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Local Unitary Equivalence of Quantum States Based on the Tensor Decompositions of Unitary Matrices

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Abstract: Since two quantum states that are local unitary (LU) equivalent have the same amount of entanglement, it is meaningful to find a practical method to determine the LU equivalence of given quantum states. In this paper, we present a valid process to find the unitary tensor product decomposition for an arbitrary unitary matrix. Then, by using this process, the conditions for determining the local unitary equivalence of quantum states are obtained. A numerical verification is carried out, which shows the practicability of our protocol. We also present a property of LU invariants by using the universality of quantum gates which can be used to construct the complete set of LU invariants.

Keywords: local unitary equivalence; quantum states; quantum entanglement

1. Introduction

Quantum entanglement is one of the most extraordinary features in quantum information science, and quantum entangled states have become the most important physical resource [1]. In particular, multipartite quantum entanglement plays key roles in the rapidly developing field of quantum information science, for example, in one-way quantum computing, quantum error correction, and quantum secret sharing [2,3]. However, it is more difficult to understand multipartite mixed states with nonlocal properties. Fortunately, the entanglement (or the local hidden variable models) of quantum states remains unchanged under local unitary (LU) transformations. In addition, local operations and classical communication (LOCC) equivalence states are interconvertible also by local unitary transformations [4]. Therefore, it is very important to determine whether or not two states are LU equivalent.

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Definition 1. *Let ρ and $\tilde{\rho}$ be two states in general $H_1 \otimes H_2 \otimes \cdots H_N$ quantum systems with dim $H_i = d_i$, $i = 1, 2, \ldots, N$. They are LU equivalent if*

$$\tilde{\rho} = (U_1 \otimes U_2 \otimes \cdots U_N)\rho(U_1 \otimes U_2 \otimes \cdots U_N)^\dagger$$

for some unitary operators U_i, $i = 1, 2, \ldots, N$, where $\dagger$ denotes transpose and conjugate.

At present, there are many results on LU equivalence and LU invariants [4–24]. In this paper, we present a practical method to find the unitary tensor decomposition of an arbitrary unitary matrix. Then, we derive a local unitary equivalence strategy for arbitrary quantum states with non-degenerate density matrices from the point of view of block matrix and unitary matrix tensor decomposition. Exact examples are analyzed numerically. We also present a property of LU invariants which can lead to the construction of a complete set of LU invariants.

2. Unitary Tensor Decomposition of an Arbitrary Unitary Matrix

In this section, we will present a sufficient and necessary condition for the existence of the unitary tensor decomposition of an arbitrary unitary matrix. We start with the bipartite decomposition.

2.1. The Decomposition Scheme of $W = U \otimes V$

Set $W = U \otimes V$, with U and V as 2×2 and $d \times d$ matrices, respectively. Set

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

with u_{kl} as the entries, and

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{1}$$

as the block representation of W. Then, according to the tensor product of matrices, one obtains

$$W = U \otimes V = \begin{bmatrix} u_{11}V & u_{12}V \\ u_{21}V & u_{22}V \end{bmatrix}$$

We have $W_{kl}W_{mn}^{\dagger} = u_{kl}VV^{\dagger}u_{mn}^{\dagger}$, where $k, l, m, n \in \{1, 2\}$. Furthermore, we can obtain $tr(W_{kl}W_{kl}^{\dagger}) = d|u_{kl}|^2$, when $k = m$, $l = n$. Thus,

$$u_{kl} = \sqrt{\frac{tr(W_{kl}W_{kl}^{\dagger})}{d}} \cdot e^{i\theta_{kl}}, \tag{2}$$

where $e^{i\theta_{kl}}$ is the complex phase of u_{kl}, $k, l = \{1, 2\}$.

Thus, we set

$$W = \begin{bmatrix} \tilde{u}_{11} & \tilde{u}_{12} \\ \tilde{u}_{21} & \tilde{u}_{22} \end{bmatrix} \otimes e^{i\theta_{11}}V \equiv \tilde{U} \otimes \tilde{V}, \tag{3}$$

with $\tilde{U} = e^{-i\theta_{11}}U$ and $\tilde{V} = e^{i\theta_{11}}V$. Without loss of generality, we set $tr(W_{11}W_{11}^{\dagger}) \neq 0$. Combining (1) and (3), we can obtain

$$\tilde{V} = e^{i\theta_{11}} \cdot V = \frac{W_{11}}{\tilde{u}_{11}}, \tag{4}$$

where $\tilde{u}_{11} = \sqrt{\frac{tr(W_{11}W_{11}^{\dagger})}{d}}$. We can further derive that for the remaining entries of $\tilde{u}_{kl}$ with $kl = \{12, 21, 22\}$,

$$\tilde{u}_{kl} = \frac{(W_{kl})_{11}}{\tilde{v}_{11}}, \tag{5}$$

where $\tilde{v}_{11}$ is the $(1, 1)$ entry of $\tilde{V}$, which is assumed to be nonzero. We also obtain that matrices W_{kl} must be the scalar multiplications of $\tilde{V}$ for any $k, l \in \{1, 2\}$.

For unitary matrix $W = U \otimes V$, where U and V are arbitrary $d_1 \times d_1$ and $d_2 \times d_2$ unitary matrices, we can also find unitary matrices $\tilde{U}$ and $\tilde{V}$ such that $W = \tilde{U} \otimes \tilde{V}$ by using the same method.

To summarize, let W be any $d_1d_2 \times d_1d_2$ unitary matrix with block representation $W = (W_{kl})$, where $k, l \in \{1, 2, \cdots, d_1\}$ and W_{kl} are $d_2 \times d_2$ matrices. According to the above analysis, we directly derive the following theorem.

Theorem 1. *If W_{kl} is a scalar multiplication of a unitary matrix for any $k, l \in \{1, 2, \cdots, d_1\}$, then we can always derive the tensor product decomposition $W = \tilde{U} \otimes \tilde{V}$, where $\tilde{U}$ and $\tilde{V}$ are $d_1 \times d_1$ and $d_2 \times d_2$ unitary matrices, respectively. We can always select one of $tr(W_{kl}W_{kl}^\dagger) \neq 0$. Without loss of generality, we set $tr(W_{11}W_{11}^\dagger) \neq 0$. The entries of $\tilde{U}$ and $\tilde{V}$ are given by*

$$\begin{aligned}
\tilde{u}_{11} &= \sqrt{\frac{tr(W_{11}W_{11}^\dagger)}{d_1}}; \\
\tilde{V} &= \frac{W_{11}}{\tilde{u}_{11}}; \\
\tilde{u}_{kl} &= \frac{(W_{kl})_{11}}{\tilde{v}_{11}}, k, l \in \{1, 2, \cdots d_1\}, kl \neq 11.
\end{aligned} \tag{6}$$

2.2. The Decomposition Scheme of $W = Q \otimes U \otimes V$

To simplify the expression, we set Q, U and V as 2×2 matrices with entries q_{kl}, u_{kl} and v_{kl}, respectively. We then consider $W = Q \otimes U \otimes V = Q \otimes P$, and set $P = U \otimes V$. We obtain

$$W = \begin{bmatrix} q_{11}P & q_{12}P \\ q_{21}P & q_{22}P \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

where W_{kl} are the block matrices of W, and

$$P = \begin{bmatrix} u_{11}V & u_{12}V \\ u_{21}V & u_{22}V \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}, V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}.$$

We can obtain $W_{kl}W_{mn}^\dagger = q_{kl}PP^\dagger q_{mn}^\dagger$, and $tr(W_{kl}W_{kl}^\dagger) = 2|q_{kl}|$, for $k, l, m, n \in \{1, 2\}$. Thus,

$$q_{kl} = \sqrt{\frac{tr(W_{kl}W_{kl}^\dagger)}{4}} \cdot e^{i\theta_{kl}},$$

where $e^{i\theta_{kl}}$ is the complex phase of q_{kl}, $k, l \in \{1, 2\}$.

Without loss of generality, one still sets $tr(W_{11}W_{11}^\dagger) \neq 0$. We then set

$$\begin{aligned}
\tilde{q}_{11} &= \sqrt{\frac{tr(W_{11}W_{11}^\dagger)}{4}}; \\
\tilde{P} &= \frac{W_{11}}{\tilde{q}_{11}}; \\
\tilde{q}_{kl} &= \frac{(W_{kl})_{11}}{\tilde{p}_{11}}, kl = \{12, 21, 22\}.
\end{aligned} \tag{7}$$

Thus, we have

$$W = e^{i\theta_{11}}\tilde{Q} \otimes P = \tilde{Q} \otimes e^{i\theta_{11}}P = \tilde{Q} \otimes \tilde{P}.$$

Then, by using the decomposition scheme in Section 2.1 for $\tilde{P}$, one finally obtains

$$W = \tilde{Q} \otimes \tilde{U} \otimes \tilde{V}.$$

One can then derive the decomposition scheme of $W = U_1 \otimes U_2 \otimes \cdots \otimes U_N$ in the same way.

2.3. Numerical Verification

We use the rand() function in MATLAB to generate a random matrix [25,26]. This function can generate random numbers between 0 and 1 according to a uniform distribution.

Then, using the singular value decomposition of this matrix, a random unitary matrix can be obtained.

Example 1. *As an example, let us consider the tensor decomposition of the following unitary matrix:*

$$W = \begin{bmatrix} 0.439154 - 0.260657i & 0.489398 + 0.00346013i & 0.43702 - 0.263586i & 0.489097 + 0i \\ 0.42086 - 0.249799i & -0.510671 - 0.00361053i & 0.418815 - 0.252606i & -0.510356 + 0i \\ 0.438872 - 0.26049i & 0.489085 + 0.00345791i & -0.4373 + 0.263755i & -0.489411 + 0i \\ 0.42059 - 0.249639i & -0.510344 - 0.00360822i & -0.419084 + 0.252768i & 0.510684 + 0i \end{bmatrix},$$

which is generated randomly by U_1 and U_2, i.e., $W = U_1 \otimes U_2$, where

$$U_1 = \begin{bmatrix} 0.7073157717804006 + 0.005000843737265154i & 0.70688003971863 \\ 0.7068623728603339 + 0.004997638129189724i & -0.7073334499706543 \end{bmatrix},$$

and

$$U_2 = \begin{bmatrix} 0.6182373567761775 - 0.3728863629370793i & 0.69190919275723i \\ 0.5924837578103638 - 0.3573532254686829i & -0.721984535137726 \end{bmatrix}.$$

According to the previous analysis, W can be decomposed as $W = U \otimes V$ with

$$U = \begin{bmatrix} 0.7073 + 0.0000i & 0.7069 - 0.0050i \\ 0.7069 - 0.0000i & -0.7073 + 0.0050i \end{bmatrix},$$

and

$$V = \begin{bmatrix} 0.6209 - 0.3685i & 0.6919 + 0.0049i \\ 0.5950 - 0.3532i & -0.7220 - 0.0051i \end{bmatrix}.$$

Example 2. *We consider a 8×8 unitary matrix $W = Y_1 + Y_2 i$, with*

$$Y_1 = \begin{bmatrix} 0.4652 & 0.1795 & 0.4922 & 0.1948 & 0.4231 & 0.1783 & 0.4355 & 0.1881 \\ 0.1932 & -0.430 & 0.2045 & -0.4691 & 0.1758 & -0.4292 & 0.1809 & -0.4529 \\ 0.4890 & 0.1886 & -0.4683 & -0.1854 & 0.4447 & 0.1874 & -0.4144 & -0.1790 \\ 0.2031 & -0.4541 & -0.1945 & 0.4463 & 0.1847 & -0.4511 & -0.1721 & 0.4309 \\ 0.4278 & 0.1650 & 0.4526 & 0.1792 & -0.4601 & -0.1939 & -0.4736 & -0.2046 \\ 0.1777 & -0.3973 & 0.1880 & -0.4313 & -0.1911 & 0.4668 & -0.1967 & 0.4925 \\ 0.4496 & 0.1735 & -0.4307 & -0.1705 & -0.4836 & -0.2038 & 0.4506 & 0.1946 \\ 0.1868 & -0.4176 & -0.1789 & 0.4104 & -0.2009 & 0.4906 & 0.1872 & -0.4686 \end{bmatrix},$$

and

$$Y_2 = \begin{bmatrix} -0.0560 & -0.0753 & -0.0158 & -0.0624 & 0.0814 & -0.0157 & 0.1241 & 0 \\ -0.0233 & 0.1813 & -0.0066 & 0.1501 & 0.0338 & 0.0377 & 0.0516 & 0 \\ -0.0588 & -0.0792 & 0.0151 & 0.0593 & 0.0855 & -0.0165 & -0.1181 & 0 \\ -0.0244 & 0.1906 & 0.0063 & -0.1428 & 0.0355 & 0.0396 & -0.0490 & 0 \\ -0.0515 & -0.0693 & -0.0146 & -0.0573 & -0.0885 & 0.0170 & -0.1350 & 0 \\ -0.0214 & 0.1667 & -0.0060 & 0.1380 & -0.0368 & -0.0410 & -0.0561 & 0 \\ -0.0541 & -0.0728 & 0.0138 & 0.0546 & -0.0930 & 0.0179 & 0.1284 & 0 \\ -0.0225 & 0.1753 & 0.0058 & -0.1313 & -0.0386 & -0.0431 & 0.0533 & 0 \end{bmatrix},$$

which is randomly generated by

$$U_1 = \begin{bmatrix} 0.701056 - 0.224367i & 0.67689 \\ 0.644678 - 0.206324i & -0.736084 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} 0.686641 - 0.06032i & 0.72449 \\ 0.721711 - 0.0634009i & -0.689285 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} 0.888143 + 0.253079i & 0.383606 \\ 0.36892 + 0.105125i & -0.923497 \end{bmatrix}.$$

After numerical verification, the unitary tensor decomposition $W = Q \otimes U \otimes V$ can be obtained with

$$Q = \begin{bmatrix} 0.7361 - 0.0000i & 0.6447 + 0.2063i \\ 0.6769 - 0.0000i & -0.7011 - 0.2244i \end{bmatrix}.$$

$$U = \begin{bmatrix} 0.6893 + 0.0000i & 0.7217 + 0.0634i \\ 0.7245 - 0.0000i & -0.6866 - 0.0603i \end{bmatrix},$$

$$V = \begin{bmatrix} 0.9169 - 0.1103i & 0.3537 - 0.1485i \\ 0.3809 - 0.0458i & -0.8515 + 0.3574i \end{bmatrix}.$$

The Matlab code is supplied in the Supplementary Materials.

3. Determine the LU Equivalence of Non-Degenerate Quantum States

The key to investigating the local unitary equivalence of quantum states lies in the unitary tensor decomposition of the corresponding unitary matrix. In this section, we present a general method to determine the LU equivalence of any pair of non-degenerate quantum states by the unitary tensor decomposition protocol derived in the above section.

Let ρ and $\tilde{\rho}$ be the density matrices of two states in quantum systems $H_1 \otimes H_2 \otimes \cdots H_N$ with $\dim H_i = d$, $i = 1, 2, \ldots, N$. We assume that both ρ and $\tilde{\rho}$ are non-degenerate. We further set that ρ and $\tilde{\rho}$ have the same eigenvalues, which is the necessary condition for the LU equivalence of the two density matrices. Let $\rho = X\Sigma X^\dagger$ and $\tilde{\rho} = Y\Sigma Y^\dagger$ be the spectral decomposition of ρ and $\tilde{\rho}$. Thus, there is a unitary matrix $W = YX^\dagger$ such that

$$\tilde{\rho} = W\rho W^\dagger.$$

To certify that ρ and $\tilde{\rho}$ are local unitary equivalent, one needs to further find unitary matrices U_i, $i = 1, 2, \ldots, N$ such that

$$W = U_1 \otimes U_2 \otimes \cdots U_N.$$

In the following, we consider bipartite quantum systems as an example. The processes of judging the local unitary equivalence of quantum states are as follows:

[1] Check whether the density matrices ρ and σ of quantum states are non-degenerate and whether they have the same eigenvalues;

[2] Find the spectral decompositions $\rho = X\Sigma X^\dagger$ and $\tilde{\rho} = Y\Sigma Y^\dagger$. Compute $W = YX^\dagger$;

[3] Determine whether W can be decomposed into the tensor product of two unitary matrices, such as $W = U \otimes V$.

Example 3. *We consider two quantum states*

$$\rho = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.2743 & 0.0429 & -0.0086 \\ 0 & 0.0429 & 0.2286 & 0.0143 \\ 0 & -0.0086 & 0.0143 & 0.2971 \end{bmatrix},$$

$$\sigma = \begin{bmatrix} 0.2648 & -0.0360 & -0.0228 & -0.0217 \\ -0.0360 & 0.2225 & 0.0212 & -0.0011 \\ -0.0228 & 0.0212 & 0.2369 & -0.0369 \\ -0.0217 & -0.0011 & -0.0369 & 0.2759 \end{bmatrix}$$

with spectral decompositions $\rho = X\Sigma X^\dagger$ *and* $\sigma = Y\Sigma Y^\dagger$*, respectively, where*

$$\Sigma = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix},$$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.507092552837110 & 0.609449400220044 & 0.609449400220044 \\ 0 & 0.845154254728517 & 0.212930374147564 & 0.490280472260179 \\ 0 & -0.169030850945703 & -0.763696329922311 & 0.623054160640762 \end{bmatrix},$$

$$Y = \begin{bmatrix} -0.585389075528045 & -0.098738906989822 & -0.087850024051642 & -0.799907889555417 \\ -0.490973763578605 & -0.730809679765713 & -0.109038680416598 & 0.461489481582411 \\ -0.494334763695753 & 0.621997602780637 & -0.502988683425162 & 0.340227141602823 \\ -0.414605276295110 & 0.263223901540746 & 0.852874740975106 & 0.177257774815572 \end{bmatrix}.$$

Then, the matrix $W = YX^\dagger$ *can be decomposed into the tensor product of two unitary matrices, i.e.,* $W = U \otimes V$*, where*

$$U = \begin{bmatrix} 0.7640, 0.6452 \\ 0.6452, -0.7640 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.7662, -0.6426 \\ -0.6426, 0.7662 \end{bmatrix}.$$

Therefore, the quantum states ρ *and* σ *are local unitary equivalent.*

4. LU Invariants

Let $f(\rho)$ be a function of density matrix $\rho \in H_A \otimes H_B$. In this section, we set $dimH_A = dimH_B = d = 2^n$. If f is an LU invariant, then for any quantum state ρ and unitary matrices U and V, it satisfies

$$f(\rho') = f(\rho),$$

where $\rho' = U \otimes V \rho U^\dagger \otimes V^\dagger$. Such kinds of functions contain polynomial local unitary invariants, rational local unitary invariants and so on [24]. Let G be the set of all "single-qubit" and $CNOT$ gates. According to the universality of quantum gates, an arbitrary unitary operation on n qubits can be implemented using a circuit containing $O(n^2 4^n)$ unitary operators in G. Then, for any unitary matrices U and V, there exist unitary matrices $U_i \in G$ and $V_i \in G$, $i = 1, 2, \dots, k$, such that $U = U_1 \otimes U_2 \otimes \cdots \otimes U_k$ and $V = V_1 \otimes V_2 \otimes \cdots \otimes V_k$, respectively.

Theorem 2. *A function* f *is an LU invariant if and only if* f *is invariable under LU operations of the form* $U_i \otimes I$ *and* $I \otimes V_i$ *for all* $U_i \in G$ *and* $V_i \in G$.

Proof. The "only if" is obvious. For the if part, one has to prove that the following equation

$$f(\rho') = f(U \otimes V \rho U^\dagger \otimes V^\dagger) = f(\rho)$$

holds. Suppose U and V can be rewritten as $U = U_1 \otimes U_2 \otimes \cdots \otimes U_{k_1}$ and $V = V_1 \otimes V_2 \otimes \cdots \otimes V_{k_2}$. One can always set $k_1 = k_2 = k$. Otherwise, the identity matrix I can be used as a complement. We have

$$
\begin{aligned}
f(\rho') &= f(U \otimes V \rho U^\dagger \otimes V^\dagger) \\
&= f((U_1 \otimes V_1)(U_2 \otimes V_2) \cdots (U_k \otimes V_k)\rho(U_k \otimes V_k)^\dagger \cdots (U_2 \otimes V_2)^\dagger (U_1 \otimes V_1)^\dagger) \\
&= f((U_1 \otimes I)(I \otimes V_1)(U_2 \otimes V_2) \cdots (U_k \otimes V_k)\rho(U_k \otimes V_k)^\dagger \cdots (U_2 \otimes V_2)^\dagger (I \otimes V_1)^\dagger (U_1 \otimes I)^\dagger) \\
&= f((I \otimes V_1)(U_2 \otimes V_2) \cdots (U_k \otimes V_k)\rho(U_k \otimes V_k)^\dagger \cdots (U_2 \otimes V_2)^\dagger (I \otimes V_1)^\dagger) \\
&= \cdots \\
&= f(\rho),
\end{aligned}
$$

which ends the proof of the theorem. □

5. Conclusions

In this paper, we have studied the local unitary equivalence of quantum systems from the perspective of unitary matrix tensor decomposition. We have presented a detailed process to find the unitary matrices in the tensor decomposition of an arbitrary tensor-factorable unitary matrix. We have also derived a property of LU invariants that may be used to find a complete set of LU invariants.

It should be noted that our schemes are convenient to discuss the local unitary equivalence when the number and the dimension of the subsystems are small. As quantum systems get more complex, the amount of computations increases exponentially. Therefore, we need to further find more convenient and efficient strategies to judge the local unitary equivalence of multipartite high-dimensional quantum systems.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/e25081139/s1.

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