin{aligned} I_{1,1} & \leq P\left(\sum_{l=m+1}^{\infty} \lambda_l \sum_{i=1}^n g_l^2(X_i) \leq \frac{n}{2} \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})\right) \\ & \leq \exp\left(-\frac{(n \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})/2)^2}{2nE(\sum_{l=m+1}^{\infty} \lambda_l g_l^2(X_1))^2}\right). \end{aligned} \quad (27)$$

By Minkowski's integral inequality, (14) and (15),

$$\begin{aligned} E\left(\sum_{l=m+1}^{\infty} \lambda_l g_l^2(X_1)\right)^2 & \leq \left\{\sum_{l=m+1}^{\infty} \lambda_l (Eg_l^4(X_1))^{1/2}\right\}^2 \\ & = o\left(\frac{\sqrt{n}}{x_n} \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})\right)^2. \end{aligned} \quad (28)$$

Therefore, (23) follows from (27) and (28). To show (24), notice that by (17),

$$m \max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \geq (1 - \eta) \sum_{l=1}^{\infty} \lambda_l V_{n,l}^2 \geq (1 - \eta) \sum_{l=1}^{\infty} \lambda_l \sum_{i=1}^n \bar{g}_l^2(X_i).$$

Then,

$$\begin{aligned} & P \left(2m \max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq (1 - \eta) n \sum_{1 \leq l < \infty} \lambda_l L_l(z_{n,l}) \right) \\ & \leq P \left(\sum_{l=1}^{\infty} \lambda_l \sum_{i=1}^n \bar{g}_l^2(X_i) \leq \frac{n}{2} \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l}) \right) \\ & \leq \exp \left(- \frac{((\frac{n}{2} \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l}))^2)}{2nE(\sum_{l=1}^{\infty} \lambda_l \bar{g}_l^2(X_1))^2} \right). \end{aligned}$$

Similar to the proof of $I_{1,1}$ as in (27) and (28), we have (24).

To show (25), let

$$l_n = \min \left\{ l : \lambda_l L_l(z_{n,l}) = \max_{1 \leq l < \infty} \lambda_l L_l(z_{n,l}) \right\}.$$

Then,

$$\begin{aligned} & P \left(\max_{1 \leq l < \infty} \lambda_l \sum_{i=1}^n \bar{g}_l^2(X_i) I(|g_l(X_i)| \leq \delta z_{n,l}) \leq (1 - \eta) n \max_{1 \leq l < \infty} \lambda_l L_l(z_{n,l}) \right) \\ & \leq P \left(\sum_{i=1}^n \lambda_{l_n} \bar{g}_{l_n}^2(X_i) I(|g_{l_n}(X_i)| \leq \delta z_{n,l_n}) \leq (1 - \eta) n \lambda_{l_n} L_{l_n}(z_{n,l_n}) \right) \\ & \leq \exp \left(- \frac{(1 - (1 - \eta))^2 (n \lambda_{l_n} L_{l_n}(z_{n,l_n}))^2}{2n \lambda_{l_n}^2 E \bar{g}_{l_n}^4(X_1) I(|g_{l_n}(X_1)| \leq \delta z_{n,l_n})} \right) \\ & = \exp \left(- \frac{\eta^2 (n \lambda_{l_n} L_{l_n}(z_{n,l_n}))^2}{2n \lambda_{l_n}^2 o(n L_{l_n}^2(\delta z_{n,l_n}) / x_n^2)} \right). \end{aligned}$$

Since $L_l(\delta z_{n,l}) / L_l(z_{n,l}) \rightarrow 1$, (25) follows. $\square$

2.3. Estimation of I_2

Proposition 2.

$$I_2 = P \left(\frac{\sum_{l=1}^{\infty} \lambda_l \{ \sum_{i=1}^n \bar{g}_l(X_i) I(|g_l(X_i)| > z_{n,l}) \}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq \varepsilon^2 x_n^2 \right) \leq \exp(-2x_n^2).$$

Proof. Via the Cauchy–Schwarz inequality,

$$\begin{aligned} & \frac{\sum_{l=1}^{\infty} \lambda_l \{ \sum_{i=1}^n |g_l(X_i)| I(|g_l(X_i)| > z_{n,l}) \}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \\ & \leq \frac{\sum_{l=1}^{\infty} \lambda_l \sum_{i=1}^n \bar{g}_l^2(X_i) \sum_{i=1}^n I(|g_l(X_i)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2}. \end{aligned}$$

By (17), the sum of the diagonal terms is as follows:

$$\frac{\sum_{l=1}^{\infty} \lambda_l \sum_{i=1}^n \bar{g}_l^2(X_i) I(|g_l(X_i)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \leq \frac{\varepsilon^2 x_n^2}{2}.$$

$$\begin{aligned}
I_2 &\leq P\left(\frac{\sum_{1 \leq i \neq j \leq n} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{i=1}^n g_l^2(X_i)} \geq \frac{\varepsilon^2 x_n^2}{2}\right) \\
&\leq P\left(\frac{\sum_{1 \leq i < j \leq n} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{i=1}^n g_l^2(X_i)} \geq \frac{\varepsilon^2 x_n^2}{4}\right) \\
&+ P\left(\frac{\sum_{1 \leq j < i \leq n} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{i=1}^n g_l^2(X_i)} \geq \frac{\varepsilon^2 x_n^2}{4}\right) \\
&\leq P\left(\sum_{2 \leq j \leq n} \frac{\sum_{1 \leq i < j} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{1 \leq i < j} g_l^2(X_i)} \geq \frac{\varepsilon^2 x_n^2}{4}\right) \\
&+ P\left(\sum_{1 \leq j < n} \frac{\sum_{j < i \leq n} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{j < i \leq n} g_l^2(X_i)} \geq \frac{\varepsilon^2 x_n^2}{4}\right) \\
&= I_{2,1} + I_{2,2}.
\end{aligned} \tag{29}$$

Let

$$\phi_j = \frac{\sum_{1 \leq i < j} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{1 \leq i < j} g_l^2(X_i)}.$$

Then, for any constant $t > 0$,

$$I_{2,1} \leq E e^{t \sum_{j=2}^n \phi_j} e^{-t \varepsilon^2 x_n^2 / 4}. \tag{30}$$

Let E_j be the expectation of X_j for $2 \leq j \leq n$. Then,

$$E e^{t \sum_{j=2}^n \phi_j} = E(e^{t \sum_{j=2}^{n-1} \phi_j} E_n e^{t \phi_n}). \tag{31}$$

Since $|e^s - 1| \leq e^{0 \vee s} |s|$ for any $s \in \mathbb{R}$ and $0 \leq \phi_n \leq m$ for some sufficiently large m value, then

$$\begin{aligned}
|E_n e^{t \phi_n} - 1| &\leq e^{mt} t E_n \phi_n \\
&= \frac{e^{mt} t \sum_{1 \leq i < n} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) P(|g_l(X_n)| > z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{1 \leq i < n} g_l^2(X_i)}.
\end{aligned}$$

By (12) and (15), we have $P(|g_l(X_n)| > z_{n,l}) = o(x_n^2/n)$. Then, together with (17),

$$E_n e^{t \phi_n} = 1 + o(x_n^2/n) = e^{o(x_n^2/n)}. \tag{32}$$

Applying (32) to (31), we have

$$E e^{t \sum_{j=2}^n \phi_j} = e^{o(x_n^2/n)} E e^{t \sum_{j=2}^{n-1} \phi_j}.$$

Similarly,

$$\begin{aligned}
E e^{t \sum_{j=2}^{n-1} \phi_j} &= E(e^{t \sum_{j=2}^{n-2} \phi_j} E_{n-1} e^{t \phi_{n-1}}) \\
&= e^{o(x_n^2/n)} E e^{t \sum_{j=2}^{n-2} \phi_j}.
\end{aligned}$$

Continue this process from X_n to X_1 . We conclude that

$$E e^{t \sum_{j=2}^n \phi_j} = e^{n \times o(x_n^2/n)} = e^{o(x_n^2)}. \tag{33}$$

Applying (33) to (30) and letting $t = 16/\varepsilon^2$, we have

$$I_{2,1} \leq \exp(-3x_n^2). \quad (34)$$

By the same argument,

$$I_{2,2} \leq \exp(-3x_n^2). \quad (35)$$

Combining (29), (34) and (35), we obtain the proposition. $\square$

2.4. Estimation of I_3

Let $Y_1, \dots, Y_n$ be an independent copy of $X_1, \dots, X_n$. We will use the following lemma which is a Bernstein-type exponential inequality for degenerate U -statistics.

Lemma 1 ((3.5) of Giné, Latała and Zinn [19]). *For bounded degenerate kernel $h_{i,j}(X_i, Y_j)$, let*

$$A = \max_{i,j} \|h_{i,j}(X_i, Y_j)\|_\infty, \quad C^2 = \sum_{i,j} E h_{i,j}^2(X_i, Y_j),$$

$$B^2 = \max_{i,j} \left\{ \left\| \sum_i E h_{i,j}^2(X_i, y) \right\|_\infty, \left\| \sum_j E h_{i,j}^2(x, Y_j) \right\|_\infty \right\}.$$

Then, there is a universal constant K such that

$$\Pr \left\{ \left| \sum_{i,j} h_{i,j}(X_i, Y_j) \right| > x \right\} \leq K \exp \left\{ -\frac{1}{K} \min \left[\frac{x}{C}, \left(\frac{x}{B} \right)^{2/3}, \left(\frac{x}{A} \right)^{1/2} \right] \right\}.$$

Recall (16). Hence, by (19) and the definition of I_3 in (22),

$$I_3 \leq P \left(\frac{\sum_{l=m+1}^{\infty} \lambda_l (\bar{Y}_{n,l} - E \bar{Y}_{n,l})^2}{\sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})} \geq C_1 n (1 - 2\varepsilon)^2 x_n^2 \right). \quad (36)$$

Let

$$h_m(X_i, Y_j) = \sum_{l=m+1}^{\infty} \lambda_l (\bar{g}_l(X_i) - E \bar{g}_l(X_i)) (\bar{g}_l(Y_j) - E \bar{g}_l(Y_j)). \quad (37)$$

In addition to the estimate of I_3 , we include (40) in the following proposition which will be used in the proof of Theorem 2, where

$$h_{(\beta)}(X_i, Y_j)$$

$$= \sum_{l=1}^{\infty} \lambda_l \{ (g_l(X_i) I(|g_l(X_i)| \leq \beta z_{n,l}) - E g_l(X_i) I(|g_l(X_i)| \leq \beta z_{n,l})) \}$$

$$\times \{ (g_l(Y_j) I(|g_l(Y_j)| \leq \beta z_{n,l}) - E g_l(Y_j) I(|g_l(Y_j)| \leq \beta z_{n,l})) \}. \quad (38)$$

Proposition 3. *For a sufficiently large constant $C_2 > 0$,*

$$P \left(\frac{\sum_{1 \leq i,j \leq n} h_m(X_i, Y_j)}{n \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})} \geq C_2 x_n^2 \right) \leq \exp(-3x_n^2), \quad (39)$$

Then, by (36) and the decoupling inequalities of de la Peña and Montgomery-Smith [20], for a sufficiently large $C_1 > 0$,

$$I_3 \leq P\left(\frac{\sum_{1 \leq i, j \leq n} h_m(X_i, X_j)}{n \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})} \geq C_1(1 - 2\varepsilon)^2 x_n^2\right) \leq \exp(-2x_n^2).$$

Suppose that $\lambda_l > 0$ for all $1 \leq l < \infty$ and $d > 0$ is a constant. For constants $0 < \alpha, \beta \leq 1$ that are sufficiently small,

$$P\left(\frac{\sum_{1 \leq i, j \leq [an]} h_{(\beta)}(X_i, Y_j)}{n \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})} \geq dx_n^2\right) \leq \exp(-2x_n^2). \quad (40)$$

Proof. We will prove (39) and (40) simultaneously. By (15) and (37),

$$A_n := \|h_m(X_i, Y_j)\|_{\infty} \leq 4 \sum_{l=m+1}^{\infty} \lambda_l z_{n,l}^2 = 4 \sum_{l=m+1}^{\infty} \lambda_l \frac{n L_l(z_{n,l})}{x_n^2}. \quad (41)$$

By (15) and (38),

$$A_{n,(\beta)} := \|h_{(\beta)}(X_i, Y_j)\|_{\infty} \leq 4 \sum_{l=1}^{\infty} \lambda_l \beta^2 z_{n,l}^2 = \frac{4\beta^2 n \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})}{x_n^2}. \quad (42)$$

Let

$$B_n^2 := \max \left\{ \left\| \sum_{1 \leq i \leq n} E h_m^2(X_i, y) \right\|_{\infty}, \left\| \sum_{1 \leq j \leq n} E h_m^2(x, Y_j) \right\|_{\infty} \right\}.$$

Since $|\bar{g}_l(Y_j)| \leq z_{n,l}$, then by Cauchy–Schwarz inequality, (15) and (37),

$$\begin{aligned} \left\| \sum_{1 \leq i \leq n} E h_m^2(X_i, y) \right\|_{\infty} &\leq n E \left(2 \sum_{l=m+1}^{\infty} \lambda_l |\bar{g}_l(X_1) - E \bar{g}_l(X_1)| z_{n,l} \right)^2 \\ &\leq 4n E \sum_{l=m+1}^{\infty} \lambda_l (\bar{g}_l(X_1) - E \bar{g}_l(X_1))^2 \sum_{l=m+1}^{\infty} \lambda_l z_{n,l}^2 \\ &\leq 4n \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l}) \sum_{l=m+1}^{\infty} \lambda_l \frac{n L_l(z_{n,l})}{x_n^2}. \end{aligned}$$

The same result can be obtained for $\|\sum_{1 \leq j \leq n} E h_m^2(x, Y_j)\|_{\infty}$. Therefore,

$$B_n^2 \leq \frac{4n^2 (\sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l}))^2}{x_n^2}. \quad (43)$$

Similarly,

$$\begin{aligned} B_{n,\alpha,(\beta)}^2 &:= \max \left\{ \left\| \sum_{1 \leq i \leq [an]} E h_{(\beta)}^2(X_i, y) \right\|_{\infty}, \left\| \sum_{1 \leq j \leq [an]} E h_{(\beta)}^2(x, Y_j) \right\|_{\infty} \right\} \\ &\leq 4\alpha n \sum_{l=1}^{\infty} \lambda_l L_l(\beta z_{n,l}) \sum_{l=1}^{\infty} \lambda_l \beta^2 \frac{n L_l(z_{n,l})}{x_n^2}. \end{aligned}$$

Since $0 < \beta \leq 1$, then $L_l(\beta z_{n,l})/L_l(z_{n,l}) \leq 1$. Hence,

$$B_{n,\alpha,(\beta)}^2 \leq \frac{4\alpha \beta^2 n^2 (\sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l}))^2}{x_n^2}.$$

By (37) and the Cauchy–Schwarz inequality,

$$\begin{aligned} C_n^2 &:= \sum_{1 \leq i, j \leq n} E h_m^2(X_i, Y_j) \\ &\leq \sum_{1 \leq i, j \leq n} \sum_{l=m+1}^{\infty} \lambda_l E(\bar{g}_l(X_i) - E\bar{g}_l(X_i))^2 \sum_{l=m+1}^{\infty} \lambda_l E(\bar{g}_l(Y_j) - E\bar{g}_l(Y_j))^2 \\ &\leq n^2 \left(\sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l}) \right)^2. \end{aligned} \quad (44)$$

Similarly,

$$\begin{aligned} C_{n,\alpha,(\beta)}^2 &:= \sum_{1 \leq i \neq j \leq [\alpha n]} E h_{(\beta)}^2(X_i, Y_j) \\ &\leq \alpha^2 n^2 \left(\sum_{l=1}^{\infty} \lambda_l L_l(\beta z_{n,l}) \right)^2 \\ &\leq \alpha^2 n^2 \left(\sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l}) \right)^2. \end{aligned} \quad (45)$$

Now let

$$x = C_2 n x_n^2 \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l}). \quad (46)$$

By (41) and (46),

$$\left(\frac{x}{A_n} \right)^{1/2} \geq \left(\frac{C_2 n x_n^2 \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})}{4n \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l}) / x_n^2} \right)^{1/2} = (C_2/4)^{1/2} x_n^2.$$

By (43) and (46),

$$\left(\frac{x}{B_n} \right)^{2/3} \geq \left(\frac{C_2 n x_n^2 \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})}{2n \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l}) / x_n} \right)^{2/3} = (C_2/2)^{2/3} x_n^2.$$

By (44) and (46),

$$\frac{x}{C_n} \geq \frac{C_2 n x_n^2 \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})}{n \sum_{l=m+1}^{\infty} \lambda_l L_l(z_{n,l})} = C_2 x_n^2.$$

Then, (39) follows from Lemma 1 for a sufficiently large C_2 value. Similarly, let

$$x_d = d n x_n^2 \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l}). \quad (47)$$

By (42) and (47),

$$\left(\frac{x_d}{A_{n,(\beta)}} \right)^{1/2} \geq \left(\frac{d n x_n^2 \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})}{4\beta^2 n \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l}) / x_n^2} \right)^{1/2} = \frac{\sqrt{d}}{2\beta} x_n^2.$$

By (44) and (47),

$$\left(\frac{x_d}{B_{n,\alpha,(\beta)}} \right)^{2/3} \geq \left(\frac{d n x_n^2 \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})}{2\sqrt{\alpha}\beta n \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l}) / x_n} \right)^{2/3} = \left(\frac{d}{2\sqrt{\alpha}\beta} \right)^{2/3} x_n^2.$$

By (45) and (47),

$$\frac{x_d}{C_{n,\alpha}(\beta)} \geq \frac{dnx_n^2 \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})}{\alpha n \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})} = \frac{d}{\alpha} x_n^2.$$

Therefore, (40) follows from Lemma 1 for α and β values that are sufficiently small. $\square$

2.5. Estimation of I_4

Lemma 2 below follows Corollary 1(b) of Einmahl [21] and Lemma 4.2 of Lin and Liu [22]. However, we add the condition $\sum_{i=1}^{k_n} E\|\xi_{n,i}\|^2 \leq b_n^2$, and our result is in a form of an exponential inequality for independent random vectors. We use the same positive constants c_{17} , c_{20} and c_{22} (depending only on the vector dimension d) in Einmahl [21].

Lemma 2. Let $\xi_{n,1}, \dots, \xi_{n,k_n}$ be independent random vectors with a mean of zero and values in $\mathbb{R}^d$ such that $\|\xi_{n,i}\| \leq A_n$ and $\sum_{i=1}^{k_n} E\|\xi_{n,i}\|^2 \leq b_n^2$, where $\|\cdot\|$ denotes the Euclidean norm. Let $S_n = \sum_{i=1}^{k_n} \xi_{n,i}$. Suppose that

$$\text{Cov}(S_n) = B_n I_d \quad (48)$$

where $B_n > 0$, I_d is a $d \times d$ identity matrix, and α_n is a positive sequence such that $\alpha_n B_n^{1/2} \rightarrow \infty$ and

$$\alpha_n \sum_{i=1}^{k_n} E\left\{\|\xi_{n,i}\|^3 \exp(\alpha_n \|\xi_{n,i}\|)\right\} \leq B_n. \quad (49)$$

Let

$$\beta_n = B_n^{-3/2} \sum_{i=1}^{k_n} E\left\{\|\xi_{n,i}\|^3 \exp(\alpha_n \|\xi_{n,i}\|)\right\}. \quad (50)$$

Then, for any $0 < \gamma < 1$, there exists n_γ such that for all $n \geq n_\gamma$,

$$\begin{aligned} P(\|S_n\| \geq x) &\leq \exp\left\{c_{20}\beta_n\left(c_{17}^3\alpha_n^3 B_n^{3/2} + 1\right)\right\} \\ &\times \left\{\exp\left(-\frac{(1-\gamma)^6 x^2}{2B_n}\right) + \exp\left(-\frac{\gamma^3(1-\gamma)^3 x^2}{2c_{22}B_n\beta_n^2 \log(1/\beta_n)}\right)\right\} \\ &+ 2d \exp\left(-\frac{(1-\gamma)^2 c_{17}^2 \alpha_n^2 B_n^2}{2(d^2 b_n^2 + d c_{17} A_n \alpha_n B_n)}\right) \end{aligned}$$

uniformly for $x \in [e_n B_n^{1/2}, c_{17} \alpha_n B_n]$, where $\{e_n\}_{n \geq 1}$ can be any sequence with $e_n \rightarrow \infty$ and $e_n \leq c_{17} \alpha_n B_n^{1/2}$.

Proof. Let $\eta_{n,i}$, $1 \leq i \leq k_n$, be independent $N(0, \sigma^2 \text{Cov}(\xi_{n,i}))$ random vectors, which are independent of the $\xi_{n,i}$ s, where

$$\sigma^2 = c_{22} \beta_n^2 \log(1/\beta_n). \quad (51)$$

By (49) and (50), we have $\beta_n \leq \alpha_n^{-1} B_n^{-1/2} \rightarrow 0$ as $n \rightarrow \infty$. Hence, $\sigma \rightarrow 0$ as $n \rightarrow \infty$. Let $p_n(y)$ be the probability density of $B_n^{-1/2} \sum_{i=1}^{k_n} (\xi_{n,i} + \eta_{n,i})$ and $\phi_{(1+\sigma^2)I_d}$ be the density of $N(0, (1+\sigma^2)I_d)$. By Corollary 1(b) in Einmahl [21] (together with the Remark on page 32), for $\|y\| \leq c_{17} \alpha_n B_n^{1/2}$,

$$p_n(y) = \phi_{(1+\sigma^2)I_d}(y) \exp(T_n(y)) \quad \text{with} \quad |T_n(y)| \leq c_{20} \beta_n (\|y\|^3 + 1). \quad (52)$$

For any $0 < \gamma < 1$ and $x \in [e_n B_n^{1/2}, c_{17} \alpha_n B_n]$,

$$\begin{aligned}
 & P(\|S_n\| \geq x) \\
 & \leq P\left(\left\|S_n + \sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq (1-\gamma)x\right) + P\left(\left\|\sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq \gamma x\right) \\
 & = P\left((1-\gamma)x \leq \left\|S_n + \sum_{i=1}^{k_n} \eta_{n,i}\right\| < c_{17} \alpha_n B_n\right) \\
 & \quad + P\left(\left\|S_n + \sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq c_{17} \alpha_n B_n\right) + P\left(\left\|\sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq \gamma x\right) \\
 & \leq P\left((1-\gamma)x \leq \left\|S_n + \sum_{i=1}^{k_n} \eta_{n,i}\right\| < c_{17} \alpha_n B_n\right) \\
 & \quad + P(\|S_n\| \geq (1-\gamma)c_{17} \alpha_n B_n) \\
 & \quad + P\left(\left\|\sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq \gamma c_{17} \alpha_n B_n\right) + P\left(\left\|\sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq \gamma x\right) \\
 & \leq P\left((1-\gamma)x \leq \left\|S_n + \sum_{i=1}^{k_n} \eta_{n,i}\right\| < c_{17} \alpha_n B_n\right) + 2P\left(\left\|\sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq \gamma x\right) \\
 & \quad + P(\|S_n\| \geq (1-\gamma)c_{17} \alpha_n B_n) \\
 & := J_1 + J_2 + J_3.
 \end{aligned} \tag{53}$$

Let N denote a centered normal random vector with covariance matrix I_d . Then, by (52),

$$\begin{aligned}
 J_1 &= \int_{(1-\gamma)x/B_n^{1/2} < \|y\| \leq c_{17} \alpha_n B_n^{1/2}} \phi_{(1+\sigma^2)I_d}(y) \exp(T_n(y)) dy \\
 &\leq \exp\left\{c_{20} \beta_n \left(c_{17}^3 \alpha_n^3 B_n^{3/2} + 1\right)\right\} \int_{\|y\| \geq (1-\gamma)x/B_n^{1/2}} \phi_{(1+\sigma^2)I_d}(y) dy \\
 &\leq \exp\left\{c_{20} \beta_n \left(c_{17}^3 \alpha_n^3 B_n^{3/2} + 1\right)\right\} \times \\
 &\quad \left\{P\left(\|N\| \geq (1-\gamma)^2 x / B_n^{1/2}\right) + P\left(\sigma \|N\| \geq \gamma(1-\gamma)x / B_n^{1/2}\right)\right\}.
 \end{aligned} \tag{54}$$

Observe that $\|N\|^2$ has a χ_d^2 distribution. It is well known that for a χ_d^2 random variable Y , $P(Y > y) \leq (ye^{1-y/d}/d)^{d/2}$ for $y > d$. Hence,

$$\begin{aligned}
 & P\left(\|N\| \geq (1-\gamma)^2 x / B_n^{1/2}\right) \\
 & \leq \left(\frac{(1-\gamma)^4 x^2}{dB_n} \exp\left(1 - \frac{(1-\gamma)^4 x^2}{dB_n}\right)\right)^{d/2} \\
 & = \frac{(1-\gamma)^{2d} x^d}{d^{d/2} B_n^{d/2}} \exp\left(d/2 - \frac{(1-\gamma)^4 x^2}{2B_n}\right) \\
 & \leq \exp\left(-\frac{(1-\gamma)^6 x^2}{2B_n}\right)
 \end{aligned} \tag{55}$$

by $x^2/B_n \rightarrow \infty$. Similarly,

$$\begin{aligned}
 & P\left(\sigma \|N\| \geq \gamma(1-\gamma)x / B_n^{1/2}\right) \leq \exp\left(-\frac{\gamma^3(1-\gamma)^3 x^2}{2B_n \sigma^2}\right) \\
 & = \exp\left(-\frac{\gamma^3(1-\gamma)^3 x^2}{2c_{22} B_n \beta_n^2 \log(1/\beta_n)}\right)
 \end{aligned} \tag{56}$$

by (51). Then, by (54)–(56), we have

$$J_1 \leq \exp\left\{c_{20}\beta_n\left(c_{17}^3\alpha_n^3B_n^{3/2}+1\right)\right\} \times \left\{\exp\left(-\frac{(1-\gamma)^6x^2}{2B_n}\right)+\exp\left(-\frac{\gamma^3(1-\gamma)^3x^2}{2c_{22}B_n\beta_n^2\log(1/\beta_n)}\right)\right\}. \quad (57)$$

Since the distribution of $\eta_{n,i}$ is $N(0, \sigma^2 \text{Cov}(\xi_{n,i}))$, the distribution of $\sum_{i=1}^{k_n} \eta_{n,i}$ is $N(0, \sigma^2 \sum_{i=1}^{k_n} \text{Cov}(\xi_{n,i}))$. Since the $\xi_{n,i}$ s are independent, $\sum_{i=1}^{k_n} \text{Cov}(\xi_{n,i}) = \text{Cov}(\sum_{i=1}^{k_n} \xi_{n,i}) = B_n I_d$ by (48). Hence, the distribution of $\sum_{i=1}^{k_n} \eta_{n,i}$ is $N(0, \sigma^2 B_n I_d)$. Then, similar to (56),

$$J_2 = 2P\left(\left\|\sum_{i=1}^{k_n} \eta_{n,i}\right\| \geq \gamma x\right) = 2P\left(\sigma\|N\| \geq \gamma x/B_n^{1/2}\right) \leq 2\exp\left(-\frac{\gamma^3x^2}{2B_n c_{22}\beta_n^2\log(1/\beta_n)}\right). \quad (58)$$

By (57) and (58), we have

$$J_1 + J_2 \leq \exp\left\{c_{20}\beta_n\left(c_{17}^3\alpha_n^3B_n^{3/2}+1\right)\right\} \times \left\{\exp\left(-\frac{(1-\gamma)^6x^2}{2B_n}\right)+3\exp\left(-\frac{\gamma^3(1-\gamma)^3x^2}{2c_{22}B_n\beta_n^2\log(1/\beta_n)}\right)\right\}. \quad (59)$$

Next, we estimate J_3 . For each $1 \leq i \leq n$, let $\xi_{n,i} = (\xi_{n,i}^{(1)}, \dots, \xi_{n,i}^{(d)})^T$, where $\mathbf{a}^T$ denote the transpose of a vector $\mathbf{a}$. Then

$$\|S_n\| = \left\|\sum_{i=1}^{k_n} \xi_{n,i}\right\| = \left(\sum_{l=1}^d \left(\sum_{i=1}^{k_n} \xi_{n,i}^{(l)}\right)^2\right)^{1/2} \leq \sum_{l=1}^d \left|\sum_{i=1}^{k_n} \xi_{n,i}^{(l)}\right|.$$

Hence,

$$\begin{aligned} J_3 &= P(\|S_n\| \geq (1-\gamma)c_{17}\alpha_n B_n) \\ &\leq P\left(\sum_{l=1}^d \left|\sum_{i=1}^{k_n} \xi_{n,i}^{(l)}\right| \geq (1-\gamma)c_{17}\alpha_n B_n\right) \\ &\leq \sum_{l=1}^d P\left(\left|\sum_{i=1}^{k_n} \xi_{n,i}^{(l)}\right| \geq \frac{(1-\gamma)c_{17}\alpha_n B_n}{d}\right). \end{aligned}$$

Since $\|\xi_{n,i}\| \leq A_n$ and $\sum_{i=1}^{k_n} E\|\xi_{n,i}\|^2 \leq b_n^2$, then $|\xi_{n,i}^{(l)}| \leq \|\xi_{n,i}\| \leq A_n$ and $\sum_{i=1}^{k_n} E(\xi_{n,i}^{(l)})^2 \leq \sum_{i=1}^{k_n} E\|\xi_{n,i}\|^2 \leq b_n^2$ for each $1 \leq l \leq d$. By Bernstein's inequality (e.g., (2.17) of de la Peña, Lai and Shao [18]),

$$J_3 \leq 2d \exp\left(-\frac{(1-\gamma)^2 c_{17}^2 \alpha_n^2 B_n^2}{2d^2(b_n^2 + c_{17} A_n \alpha_n B_n/d)}\right). \quad (60)$$

Then, the lemma follows by applying (59) and (60) to (53). $\square$

Now, we estimate I_4 in the following proposition, which uses some ideas in Liu and Shao [23]:

Proposition 4.

$$\begin{aligned}
I_4 &\leq P\left(\sum_{l=1}^m \lambda_l (\bar{Y}_{n,l} - E\bar{Y}_{n,l})^2 \geq (1 - C_1\varepsilon)(1 - 2\varepsilon)^3 n x_n^2 \max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l})\right) \\
&\leq \exp\left(-\frac{(1 - C_1\varepsilon)(1 - 2\varepsilon)^4 x_n^2}{2(1 + \varepsilon)}\right).
\end{aligned}$$

Proof. For each $1 \leq i \leq n$, let

$$\mathbf{G}_{n,i} = \left(\sqrt{\lambda_1}(\bar{g}_1(X_i) - E\bar{g}_1(X_i)), \dots, \sqrt{\lambda_m}(\bar{g}_m(X_i) - E\bar{g}_m(X_i))\right)^T.$$

Let $B_n = n$ and Σ be the covariance matrix of $\mathbf{G}_{n,1}$. For $1 \leq i \leq n$, let

$$\xi_{n,i} = \Sigma^{-1/2} \mathbf{G}_{n,i}. \quad (61)$$

Then,

$$\begin{aligned}
&\text{Cov}(\xi_{n,1} + \dots + \xi_{n,n}) \\
&= E\left\{\left(\Sigma^{-1/2}(\mathbf{G}_{n,1} + \dots + \mathbf{G}_{n,n})\right)\left(\Sigma^{-1/2}(\mathbf{G}_{n,1} + \dots + \mathbf{G}_{n,n})\right)^T\right\} \\
&= \Sigma^{-1/2} E\left\{(\mathbf{G}_{n,1} + \dots + \mathbf{G}_{n,n})(\mathbf{G}_{n,1} + \dots + \mathbf{G}_{n,n})^T\right\} \Sigma^{-1/2} \\
&= \Sigma^{-1/2} \sum_{1 \leq i,j \leq n} E\left\{\mathbf{G}_{n,i} \mathbf{G}_{n,j}^T\right\} \Sigma^{-1/2}.
\end{aligned}$$

Since the X_i s are independent, then

$$\begin{aligned}
\text{Cov}(\xi_{n,1} + \dots + \xi_{n,n}) &= \Sigma^{-1/2} \sum_{i=1}^n E\left\{\mathbf{G}_{n,i} \mathbf{G}_{n,i}^T\right\} \Sigma^{-1/2} \\
&= n I_m = B_n I_m.
\end{aligned}$$

Hence, Condition (48) in Lemma 2 is satisfied. Let

$$\alpha_n = \frac{C_m x_n}{n^{1/2}}$$

where $C_m > 0$ is a finite constant depending only on m . We shall verify Condition (49). By (61),

$$\|\xi_{n,i}\|^2 = \left(\Sigma^{-1/2} \mathbf{G}_{n,i}\right)^T \left(\Sigma^{-1/2} \mathbf{G}_{n,i}\right) = \mathbf{G}_{n,i}^T \Sigma^{-1} \mathbf{G}_{n,i}. \quad (62)$$

Observe that Σ is positive definite by Assumption (9). Then, by the identity

$$\mathbf{x}^T \mathbf{A}^{-1} \mathbf{x} = \max_{\|\boldsymbol{\vartheta}\|=1} \frac{(\mathbf{x}^T \boldsymbol{\vartheta})^2}{\boldsymbol{\vartheta}^T \mathbf{A} \boldsymbol{\vartheta}} \quad (63)$$

for any $m \times m$ positive definite matrix $\mathbf{A}$, we have

$$\|\xi_{n,i}\|^2 = \mathbf{G}_{n,i}^T \Sigma^{-1} \mathbf{G}_{n,i} = \max_{\|\boldsymbol{\vartheta}\|=1} \frac{(\mathbf{G}_{n,i}^T \boldsymbol{\vartheta})^2}{\boldsymbol{\vartheta}^T \Sigma \boldsymbol{\vartheta}}. \quad (64)$$

Let $\vartheta^* = (\vartheta_1^*, \dots, \vartheta_m^*)$ such that $\|\vartheta^*\| = 1$ and $(G_{n,i}^T \vartheta^*)^2 = \max_{\|\vartheta\|=1} (G_{n,i}^T \vartheta)^2$. Then, for any $\vartheta = (\vartheta_1, \dots, \vartheta_m) \in l^2$, by the Cauchy–Schwarz inequality,

$$\begin{aligned} (G_{n,i}^T \vartheta)^2 &= \left(\sum_{l=1}^m \sqrt{\lambda_l} (\bar{g}_l(X_i) - E\bar{g}_l(X_i)) \vartheta_l \right)^2 \\ &= \left(\sum_{l=1}^m \frac{\bar{g}_l(X_i) - E\bar{g}_l(X_i)}{\sqrt{L_l(z_{n,l})}} \vartheta_l \sqrt{\lambda_l L_l(z_{n,l})} \right)^2 \\ &\leq \sum_{l=1}^m \frac{(\bar{g}_l(X_i) - E\bar{g}_l(X_i))^2}{L_l(z_{n,l})} \sum_{l=1}^m \vartheta_l^2 \lambda_l L_l(z_{n,l}). \end{aligned} \quad (65)$$

Since $E\bar{g}_l(X_1) = 0$ for all $l \geq 1$, then $E\bar{g}_l(X_1) = o(x_n \sqrt{L_l(z_{n,l})})/\sqrt{n}$ by (13) and (15). By Assumption (9),

$$\begin{aligned} \vartheta^T \Sigma \vartheta &= \sum_{1 \leq l, l' \leq m} \vartheta_l \vartheta_{l'} \sqrt{\lambda_l \lambda_{l'}} E(\bar{g}_l(X_1) - E\bar{g}_l(X_1)) (\bar{g}_{l'}(X_1) - E\bar{g}_{l'}(X_1)) \\ &= \sum_{l=1}^m \vartheta_l^2 \lambda_l E\bar{g}_l^2(X_1) - \sum_{l=1}^m \vartheta_l^2 \lambda_l (E\bar{g}_l(X_1))^2 \\ &\quad + \sum_{1 \leq l \neq l' \leq m} \vartheta_l \vartheta_{l'} \sqrt{\lambda_l \lambda_{l'}} E\bar{g}_l(X_1) \bar{g}_{l'}(X_1) \\ &\quad - \sum_{1 \leq l \neq l' \leq m} \vartheta_l \vartheta_{l'} \sqrt{\lambda_l \lambda_{l'}} E\bar{g}_l(X_1) E\bar{g}_{l'}(X_1) \\ &= \sum_{l=1}^m \vartheta_l^2 \lambda_l L_l(z_{n,l}) - \sum_{l=1}^m \vartheta_l^2 \lambda_l \times o\left(\frac{x_n^2 L_l(z_{n,l})}{n}\right) \\ &\quad + o(1) \sum_{1 \leq l \neq l' \leq m} \vartheta_l \vartheta_{l'} \sqrt{\lambda_l \lambda_{l'} L_l(z_{n,l}) L_{l'}(z_{n,l'})} \\ &\quad - \sum_{1 \leq l \neq l' \leq m} \vartheta_l \vartheta_{l'} \sqrt{\lambda_l \lambda_{l'}} \times o\left(\frac{x_n \sqrt{L_l(z_{n,l})}}{\sqrt{n}}\right) o\left(\frac{x_n \sqrt{L_{l'}(z_{n,l'})}}{\sqrt{n}}\right). \end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\sum_{1 \leq l \neq l' \leq m} \vartheta_l \vartheta_{l'} \sqrt{\lambda_l \lambda_{l'} L_l(z_{n,l}) L_{l'}(z_{n,l'})} \leq m \sum_{l=1}^m \vartheta_l^2 \lambda_l L_l(z_{n,l}).$$

Hence,

$$\vartheta^T \Sigma \vartheta = (1 + o(1)) \sum_{l=1}^m \vartheta_l^2 \lambda_l L_l(z_{n,l}). \quad (66)$$

Applying (65) and (66) to (64), we have

$$\|\xi_{n,i}\|^2 \leq 2 \sum_{l=1}^m \frac{(\bar{g}_l(X_i) - E\bar{g}_l(X_i))^2}{L_l(z_{n,l})}. \quad (67)$$

Since $|\bar{g}_l(X_i)| \leq z_{n,l} = \sqrt{n L_l(z_{n,l})}/x_n$ by (15), and (16), we have

$$\|\xi_{n,i}\|^2 \leq \frac{4mn}{x_n^2}. \quad (68)$$

By (67),

$$E\|\xi_{n,i}\|^2 \leq 2 \sum_{l=1}^m \frac{E(\bar{g}_l(X_i) - E\bar{g}_l(X_i))^2}{L_l(z_{n,l})} \leq 2m \quad (69)$$

and

$$E\|\xi_{n,i}\|^3 \leq 2^{3/2} E \left(\sum_{l=1}^m \frac{(\bar{g}_l(X_i) - E\bar{g}_l(X_i))^2}{L_l(z_{n,l})} \right)^{3/2}. \quad (70)$$

By Hölder's inequality,

$$\left(\sum_{l=1}^m \frac{(\bar{g}_l(X_i) - E\bar{g}_l(X_i))^2}{L_l(z_{n,l})} \right)^{3/2} \leq m^{1/2} \sum_{l=1}^m \frac{|\bar{g}_l(X_i) - E\bar{g}_l(X_i)|^3}{L_l^{3/2}(z_{n,l})}. \quad (71)$$

Combining (70) and (71), we have

$$\begin{aligned} E\|\xi_{n,i}\|^3 &\leq 2^{3/2} m^{1/2} \sum_{l=1}^m \frac{8E|\bar{g}_l(X_i)|^3}{L_l^{3/2}(z_{n,l})} \\ &= \sum_{l=1}^m \frac{o(z_{n,l} L_l(z_{n,l}))}{L_l^{3/2}(z_{n,l})} = o\left(\frac{n^{1/2}}{x_n}\right) \end{aligned} \quad (72)$$

by (14) and (15). Since $B_n = n$ and $\alpha_n = C_m x_n / n^{1/2}$, then by (68) and (72), we have

$$\begin{aligned} &\alpha_n \sum_{i=1}^n E \left\{ \|\xi_{n,i}\|^3 \exp(\alpha_n \|\xi_{n,i}\|) \right\} \\ &\leq \frac{C_m x_n}{n^{1/2}} n \times o\left(\frac{n^{1/2}}{x_n}\right) \exp\left(\frac{C_m x_n}{n^{1/2}} \left(\frac{4mn}{x_n^2}\right)^{1/2}\right) = o(n) = o(B_n). \end{aligned}$$

Hence, Condition (49) in Lemma 2 is satisfied. Similarly,

$$\begin{aligned} \beta_n &:= B_n^{-3/2} \sum_{i=1}^n E \left\{ \|\xi_{n,i}\|^3 \exp(\alpha_n \|\xi_{n,i}\|) \right\} \\ &= n^{-3/2} n \times o\left(\frac{n^{1/2}}{x_n}\right) \exp\left(\frac{C_m x_n}{n^{1/2}} \left(\frac{4mn}{x_n^2}\right)^{1/2}\right) \\ &= o(1/x_n). \end{aligned} \quad (73)$$

Then, $\beta_n^2 \log(1/\beta_n) = o(1/x_n)$. By (68), we have $\|\xi_{n,i}\| \leq (4mn/x_n^2)^{1/2} := A_n$. By (69), we have $\sum_{i=1}^n E\|\xi_{n,i}\|^2 \leq 2mn := b_n^2$. Then, by Lemma 2 and (73) with $B_n = n$ and $\alpha_n = C_m x_n / n^{1/2}$ for a sufficiently large C_m value, we have

$$\begin{aligned} &P(\|S_n\| \geq n^{1/2} x_n) \\ &\leq \exp\{o(x_n^2)\} \left\{ \exp\left(-\frac{(1-\gamma)^6 n x_n^2}{2n}\right) + \exp\left(-\frac{\gamma^3 (1-\gamma)^3 n x_n^2}{n \times o(1/x_n)}\right) \right\} \\ &\quad + 2m \exp\left(-\frac{(1-\gamma)^2 c_{17}^2 C_m^2 x_n^2 n}{2(2m^3 n + m(4mn/x_n^2)^{1/2} c_{17} C_m x_n n^{1/2})}\right) \\ &\leq \exp\{o(x_n^2)\} \left\{ \exp\left(-\frac{(1-\gamma)^6 x_n^2}{2}\right) + \exp(-4x_n^2) \right\} + \exp(-4x_n^2) \\ &\leq \exp\left(-\frac{(1-\gamma)^7 x_n^2}{2}\right). \end{aligned}$$

Letting $\gamma = 1 - (1 - \varepsilon)^{1/7}$, we have

$$P\left(\|S_n\| \geq n^{1/2}x_n\right) \leq \exp\left(-\frac{(1-\varepsilon)x_n^2}{2}\right). \quad (74)$$

Similar to (62),

$$\begin{aligned} \|S_n\|^2 &= \left\| \sum_{i=1}^n \xi_{n,i} \right\|^2 = \left(\Sigma^{-1/2} \sum_{i=1}^n G_{n,i} \right)^T \left(\Sigma^{-1/2} \sum_{i=1}^n G_{n,i} \right) \\ &= \left(\sum_{i=1}^n G_{n,i} \right)^T \Sigma^{-1} \left(\sum_{i=1}^n G_{n,i} \right). \end{aligned} \quad (75)$$

We will use Identity (63) to estimate (75). Let $\vartheta^* = (\vartheta_1^*, \dots, \vartheta_m^*)$ be such that $\|\vartheta^*\| = 1$ and

$$\left(\sum_{i=1}^n G_{n,i} \right)^T \vartheta^* = \max_{\|\vartheta\|=1} \left(\sum_{i=1}^n G_{n,i} \right)^T \vartheta. \quad (76)$$

Observe that

$$\max_{\|\vartheta\|=1} \left(\sum_{i=1}^n G_{n,i} \right)^T \vartheta = \left(\sum_{l=1}^m \left(\sum_{i=1}^n \sqrt{\lambda_l} (\bar{g}_l(X_i) - E\bar{g}_l(X_i)) \right)^2 \right)^{1/2}. \quad (77)$$

By (66),

$$\begin{aligned} (\vartheta^*)^T \Sigma \vartheta^* &= (1 + o(1)) \sum_{l=1}^m (\vartheta_l^*)^2 \lambda_l L_l(z_{n,l}) \\ &\leq (1 + o(1)) \max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l}) \end{aligned} \quad (78)$$

because $\|\vartheta^*\| = 1$. By Identity (63) and by (76)–(78),

$$\left(\sum_{i=1}^n G_{n,i} \right)^T \Sigma^{-1} \left(\sum_{i=1}^n G_{n,i} \right) \geq \frac{\sum_{l=1}^m \left(\sum_{i=1}^n \sqrt{\lambda_l} (\bar{g}_l(X_i) - E\bar{g}_l(X_i)) \right)^2}{(1 + \varepsilon) \max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l})}. \quad (79)$$

By (74), (75) and (79), with the application of (16) and (19),

$$\begin{aligned} I_4 &\leq P\left(\frac{\sum_{l=1}^m \lambda_l \left(\sum_{i=1}^n (\bar{g}_l(X_i) - E\bar{g}_l(X_i))\right)^2}{\max_{1 \leq l \leq m} \lambda_l L_l(z_{n,l})} \geq (1 - C_1\varepsilon)(1 - 2\varepsilon)^3 n x_n^2\right) \\ &\leq P\left(\left(\sum_{i=1}^n G_{n,i}\right)^T \Sigma^{-1} \left(\sum_{i=1}^n G_{n,i}\right) \geq \frac{(1 - C_1\varepsilon)(1 - 2\varepsilon)^3 n x_n^2}{1 + \varepsilon}\right) \\ &= P\left(\|S_n\|^2 \geq \frac{(1 - C_1\varepsilon)(1 - 2\varepsilon)^3 n x_n^2}{1 + \varepsilon}\right) \\ &\leq \exp\left(-\frac{(1 - C_1\varepsilon)(1 - 2\varepsilon)^4 x_n^2}{2(1 + \varepsilon)}\right). \end{aligned}$$

□

Since ε is arbitrary, then the upper bound of Theorem 1 follows from (22) and the estimates of I_1 , I_2 , I_3 and I_4 .

3. The Lower Bound of Theorem 1

Let $0 < \varepsilon < 1$ be sufficiently small. For $1 \leq m < \infty$ that is sufficiently large, by (26), $\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 = \max_{1 \leq l \leq m} \lambda_l V_{n,l}^2$. Together with (17), we have

$$\begin{aligned} & P(W_n \geq (1 - \varepsilon)x_n^2) \\ &= P\left(\frac{\sum_{l=1}^{\infty} \lambda_l \left(\left\{\sum_{i=1}^n g_l(X_i)\right\}^2 - \sum_{i=1}^n g_l^2(X_i)\right)}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq (1 - \varepsilon)x_n^2\right) \\ &\geq P\left(\frac{\sum_{l=1}^m \lambda_l \left\{\sum_{i=1}^n g_l(X_i)\right\}^2}{\max_{1 \leq l \leq m} \lambda_l V_{n,l}^2} \geq x_n^2\right). \end{aligned} \quad (80)$$

Let $\tilde{g}_l(X_i)$ be the random variable with distribution which is of the distribution of $g_l(X_i)$ conditioned on $|g_l(X_i)| \leq z_{n,l}$. Define $\tilde{Y}_{n,l} = \sum_{i=1}^n \tilde{g}_l(X_i)$ and $\tilde{V}_{n,l}^2 = \sum_{i=1}^n \tilde{g}_l^2(X_i)$. By the definition of $L_l(x)$ and (12),

$$\begin{aligned} E\tilde{g}_l^2(X_i) &= E\tilde{g}_l^2(X_i) / P(|g_l(X_i)| \leq z_{n,l}) \\ &= L_l(z_{n,l}) / P(|g_l(X_i)| \leq z_{n,l}) = L_l(z_{n,l})(1 + o(1)). \end{aligned}$$

Notice that (13) implies $E\tilde{g}_l(X_1) = o(L_l(z_{n,l})/z_{n,l})$. Then, we have

$$\begin{aligned} \sigma_l^2 &:= E(\tilde{Y}_{n,l} - E\tilde{Y}_{n,l})^2 = nE(\tilde{g}_l(X_1) - E\tilde{g}_l(X_1))^2 \\ &= nE\tilde{g}_l^2(X_1)(1 + o(1)) = nL_l(z_{n,l})(1 + o(1)) \end{aligned} \quad (81)$$

and

$$E\tilde{V}_{n,l}^2 = nL_l(z_{n,l})(1 + o(1)).$$

Then, for $0 < \delta < 1$,

$$\begin{aligned} & P\left(\frac{\sum_{l=1}^m \lambda_l \left\{\sum_{i=1}^n g_l(X_i)\right\}^2}{\max_{1 \leq l \leq m} \lambda_l V_{n,l}^2} \geq x_n^2\right) \\ &\geq P\left(\frac{\sum_{l=1}^m \lambda_l \left\{\sum_{i=1}^n g_l(X_i)\right\}^2}{\max_{1 \leq l \leq m} \lambda_l V_{n,l}^2} \geq x_n^2, \max_{1 \leq i \leq n} |g_l(X_i)| \leq z_{n,l}, 1 \leq l \leq m\right) \\ &= P\left(\frac{\sum_{l=1}^m \lambda_l \left\{\sum_{i=1}^n \tilde{g}_l(X_i)\right\}^2}{\max_{1 \leq l \leq m} \lambda_l \tilde{V}_{n,l}^2} \geq x_n^2\right) P\left(\max_{1 \leq i \leq n} |g_l(X_i)| \leq z_{n,l}, 1 \leq l \leq m\right) \\ &\geq P\left(\frac{\sum_{l=1}^m \lambda_l \left\{\sum_{i=1}^n \tilde{g}_l(X_i)\right\}^2}{\max_{1 \leq l \leq m} \lambda_l \tilde{V}_{n,l}^2} \geq x_n^2, \tilde{V}_{n,l}^2 \leq (1 + 2\delta)\sigma_l^2, 1 \leq l \leq m\right) \\ &\quad \times P\left(\max_{1 \leq i \leq n} |g_l(X_i)| \leq z_{n,l}, 1 \leq l \leq m\right) \\ &\geq P\left(\frac{\sum_{l=1}^m \lambda_l \left\{\sum_{i=1}^n \tilde{g}_l(X_i)\right\}^2}{\max_{1 \leq l \leq m} \lambda_l \sigma_l^2} \geq (1 + 2\delta)x_n^2\right) \\ &\quad \times P\left(\max_{1 \leq i \leq n} |g_l(X_i)| \leq z_{n,l}, 1 \leq l \leq m\right) - \sum_{l=1}^m P(\tilde{V}_{n,l}^2 \geq (1 + 2\delta)\sigma_l^2). \end{aligned} \quad (82)$$

Without the loss of generality, assume that $\max_{1 \leq l \leq m} \lambda_l \sigma_l^2 = \lambda_1 \sigma_1^2$. Then,

$$\begin{aligned} & P\left(\frac{\sum_{l=1}^m \lambda_l \{\sum_{i=1}^n \tilde{g}_l(X_i)\}^2}{\max_{1 \leq l \leq m} \lambda_l \sigma_l^2} \geq (1+2\delta)x_n^2\right) \\ & \geq P\left(\frac{\lambda_1 \{\sum_{i=1}^n \tilde{g}_1(X_i)\}^2}{\lambda_1 \sigma_1^2} \geq (1+2\delta)x_n^2\right) \\ & \geq P\left(\sum_{i=1}^n \tilde{g}_1(X_i) \geq (1+2\delta)^{1/2} \sigma_1 x_n\right). \end{aligned}$$

Recall (15) and (81). Take $c = 1/x_n$; thus, we have $|\tilde{g}_1(X_1)| \leq z_{n,1} = c\sigma_1$. Therefore, by Theorem 5.2.2 in Stout [24], for any $\gamma > 0$, we have

$$P\left(\sum_{i=1}^n \tilde{g}_1(X_i) \geq (1+2\delta)^{1/2} \sigma_1 x_n\right) \geq \exp(-(x_n^2/2)(1+2\delta)(1+\gamma)). \quad (83)$$

On the other hand,

$$\begin{aligned} & P\left(\max_{1 \leq i \leq n} |g_l(X_i)| \leq z_{n,l}, 1 \leq l \leq m\right) = [P(|g_l(X_1)| \leq z_{n,l}, 1 \leq l \leq m)]^n \\ & = [1 - P(|g_l(X_1)| \geq z_{n,l}, \exists 1 \leq l \leq m)]^n \geq [1 - \sum_{l=1}^m P(|g_l(X_1)| \geq z_{n,l})]^n \\ & \geq \exp(-2n \sum_{l=1}^m P(|g_l(X_1)| \geq z_{n,l})) = \exp(-o(x_n^2)). \end{aligned} \quad (84)$$

We apply the following exponential inequality (see Lemma 2.1, Csörgő, Lin and Shao [25]; see also Pruitt [26] and Griffin and Kuelbs [1]) for the rest of the proof.

Lemma 3. Let $\xi, \xi_1, \dots, \xi_n$ be i.i.d. random variables. Then, for any $b, v, s > 0$,

$$P\left(\left|\sum_{i=1}^n (\xi_i I(|\xi_i| \leq b) - E\xi_i I(|\xi_i| \leq b))\right| \geq \frac{ve^v n E\xi_i^2 I(|\xi_i| \leq b)}{2b} + \frac{sb}{v}\right) \leq 2e^{-s}.$$

By (14),

$$E\tilde{g}_1^4(X_i) = o(z_{n,1}^2 L_1(z_{n,1})). \quad (85)$$

In Lemma 3, we take $\xi_i = \tilde{g}_1^2(X_i)$, $s = x_n^2$, $b = z_{n,1}^2$ and $v = 1/\delta$. Notice that $sb/v = \delta\sigma_1^2$ and $\frac{ve^v n E\xi_i^2 I(|\xi_i| \leq b)}{2b} = o(\sigma_1^2)$ by (81) and (85). Then,

$$\begin{aligned} & P(\tilde{V}_{n,1}^2 \geq (1+2\delta)\sigma_1^2) = P(\tilde{V}_{n,1}^2 - E\tilde{V}_{n,1}^2 \geq (1+2\delta)\sigma_1^2 - E\tilde{V}_{n,1}^2) \\ & \leq P\left(\sum_{i=1}^n (\tilde{g}_1^2(X_i) - E\tilde{g}_1^2(X_i)) \geq \delta(1+\delta)\sigma_1^2\right) \\ & \leq 2\exp(-x_n^2). \end{aligned} \quad (86)$$

Combining (80), (82), (83), (84) and (86) and letting $\lambda, \delta \rightarrow 0$, we have

$$P(W_n \geq (1-\varepsilon)x_n^2) \geq \exp(-x_n^2/2).$$

4. The Upper Bound of Theorem 2

Lemma 4 (Lemma 2.3 of Giné, Kwapień, Latała, and Zinn [15]). *There exists a universal constant $C_3 < \infty$ such that for any kernel h and any two sequences of i.i.d. random variables, we have*

$$P\left(\max_{k \leq m, l \leq n} \left| \sum_{i \leq k, j \leq l} h(X_i, Y_j) \right| \geq t\right) \leq C_3 P\left(\left| \sum_{i \leq m, j \leq n} h(X_i, Y_j) \right| \geq t/C_3\right)$$

for all $m, n \in \mathbb{N}$ and all $t > 0$.

Proposition 5. *Under the assumptions of Theorem 1,*

$$\limsup_{n \rightarrow \infty} \frac{\sum_{l=1}^{\infty} \lambda_l (\sum_{i=1}^n g_l(X_i))^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \log \log n} \leq 2 \text{ a.s.}$$

Consequently,

$$\limsup_{n \rightarrow \infty} \frac{W_n}{\log \log n} \leq 2 \text{ a.s.}$$

Proof. Let $x_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $\theta > 1$ with $\theta - 1$ be sufficiently small. For any positive integer $k \in (n, \theta n]$, via a similar idea as in (19) with $0 < \eta < 1$,

$$\begin{aligned} & P\left(\max_{n < k \leq \theta n} \frac{\sum_{l=1}^{\infty} \lambda_l (\sum_{i=1}^k g_l(X_i))^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \geq 2(1 + \eta)^3 x_n^2\right) \\ & \leq P\left(\frac{\sum_{l=1}^{\infty} \lambda_l (\sum_{i=1}^n g_l(X_i))^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \geq 2(1 - \eta)(1 + \eta)^3 x_n^2\right) \\ & \quad + P\left(\max_{n < k \leq \theta n} \frac{\sum_{l=1}^{\infty} \lambda_l (\sum_{i=n+1}^k g_l(X_i))^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \geq \frac{\eta^2(1 + \eta)^3 x_n^2}{2}\right) \\ & := H_1 + H_2. \end{aligned} \tag{87}$$

Notice that (10) implies (8). By (17) and the upper bound of Theorem 1,

$$H_1 \leq \exp\left(-(1 - \eta)^{3/2}(1 + \eta)^3 x_n^2\right). \tag{88}$$

Let $0 < \delta < 1$ be a sufficiently small constant. By (19),

$$\begin{aligned} H_2 & \leq P\left(2m \max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \leq (1 - \eta)n \sum_{1 \leq l < \infty} \lambda_l L_l(z_{n,l})\right) \\ & \quad + P\left(\max_{n < k \leq \theta n} \frac{\sum_{l=1}^{\infty} \lambda_l (\sum_{i=n+1}^k g_l(X_i) I(|g_l(X_i)| > \delta \eta z_{n,l}))^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \geq \frac{\eta^4(1 + \eta)^3 x_n^2}{2}\right) \\ & \quad + P\left(\frac{\max_{n < k \leq \theta n} \sum_{l=1}^{\infty} \lambda_l (\sum_{i=n+1}^k g_l(X_i) I(|g_l(X_i)| \leq \delta \eta z_{n,l}))^2}{n \sum_{1 \leq l < \infty} \lambda_l L_l(z_{n,l})} \geq \frac{(1 - \eta)(1 - 2\eta)\eta^2(1 + \eta)^3 x_n^2}{4m}\right) \\ & := H_{2,1} + H_{2,2} + H_{2,3}. \end{aligned} \tag{89}$$

By (24) in Proposition 1,

$$H_{2,1} \leq \exp(-2x_n^2). \quad (90)$$

By the Cauchy–Schwarz inequality, for each k ,

$$\begin{aligned} & \frac{\sum_{l=1}^{\infty} \lambda_l \left\{ \sum_{i=n+1}^k |g_l(X_i)| I(|g_l(X_i)| > \delta \eta z_{n,l}) \right\}^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \\ & \leq \frac{\sum_{l=1}^{\infty} \lambda_l \sum_{i=n+1}^k g_l^2(X_i) \sum_{i=n+1}^k I(|g_l(X_i)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2}. \end{aligned} \quad (91)$$

By (17), for some m value that is sufficiently large, the sum of the diagonal terms is as follows:

$$\frac{\sum_{l=1}^{\infty} \lambda_l \sum_{i=n+1}^k g_l^2(X_i) I(|g_l(X_i)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \leq m. \quad (92)$$

By (91) and (92),

$$\begin{aligned} H_{2,2} & \leq P \left(\max_{n < k \leq \theta n} \frac{\sum_{n+1 \leq i \neq j \leq k} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{i=n+1}^k g_l^2(X_i)} \geq \frac{\eta^5(1+\eta)^3 x_n^2}{2} \right) \\ & \leq P \left(\max_{n < k \leq \theta n} \frac{\sum_{n+1 \leq i < j \leq k} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{i=n+1}^k g_l^2(X_i)} \geq \frac{\eta^5(1+\eta)^3 x_n^2}{4} \right) \\ & + P \left(\max_{n < k \leq \theta n} \frac{\sum_{n+1 \leq j < i \leq k} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{i=n+1}^k g_l^2(X_i)} \geq \frac{\eta^5(1+\eta)^3 x_n^2}{4} \right) \\ & \leq P \left(\max_{n < k \leq \theta n} \sum_{n+2 \leq j \leq k} \frac{\sum_{n+1 \leq i < j} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{n+1 \leq i < j} g_l^2(X_i)} \geq \frac{\eta^5(1+\eta)^3 x_n^2}{4} \right) \\ & + P \left(\max_{n < k \leq \theta n} \sum_{n+1 \leq j < k} \frac{\sum_{j < i \leq k} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{j < i \leq k} g_l^2(X_i)} \geq \frac{\eta^5(1+\eta)^3 x_n^2}{4} \right) \\ & = H_{2,2,1} + H_{2,2,2}. \end{aligned} \quad (93)$$

Let

$$\phi_j = \frac{\sum_{n+1 \leq i < j} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{n+1 \leq i < j} g_l^2(X_i)}.$$

Then, for any constant $t > 0$,

$$\begin{aligned} H_{2,2,1} & \leq \left(\sum_{n+2 \leq j \leq [\theta n]} \frac{\sum_{n+1 \leq i < j} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{n+1 \leq i < j} g_l^2(X_i)} \geq \frac{\eta^5(1+\eta)^3 x_n^2}{4} \right) \\ & \leq E e^{t \sum_{j=n+2}^{[\theta n]} \phi_j} e^{-t \eta^5(1+\eta)^3 x_n^2/4}. \end{aligned} \quad (94)$$

Let E_j be the expectation of X_j for $n+2 \leq j \leq [\theta n]$. Then,

$$E e^{t \sum_{j=n+2}^{[\theta n]} \phi_j} = E(e^{t \sum_{j=n+2}^{[\theta n]-1} \phi_j} E_{[\theta n]} e^{t \phi_{[\theta n]}}). \quad (95)$$

Since $|e^s - 1| \leq e^{0 \vee s} |s|$ for any $s \in \mathbb{R}$ and $0 \leq \phi_{[\theta n]} \leq m$ for some m value that is sufficiently large, then

$$\begin{aligned} |E_{[\theta n]} e^{t\phi_{[\theta n]}} - 1| &\leq e^{mt} t E_{[\theta n]} \phi_{[\theta n]} \\ &= \frac{e^{mt} t \sum_{n+1 \leq i < [\theta n]} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) P(|g_l(X_{[\theta n]})| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{n+1 \leq i < [\theta n]} g_l^2(X_i)}. \end{aligned}$$

By (12) and (15), we have $P(|g_l(X_{[\theta n]})| > \delta \eta z_{n,l}) = o(x_n^2/n)$. Then, together with (17),

$$E_{[\theta n]} e^{t\phi_{[\theta n]}} = 1 + o(x_n^2/n) = e^{o(x_n^2/n)}. \quad (96)$$

Applying (96) to (95), we have

$$E e^{t \sum_{j=2}^{[\theta n]} \phi_j} = e^{o(x_n^2/n)} E e^{t \sum_{j=n+2}^{[\theta n]-1} \phi_j}.$$

Similarly,

$$\begin{aligned} E e^{t \sum_{j=n+2}^{[\theta n]-1} \phi_j} &= E(e^{t \sum_{j=n+2}^{[\theta n]-2} \phi_j} E_{n-1} e^{t\phi_{[\theta n]-1}}) \\ &= e^{o(x_n^2/n)} E e^{t \sum_{j=n+2}^{[\theta n]-2} \phi_j}. \end{aligned}$$

Continue this process from $X_{[\theta n]}$ to X_{n+1} . Thus, we conclude that

$$E e^{t \sum_{j=n+2}^{[\theta n]} \phi_j} = e^{[(\theta-1)n] \times o(x_n^2/n)} = e^{o(x_n^2)}. \quad (97)$$

Applying (97) to (94) and letting $t = 8/(\eta^5(1+\eta)^3)$, we have

$$H_{2,2,1} \leq \exp(-2x_n^2). \quad (98)$$

To estimate $H_{2,2,2}$, let

$$\psi_{j,k} = \frac{\sum_{j < i \leq k} \sum_{l=1}^{\infty} \lambda_l g_l^2(X_i) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{j < i \leq k} g_l^2(X_i)}.$$

Then, for any constant $t > 0$,

$$\begin{aligned} H_{2,2,2} &\leq P\left(\sum_{n+1 \leq j < [\theta n]} \max_{j < k \leq \theta n} \psi_{j,k} \geq \frac{\eta^5(1+\eta)^3 x_n^2}{4}\right) \\ &\leq E e^{t \sum_{j=n+1}^{[\theta n]-1} \max_{j < k \leq \theta n} \psi_{j,k}} e^{-t \eta^5(1+\eta)^3 x_n^2/4}. \end{aligned} \quad (99)$$

Let E_j be the expectation of X_j for $n+1 \leq j \leq [\theta n]$. Note $k > j$. Then,

$$\begin{aligned} &E e^{t \sum_{j=n+1}^{[\theta n]-1} \max_{j < k \leq \theta n} \psi_{j,k}} \\ &= E(e^{t \sum_{j=n+2}^{[\theta n]-1} \max_{j < k \leq \theta n} \psi_{j,k}} E_{n+1} e^{t \max_{n+1 < k \leq \theta n} \psi_{n+1,k}}). \end{aligned} \quad (100)$$

Observe that

$$\psi_{j,k} = \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{j < i \leq k} g_l^2(X_i)\right) I(|g_l(X_j)| > \delta \eta z_{n,l})}{\max_{1 \leq l < \infty} \lambda_l \sum_{j < i \leq k} g_l^2(X_i)}. \quad (101)$$

Then, by (17), $0 \leq \psi_{n+1,k} \leq m$ for some m value that is sufficiently large. Since $|e^s - 1| \leq e^{0 \vee s}|s|$ for any $s \in \mathbb{R}$,

$$|E_{n+1} e^{t \max_{n+1 < k \leq \theta n} \psi_{n+1,k}} - 1| \leq e^{mt} E_{n+1} \max_{n+1 < k \leq \theta n} \psi_{n+1,k}. \quad (102)$$

Under Assumption (10), for each $l \in [1, \infty)$,

$$\lambda_l V_{n,l}^2 = \sum_{i=1}^n \lambda_l g_l^2(X_i) \leq c_l \sum_{v=1}^{\infty} \sum_{i=1}^n \lambda_v g_v^2(X_i) = c_l \sum_{v=1}^{\infty} \lambda_v V_{n,v}^2.$$

Recall that (17); then, for each $l \in [1, \infty)$,

$$\frac{\lambda_l V_{n,l}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2} \leq \frac{m c_l}{1 - \varepsilon}.$$

Hence, by (101),

$$\psi_{j,k} \leq \frac{m}{1 - \varepsilon} \sum_{l=1}^{\infty} c_l I(|g_l(X_j)| > \delta \eta z_{n,l}).$$

Then,

$$E_{n+1} \max_{n+1 < k \leq \theta n} \psi_{n+1,k} \leq \frac{m}{1 - \varepsilon} \sum_{l=1}^{\infty} c_l P(|g_l(X_{n+1})| > \delta \eta z_{n,l}).$$

By (12) and (15), we have $P(|g_l(X_{n+1})| > \delta \eta z_{n,l}) = o(x_n^2/n)$. Then, together with (10),

$$E_{n+1} \max_{n+1 < k \leq \theta n} \psi_{n+1,k} = o(x_n^2/n). \quad (103)$$

Then, by (102) and (103),

$$E_{n+1} e^{t \max_{n+1 < k \leq \theta n} \psi_{n+1,k}} = 1 + o(x_n^2/n) = e^{o(x_n^2/n)}.$$

Continue this process from $j = n + 2$ to $j = [\theta n] - 1$ and by (100),

$$E e^{t \sum_{j=n+1}^{[\theta n]-1} \max_{j < k \leq \theta n} \psi_{j,k}} = e^{o(x_n^2)}. \quad (104)$$

Applying (104) to (99) and letting $t = 8/(\eta^5(1 + \eta)^3)$, we have

$$H_{2,2,2} \leq \exp(-2x_n^2). \quad (105)$$

By (93), (98) and (105),

$$H_{2,2} \leq 2 \exp(-2x_n^2). \quad (106)$$

By the definition of $H_{2,3}$ in (89), and by Lemma 4, there is a constant $0 < C' < \infty$ such that

$$H_{2,3} \leq C' P \left(\frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=n+1}^{[\theta n]} g_l(X_i) I(|g_l(X_i)| \leq \delta \eta z_{n,l}) \right)^2}{n \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})} \geq \frac{(1 - 3\eta)\eta^2 x_n^2}{4mC'} \right).$$

Similar to (36),

$$\begin{aligned} H_{2,3} &\leq C' P \left(\frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=n+1}^{[\theta n]} \{g_l(X_i) I(|g_l(X_i)| \leq \delta \eta z_{n,l}) - E g_l(X_i) I(|g_l(X_i)| \leq \delta \eta z_{n,l})\} \right)^2}{n \sum_{l=1}^{\infty} \lambda_l L_l(z_{n,l})} \right)^2 \\ &\geq \frac{(1-3\eta)^2 \eta^2 x_n^2}{4mC'} \end{aligned}$$

By the decoupling version of (40) in Proposition 3,

$$H_{2,3} \leq C' \exp(-2x_n^2). \quad (107)$$

Combining (89), (90), (106) and (107), we have

$$H_2 \leq (3 + C') \exp(-2x_n^2). \quad (108)$$

By (87), (88) and (108),

$$\begin{aligned} &P \left(\max_{n < k \leq \theta n} \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^k g_l(X_i) \right)^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \geq 2(1+\eta)^3 x_n^2 \right) \\ &\leq \exp(-(1-\eta)^{7/4} (1+\eta)^3 x_n^2). \end{aligned}$$

Let $n = [\theta^j]$ for some $j \in \mathbb{N}$. We have

$$\begin{aligned} &P \left(\max_{\theta^j < k \leq \theta^{j+1}} \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^k g_l(X_i) \right)^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \geq 2(1+\eta)^3 x_{[\theta^j]}^2 \right) \\ &\leq \exp(-(1-\eta)^{7/4} (1+\eta)^3 x_{[\theta^j]}^2). \end{aligned}$$

Let $x_n^2 = \log \log n$. Then,

$$\begin{aligned} &\sum_{j=1}^{\infty} P \left(\max_{\theta^j < k \leq \theta^{j+1}} \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^k g_l(X_i) \right)^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2 \log \log k} \geq 2(1+\eta)^3 \right) \\ &\leq \sum_{j=1}^{\infty} P \left(\max_{\theta^j < k \leq \theta^{j+1}} \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^k g_l(X_i) \right)^2}{\max_{1 \leq l < \infty} \lambda_l V_{k,l}^2} \geq 2(1+\eta)^3 x_{[\theta^j]}^2 \right) \\ &\leq \sum_{j=1}^{\infty} \exp(-(1-\eta)^{7/4} (1+\eta)^3 \log \log [\theta^j]) \\ &\leq K \sum_{j=1}^{\infty} \exp(-(1-\eta)^2 (1+\eta)^3 \log j) < \infty. \end{aligned}$$

By the Borel–Cantelli lemma,

$$\limsup_{n \rightarrow \infty} \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^n g_l(X_i) \right)^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \log \log n} \leq 2 \text{ a.s.}$$

□

5. The Lower Bound of Theorem 2

Proof. By the definition of W_n ,

$$\begin{aligned}\frac{W_n}{\log \log n} &= \frac{\sum_{l=1}^{\infty} \lambda_l \left(\left(\sum_{i=1}^n g_l(X_i) \right)^2 - \sum_{i=1}^n g_l^2(X_i) \right)}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \log \log n} \\ &= \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^n g_l(X_i) \right)^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \log \log n} - \frac{\sum_{l=1}^{\infty} \lambda_l V_{n,l}^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \log \log n}.\end{aligned}$$

By (17), $\sum_{l=1}^{\infty} \lambda_l V_{n,l}^2 / (\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \log \log n) \leq m / ((1 - \varepsilon) \log \log n) \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\frac{W_n}{\log \log n} \geq \frac{\sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^n g_l(X_i) \right)^2}{\max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 \log \log n}.$$

Then,

$$\frac{W_n}{\log \log n} \geq \sum_{k=1}^{\infty} \frac{\left(\sum_{i=1}^n g_k(X_i) \right)^2}{V_{n,k}^2 \log \log n} I_{k=\min\{j: \max_{1 \leq l < \infty} \lambda_l V_{n,l}^2 = \lambda_j V_{n,j}^2\}}.$$

Hence, by (6),

$$\limsup_{n \rightarrow \infty} \frac{W_n}{\log \log n} \geq 2 \text{ a.s.}$$

□

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