

Article

# Detecting Tripartite Steering via Quantum Entanglement

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**Abstract:** Einstein-Podolsky-Rosen steering is a kind of powerful nonlocal quantum resource in quantum information processing such as quantum cryptography and quantum communication. Many criteria have been proposed in the past few years to detect steerability, both analytically and numerically, for bipartite quantum systems. We propose effective criteria for tripartite steerability and genuine tripartite steerability of three-qubit quantum states by establishing connections between the tripartite steerability (resp. genuine tripartite steerability) and the tripartite entanglement (resp. genuine tripartite entanglement) of certain corresponding quantum states. From these connections, tripartite steerability and genuine tripartite steerability can be detected without using any steering inequalities. The “complex cost” of determining tripartite steering and genuine tripartite steering can be reduced by detecting the entanglement of the newly constructed states in the experiment. Detailed examples are given to illustrate the power of our criteria in detecting the (genuine) tripartite steerability of tripartite states.

**Keywords:** tripartite steerability; genuine tripartite steerability; tripartite entanglement; genuine tripartite entanglement



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## 1. Introduction

Originally introduced by Schrödinger [1] the Einstein-Podolsky-Rosen (EPR) steering for bipartite systems was considered as a ‘spooky action at distance’ [2] in the sense that one party can steer another distant party’s state instantly. The concept of EPR steering was proposed by Wiseman, Jones, and Doherty in 2007 [3]. Since then the EPR steering has been systematically studied. Many different methods were proposed to detect and quantify the steerability of bipartite quantum states [4–15], together with many applications in quantum information processing tasks including one-sided device-independent quantum key distribution, random generation and one-sided device-independent quantum self-testing of pure quantum states, subchannel discrimination, quantum communication et al. [16–25].

The EPR steering lies between quantum nonlocality and quantum entanglement. A bipartite state is quantum nonlocal if it does not admit a local hidden variable model [26], while it is EPR steerable if it does not admit a hidden state model [3].

Bipartite steering is defined as follows. Alice and Bob share a quantum state  $\rho_{AB}$ . Alice performs black-box measurements  $A$  with outcomes  $a$ , denoted by  $M_A^a$  ( $M_A^a \geq 0 \forall A, a$  and  $\sum_a M_A^a = I \forall A$ , with  $I$  denoting the identity operator). The set of unnormalized conditional states  $\{\delta_A^a\}$  on Bob’s side is called an assemblage. Each element in this assemblage is given by

$$\delta_A^a = \text{Tr}[(M_A^a \otimes I) \cdot \rho_{AB}]. \quad (1)$$

Alice can not steer Bob if  $\delta_A^a$  admits a local hidden state model (LHS), i.e.,  $\delta_A^a$  admits the decomposition

$$\delta_A^a = \sum_{\lambda} p(\lambda) p(a|A, \lambda) \rho_{\lambda}^{\beta}, \quad (2)$$

where  $\lambda$  denotes classical random variable which occurs with probability  $p(\lambda)$  satisfying  $\sum_{\lambda} p(\lambda) = 1$ ,  $p(a|A, \lambda)$  is the probability given by the black-box measurement on Alice’s side,  $\rho_{\lambda}^{\beta}$  are some local hidden states. Bob performs measurement  $B$  with outcomes  $b$ , denoted by  $M_B^b$ , on the assemblage. The joint probability is  $p(a, b|A, B) = \text{Tr}[M_B^b \delta_A^a]$ .  $\rho_{AB}$  is said to be a steerable state from Alice to Bob if  $p(a, b|A, B)$  does not admit a local hidden variable-local hidden state (LHV-LHS) model of the form,

$$p(a, b|A, B) = \sum_{\lambda} p(\lambda) p(a|A, \lambda) p_Q(b|B, \rho_{\lambda}^{\beta}). \tag{3}$$

Different from quantum entanglement and quantum nonlocality, EPR steering is asymmetric in general, which means that Alice can steer Bob but not vice versa for some bipartite quantum states  $\rho_{AB}$  [27]. The bipartite quantum nonlocality and EPR steering can be detected by detecting the EPR steering and quantum entanglement of some newly constructed quantum states, respectively [28–30].

The multipartite steering is an important resource in quantum communication networks [31–33] and in one-sided or two-sided device-independent entanglement detections [34,35]. Some ambiguities exist in the definition of multipartite steering. With respect to the typical spooky action at a distance [31–33], and the semi-device independent entanglement verification scheme [34,35], two different approaches have been introduced to define the multipartite steering [31,34,35]. One approach is to define genuine multipartite steering in terms of the steering under bi-partitions. A tripartite state  $\rho_{ABC}$  is defined to be genuine tripartite steerable if the state does not admit the mixtures of bi-partitions where in each partition (e.g.,  $A|BC$ ) the two-party state (e.g.,  $BC$ ) is allowed to be steerable. Linear inequalities have been derived to detect this kind of genuine multipartite steering [31] and used in experimental demonstrations [32,33].

Another approach to defining tripartite steering and genuine tripartite steering is given as follows [34,35]. Let  $p(a, b, c|A, B, C)$  be the joint probability that Alice, Bob and Charlie perform measurements  $A, B$  and  $C$  with outcomes  $a, b$  and  $c$ , given by measurements operators  $M_A^a, M_B^b$  and  $M_C^c$ , respectively. A quantum state  $\rho_{ABC}$  is said to be tripartite steerable from Alice (untrusted party) to Bob and Charlie (trusted parties) if  $p(a, b, c|A, B, C)$  does not admit a fully LHV-LHS model such that

$$p(a, b, c|A, B, C) = \sum_{\lambda} p(\lambda) p(a|A, \lambda) p_Q(b|B, \tau_{\lambda}^{\beta}) p_Q(c|C, \tau_{\lambda}^{\gamma}), \tag{4}$$

where  $p_Q(b|B, \tau_{\lambda}^{\beta}) = \text{Tr}[M_B^b \tau_{\lambda}^{\beta}]$  and  $p_Q(c|C, \tau_{\lambda}^{\gamma}) = \text{Tr}[M_C^c \tau_{\lambda}^{\gamma}]$  are the distributions from the local hidden states  $\tau_{\lambda}^{\beta}$  and  $\tau_{\lambda}^{\gamma}$ , see Equation (13) in [35] and Equation (2) in [36].

The genuine tripartite steering has been defined in [34–36]. Alice measures her system so as to nonlocally influence the state of the other two parties. The ensemble of the unnormalized states is given by

$$\{\delta_{M_A^a}^{BC} = \text{Tr}[(M_A^a \otimes I \otimes I) \cdot \rho_{ABC}]\}. \tag{5}$$

If the ensemble prepared on Bob’s and Charlie’s sides cannot be reproduced by a biseparable state as Equation (6),

$$\rho_{ABC} = \sum_{\lambda} p_1(\lambda) \rho_{\lambda}^{\alpha} \otimes \rho_{\lambda}^{\beta\gamma} + \sum_{\lambda} p_2(\lambda) \rho_{\lambda}^{\alpha\beta} \otimes \rho_{\lambda}^{\gamma} + \sum_{\lambda} p_3(\lambda) \rho_{\lambda}^{\beta} \otimes \rho_{\lambda}^{\alpha\gamma}, \tag{6}$$

with  $\sum_{\lambda} p_1(\lambda) + p_2(\lambda) + p_3(\lambda) = 1$ , then  $\rho_{ABC}$  is not genuine tripartite steerable from Alice to Bob and Charlie. Therefore, if  $\rho_{ABC}$  is genuine tripartite steerable from Alice to Bob and Charlie, then each member of the ensemble (5) can not be expressed as [34,36],

$$\begin{aligned} \delta_{M_A^a}^{BC} &= \sum_{\lambda} p_1(\lambda) p_1(a|A, \lambda) \rho_{\lambda}^{\beta\gamma} \\ &+ \sum_{\lambda} p_2(\lambda) \delta_{M_A^a, \lambda}^{\beta} \otimes \rho_{\lambda}^{\gamma} \\ &+ \sum_{\lambda} p_3(\lambda) \rho_{\lambda}^{\beta} \otimes \delta_{M_A^a, \lambda}^{\gamma} \end{aligned} \tag{7}$$

with  $\delta_{M_A^a, \lambda}^{\beta} = \text{Tr}_{\mathcal{A}}[(M_A^a \otimes \mathbb{I})\rho_{\lambda}^{\alpha\beta}]$  and  $\delta_{M_A^a, \lambda}^{\gamma} = \text{Tr}_{\mathcal{A}}[(M_A^a \otimes \mathbb{I})\rho_{\lambda}^{\alpha\gamma}]$ . The first term on the right-hand side of (7) stands for that Alice cannot steer Bob and Charlie. Bob and Charlie share entanglement and a local hidden entangled state  $\rho_{\lambda}^{\beta\gamma}$ . The other two terms imply that there is no entanglement between Bob and Charlie, and Alice can steer one of the two systems but not both: the second (third) term stands for that Alice can steer Bob (Charlie) but not Charlie (Bob).

A state is genuine tripartite steerable from Alice to Bob and Charlie if the joint probability  $p(a, b, c|A, B, C) = \text{Tr}[(M_B^b \otimes M_C^c)\delta_{M_A^a}^{BC}]$  does not admit a hybrid LHV-LHS model [35,36],

$$\begin{aligned} p(a, b, c|A, B, C) &= \sum_{\lambda} p_1(\lambda) p(a|A, \lambda) p_Q(b, c|B, C, \rho_{\lambda}^{\beta\gamma}) \\ &+ \sum_{\lambda} p_2(\lambda) p_Q(a, b|A, B) p_Q(c|C, \rho_{\lambda}^{\gamma}) \\ &+ \sum_{\lambda} p_3(\lambda) p_Q(a, c|A, C) p_Q(b|B, \rho_{\lambda}^{\beta}), \end{aligned} \tag{8}$$

where  $p(a|A, \lambda)$  is the distribution on Alice’s side from black-box measurements performed on a quantum state,  $p_Q(c|C, \rho_{\lambda}^{\gamma})$  and  $p_Q(b|B, \rho_{\lambda}^{\beta})$  are the distributions from measurements on quantum states  $\rho_{\lambda}^{\gamma}$  and  $\rho_{\lambda}^{\beta}$ .  $p_Q(b, c|B, C, \rho_{\lambda}^{\beta\gamma})$  can be reproduced by quantum state  $\rho_{\lambda}^{\beta\gamma}$  shared by Bob and Charlie.  $p_Q(a, b|A, B) = \text{Tr}[(M_A^a \otimes M_B^b)\rho_{\lambda}^{\alpha\beta}]$  and  $p_Q(a, c|A, C) = \text{Tr}[(M_A^a \otimes M_C^c)\rho_{\lambda}^{\alpha\gamma}]$  are distributions from a quantum state with untrusted  $\mathcal{A}$  and trusted  $\mathcal{B}$  and  $\mathcal{C}$ . When  $M_A^a = \mathbb{I}$ ,  $p_Q(b|B)$  and  $p_Q(c|C)$  are probabilities from the local hidden states  $\text{Tr}_{\mathcal{A}}[\rho_{\lambda}^{\alpha\beta}]$  and  $\text{Tr}_{\mathcal{A}}[\rho_{\lambda}^{\alpha\gamma}]$ , respectively, since  $\mathcal{B}$  and  $\mathcal{C}$  are the trusted parties. We always use  $p_Q(x, y|X, Y)$  ( $x, y = a, b$  or  $c$  and  $X, Y = A, B$  or  $C$ ) to represent the distribution from measurements on two parties with one party trusted and the other two untrusted in this paper.

A quantum state  $\rho_{ABC}$  is said to be tripartite steerable from (untrusted) Alice and Bob to (trusted) Charlie if the joint probability  $p(a, b, c|A, B, C)$  does not admit a fully LHV-LHS model such that

$$p(a, b, c|A, B, C) = \sum_{\lambda} p(\lambda) p(a|A, \lambda) p(b|B, \lambda) p_Q(c|C, \tau_{\lambda}^{\gamma}), \tag{9}$$

where  $p(a|A, \lambda)$  and  $p(b|B, \lambda)$  are the probabilities from the black-box measurements,  $p_Q(c|C, \tau_{\lambda}^{\gamma})$  is the distribution from local hidden state  $\tau_{\lambda}^{\gamma}$ , see also the definition given in [35,36].

The genuine tripartite steering from Alice and Bob to Charlie has also been defined in [34–36]. Alice and Bob measure their systems so as to nonlocally influence the state of Charlie’s. The ensemble prepared on Charlie’s side cannot be reproduced by a biseparable

state as Equation (6). Each member in the ensemble of unnormalized states can not be given by

$$\begin{aligned} \delta_{M_A^a, M_B^b}^C &= \text{Tr}[(M_A^a \otimes M_B^b \otimes I) \cdot \rho_{ABC}] \\ &= \sum_{\lambda} p_1(\lambda) p_1(a, b|A, B, \lambda) \rho_{\lambda}^{\gamma} \\ &\quad + \sum_{\lambda} p_2(\lambda) p(a|A, \lambda) \delta_{M_B^b, \lambda}^{\gamma} \\ &\quad + \sum_{\lambda} p_3(\lambda) p(b|B, \lambda) \delta_{M_A^a, \lambda}^{\gamma} \end{aligned} \tag{10}$$

with  $\delta_{M_B^b, \lambda}^{\gamma} = \text{Tr}_B[(M_B^b \otimes I) \rho_{\lambda}^{\beta\gamma}]$  and  $\delta_{M_A^a, \lambda}^{\gamma} = \text{Tr}_A[(M_A^a \otimes I) \rho_{\lambda}^{\alpha\gamma}]$ . The first term on the right-hand side of (10) stands for that Alice and Bob cannot jointly steer Charlie, and the second (third) term stands for that only Bob (Alice) can steer the state of Charlie. A state is genuine tripartite steerable from Alice and Bob to Charlie if the joint probability  $p(a, b, c|A, B, C)$  does not admit a hybrid LHV-LHS model such that

$$\begin{aligned} p(a, b, c|A, B, C) &= \sum_{\lambda} p_1(\lambda) p(a, b|A, B, \lambda) p_Q(c|C, \rho_{\lambda}^{\gamma}) \\ &\quad + \sum_{\lambda} p_2(\lambda) p(a|A, \lambda) p_Q(b, c|B, C) \\ &\quad + \sum_{\lambda} p_3(\lambda) p(b|B, \lambda) p_Q(a, c|A, C), \end{aligned} \tag{11}$$

where  $\sum_{\lambda} p_1(\lambda) + \sum_{\lambda} p_2(\lambda) + \sum_{\lambda} p_3(\lambda) = 1$ .  $p(a|A, \lambda)$  and  $p(b|B, \lambda)$  are the distributions on Alice’s and Bob’s sides, respectively, arising from black-box measurements performed on a quantum state.  $p(a, b|A, B, \lambda)$  is the distribution produced from black-box measurements performed on a quantum state.  $p_Q(c|C, \rho_{\lambda}^{\gamma})$  is the distribution from the state  $\rho_{\lambda}^{\gamma}$ .  $p_Q(b, c|B, C) = \text{Tr}[(M_B^b \otimes M_C^c) \rho_{\lambda}^{\beta\gamma}]$  and  $p_Q(a, c|A, C) = \text{Tr}[(M_A^a \otimes M_C^c) \rho_{\lambda}^{\alpha\gamma}]$  are probabilities from a 2-qubit quantum state with untrusted  $\mathcal{A}$  and  $\mathcal{B}$  and trusted  $\mathcal{C}$ . When  $M_A^a = M_B^b = I$ ,  $p_Q(c|C)$  are probabilities from the local hidden states  $\text{Tr}_B[\rho_{\lambda}^{\beta\gamma}]$  and  $\text{Tr}_A[\rho_{\lambda}^{\alpha\gamma}]$ , respectively, since  $\mathcal{C}$  is the trusted party.

Entropic steering inequalities and semi-definite-program have been adopted to investigate the detection of multipartite steering [34,36,37]. In the following, we construct new quantum states with respect to given three-qubit states and detect the tripartite steering and genuine tripartite steering analytically in terms of the tripartite entanglement and the genuine tripartite entanglement of the newly constructed quantum states. The entanglement of the newly constructed states can be detected by using the entanglement witness without full tomography of the states. By detecting the entanglement of the newly constructed states, the tripartite steering and genuine tripartite steering can be detected without using any steering inequalities. Since the “complexity cost” (the number of possible patterns of joint detection outcomes that can occur, see [38]) for the least complex demonstration of entanglement is less than the “complexity cost” for the least complex demonstration of EPR steering [29,38], our scheme reduces the “complex cost” in experimental steering demonstration.

## 2. Main Results

A quantum state is fully separable if the joint probability  $p(a, b, c|A, B, C)$  satisfies the condition,

$$p(a, b, c|A, B, C) = \sum_{\lambda} p_{\lambda} p_Q(a|A, \tau_{\lambda}^{\alpha}) p_Q(b|B, \tau_{\lambda}^{\beta}) p_Q(c|C, \tau_{\lambda}^{\gamma}). \tag{12}$$

Fully separable states are neither tripartite steerable states from Alice to Bob and Charlie nor from Alice and Bob to Charlie. From (4) and (9) a state which is not tripartitely steerable from Alice to Bob and Charlie is not tripartitely steerable from Alice and Bob to

Charlie, i.e., tripartite steering from Alice and Bob to Charlie is stronger than that from Alice to Bob and Charlie.

A quantum state is bi-separable if the joint probability  $p(a, b, c|A, B, C)$  satisfies the condition,

$$\begin{aligned}
 p(a, b, c|A, B, C) &= \sum_{\lambda} p_1(\lambda) p_Q(a, b|A, B, \rho_{\lambda}^{\alpha\beta}) p_Q(c|C, \rho_{\lambda}^{\gamma}) \\
 &+ \sum_{\lambda} p_2(\lambda) p_Q(a|A, \rho_{\lambda}^{\alpha}) p_Q(b, c|B, C, \rho_{\lambda}^{\beta\gamma}) \\
 &+ \sum_{\lambda} p_3(\lambda) p(b|B, \rho_{\lambda}^{\beta}) p_Q(a, c|A, C, \rho_{\lambda}^{\alpha\gamma}),
 \end{aligned} \tag{13}$$

where  $\sum_{\lambda} p_1(\lambda) + \sum_{\lambda} p_2(\lambda) + \sum_{\lambda} p_3(\lambda) = 1$ . A bi-separable quantum state must not be a genuine tripartite steerable state from Alice to Bob and Charlie or from Alice and Bob to Charlie. From (8) and (11) a state which is not genuine tripartite steerable from Alice to Bob and Charlie is not genuine tripartite steerable from Alice and Bob to Charlie. As a result, given in [34], the noisy GHZ state demonstrates the genuine tripartite steering from Alice to Bob and Charlie in a larger region compared to that from Alice and Bob to Charlie. For general tripartite quantum states, the genuine tripartite steering from Alice and Bob to Charlie is also stronger than that from Alice to Bob and Charlie.

**Theorem 1.** Let  $\rho_{ABC}$  be a three-qubit quantum state and

$$\tau_{ABC}^1 = \mu \rho_{ABC} + (1 - \mu) \frac{I_2}{2} \otimes \rho_{BC} \tag{14}$$

with  $\rho_{BC} = \text{Tr}_A \rho_{ABC}$  and  $I_2$  the  $2 \times 2$  identity matrix. We have

- (i) If  $\tau_{ABC}^1$  is genuine tripartite entangled, then  $\rho_{ABC}$  is genuine tripartite steerable from Alice to Bob and Charlie for  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ ;
- (ii) If  $\tau_{ABC}^1$  is tripartite entangled, then  $\rho_{ABC}$  is tripartite steerable from Alice to Bob and Charlie for  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ .

The statements in Theorem 1 are equivalent to the following:

- (i') If  $\rho_{ABC}$  is not genuine tripartite steerable from Alice to Bob and Charlie, then  $\tau_{ABC}^1$  is bi-separable for  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ ;
- (ii') If  $\rho_{ABC}$  is not tripartite steerable from Alice to Bob and Charlie, then  $\tau_{ABC}^1$  is fully separable for  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ .

**Proof of Theorem 1.** We prove the theorem by proving its converse negative proposition: if  $\rho_{ABC}$  is not a genuine tripartite steerable state from Alice to Bob and Charlie, then  $\tau_{ABC}^1$  is a bi-separable state; if  $\rho_{ABC}$  is not a tripartite steerable state from Alice to Bob and Charlie, then  $\tau_{ABC}^1$  is a fully-separable state.

Firstly we give the (unnormalized) conditional quantum state  $\delta_{BC}^{\beta\gamma}$  on Alice's side after Bob and Charlie perform measurements  $M_B^b$  and  $M_C^c$  on  $\tau_{ABC}^1$ . Then the Bloch sphere representation of the conditional state can be expressed according to the joint probabilities. Lastly from the condition that  $\rho_{ABC}$  is not genuine steering or steering from Alice to Bob and Charlie, we prove that  $\delta_{BC}^{\beta\gamma}$  is the convex combination of some qubit quantum states if  $\mu$  satisfies certain conditions.

**Step 1.** From (14) we have the (unnormalized) conditional state on Alice's side when Bob and Charlie perform measurements  $M_B^b$  and  $M_C^c$  on  $\tau_{ABC}^1$ ,

$$\begin{aligned}
 \delta_{BC}^{\beta\gamma} &:= \text{Tr}_{BC}[(I_2 \otimes M_B^b \otimes M_C^c) \cdot \tau_{ABC}^1] = \mu \text{Tr}_{BC}[(I_2 \otimes M_B^b \otimes M_C^c) \cdot \rho_{ABC}] \\
 &+ (1 - \mu) p(b, c|B, C, \rho_{ABC}) \frac{I_2}{2} = \frac{1}{2} (y I_2 + \sum_i r_i \sigma_i),
 \end{aligned}$$

where  $\sigma_i$  ( $i = 1, 2, 3$ ) are Pauli matrices  $\sigma_x, \sigma_y$  and  $\sigma_z$ , respectively.

**Step 2.**  $y$  and  $r_i$  ( $i = 1, 2, 3$ ) are given by the joint probabilities,

$$\begin{aligned} y &= \text{Tr}[\delta_{BC}^{\beta\gamma}] = \mu \text{Tr}[\text{Tr}_{BC}[(\mathbb{I}_2 \otimes M_B^b \otimes M_C^c) \cdot \rho_{ABC}]] + (1 - \mu)p(b, c|B, C, \rho_{ABC}) \\ &= p(b, c|B, C, \rho_{ABC}), \end{aligned}$$

$$\begin{aligned} r_i &= \text{Tr}[\delta_{BC}^{\beta\gamma} \cdot \sigma_i] = \text{Tr}[\delta_{BC}^{\beta\gamma} \cdot (\sigma_i^+ - \sigma_i^-)] = \mu \text{Tr}[\text{Tr}_{BC}[(\mathbb{I}_2 \otimes M_B^b \otimes M_C^c) \cdot \rho_{ABC}] \cdot (\sigma_i^+ - \sigma_i^-)] \\ &= \mu(p(+, b, c, |\sigma_i, B, C, \rho_{ABC}) - p(-, b, c, |\sigma_i, B, C, \rho_{ABC})), \end{aligned}$$

with  $\sigma_i^+$  and  $\sigma_i^-$  the eigenvectors of  $\sigma_i$  with respect to the eigenvalues 1 and  $-1$  of  $\sigma_i$  ( $i = 1, 2, 3$ ), respectively.

**Step 3.** (I). If  $\rho_{ABC}$  is not a genuine tripartite steerable state from Alice to Bob and Charlie, the joint probabilities admit a hybrid LHV-LHS model as follows,

$$\begin{aligned} p(a, b, c|A, B, C) &= \sum_{\lambda} p_1(\lambda)p_1(a|A, \lambda)p_Q(b, c|B, C, \rho_{\lambda}^{\beta\gamma}) \\ &\quad + p_2(\lambda)p_Q(a, b|A, B)p_Q(c|C, \rho_{\lambda}^{\gamma}) \\ &\quad + p_3(\lambda)p_Q(a, c|A, C)p_Q(b|B, \rho_{\lambda}^{\beta}), \\ p(b, c|B, C, \rho_{ABC}) &= \sum_{\lambda} p_1(\lambda)p_Q(b, c|B, C, \rho_{\lambda}^{\beta\gamma}) \\ &\quad + \sum_{\lambda} p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma})p_Q(b|B) \\ &\quad + \sum_{\lambda} p_3(\lambda)p_Q(b|B, \rho_{\lambda}^{\beta})p_Q(c|C), \end{aligned}$$

where  $p_Q(b|B)$  and  $p_Q(c|C)$  are probabilities from qubit states  $\rho_{\lambda}^{\beta'} = \text{Tr}_A[\rho_{\lambda}^{\alpha\beta}]$  and  $\rho_{\lambda}^{\gamma'} = \text{Tr}_A[\rho_{\lambda}^{\alpha\gamma}]$  on Bob's and Charlie's sides, respectively.

$$\begin{aligned} p(\pm, b, c|\sigma_i, B, C) &= \sum_{\lambda} p_1(\lambda)p_1(\pm|\sigma_i, \lambda)p_Q(b, c|B, C, \rho_{\lambda}^{\beta\gamma}) \\ &\quad + p_2(\lambda)p_Q(\pm, b|\sigma_i, B)p_Q(c|C, \rho_{\lambda}^{\gamma}) \\ &\quad + p_3(\lambda)p_Q(\pm, c|\sigma_i, C)p_Q(b|B, \rho_{\lambda}^{\beta}). \end{aligned}$$

**Step 4.** We now prove that the following conditional state  $\delta_{BC}^{\beta\gamma}$  is the convex combination of qubit quantum states when  $\mu$  satisfies certain conditions,

$$\begin{aligned} \delta_{BC}^{\beta\gamma} &= \sum_{\lambda} p_1(\lambda)p_Q(b, c|B, C, \rho_{\lambda}^{\beta\gamma})\rho_{\lambda}^1 + \sum_{\lambda} p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma})p_Q(b|B)\rho_{\lambda}^2 \\ &\quad + \sum_{\lambda} p_3(\lambda)p_Q(b|B, \rho_{\lambda}^{\beta})p_Q(c|C)\rho_{\lambda}^3, \end{aligned}$$

where

$$\begin{aligned} \rho_{\lambda}^1 &= \frac{1}{2}(\mathbb{I}_2 + \mu \sum_i (p_1(+|\sigma_i, \lambda) - p_1(-|\sigma_i, \lambda))\sigma_i), \\ \rho_{\lambda}^2 &= \frac{1}{2}(\mathbb{I}_2 + \mu \sum_i \frac{p_Q(+, b|\sigma_i, B) - p_Q(-, b|\sigma_i, B)}{p_Q(b|B)}\sigma_i), \\ \rho_{\lambda}^3 &= \frac{1}{2}(\mathbb{I}_2 + \mu \sum_i \frac{p_Q(+, c|\sigma_i, C) - p_Q(-, c|\sigma_i, C)}{p_Q(c|C)}\sigma_i). \end{aligned}$$

Since  $|p_1(+|\sigma_i, \lambda) - p_1(-|\sigma_i, \lambda)| \leq 1$ ,  $|\frac{p_Q(+,b|\sigma_i,B) - p_Q(-,b|\sigma_i,B)}{p_Q(b|B)}| \leq 1$  and  $|\frac{p_Q(+,c|\sigma_i,C) - p_Q(-,c|\sigma_i,C)}{p_Q(c|C)}| \leq 1$ , when  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ ,  $\rho_\lambda^1, \rho_\lambda^2$  and  $\rho_\lambda^3$  are semi-definite positive matrices with trace one. They are quantum states when  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ . Therefore,

$$\begin{aligned} p(a, b, c|A, B, C, \tau_{ABC}^1) &= \text{Tr}[M_A^a \delta_{BC}^{\beta\gamma}] = \sum_\lambda p_1(\lambda) p_Q(b, c|B, C, \rho_\lambda^{\beta\gamma}) p_Q(a|A, \rho_\lambda^1) \\ &\quad + \sum_\lambda p_2(\lambda) p_Q(c|C, \rho_\lambda^\gamma) p_Q(b|B) p_Q(a|A, \rho_\lambda^2) \\ &\quad + \sum_\lambda p_3(\lambda) p_Q(b|B, \rho_\lambda^\beta) p_Q(c|C) p_Q(a|A, \rho_\lambda^3) \\ &= \sum_\lambda p_1(\lambda) p_Q(b, c|B, C, \rho_\lambda^{\beta\gamma}) p_Q(a|A, \rho_\lambda^1) \\ &\quad + \sum_\lambda p_2(\lambda) p_Q(c|C, \rho_\lambda^\gamma) p_Q(a, b|A, B, \rho_\lambda^{\alpha\beta'}) \\ &\quad + \sum_\lambda p_3(\lambda) p_Q(b|B, \rho_\lambda^\beta) p_Q(a, c|A, C, \rho_\lambda^{\alpha\gamma'}), \end{aligned}$$

with  $p_Q(b|B) = \text{Tr}[M_B^b \cdot \rho_\lambda^{\beta'}]$  and  $p_Q(c|C) = \text{Tr}[M_C^c \cdot \rho_\lambda^{\gamma'}]$ , and  $\rho_\lambda^{\alpha\beta'} = \rho_\lambda^2 \otimes \rho_\lambda^{\beta'}$  and  $\rho_\lambda^{\alpha\gamma'} = \rho_\lambda^3 \otimes \rho_\lambda^{\gamma'}$ . From (13)  $\tau_{ABC}^1$  is a bi-separable state. Namely, if  $\tau_{ABC}^1$  is genuine tripartite entangled, then  $\rho_{ABC}$  is genuine tripartite steerable from Alice to Bob and Charlie for  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ .

**Step 3'.** (II). If  $\rho_{ABC}$  is not tripartite steerable from Alice to Bob and Charlie, the joint probabilities admit LHV-LHS model,

$$p(a, b, c|A, B, C) = \sum_\lambda p(\lambda) p(a|A, \lambda) p_Q(b|B, \tau_\lambda^\beta) p_Q(c|C, \tau_\lambda^\gamma),$$

$$p(b, c|B, C, \rho_{ABC}) = \sum_\lambda p(\lambda) p_Q(b|B, \tau_\lambda^\beta) p_Q(c|C, \tau_\lambda^\gamma)$$

and

$$p(\pm, b, c|\sigma_i, B, C) = \sum_\lambda p(\lambda) p(\pm|\sigma_i, \lambda) p_Q(b|B, \tau_\lambda^\beta) p_Q(c|C, \tau_\lambda^\gamma). \tag{15}$$

**Step 4'.** Therefore,  $\delta_{BC}^{\beta\gamma}$  is given by the convex combination of some qubit quantum states when  $\mu$  satisfies certain condition,

$$\delta_{BC}^{\beta\gamma} = \sum_\lambda p(\lambda) p_Q(b|B, \tau_\lambda^\beta) p_Q(c|C, \tau_\lambda^\gamma) \tau_\lambda^\alpha,$$

where  $\tau_\lambda^\alpha = \frac{1}{2}(\mathbf{I}_2 + \mu \sum_i (p_1(+|\sigma_i, \lambda) - p_1(-|\sigma_i, \lambda)) \sigma_i)$ . Since  $|p_1(+|\sigma_i, \lambda) - p_1(-|\sigma_i, \lambda)| \leq 1$  for  $i = 1, 2, 3$ , when  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ ,  $\tau_\lambda^\alpha$  is a semi-definite positive matrix when  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ . Therefore,  $\tau_\lambda^\alpha$  is a quantum state when  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ . Since

$$p(a, b, c|A, B, C, \tau_{ABC}^1) = \text{Tr}[M_A^a \delta_{BC}^{\beta\gamma}] = \sum_\lambda p(\lambda) p_Q(a|A, \tau_\lambda^\alpha) p_Q(b|B, \tau_\lambda^\beta) p_Q(c|C, \tau_\lambda^\gamma),$$

from (12),  $\tau_{ABC}^1$  is fully separable. Hence, if  $\tau_{ABC}^1$  is tripartite entangled,  $\rho_{ABC}$  must be tripartite steerable from Alice to Bob and Charlie for  $0 \leq \mu \leq \frac{1}{\sqrt{3}}$ .  $\square$

**Theorem 2.** Let  $\rho_{ABC}$  be a three-qubit state and

$$\tau_{ABC}^2 = \mu\rho_{ABC} + (1 - \mu)\frac{I_4}{4} \otimes \rho_C, \quad (16)$$

where  $\rho_C = \text{Tr}_{AB}\rho_{ABC}$  and  $I_4$  is the  $4 \times 4$  identity matrix. We have

- a) If  $\tau_{ABC}^2$  is genuine tripartite entangled, then  $\rho_{ABC}$  is genuine tripartite steerable from Alice and Bob to Charlie for  $0 \leq \mu \leq \frac{1}{9}$ ;
- b) If  $\tau_{ABC}^2$  is tripartite entangled, then  $\rho_{ABC}$  is tripartite steerable from Alice to Bob and Charlie for  $0 \leq \mu \leq \frac{1}{3}$ .

The proof of Theorem 2 is given in Appendix A. The statements in Theorem 2 are also equivalent to the following:

- (a') If  $\rho_{ABC}$  is not genuine tripartite steerable from Alice and Bob to Charlie, then  $\tau_{ABC}^2$  is bi-separable for  $0 \leq \mu \leq \frac{1}{9}$ ;
- (b') If  $\rho_{ABC}$  is not tripartite steerable from Alice to Bob and Charlie, then  $\tau_{ABC}^2$  is fully separable for  $0 \leq \mu \leq \frac{1}{3}$ .

Next, we illustrate our theorems with detailed examples.

**Example 1.** Consider  $\rho_{ABC} = |\text{GHZ}\rangle\langle\text{GHZ}|$ , where  $|\text{GHZ}\rangle = a|000\rangle + \sqrt{1-a^2}|111\rangle$ . The  $\tau_{ABC}^1$  defined in Theorem 1 is a  $8 \times 8$  matrix with entries  $\tau_{ij}$ ,  $i, j = 1, 2, \dots, 8$ . The state  $\tau_{ABC}^1$  is genuine entangled if  $|\tau_{18}| > \sqrt{\tau_{22}\tau_{77}} + \sqrt{\tau_{33}\tau_{66}} + \sqrt{\tau_{44}\tau_{55}}$  [39], and  $\tau_{ABC}^1$  is entangled if one of the following three inequalities is satisfied:  $|\tau_{18}| > \sqrt{\tau_{22}\tau_{77}}$ ,  $|\tau_{18}| > \sqrt{\tau_{33}\tau_{66}}$  or  $|\tau_{18}| > \sqrt{\tau_{44}\tau_{55}}$  [40]. Therefore, from Theorem 1 we have that when  $0 < a < 1$  this state  $\rho_{ABC}$  is tripartite steerable and also genuine tripartite steerable from Alice to Bob and Charlie. Similarly, according to the entanglement of  $\tau_{ABC}^2$ , from Theorem 2 we obtain that  $\rho_{ABC}$  is tripartitely steerable from Alice and Bob to Charlie when  $0 < a < 1$ . While in [36], genuine tripartite steering from Alice to Bob and Charlie is detected only when  $0.5 < a < 0.85$ .

**Example 2.** Consider  $\rho_{ABC} = \frac{1-p}{8}I_8 + p|\text{GHZ}\rangle\langle\text{GHZ}|$  with  $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ . Similar to Example 1, by using the entanglement criteria given in [39,40] and Theorem 1, we have that  $\tau_{ABC}^1$  is genuine tripartite entangled when  $p > 0.672$ , and thus  $\rho_{ABC}$  is genuine tripartite steerable from Alice to Bob and Charlie. When  $p > 0.406$   $\tau_{ABC}^1$  is an entangled state, and  $\rho_{ABC}$  is tripartite steerable from Alice to Bob and Charlie. Furthermore, from the entanglement of  $\tau_{ABC}^2$  and Theorem 2, we have that  $\rho_{ABC}$  is tripartitely steerable from Alice and Bob to Charlie when  $p > 0.6$ . While in [36]  $\rho_{ABC}$  is proved to be tripartite steerable from Alice to Bob and Charlie when  $p > 0.74$  and genuine tripartite steerable from Alice to Bob and Charlie when  $p > 0.95$ . In [35]  $\rho_{ABC}$  is shown to be tripartite steerable from Alice to Bob and Charlie when  $p > 0.35$  and genuine steerable when  $p > 0.71$ .  $\rho_{ABC}$  is tripartite steerable from Alice and Bob to Charlie when  $p > 0.5$  and genuine steerable when  $p > 0.71$ . In [37]  $\rho_{ABC}$  is shown to be tripartite steering from Alice to Bob and Charlie when  $p > 0.8631$  for two measurement settings, and  $p > 0.7642$  for three measurement settings.  $\rho_{ABC}$  is tripartite steering from Alice and Bob to Charlie when  $p > 0.6751$  for two measurement settings, and  $p > 0.5514$  for three measurement settings. Hence, in the case of detecting genuine tripartite steering from Alice to Bob and Charlie, our proposed method is stronger compared with the criteria given in [35–37], and in the case of tripartite steering from Alice to Bob and Charlie, our proposed method is stronger with respect to the criteria in [36,37]. The results are listed in Table 1.

**Table 1.** Critical values to white noise  $p$  for example 2 by our theorems and the criteria in [35–37], here, S represents tripartite steering and GMS represents genuine tripartite steering.

Steering	$\mathcal{A}$ to $\mathcal{B}$ & $\mathcal{C}$ (S)	$\mathcal{A}$ to $\mathcal{B}$ & $\mathcal{C}$ (GMS)	$\mathcal{A}$ & $\mathcal{B}$ to $\mathcal{C}$ (S)	$\mathcal{A}$ & $\mathcal{B}$ to $\mathcal{C}$ (GMS)
our result	0.406	0.672	0.6	
[35]	0.35	0.71	0.5	0.71
[36]	0.74	0.95		
[37]	0.7642		0.5514	

Next, instead of the criteria given in [39,40] we first present improved separability criteria. Consider a three-qubit state  $|\psi\rangle$ . Let  $\sigma_{ij}$  be the entries of the matrix  $\sigma = |\psi\rangle\langle\psi|$ . If the state  $\sigma = |\psi\rangle\langle\psi|$  is bi-separable, we have  $|\sigma_{25}| \leq \frac{1}{2}(\sigma_{11} + \sigma_{66})$ ,  $|\sigma_{35}| \leq \frac{1}{2}(\sigma_{11} + \sigma_{77})$  and  $|\sigma_{23}| \leq \frac{1}{2}(\sigma_{22} + \sigma_{33})$  under the bipartition  $\mathcal{A}|\mathcal{B}\mathcal{C}$ ;  $|\sigma_{23}| \leq \frac{1}{2}(\sigma_{11} + \sigma_{44})$ ,  $|\sigma_{35}| \leq \frac{1}{2}(\sigma_{11} + \sigma_{77})$  and  $|\sigma_{25}| \leq \frac{1}{2}(\sigma_{22} + \sigma_{55})$  under the bi-partition  $\mathcal{B}|\mathcal{A}\mathcal{C}$ ;  $|\sigma_{23}| \leq \frac{1}{2}(\sigma_{11} + \sigma_{44})$ ,  $|\sigma_{25}| \leq \frac{1}{2}(\sigma_{11} + \sigma_{66})$  and  $|\sigma_{35}| \leq \frac{1}{2}(\sigma_{33} + \sigma_{55})$  under the bi-partition  $\mathcal{C}|\mathcal{A}\mathcal{B}$ . Hence for any pure bi-separable quantum state  $\sigma$ , we have  $|\sigma_{23}| + |\sigma_{25}| + |\sigma_{35}| \leq \frac{1}{2}(2\sigma_{11} + \sigma_{44} + \sigma_{66} + \sigma_{77}) + \frac{1}{2}(\sigma_{22} + \sigma_{33} + \sigma_{55})$ . The above inequalities are also satisfied for bi-separable mixed states by the convex roof construction. Therefore, we have

**Proposition 1.** Let  $\tau_{ABC}$  be any three-qubit state and  $\tau_{ij}$  the entries of the  $8 \times 8$  matrix  $\tau_{ABC}$ . Then  $\tau_{ABC}$  is genuine tripartite entangled if

$$|\tau_{23}| + |\tau_{25}| + |\tau_{35}| > \frac{1}{2}(2\tau_{11} + \tau_{44} + \tau_{66} + \tau_{77}) + \frac{1}{2}(\tau_{22} + \tau_{33} + \tau_{55}). \tag{17}$$

**Example 3.** Let us consider now  $\rho_{ABC} = \frac{1-p}{8}\mathbb{I}_8 + p|W\rangle\langle W|$  with  $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ . Using the inequality (17), we have that the state  $\tau_{ABC}^1$  defined in Theorem 1 is genuine tripartite entangled when  $p > 0.816$ , whereas from the result  $|\tau_{23}| + |\tau_{25}| + |\tau_{35}| > \sqrt{\tau_{11}\tau_{44}} + \sqrt{\tau_{11}\tau_{66}} + \sqrt{\tau_{11}\tau_{77}} + \frac{1}{2}(\tau_{22} + \tau_{33} + \tau_{55})$  given in [39,40],  $\tau_{ABC}^1$  is genuine tripartite entangled when  $p > 0.862$ . Hence, from Theorem 1 when  $p > 0.816$ ,  $\rho_{ABC}$  is genuine tripartite steerable from Alice to Bob and Charlie. Concerning the tripartite steerability, it has been shown in [41] that  $\tau_{ABC}^1$  is tripartite entangled if  $(\Gamma \otimes \mathbb{I}_4)\tau_{ABC}^1$  is not a positive semi-definite matrix, where  $\Gamma$  is the transpose with respect to subsystems  $\mathcal{A}$ ,  $\mathcal{B}$  or  $\mathcal{C}$ . From this criterion we have that  $\tau_{ABC}^1$  is tripartite entangled when  $p > 0.31$ , i.e.,  $\rho_{ABC}$  is tripartite steerable from Alice to Bob and Charlie for  $p > 0.31$ . Similarly from the  $\tau_{ABC}^2$  given in Theorem 2 and the criteria given [41], we have that  $\rho_{ABC}$  is tripartite steerable from Alice and Bob to Charlie when  $p > 0.621$ . While in [36],  $\rho_{ABC}$  is proved to be tripartitely steerable from Alice to Bob and Charlie when  $p > 0.85$  and no genuine tripartite steerability is detected. In [37],  $\rho_{ABC}$  is shown to be tripartite steering from Alice to Bob and Charlie when  $p > 0.9814$  for two measurement settings, and  $p > 0.8366$  for three measurement settings.  $\rho_{ABC}$  is tripartite steering from Alice and Bob to Charlie when  $p > 0.75$  for two measurement settings, and  $p > 0.623$  for three measurement settings. Hence, in the case of detecting tripartite steering and genuine tripartite steering from Alice to Bob and Charlie, our proposed method is stronger with respect to the criteria in [35–37]. The results are listed in Table 2.

**Table 2.** Critical values to white noise  $p$  for example 3 by our theorems and the criteria in [35–37], here, S represents tripartite steering and GMS represents genuine tripartite steering

Steering	$\mathcal{A}$ to $\mathcal{B}$ & $\mathcal{C}$ (S)	$\mathcal{A}$ to $\mathcal{B}$ & $\mathcal{C}$ (GMS)	$\mathcal{A}$ & $\mathcal{B}$ to $\mathcal{C}$ (S)	$\mathcal{A}$ & $\mathcal{B}$ to $\mathcal{C}$ (GMS)
our result	0.31	0.816	0.621	
[35]				
[36]	0.85			
[37]	0.8366		0.623	

One point to be stressed here is that, instead of the numerical results based on a semi-definite program in [34], our results are derived analytically. For the GHZ state and W state mixed white noise, our criteria are powerful in detecting the genuine tripartite steering from Alice to Bob and Charlie. Nevertheless, the criteria can not detect any genuine tripartite steering from Alice and Bob to Charlie, which illustrates that the genuine multipartite steering from Alice and Bob to Charlie is a kind of stronger quantum correlation and some more powerful criteria are needed.

### 3. Conclusions

The tripartite steerability and genuine tripartite steerability can be detected by detecting the multipartite entanglement and genuine multipartite entanglement of the newly constructed state analytically. Some examples show that the criteria are powerful to detect tripartite steering from Alice to Bob and Charlie, Alice and Bob to Charlie, and genuine tripartite steering from Alice to Bob and Charlie. Besides, we give the relationship of fully separable states, non-tripartite steerable states in a one-to-two scenario and a two-to-one scenario, bi-separable states, and non-GMS states in two scenarios. More analytical powerful criteria will be studied to detect genuine multipartite steering from Alice and Bob to Charlie in future research.

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### Appendix A

**Proof of Theorem 2.** The proof of Theorem 2 is similar to that of Theorem 1. We prove Theorem 2 also by proving the converse negative proposition: if  $\rho_{ABC}$  is not a genuine tripartite steerable state from Alice and Bob to Charlie, then  $\tau_{ABC}^2$  is a bi-separable state; if  $\rho_{ABC}$  is not a tripartite steerable state from Alice and Bob to Charlie, then  $\tau_{ABC}^2$  is a fully-separable state.

Firstly we give the (unnormalized) conditional quantum state  $\delta_C^\gamma$  on Alice's and Bob's sides after Charlie performs measurements  $M_C^c$  on  $\tau_{ABC}^2$ . Then the Bloch sphere representation of the conditional state  $\delta_C^\gamma$  can be expressed by the joint probabilities. Lastly from the condition that  $\rho_{ABC}$  is not genuine steering or steering from Alice and Bob to Charlie, the theorem is proved by proving that  $\delta_C^\gamma$  is the convex combination of some 2-qubit quantum states when  $\mu$  satisfies certain condition.

**Step 1.** From (16) we have the (unnormalized) conditional quantum state on Alice's and Bob's sides after Charlie performs measurements  $M_C^c$  on  $\tau_{ABC}^2$ ,

$$\begin{aligned}\delta_C^\gamma &= \text{Tr}_C[(I_4 \otimes M_C^c)\tau_{ABC}^2] \\ &= \mu \text{Tr}_C[(I_4 \otimes M_C^c)\rho_{ABC}] + (1 - \mu)p(c|C, \rho_{ABC})\frac{I_4}{4} \\ &= \frac{1}{4}(xI_4 + \sum_i a_i \sigma_i \otimes I_2 + I_2 \otimes \sum_i b_i \sigma_i + \sum_{ij} c_{ij} \sigma_i \otimes \sigma_j).\end{aligned}$$

**Step 2.** The bloch representation are given by the joint probabilities,  $x = \text{Tr}[\delta_C] = \mu \text{Tr}[(\mathbb{I}_4 \otimes M_C^c) \rho_{ABC}] + (1 - \mu) p(c|C, \rho_{ABC}) = p(c|C, \rho_{ABC})$ ,

$$\begin{aligned} a_i &= \text{Tr}[(\sigma_i^+ - \sigma_i^-) \otimes \mathbb{I}_2] \cdot \delta_C^\gamma \\ &= \mu \text{Tr}[(\sigma_i^+ \otimes \mathbb{I}_2 \otimes M_C^c) \cdot \rho_{ABC}] - \mu \text{Tr}[(\sigma_i^- \otimes \mathbb{I}_2 \otimes M_C^c) \cdot \rho_{ABC}] \\ &= \mu(p(+, c|\sigma_i^A, C, \rho_{ABC}) - p(-, c|\sigma_i^A, C, \rho_{ABC})) \\ &= \mu(2p(+, c|\sigma_i^A, C, \rho_{ABC}) - p(c|C, \rho_{ABC})), \end{aligned}$$

$$\begin{aligned} b_i &= \text{Tr}[\mathbb{I}_2 \otimes (\sigma_i^+ - \sigma_i^-)] \cdot \delta_C^\gamma \\ &= \mu \text{Tr}[(\mathbb{I}_2 \otimes \sigma_i^+ \otimes M_C^c) \cdot \rho_{ABC}] - \mu \text{Tr}[(\mathbb{I}_2 \otimes \sigma_i^- \otimes M_C^c) \cdot \rho_{ABC}] \\ &= \mu(p(+, c|\sigma_i^B, C, \rho_{ABC}) - p(-, c|\sigma_i^B, C, \rho_{ABC})) \\ &= \mu(2p(+, c|\sigma_i^B, C, \rho_{ABC}) - p(c|C, \rho_{ABC})) \end{aligned}$$

and

$$\begin{aligned} c_{ij} &= \text{Tr}[(\sigma_i^+ - \sigma_i^-) \otimes (\sigma_j^+ - \sigma_j^-)] \delta_C^\gamma \\ &= \mu \text{Tr}[(\sigma_i^+ - \sigma_i^-) \otimes (\sigma_j^+ - \sigma_j^-) \otimes M_C^c] \cdot \rho_{ABC} \\ &= \mu[p(+, +, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC}) - p(+, -, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC}) \\ &\quad - p(-, +, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC}) + p(-, -, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC})] \\ &= \mu(2p(+, +, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC}) + 2p(-, -, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC}) - p(c|C, \rho_{ABC})). \end{aligned}$$

Therefore,

$$\begin{aligned} \delta_C^\gamma &= \frac{1}{4} [p(c|C, \rho_{ABC}) \mathbb{I}_4 + \mu \sum_i (2p(+, c|\sigma_i^A, C, \rho_{ABC}) - p(c|C, \rho_{ABC})) \sigma_i \otimes \mathbb{I}_2 \\ &\quad + \mu \sum_i (2p(+, c|\sigma_i^B, C, \rho_{ABC}) - p(c|C, \rho_{ABC})) \mathbb{I}_2 \otimes \sigma_i \\ &\quad + \mu \sum_{ij} (2p(+, +, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC}) + 2p(-, -, c|\sigma_i^A, \sigma_j^B, C, \rho_{ABC}) \\ &\quad - p(c|C, \rho_{ABC})) \sigma_i \otimes \sigma_j]. \end{aligned}$$

**Step 3. (I)** If  $\rho_{ABC}$  is not genuine steerable from Alice and Bob to Charlie, we have that the joint probabilities admit a hybrid LHV-LHS model as follows,

$$\begin{aligned} p(a, b, c|A, B, C, \rho_{ABC}) &= \sum_\lambda [p_1(\lambda) p(a, b|A, B, \lambda) p_Q(c|C, \rho_\lambda^\gamma) \\ &\quad + p_2(\lambda) p(a|A, \lambda) p_Q(b, c|B, C) \\ &\quad + p_3(\lambda) p(b|B, \lambda) p_Q(a, c|A, C)]. \end{aligned} \tag{A1}$$

Specially,

$$\begin{aligned} p(c|C, \rho_{ABC}) &= \sum_\lambda [p_1(\lambda) p_Q(c|C, \rho_\lambda^\gamma) + p_2(\lambda) P_Q(c|C, \rho_\lambda^{\gamma'}) + p_3(\lambda) P_Q(c|C, \rho_\lambda^{\gamma''})] \end{aligned} \tag{A2}$$

with  $P_Q(c|C, \rho_\lambda^{\gamma'})$  and  $P_Q(c|C, \rho_\lambda^{\gamma''})$  the distributions from Charlie’s measurement  $M_C^c$  on  $\rho_\lambda^{\gamma'} = \text{Tr}_\beta[\rho_\lambda^{\beta\gamma}]$  and  $\rho_\lambda^{\gamma''} = \text{Tr}_\alpha[\rho_\lambda^{\alpha\gamma}]$ , respectively.

$$\begin{aligned}
 p(\pm, c | \sigma_i^A, C, \rho_{ABC}) &= \sum_{\lambda} [p_1(\lambda) p(\pm | \sigma_i^A, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) \\
 &\quad + p_2(\lambda) p(\pm | \sigma_i^A, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) \\
 &\quad + p_3(\lambda) p_Q(\pm, c | \sigma_i^A, C)]
 \end{aligned}
 \tag{A3}$$

$$\begin{aligned}
 p(\pm, c | \sigma_i^B, C, \rho_{ABC}) &= \sum_{\lambda} [p_1(\lambda) p(\pm | \sigma_i^B, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) \\
 &\quad + p_2(\lambda) p_Q(\pm, c | \sigma_i^B, C) \\
 &\quad + p_3(\lambda) p(\pm | \sigma_i^B, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''})]
 \end{aligned}
 \tag{A4}$$

and

$$\begin{aligned}
 p(\pm, \pm, c | \sigma_i^A, \sigma_j^B, C) &= \sum_{\lambda} [p_1(\lambda) p(\pm, \pm | \sigma_i^A, \sigma_j^B, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) \\
 &\quad + p_2(\lambda) p(\pm | \sigma_i^A, \lambda) p_Q(\pm, c | \sigma_j^B, C) \\
 &\quad + p_3(\lambda) p(\pm | \sigma_j^B, \lambda) p_Q(\pm, c | \sigma_i^A, C)].
 \end{aligned}
 \tag{A5}$$

Substituting Equations (A2)–(A5) into the expressions of  $x, a_i, b_i$  and  $c_{ij}$  ( $i, j = 1, 2, 3$ ), we have

$$\begin{aligned}
 x &= \sum_{\lambda} p_1(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) + p_2(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) + p_3(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''}), \\
 a_i &= \mu \sum_{\lambda} [2p_1(\lambda) p(+ | \sigma_i^A, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) + 2p_2(\lambda) p(+ | \sigma_i^A, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) \\
 &\quad + 2p_3(\lambda) p_Q(+, c | \sigma_i^A, C) \\
 &\quad - p_1(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) - p_2(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) - p_3(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''})], \\
 &= \mu \sum_{\lambda} [p_1(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) (2p(+ | \sigma_i^A, \lambda) - 1) + p_2(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) (2p(+ | \sigma_i^A) - 1) \\
 &\quad + p_3(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''}) \times (2 \frac{p_Q(+, c | \sigma_i^A, C)}{p_Q(c | C, \rho_{\lambda}^{\gamma''})} - 1)] \\
 &= \mu \sum_{\lambda} [p_1(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) (2p(+ | \sigma_i^A, \lambda) - 1) + p_2(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) (2p(+ | \sigma_i^A) - 1) \\
 &\quad + p_3(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''}) \times \frac{p_Q(+, c | \sigma_i^A, C) - p_Q(-, c | \sigma_i^A, C)}{p_Q(c | C, \rho_{\lambda}^{\gamma''})}], \\
 b_i &= \mu \sum_{\lambda} [2p_1(\lambda) p(+ | \sigma_i^B, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) + 2p_2(\lambda) p_Q(+, c | \sigma_i^B, C) \\
 &\quad + 2p_3(\lambda) p(+ | \sigma_i^B, \lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''}) \\
 &\quad - p_1(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) - p_2(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) - p_3(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''})] \\
 &= \mu \sum_{\lambda} [p_1(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma}) (2p(+ | \sigma_i^B, \lambda) - 1) + p_3(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma''}) (2p(+ | \sigma_i^B, \lambda) - 1) \\
 &\quad + p_2(\lambda) p_Q(c | C, \rho_{\lambda}^{\gamma'}) \times \frac{p_Q(+, c | \sigma_i^B, C) - p_Q(-, c | \sigma_i^B, C)}{p_Q(c | C, \rho_{\lambda}^{\gamma'})}]
 \end{aligned}$$

$$\begin{aligned}
 c_{ij} &= \mu \sum_{\lambda} \{ [2p_1(\lambda)p(+, +|\sigma_i^A, \sigma_j^B, \lambda)p_Q(c|C, \rho_{\lambda}^{\gamma}) + 2p_2(\lambda)p(+|\sigma_i^A, \lambda) \\
 &\quad \times p_Q(+, c|\sigma_j^B, C) + 2p_3(\lambda)p(+|\sigma_j^B, \lambda)p_Q(+, c|\sigma_i^A, C)] \\
 &\quad + \sum_{\lambda} [2p_1(\lambda)p(-, -|\sigma_i^A, \sigma_j^B, \lambda)p_Q(c|C, \rho_{\lambda}^{\gamma}) + 2p_2(\lambda)p(-|\sigma_i^A, \lambda)p_Q(-, c|\sigma_j^B, C) \\
 &\quad + 2p_3(\lambda)p(-|\sigma_j^B, \lambda)p_Q(-, c|\sigma_i^A, C)] \\
 &\quad - \sum_{\lambda} [p_1(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma}) + p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma'}) + p_3(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma''})] \} \\
 &= \mu \sum_{\lambda} \{ p_1(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma})(2p(+, +|\sigma_i^A, \sigma_j^B, \lambda) + 2p(-, -|\sigma_i^A, \sigma_j^B, \lambda) - 1) \\
 &\quad + p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma'}) \\
 &\quad \times [2 \frac{p(+|\sigma_i^A, \lambda)p_Q(+, c|\sigma_j^B, C) + p(-|\sigma_i^A, \lambda)p_Q(-, c|\sigma_j^B, C)}{p_Q(c|C, \rho_{\lambda}^{\gamma'})} - 1] \\
 &\quad + p_3(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma''}) \\
 &\quad \times [2 \frac{p(+|\sigma_j^B, \lambda)p_Q(+, c|\sigma_i^A, C) + p(-|\sigma_j^B, \lambda)p_Q(-, c|\sigma_i^A, C)}{p_Q(c|C, \rho_{\lambda}^{\gamma''})} - 1] \} \\
 &= \mu \sum_{\lambda} \{ p_1(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma})(2p(+, +|\sigma_i^A, \sigma_j^B, \lambda) + 2p(-, -|\sigma_i^A, \sigma_j^B, \lambda) - 1) \\
 &\quad + p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma'})(2p(+|\sigma_i^A) - 1) \frac{p_Q(+, c|\sigma_j^B, C) - p_Q(+, c|\sigma_j^B, C)}{p_Q(c|C, \rho_{\lambda}^{\gamma'})} \\
 &\quad + p_3(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma''}) [(2p(+|\sigma_j^B) - 1) \frac{p_Q(+, c|\sigma_i^A, C) - p_Q(-, c|\sigma_i^A, C)}{p_Q(c|C, \rho_{\lambda}^{\gamma''})}] \}.
 \end{aligned}$$

**Step 4.** Denoting  $\Delta_1$ ,  $\Delta_2$  and  $\Delta_3$  the terms related to  $p_1(\lambda)$ ,  $p_2(\lambda)$  and  $p_3(\lambda)$  in  $\delta_C^{\gamma}$ , respectively, we have

$$\Delta_1 = \sum_{\lambda} p_1(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma})\rho_{\lambda}^{\alpha\beta}$$

with

$$\begin{aligned}
 \rho_{\lambda}^{\alpha\beta} &= \frac{1}{4} [I_4 + \mu \sum_i (2p(+|\sigma_i^A, \lambda) - 1)\sigma_i \otimes I_2 + \mu \sum_i (2p(+|\sigma_i^B, \lambda) - 1)I_2 \otimes \sigma_i \\
 &\quad + \mu \sum_{ij} (2p(+, +|\sigma_i^A, \sigma_j^B, \lambda) + 2p(-, -|\sigma_i^A, \sigma_j^B, \lambda) - 1)\sigma_i \otimes \sigma_j].
 \end{aligned}$$

$$\begin{aligned}
 \Delta_2 &= \frac{1}{4} \sum_{\lambda} p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma'}) [I_4 + \mu \sum_i (2p(+|\sigma_i^A, \lambda) - 1)\sigma_i \otimes I_2 \\
 &\quad + \mu \sum_j \frac{p_Q(+, c|\sigma_j^B, C) - p_Q(-, c|\sigma_j^B, C)}{p_Q(c|C, \rho_{\lambda}^{\gamma'})} \\
 &\quad \times [I_2 + \sum_i (2p(+|\sigma_i^A) - 1)\sigma_i] \otimes \sigma_j] \\
 &= \frac{1}{4} \sum_{\lambda} p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma'}) [I_4 + \mu \sum_i (\alpha_i \sigma_i \otimes I_2 + \beta_i I_2 \otimes \sigma_i) + \mu \sum_i \alpha_i \sigma_i \sum_j \beta_j \sigma_j] \\
 &\equiv \sum_{\lambda} p_2(\lambda)p_Q(c|C, \rho_{\lambda}^{\gamma'}) \Omega
 \end{aligned}$$

with  $\alpha_i = 2p(+|\sigma_i^A, \lambda) - 1$  and  $\beta_i = \frac{p_Q(+,c|\sigma_i^B,C) - p_Q(-,c|\sigma_i^B,C)}{p_Q(c|C,\rho_\lambda^\gamma)}$ , and  $\Omega = \frac{1}{4}[\mathbf{I}_4 + \mu \sum_i (\alpha_i \sigma_i \otimes \mathbf{I}_2 + \beta_i \mathbf{I}_2 \otimes \sigma_i) + \mu \sum_i \alpha_i \sigma_i \sum_j \beta_j \sigma_j]$ .

$$\begin{aligned} \Delta_3 &= \frac{1}{4} \sum_\lambda p_3(\lambda) p_Q(c|C, \rho_\lambda^{\gamma''}) [\mathbf{I}_4 + \mu \sum_i (\alpha'_i \sigma_i \otimes \mathbf{I}_2 + \mathbf{I}_2 \otimes \beta'_i \sigma_i) + \mu \sum_i \alpha'_i \sigma_i \otimes \sum_j \beta'_j \sigma_j] \\ &\equiv \sum_\lambda p_3(\lambda) p_Q(c|C, \rho_\lambda^{\gamma''}) \omega \end{aligned}$$

with  $\alpha'_i = \frac{p_Q(+,c|\sigma_i^A,C) - p_Q(-,c|\sigma_i^A,C)}{p_Q(c|C,\rho_\lambda^{\gamma''})}$ ,  $\beta'_i = 2p(+|\sigma_i^B, \lambda) - 1$ , and  $\omega = \frac{1}{4}[\mathbf{I}_4 + \mu \sum_i (\alpha'_i \sigma_i \otimes \mathbf{I}_2 + \mathbf{I}_2 \otimes \beta'_i \sigma_i) + \mu \sum_i \alpha'_i \sigma_i \otimes \sum_j \beta'_j \sigma_j]$ .

**Step 5.** We now prove that  $\rho_\lambda^{\alpha\beta}$  in  $\Delta_1$ ,  $\Omega$  in  $\Delta_2$  and  $\omega$  in  $\Delta_3$  are quantum states when  $\mu$  satisfies certain conditions.

As for  $\Delta_1$ ,  $\rho_\lambda^{\alpha\beta}$  can be proved to be quantum states by decomposing  $1, 2p(+|\sigma_i^A, \lambda) - 1, 2p(+|\sigma_j^B, \lambda) - 1$  and  $2p(+, +|\sigma_i^A, \sigma_j^B, \lambda) + 2p(-, -|\sigma_i^A, \sigma_j^B, \lambda) - 1$  into joint probabilities that Alice and Bob perform the measurements  $M_A^a$  and  $M_B^b$ , respectively. Noting that

$$\begin{aligned} 1 &= \frac{1}{9} \sum_{ij} (p(+, +|\sigma_i^A, \sigma_j^B, \lambda) + p(+, -|\sigma_i^A, \sigma_j^B, \lambda) + p(-, +|\sigma_i^A, \sigma_j^B, \lambda) \\ &\quad + p(-, -|\sigma_i^A, \sigma_j^B, \lambda)), \\ 2p(+|\sigma_i^A, \lambda) - 1 &= p(+, +|\sigma_i^A, \sigma_j^B, \lambda) + p(+, -|\sigma_i^A, \sigma_j^B, \lambda) - p(-, +|\sigma_i^A, \sigma_j^B, \lambda) - p(-, -|\sigma_i^A, \sigma_j^B, \lambda), \\ 2p(+|\sigma_j^B, \lambda) - 1 &= p(+, +|\sigma_i^A, \sigma_j^B, \lambda) - p(+, -|\sigma_i^A, \sigma_j^B, \lambda) + p(-, +|\sigma_i^A, \sigma_j^B, \lambda) - p(-, -|\sigma_i^A, \sigma_j^B, \lambda), \\ 2p(+, +|\sigma_i^A, \sigma_j^B, \lambda) + 2p(-, -|\sigma_i^A, \sigma_j^B, \lambda) - 1 &= p(+, +|\sigma_i^A, \sigma_j^B, \lambda) - p(+, -|\sigma_i^A, \sigma_j^B, \lambda) - p(-, +|\sigma_i^A, \sigma_j^B, \lambda) + p(-, -|\sigma_i^A, \sigma_j^B, \lambda), \end{aligned}$$

we have

$$\begin{aligned} \Delta_1 &= \frac{1}{4} \sum_\lambda p_1(\lambda) p_Q(c|C, \rho_\lambda^\gamma) \\ &\quad \times \sum_{ij} [p(+, +|\sigma_i^A, \sigma_j^B, \lambda) (\frac{1}{9} \mathbf{I}_4 + \mu (\frac{1}{3} \sigma_i \otimes \mathbf{I}_2 + \frac{1}{3} \mathbf{I}_2 \otimes \sigma_j + \sigma_i \otimes \sigma_j)) \\ &\quad + p(+, -|\sigma_i^A, \sigma_j^B, \lambda) (\frac{1}{9} \mathbf{I}_4 + \mu (\frac{1}{3} \sigma_i \otimes \mathbf{I}_2 - \frac{1}{3} \mathbf{I}_2 \otimes \sigma_j - \sigma_i \otimes \sigma_j)) \\ &\quad + p(-, +|\sigma_i^A, \sigma_j^B, \lambda) (\frac{1}{9} \mathbf{I}_4 + \mu (-\frac{1}{3} \sigma_i \otimes \mathbf{I}_2 + \frac{1}{3} \mathbf{I}_2 \otimes \sigma_j - \sigma_i \otimes \sigma_j)) \\ &\quad + p(-, -|\sigma_i^A, \sigma_j^B, \lambda) (\frac{1}{9} \mathbf{I}_4 + \mu (-\frac{1}{3} \sigma_i \otimes \mathbf{I}_2 - \frac{1}{3} \mathbf{I}_2 \otimes \sigma_j + \sigma_i \otimes \sigma_j))]. \end{aligned}$$

When  $0 \leq \mu \leq \frac{1}{9}$ , the matrices  $\frac{1}{9} \mathbf{I}_4 + \mu (\pm \frac{1}{3} \sigma_i \otimes \mathbf{I}_2 \pm \frac{1}{3} \mathbf{I}_2 \otimes \sigma_i \pm \sigma_i \otimes \sigma_i)$ ,  $i = 1, 2, 3$ , are semi-definite positive matrices. Hence,  $\rho_\lambda^{\alpha\beta}$  is a quantum state shared by Alice and Bob. By direct numerical calculation  $\rho_\lambda^{\alpha\beta}$  is a quantum state when  $0 \leq \mu \leq 0.23$ .

$\Omega$  in  $\Delta$  can be proved to be a quantum state by decomposing  $\sum_i \alpha_i \sigma_i$ ,  $\sum_i \beta_i \sigma_i$  and  $I_4$  into the eigenvectors of  $\sum_i \alpha_i \sigma_i$  and  $\sum_i \beta_i \sigma_i$ . Since

$$\begin{aligned} \sum_i \alpha_i \sigma_i &= |\alpha_0|(|\phi\rangle_{\mathcal{A}}\langle\phi| - |\phi^\perp\rangle_{\mathcal{A}}\langle\phi^\perp|) \\ \sum_i \beta_i \sigma_i &= |\beta_0|(|\psi\rangle_{\mathcal{B}}\langle\psi| - |\psi^\perp\rangle_{\mathcal{B}}\langle\psi^\perp|) \end{aligned}$$

with  $|\alpha_0| = \sqrt{\sum_i \alpha_i^2}$ ,  $|\beta_0| = \sqrt{\sum_i \beta_i^2}$  and  $I_4 = (|\phi\rangle_{\mathcal{A}}\langle\phi| + |\phi^\perp\rangle_{\mathcal{A}}\langle\phi^\perp|) \otimes (|\psi\rangle_{\mathcal{B}}\langle\psi| + |\psi^\perp\rangle_{\mathcal{B}}\langle\psi^\perp|)$ , concerning  $\Delta_2$  we have

$$\begin{aligned} \Delta_2 &= \frac{1}{4} \sum_\lambda p_2(\lambda) p_Q(c|C, \rho_\lambda^{\gamma'}) [(1 + \mu(|\alpha_0\beta_0| + |\alpha_0| + |\beta_0|)) |\phi\rangle_{\mathcal{A}}\langle\phi| \otimes |\psi\rangle_{\mathcal{B}}\langle\psi| \\ &\quad + (1 + \mu(-|\alpha_0\beta_0| + |\alpha_0| - |\beta_0|)) |\phi\rangle_{\mathcal{A}}\langle\phi| \otimes |\psi^\perp\rangle_{\mathcal{B}}\langle\psi^\perp| \\ &\quad + (1 + \mu(-|\alpha_0\beta_0| - |\alpha_0| + |\beta_0|)) |\phi^\perp\rangle_{\mathcal{A}}\langle\phi^\perp| \otimes |\psi\rangle_{\mathcal{B}}\langle\psi| \\ &\quad + (1 + \mu(|\alpha_0\beta_0| - |\alpha_0| - |\beta_0|)) |\phi^\perp\rangle_{\mathcal{A}}\langle\phi^\perp| \otimes |\psi^\perp\rangle_{\mathcal{B}}\langle\psi^\perp|] \\ &\equiv \sum_\lambda p_2(\lambda) p_Q(c|C, \rho_\lambda^{\gamma'}) \Omega. \end{aligned}$$

Let  $q_1(\lambda) = \frac{1}{4}(1 + \mu(|\alpha_0\beta_0| + |\alpha_0| + |\beta_0|))$ ,  $q_2(\lambda) = \frac{1}{4}(1 + \mu(-|\alpha_0\beta_0| + |\alpha_0| - |\beta_0|))$ ,  $q_3(\lambda) = \frac{1}{4}(1 + \mu(-|\alpha_0\beta_0| - |\alpha_0| + |\beta_0|))$  and  $q_4(\lambda) = \frac{1}{4}(1 + \mu(|\alpha_0\beta_0| - |\alpha_0| - |\beta_0|))$ ,  $\rho_\lambda^{\alpha,i} = |\phi\rangle_{\mathcal{A}}\langle\phi|$  or  $|\phi^\perp\rangle_{\mathcal{A}}\langle\phi^\perp|$  and  $\rho_\lambda^{\beta,i} = |\psi\rangle_{\mathcal{B}}\langle\psi|$  or  $|\psi^\perp\rangle_{\mathcal{B}}\langle\psi^\perp|$ . Since  $|\alpha_i| \leq 1$  and  $|\beta_i| \leq 1$  ( $i = 1, 2, 3$ ),  $|\alpha_0| \leq \sqrt{3}$  and  $|\beta_0| \leq \sqrt{3}$ , the maximum value of  $q_1(\lambda)$  is  $\frac{1}{4}(1 + \mu(3 + 2\sqrt{3}))$ , the minimum values of  $q_2(\lambda)$  and  $q_3(\lambda)$  are  $\frac{1}{4}(1 - 3\mu)$  and the minimum value of  $q_4(\lambda)$  is  $\frac{1}{4}(1 - \sqrt{3}\mu)$ . One verifies that when  $0 \leq \mu \leq \frac{1}{3}$ , the coefficients  $q_i(\lambda)$  ( $i = 1, \dots, 4$ ) are all positive, and their summation is one. The matrix  $\Omega$  is a quantum state with the first subsystem determined by party  $\mathcal{A}$  and the second subsystem determined by parties  $\mathcal{B}$  and  $\mathcal{C}$ . Denote  $\Omega = \sum_i q_i(\lambda) \rho_\lambda^{\alpha,i} \otimes \rho_\lambda^{\beta,i}$  with  $\sum_i q_i(\lambda) = 1$ .

Similarly, we can prove that

$$\begin{aligned} \Delta_3 &= \frac{1}{4} \sum_\lambda p_3(\lambda) p_Q(c|C, \rho_\lambda^{\gamma''}) [I_4 + \mu \sum_i (\alpha'_i \sigma_i \otimes I_2 + I_2 \otimes \beta'_i \sigma_i) + \mu \sum_i \alpha'_i \sigma_i \otimes \sum_j \beta'_j \sigma_j] \\ &\equiv \sum_\lambda p_3(\lambda) p_Q(c|C, \rho_\lambda^{\gamma''}) \omega \end{aligned}$$

with  $\alpha'_i = \frac{p_Q(+,c|\sigma_i^{\mathcal{A}},\mathcal{C}) - p_Q(-,c|\sigma_i^{\mathcal{A}},\mathcal{C})}{p_Q(c|C,\rho_\lambda^{\gamma''})}$  and  $\beta'_i = 2p(+|\sigma_i^{\mathcal{B}},\lambda) - 1$ . When  $0 \leq \mu \leq \frac{1}{3}$  we verify that  $\omega$  is a quantum state determined by parties  $\mathcal{A}$ ,  $\mathcal{C}$  and  $\mathcal{B}$ . We denote  $\omega = \sum_i q'_i(\lambda) \rho_\lambda^{\alpha',i} \otimes \rho_\lambda^{\beta',i}$  satisfying  $\sum_i q'_i(\lambda) = 1$ .

Therefore,

$$\begin{aligned} p(a, b, c|M_A, M_B, M_C) &= \text{Tr}[M_A^a \otimes M_B^b \otimes M_C^c \cdot \tau_{ABC}^2] = \text{Tr}[M_A^a \otimes M_B^b \cdot \delta_C^c] \\ &= \sum_\lambda p_1(\lambda) p_Q(a, b|A, B, \rho_\lambda^{\alpha\beta}) p_Q(c|C, \rho_\lambda^\gamma) \\ &\quad + \sum_{\lambda'} p'_2(\lambda') p_Q(a|A, \rho_\lambda^{\alpha\lambda'}) p_Q(b, c|B, C, \rho_\lambda^{\beta\gamma'}) \\ &\quad + \sum_{\lambda''} p'_3(\lambda'') p_Q(a, c|A, C, \rho_\lambda^{\alpha\gamma''}) p_Q(b|B, \rho_\lambda^{\beta\lambda''}) \end{aligned}$$

with  $p'_2(\lambda') = p_2(\lambda)q_i(\lambda)$ ,  $p_Q(b, c|B, C, \rho_{\lambda'}^{\beta\gamma'}) = p_Q(c|C, \lambda, \rho_{\lambda'}^{\gamma'})p_Q(b|B, \rho_{\lambda'}^{\beta,i})$ ,  $\rho_{\lambda'}^{\alpha} = \rho_{\lambda}^{\alpha,i}$  and  $p'_3(\lambda'') = p_3(\lambda)q'_i(\lambda)$ ,  $p_Q(a, c|A, C, \rho_{\lambda''}^{\alpha\gamma'}) = p_Q(c|C, \lambda, \rho_{\lambda''}^{\gamma'})p_Q(a|A, \rho_{\lambda''}^{\alpha,i})$ ,  $\rho_{\lambda''}^{\beta} = \rho_{\lambda}^{\beta,i}$ . Since  $\sum_{\lambda} q_i(\lambda) = 1$  and  $\sum_{\lambda} q'_i(\lambda) = 1$ ,  $\sum_{\lambda} p_1(\lambda) + \sum_{\lambda'} p'_2(\lambda') + \sum_{\lambda''} p'_3(\lambda'') = 1$ . Namely, if  $\rho_{ABC}$  is not genuine tripartite steerable from Alice and Bob to Charlie, then  $\tau_{ABC}^2$  is bi-separable for  $0 \leq \mu \leq \frac{1}{9}$  analytically.

**Step 3'. (II)** We next prove that  $\delta_C^{\gamma}$  is the convex combination of some quantum states when  $\mu$  satisfies certain conditions for tripartite steering from Alice and Bob to Charlie. If  $\rho_{ABC}$  is not tripartite steerable from Alice and Bob to Charlie, we have the joint probabilities admitting LHV-LHS models,

$$p(a, b, c|A, B, C) = \sum_{\lambda} p(\lambda)p(a|A, \lambda)p(b|B, \lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}). \tag{A6}$$

Specially,

$$p(c|C) = \sum_{\lambda} p(\lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}), \tag{A7}$$

$$p(\pm, c|\sigma_i^A, C) = \sum_{\lambda} p(\lambda)p(\pm|\sigma_i^A, \lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}), \tag{A8}$$

$$p(\pm, c|\sigma_j^B, C) = \sum_{\lambda} p(\lambda)p(\pm|\sigma_j^B, \lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}), \tag{A9}$$

and

$$p(\pm, \pm, c|\sigma_i^A, \sigma_j^B, C) = \sum_{\lambda} p(\lambda)p(\pm|\sigma_i^A, \lambda)p(\pm|\sigma_j^B, \lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}) \tag{A10}$$

Substituting Equations (A7)–(A10) into the expressions of  $x$ ,  $a_i$ ,  $b_i$  and  $c_{ij}$  ( $i = 1, 2, 3$ ,  $j = 1, 2, 3$ ), we have

$$\begin{aligned} x &= \sum_{\lambda} p(\lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}), \\ a_i &= \mu \sum_{\lambda} [2p(\lambda)p(+|\sigma_i^A)p_Q(c|C, \tau_{\lambda}^{\gamma}) - p(\lambda)p_Q(c|C, \tau_{\lambda}^{\gamma})], \\ b_i &= \mu \sum_{\lambda} [2p(\lambda)p(+|\sigma_i^B)p_Q(c|C, \tau_{\lambda}^{\gamma}) - p(\lambda)p_Q(c|C, \tau_{\lambda}^{\gamma})], \\ c_{ij} &= \mu \sum_{\lambda} [2p(\lambda)p(+|\sigma_i^A, \lambda)p(+|\sigma_j^B, \lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}) \\ &\quad + 2p(\lambda)p(-|\sigma_i^A, \lambda)p(-|\sigma_j^B, \lambda)p_Q(c|C, \tau_{\lambda}^{\gamma}) - p(\lambda)p_Q(c|C, \tau_{\lambda}^{\gamma})]. \end{aligned}$$

Therefore,

$$\begin{aligned} \delta_C^{\gamma} &= \frac{1}{4} \sum_{\lambda} p(\lambda)p_Q(c|C, \tau_{\lambda}^{\gamma})(\mathbf{I}_4 + \mu \sum_i (2p(+|\sigma_i^A) - 1)\sigma_i \otimes \mathbf{I}_2 \\ &\quad + \mu \sum_i (2p(+|\sigma_i^B) - 1)\mathbf{I}_2 \otimes \sigma_i \\ &\quad + \mu \sum_{ij} (2p(+|\sigma_i^A)p(+|\sigma_j^B) + 2p(-|\sigma_i^A)p(-|\sigma_j^B) - 1)\sigma_i \otimes \sigma_j). \end{aligned}$$

Since

$$\begin{aligned} &\sum_{ij} 2p(+|\sigma_i^A, \lambda)p(+|\sigma_j^B, \lambda) + 2p(-|\sigma_i^A, \lambda)p(-|\sigma_j^B, \lambda) - 1 \\ &= \sum_i (2p(+|\sigma_i^A, \lambda) - 1) \sum_j (2p(+|\sigma_j^B, \lambda) - 1), \end{aligned}$$

$\delta_C^{\gamma}$  can be written as

$$\delta_C^{\gamma} = p(\lambda)p_Q(c|C, \tau_{\lambda}^{\gamma})\chi, \tag{A11}$$

where  $\chi = \frac{1}{4}[\mathbb{I}_4 + \sum_i a_i'' \sigma_i \otimes \mathbb{I}_2 + \sum_i b_i'' \sigma_i \mathbb{I}_2 \otimes \sigma_i + \sum_i a_i'' \sigma_i \otimes \mathbb{I}_2 \sum_j b_j'' \sigma_j \mathbb{I}_2 \otimes \sigma_i]$  with  $a_i'' = 2p(+|\sigma_i^A) - 1$  and  $b_i'' = 2p(+|\sigma_i^B) - 1 (i = 1, 2, 3)$ . With  $\Delta_2$  and  $\Delta_3$  in Step 5, we can prove that  $\delta_C^c$  is a quantum state when  $0 \leq \mu \leq \frac{1}{3}$ , which implies that  $\tau_{ABC}^2$  is fully separable. Namely, if  $\rho_{ABC}$  is not tripartite steerable from Alice to Bob and Charlie, then  $\tau_{ABC}^2$  is fully separable for  $0 \leq \mu \leq \frac{1}{3}$ .  $\square$

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