

## Article

# Some Properties of Weighted Tsallis and Kaniadakis Divergences

Răzvan-Cornel Sfetcu <sup>1,\*</sup>, Sorina-Cezarina Sfetcu <sup>1</sup> and Vasile Preda <sup>1,2,3</sup> 
<sup>1</sup> Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, 010014 Bucharest, Romania

<sup>2</sup> “Gheorghe Mihoc-Caius Iacob” Institute of Mathematical Statistics and Applied Mathematics, Calea 13 Septembrie 13, 050711 Bucharest, Romania

<sup>3</sup> “Costin C. Kirițescu” National Institute of Economic Research, Calea 13 Septembrie 13, 050711 Bucharest, Romania

\* Correspondence: razvan.sfetcu@fmi.unibuc.ro

**Abstract:** We are concerned with the weighted Tsallis and Kaniadakis divergences between two measures. More precisely, we find inequalities between these divergences and Tsallis and Kaniadakis logarithms, prove that they are limited by similar bounds with those that limit Kullback–Leibler divergence and show that are pseudo-additive.

**Keywords:** Tsallis logarithm; Kaniadakis logarithm; weighted Tsallis divergence; weighted Kaniadakis divergence



**Citation:** Sfetcu, R.-C.; Sfetcu, S.-C.; Preda, V. Some Properties of Weighted Tsallis and Kaniadakis Divergences. *Entropy* **2022**, *24*, 1616. <https://doi.org/10.3390/e24111616>

Academic Editors: Karagrigoriou Alexandros and Makrides Andreas

Received: 29 September 2022

Accepted: 2 November 2022

Published: 5 November 2022

**Publisher’s Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

Shannon entropy, in the form we know it, was introduced by Boltzmann and used by Shannon in the context of Information Theory. This entropy has applications in Statistical Thermodynamics, Combinatorics and Machine Learning. In Machine Learning, Shannon entropy represents the basis for building decision trees and fitting classification models.

In the last couple of years, many generalizations of Shannon entropy appeared: Tsallis entropy, Kaniadakis entropy, Rényi entropy, Varma entropy, weighted entropy, relative entropy, cumulative entropy, etc. These entropies have applications in areas such as Physics, Information Theory, Probabilities, Communication Theory, and Statistics.

Tsallis entropy was introduced by C. Tsallis in [1] and is applied to: Income distribution (see [2,3]), Internet (see [4]), Non-coding human DNA (see [5]), Plasma (see [6]), and Stock exchanges (see [7,8]).

Kaniadakis entropy was introduced by G. Kaniadakis in [9] and is useful in many areas like: Finance (see [10–12]), Astrophysics (see [13,14]), Networks (see [15,16]), Economics (see [17,18]), and Statistical Mechanics (see [19–21]).

S. Kullback and R.A. Leibler were concerned “to measure” “the distance” or “the divergence” between statistical populations and they generalized Shannon entropy by defining, in [22], a nonsymmetric measure, called Kullback–Leibler divergence. This divergence between two probability measures  $\mu_1$  and  $\mu_2$  on a measurable non-negligible set  $A$  is additive, non-negative and greater than  $\log\left(\frac{\mu_1(A)}{\mu_2(A)}\right)$ , where  $\log$  is the classical logarithm function. Divergences are a key tool in Information Geometry (see [23]).

The goodness of fit test is based on the Corrected Weighted Kullback–Leibler divergence (see [24]) and, as a consequence, it inherits all special characteristics of this divergence measure. Narowcki and Harding proposed the use of weighted entropy as a measure of investment risk (see [25]). Afterwards, Guiaşu used the weighted entropy to group data with respect to the importance of specific regions of the domain (see [26]), Di Crescenzo and Longobardi propose the weighted residual and past entropies (see [27]) and Suhov and

Zohren proposed the quantum version of weighted entropy and its properties in Quantum Statistical Mechanics (see [28]).

Working with the Kullback–Leibler divergence formula and using the same technique like in the cases of Tsallis and Kaniadakis entropies (i.e., classical logarithm is replaced by Tsallis logarithm, respectively, by Kaniadakis logarithm), Tsallis and Kaniadakis divergences were introduced in some papers (see [29–32]).

Motivated by the aforementioned facts and by the papers [33–35], we deal with the weighted Tsallis and Kaniadakis divergences in this article.

In the following, we briefly describe the structure of the paper. Section 2 is dedicated to preliminaries. In Section 3, using some inequalities concerning the Tsallis logarithm, we obtain inequalities between the weighted Tsallis and Kaniadakis divergences on a non-negligible measurable arbitrary set and Tsallis logarithm, respectively, Kaniadakis logarithm (see Theorem 1). Finally, we prove that the weighted Tsallis and Kaniadakis divergences are limited by bounds that are similar to those that limit Kullback–Leibler divergence (see Theorem 2). In Section 4, we define the weighted Tsallis and Kaniadakis divergences for product measure spaces and prove some pseudo-additivity properties for them (see Theorem 3).

Other interesting results related to the present topics can be found in: [36–40].

## 2. Preliminary Facts

**Definition 1.** Let  $k \in \mathbb{R}^*$ . We consider the Tsallis logarithm given by

$$\log^T x = \begin{cases} \frac{x^k - 1}{k} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and the Kaniadakis logarithm given via

$$\log^K x = \begin{cases} \frac{x^k - x^{-k}}{2k} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

**Remark 1.** It is easy to see that  $\log_k^K x = \frac{1}{2}(\log_k^T x + \log_{-k}^T x)$  for any  $x \geq 0$ .

We have  $\lim_{k \rightarrow 0} \log_k^T x = \lim_{k \rightarrow 0} \log_k^K x = \log x$  for any  $x > 0$  (“log” is the classical logarithm function).

**Definition 2.** Let  $(\Omega, \mathcal{T})$  be a measurable space and  $\mu, \nu : \mathcal{T} \rightarrow \overline{\mathbb{R}}_+ = [0, \infty) \cup \{\infty\}$  two measures. We say that  $\mu$  is absolutely continuous with respect to  $\nu$  if, for any  $A \in \mathcal{T}$  such that  $\nu(A) = 0$ , one has  $\mu(A) = 0$ .

**Notation 1.** If  $\mu$  and  $\nu$  are absolutely continuous with respect to each other, we denote this fact by  $\mu \sim \nu$ .

In the absence of other mentions, we work in the following scenario: Let  $(\Omega, \mathcal{T}, \mu_i)$ ,  $i = 1, 2$  be two measure spaces and  $\lambda$  a measure on  $(\Omega, \mathcal{T})$  such that  $\mu_i \sim \lambda$  for any  $i = 1, 2$ . With the help of Radon–Nikodým Theorem we find two non-negative measurable functions  $f_1$  and  $f_2$  defined on  $\Omega$  such that  $\mu_i(A) = \int_A f_i d\lambda$  for any  $A \in \mathcal{T}$  and any  $i = 1, 2$ . Consider  $w : \Omega \rightarrow (0, \infty)$  a weight function (i.e.,  $w$  is a non-negative measurable function).

**Definition 3.** Let  $A \in \mathcal{T}$ . The weighted Tsallis divergence on  $A$  between  $\mu_1$  and  $\mu_2$  is defined via

$$D_k^{w,T}(\mu_1|\mu_2, A) = \begin{cases} \frac{1}{\int_A w d\mu_1} \int_A w \log_k^T\left(\frac{f_1}{f_2}\right) d\mu_1 & \text{if } \mu_1(A) \neq 0 \\ 0 & \text{if } \mu_1(A) = 0 \end{cases}$$

and the weighted Kaniadakis divergence on  $A$  between  $\mu_1$  and  $\mu_2$  is given by

$$D_k^{w,K}(\mu_1|\mu_2, A) = \begin{cases} \frac{1}{\int_A w d\mu_1} \int_A w \log_k^K\left(\frac{f_1}{f_2}\right) d\mu_1 & \text{if } \mu_1(A) \neq 0 \\ 0 & \text{if } \mu_1(A) = 0. \end{cases}$$

**Remark 2.** We assume that all divergences and integrals which appear in this paper are finite.

**Remark 3.** We can see that the values of  $D_k^{w,T}(\mu_1|\mu_2, A)$  and  $D_k^{w,K}(\mu_1|\mu_2, A)$  do not depend on the choice of reference measure  $\lambda$  (because  $\frac{f_1}{f_2} = \frac{d\mu_1/d\lambda}{d\mu_2/d\lambda} = \frac{d\mu_1}{d\mu_2}$ ).

### 3. Bounds of the Weighted Tsallis and Kaniadakis Divergences

The proof of the following lemma is elementary and is omitted.

**Lemma 1.** For any  $x \in (0, \infty) \setminus \{1\}$ , let  $\varphi_x : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ ,

$$\varphi_x(t) = \frac{x^t - 1}{t}.$$

The function  $\varphi_x$  is strictly increasing.

The next two corollaries are very useful in this article.

**Corollary 1.** Let  $x > 0$  and  $k \in (-1, \infty) \setminus \{0\}$ . Then,

$$x \log_k^T x \geq x - 1$$

and the equality is valid if and only if  $x = 1$ .

**Corollary 2.** Let  $x > 0$  and  $k \in \left(-\frac{1}{2}, 1\right) \setminus \{0\}$ . Then,

$$\frac{2(\sqrt{x} - 1)}{\sqrt{x}} \leq \log_k^T x \leq x - 1.$$

We have equality in these inequalities if and only if  $x = 1$ .

**Theorem 1.** Let  $A \in \mathcal{T}$  such that  $\mu_i(A) \neq 0$  for any  $i = 1, 2$ .

(a) Assume that  $k \in (-1, \infty) \setminus \{0\}$ . Then,

$$D_k^{w,T}(\mu_1|\mu_2, A) \geq \log_k^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right).$$

(b) Assume that  $k \in (-1, 1) \setminus \{0\}$ . Then,

$$D_k^{w,K}(\mu_1|\mu_2, A) \geq \log_k^K \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right).$$

In both cases, the equality holds if and only if  $\frac{f_1(\omega)}{f_2(\omega)} = \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \lambda - a.e.$  for  $\omega \in A$ .

**Proof.** (a) We will make the proof in two steps.

*Step 1.* Assume that  $\int_A w d\mu_1 = \int_A w d\mu_2$ . Because  $\log_k^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right) = 0$ , we have to show that  $D_k^{w,T}(\mu_1|\mu_2, A) \geq 0$ .

According to Corollary 1, we have

$$\begin{aligned} D_k^{w,T}(\mu_1|\mu_2, A) &= \frac{1}{\int_A w d\mu_1} \int_A w \log_k^T \left( \frac{f_1}{f_2} \right) d\mu_1 = \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{f_2}{f_1} \cdot \frac{f_1}{f_2} \log_k^T \left( \frac{f_1}{f_2} \right) d\mu_1 \geq \\ &= \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{f_2}{f_1} \left( \frac{f_1}{f_2} - 1 \right) d\mu_1 = \frac{1}{\int_A w d\mu_1} \int_A w \left( 1 - \frac{f_2}{f_1} \right) d\mu_1 = \\ &= 1 - \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{f_2}{f_1} d\mu_1 = 1 - \frac{\int_A w d\mu_2}{\int_A w d\mu_1} = 0. \end{aligned}$$

The equality holds if and only if  $\frac{f_1(\omega)}{f_2(\omega)} = 1 \mu_1 - a.e.$ , i.e., if and only if  $\frac{f_1(\omega)}{f_2(\omega)} = 1 \lambda - a.e.$

*Step 2.* Let  $A \in \mathcal{T}$  with  $\mu_i(A) \neq 0$  for any  $i = 1, 2$ . We define the measures  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$  via

$$\tilde{\mu}_i(B) = \frac{\mu_i(B)}{\int_A w d\mu_i} \text{ for any } B \in \mathcal{T} \text{ and any } i = 1, 2.$$

We remark that  $\int_A w d\tilde{\mu}_1 = \int_A w d\tilde{\mu}_2 = 1$ .

Hence the weighted Tsallis divergence between  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$  on  $A$  is

$$\begin{aligned} D_k^T(\tilde{\mu}_1|\tilde{\mu}_2, A) &= \int_A w \cdot \frac{f_1}{\int_A w d\mu_1} \cdot \log_k^T \left( \frac{\frac{f_1}{\int_A w d\mu_1}}{\frac{f_2}{\int_A w d\mu_2}} \right) d\lambda = \\ &= \int_A w \cdot \frac{f_1}{\int_A w d\mu_1} \cdot \log_k^T \left( \frac{f_1 \int_A w d\mu_2}{f_2 \int_A w d\mu_1} \right) d\lambda. \end{aligned}$$

We deduce from Step 1 that  $D_k^T(\tilde{\mu}_1|\tilde{\mu}_2, A) \geq 0$ .

So,

$$\begin{aligned}
0 &\leq \int_A w \cdot \frac{f_1}{\int_A w d\mu_1} \cdot \log_k^T \left( \frac{f_1 \int_A w d\mu_2}{f_2 \int_A w d\mu_1} \right) d\lambda = \int_A w \cdot \frac{f_1}{\int_A w d\mu_1} \cdot \frac{\left( \frac{f_1 \int_A w d\mu_2}{f_2 \int_A w d\mu_1} \right)^k - 1}{k} d\lambda = \\
&\int_A w \cdot \frac{f_1}{\int_A w d\mu_1} \cdot \frac{\left( \frac{\int_A w d\mu_2}{\int_A w d\mu_1} \right)^k \cdot \left( \frac{f_1}{f_2} \right)^k - 1}{k} d\lambda = \\
&\int_A w \cdot \frac{f_1}{\int_A w d\mu_1} \cdot \frac{\left( \frac{\int_A w d\mu_2}{\int_A w d\mu_1} \right)^k \cdot \left[ \left( \frac{f_1}{f_2} \right)^k - 1 + 1 \right] - 1}{k} d\lambda = \\
&\int_A w \cdot \frac{f_1}{\int_A w d\mu_1} \cdot \left[ \left( \frac{\int_A w d\mu_2}{\int_A w d\mu_1} \right)^k \cdot \frac{\left( \frac{f_1}{f_2} \right)^k - 1}{k} + \frac{\left( \frac{\int_A w d\mu_2}{\int_A w d\mu_1} \right)^k - 1}{k} \right] d\lambda = \\
&\left( \frac{\int_A w d\mu_2}{\int_A w d\mu_1} \right)^k D_k^{w,T}(\mu_1|\mu_2, A) + \frac{\left( \frac{\int_A w d\mu_2}{\int_A w d\mu_1} \right)^k - 1}{k}.
\end{aligned}$$

$$\text{Hence, } D_k^{w,T}(\mu_1|\mu_2, A) \geq \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right)^k \cdot \frac{1 - \left( \frac{\int_A w d\mu_2}{\int_A w d\mu_1} \right)^k}{k} = \log_k^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right).$$

The equality holds if and only if  $\frac{f_1(\omega)}{\int_A w d\mu_1} = \frac{f_2(\omega)}{\int_A w d\mu_2}$   $\lambda$ -a.e. for  $\omega \in A$ , i.e., if and

$$\text{only if } \frac{f_1(\omega)}{f_2(\omega)} = \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \quad \lambda\text{-a.e. for } \omega \in A.$$

(b) Because  $\log_k^K x = \frac{1}{2} (\log_k^T x + \log_{-k}^T x)$ , we obtain

$$\begin{aligned}
D_k^{w,K}(\mu_1|\mu_2) &= \frac{1}{2} (D_k^{w,T}(\mu_1|\mu_2) + D_{-k}^{w,T}(\mu_1|\mu_2)) \geq \\
&\frac{1}{2} \left( \log_k^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right) + \log_{-k}^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right) \right) = \log_k^K \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right) \quad (\text{see (a)}).
\end{aligned}$$

We have equality in the preceding inequality if and only if

$$\frac{1}{2} \left( D_k^{w,T}(\mu_1|\mu_2, A) + D_{-k}^{w,T}(\mu_1|\mu_2, A) \right) = \frac{1}{2} \left( \log_k^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right) + \log_{-k}^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right) \right),$$

which is equivalent to  $D_k^{w,T}(\mu_1|\mu_2, A) = \log_k^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right)$  and  $D_{-k}^{w,T}(\mu_1|\mu_2, A) = \log_{-k}^T \left( \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \right)$  and these are equivalent to  $\frac{f_1(\omega)}{f_2(\omega)} = \frac{\int_A w d\mu_1}{\int_A w d\mu_2} \lambda - a.e.$  for  $\omega \in A$ .  $\square$

**Theorem 2.** Let  $A \in \mathcal{T}$  such that  $\mu_i(A) \neq 0$  for any  $i = 1, 2$  and  $\int_A w d\mu_1 = \int_A w d\mu_2$ .

(a) Assume that  $k \in \left(-\frac{1}{2}, 1\right) \setminus \{0\}$ . Then,

$$\frac{1}{\int_A w d\mu_1} \int_A w \left( \sqrt{\frac{d\mu_1}{d\mu_2}} - 1 \right)^2 d\mu_2 \leq D_k^{w,T}(\mu_1|\mu_2) \leq \frac{1}{\int_A w d\mu_1} \int_A w \left( \frac{d\mu_1}{d\mu_2} - 1 \right)^2 d\mu_2.$$

(b) Assume that  $k \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$ . Then,

$$\frac{1}{\int_A w d\mu_1} \int_A w \left( \sqrt{\frac{d\mu_1}{d\mu_2}} - 1 \right)^2 d\mu_2 \leq D_k^{w,K}(\mu_1|\mu_2) \leq \frac{1}{\int_A w d\mu_1} \int_A w \left( \frac{d\mu_1}{d\mu_2} - 1 \right)^2 d\mu_2.$$

**Proof.** (a) We have (see Corollary 2)

$$\begin{aligned} D_k^{w,T}(\mu_1|\mu_2) &= \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{f_1}{f_2} \cdot \log_k^T \left( \frac{f_1}{f_2} \right) d\mu_2 \geq \\ &= \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{f_1}{f_2} \cdot \frac{2 \left( \sqrt{\frac{f_1}{f_2}} - 1 \right)}{\sqrt{\frac{f_1}{f_2}}} d\mu_2 = \frac{1}{\int_A w d\mu_1} \int_A 2w \cdot \sqrt{\frac{f_1}{f_2}} \cdot \left( \sqrt{\frac{f_1}{f_2}} - 1 \right) d\mu_2 = \\ &= \frac{1}{\int_A w d\mu_1} \int_A 2w \cdot \sqrt{\frac{d\mu_1}{d\mu_2}} \cdot \left( \sqrt{\frac{d\mu_1}{d\mu_2}} - 1 \right) d\mu_2 = \frac{1}{\int_A w d\mu_1} \int_A w \cdot \left( 2 \cdot \frac{d\mu_1}{d\mu_2} - 2 \sqrt{\frac{d\mu_1}{d\mu_2}} \right) d\mu_2 = \\ &= \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{d\mu_1}{d\mu_2} d\mu_2 - 2 \cdot \frac{1}{\int_A w d\mu_1} \int_A w \cdot \sqrt{\frac{d\mu_1}{d\mu_2}} d\mu_2 + \frac{1}{\int_A w d\mu_1} \int_A w d\mu_2 = \\ &= \frac{1}{\int_A w d\mu_1} \int_A w \cdot \left( \sqrt{\frac{d\mu_1}{d\mu_2}} - 1 \right)^2 d\mu_2. \end{aligned}$$

On the other hand (see again Corollary 2),

$$\begin{aligned} D_k^{w,T}(\mu_1|\mu_2) &= \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{f_1}{f_2} \cdot \log_k^T \left( \frac{f_1}{f_2} \right) d\mu_2 \leq \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{f_1}{f_2} \cdot \left( \frac{f_1}{f_2} - 1 \right) d\mu_2 = \\ &= \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{d\mu_1}{d\mu_2} \cdot \left( \frac{d\mu_1}{d\mu_2} - 1 \right) d\mu_2 = \frac{1}{\int_A w d\mu_1} \int_A w \cdot \left( \left( \frac{d\mu_1}{d\mu_2} \right)^2 - \frac{d\mu_1}{d\mu_2} \right) d\mu_2 = \end{aligned}$$

$$\begin{aligned} & \frac{1}{\int_A w d\mu_1} \int_A w \cdot \left( \frac{d\mu_1}{d\mu_2} \right)^2 d\mu_2 - 2 \cdot \frac{1}{\int_A w d\mu_1} \int_A w \cdot \frac{d\mu_1}{d\mu_2} d\mu_2 + \frac{1}{\int_A w d\mu_1} \int_A w d\mu_2 = \\ & \frac{1}{\int_A w d\mu_1} \int_A w \left( \frac{d\mu_1}{d\mu_2} - 1 \right)^2 d\mu_2. \end{aligned}$$

(b) Using (a) we obtain

$$\begin{aligned} & \frac{1}{2} \cdot \frac{1}{\int_A w d\mu_1} \int_A w \left( \sqrt{\frac{d\mu_1}{d\mu_2}} - 1 \right)^2 d\mu_2 + \frac{1}{2} \cdot \frac{1}{\int_A w d\mu_1} \int_A w \left( \sqrt{\frac{d\mu_1}{d\mu_2}} - 1 \right)^2 d\mu_2 \leq \\ & \frac{1}{2} D_k^{w,T}(\mu_1|\mu_2) + \frac{1}{2} D_{-k}^{w,T}(\mu_1|\mu_2) \leq \\ & \frac{1}{2} \cdot \frac{1}{\int_A w d\mu_1} \int_A w \left( \frac{d\mu_1}{d\mu_2} - 1 \right)^2 d\mu_2 + \frac{1}{2} \cdot \frac{1}{\int_A w d\mu_1} \int_A w \left( \frac{d\mu_1}{d\mu_2} - 1 \right)^2 d\mu_2. \end{aligned}$$

Hence,

$$\frac{1}{\int_A w d\mu_1} \int_A w \left( \sqrt{\frac{d\mu_1}{d\mu_2}} - 1 \right)^2 d\mu_2 \leq D_k^{w,K}(\mu_1|\mu_2) \leq \frac{1}{\int_A w d\mu_1} \int_A w \left( \frac{d\mu_1}{d\mu_2} - 1 \right)^2 d\mu_2.$$

□

#### 4. Pseudo-Additivity of the Weighted Tsallis and Kaniadakis Divergences

Let  $(\Omega, \mathcal{T}, \mu_i)$ ,  $i = 1, 2$  be two measure spaces and  $\lambda_1$  a measure on  $(\Omega, \mathcal{T})$  such that  $\mu_i \sim \lambda_1$  for any  $i = 1, 2$ . We consider Radon–Nikodým derivatives  $f_1^{(1)}$  and  $f_2^{(1)}$  on  $\Omega$ , i.e.,  $f_i^{(1)} = \frac{d\mu_i}{d\lambda_1}$  for any  $i = 1, 2$ . Let also  $(S, \mathcal{S}, \nu_j)$ ,  $j = 1, 2$  be two measure spaces and  $\lambda_2$  a measure on  $(S, \mathcal{S})$  such that  $\nu_j \sim \lambda_2$  for any  $j = 1, 2$ . We apply Radon–Nikodým Theorem and find the non-negative measurable functions  $f_1^{(2)}$  and  $f_2^{(2)}$  defined on  $S$  such that  $f_j^{(2)} = \frac{d\nu_j}{d\lambda_2}$  for any  $j = 1, 2$ . We take  $w_1 : \Omega \rightarrow (0, \infty)$  and  $w_2 : S \rightarrow (0, \infty)$  two weight functions.

We consider the measure  $\lambda$  on  $(\Omega \times S, \mathcal{T} \times \mathcal{S})$  induced by  $\lambda_1$  and  $\lambda_2$ . Because  $\mu_i \times \nu_i$  is absolutely continuous with respect to  $\lambda$ , we apply Radon–Nikodým Theorem and find two non-negative measurable functions  $f_1$  and  $f_2$  on  $\Omega \times S$  such that  $f_i = \frac{d(\mu_i \times \nu_i)}{d\lambda}$  for any  $i = 1, 2$ .

The uniqueness from Radon–Nikodým Theorem assures us that  $f_i(\omega, s) = f_i^{(1)}(\omega) f_i^{(2)}(s)$  for any  $\omega \in \Omega$ ,  $s \in S$  and any  $i = 1, 2$ .

Let  $A \in \mathcal{T}$  and  $B \in \mathcal{S}$ .

We define the weighted Tsallis divergence for product measures via

$$\begin{aligned} & D_k^{w_1 w_2, T}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = \\ & \frac{1}{\int \int_{A \times B} w_1 w_2 d(\mu_1 \times \nu_1)} \int \int_{A \times B} w_1(\omega) w_2(s) \log_k^T \left( \frac{f_1(\omega, s)}{f_2(\omega, s)} \right) d(\mu_1 \times \nu_1)(\omega, s) = \\ & \frac{1}{\int_A w_1 d\mu_1 \int_B w_2 d\nu_1} \int \int_{A \times B} w_1(\omega) w_2(s) f_1(\omega, s) \log_k^T \left( \frac{f_1(\omega, s)}{f_2(\omega, s)} \right) d\lambda(\omega, s) \text{ if} \\ & (\mu_1 \times \nu_1)(A \times B) \neq 0 \end{aligned}$$

and

$$D_k^{w_1 w_2, T}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = 0 \text{ if } (\mu_1 \times \nu_1)(A \times B) = 0.$$

The weighted Kaniadakis divergence for product measures is given by

$$D_k^{w_1 w_2, K}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = \frac{1}{\iint_{A \times B} w_1 w_2 d(\mu_1 \times \nu_1)} \iint_{A \times B} w_1(\omega) w_2(s) \log_k^K \left( \frac{f_1(\omega, s)}{f_2(\omega, s)} \right) d(\mu_1 \times \nu_1)(\omega, s) = \frac{1}{\int_A w_1 d\mu_1 \int_B w_2 d\nu_1} \iint_{A \times B} w_1(\omega) w_2(s) f_1(\omega, s) \log_k^K \left( \frac{f_1(\omega, s)}{f_2(\omega, s)} \right) d\lambda(\omega, s) \text{ if } (\mu_1 \times \nu_1)(A \times B) \neq 0$$

and

$$D_k^{w_1 w_2, K}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = 0 \text{ if } (\mu_1 \times \nu_1)(A \times B) = 0.$$

**Lemma 2** (see [41]). *We have the following pseudo-additivity property for Tsallis logarithm (valid for any  $x, y > 0$ ):*

$$\log_k^T(xy) = \log_k^T x + \log_k^T y + k(\log_k^T x)(\log_k^T y).$$

**Theorem 3.** *Let  $A \in \mathcal{T}$  and  $B \in \mathcal{S}$  such that  $(\mu_1 \times \nu_1)(A \times B) \neq 0$ . The weighted Tsallis and Kaniadakis divergences for product measures satisfy the following pseudo-additivity properties:*

$$\begin{aligned} (a) \quad & D_k^{w_1 w_2, T}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = D_k^{w_1, T}(\mu_1 | \mu_2, A) + D_k^{w_2, T}(\nu_1 | \nu_2, B) + k D_k^{w_1, T}(\mu_1 | \mu_2, A) D_k^{w_2, T}(\nu_1 | \nu_2, B). \\ (b) \quad & D_k^{w_1 w_2, K}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = D_k^{w_1, K}(\mu_1 | \mu_2, A) + D_k^{w_2, K}(\nu_1 | \nu_2, B) + \frac{k}{2} \left( D_k^{w_1, T}(\mu_1 | \mu_2, A) D_k^{w_2, T}(\nu_1 | \nu_2, B) - D_{-k}^{w_1, T}(\mu_1 | \mu_2, A) D_{-k}^{w_2, T}(\nu_1 | \nu_2, B) \right). \\ (c) \quad & D_k^{w_1 w_2, K}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = \frac{1}{\int_B w_2 d\nu_1} \left( \int_B w_2(s) f_1^{(2)}(s) \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right)^k d\lambda_2(s) \right) \cdot D_k^{w_1, K}(\mu_1 | \mu_2, A) + \frac{1}{\int_A w_1 d\mu_1} \left( \int_A w_1(\omega) f_1^{(1)}(\omega) \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right)^{-k} d\lambda_1(\omega) \right) D_k^{w_2, K}(\nu_1 | \nu_2, B). \end{aligned}$$

**Proof.** (a) According to Lemma 2, we have

$$\begin{aligned} D_k^{w_1 w_2, T}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) &= \frac{1}{\int_A w_1 d\mu_1 \int_B w_2 d\nu_1} \iint_{A \times B} w_1(\omega) w_2(s) f_1(\omega, s) \log_k^T \left( \frac{f_1(\omega, s)}{f_2(\omega, s)} \right) d\lambda(\omega, s) = \\ &= \frac{1}{\int_A w_1 d\mu_1} \cdot \frac{1}{\int_B w_2 d\nu_1} \iint_{A \times B} w_1(\omega) w_2(s) f_1^{(1)}(\omega) f_1^{(2)}(s) \log_k^T \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right) d\lambda_1(\omega) d\lambda_2(s) + \\ &+ \frac{1}{\int_A w_1 d\mu_1} \cdot \frac{1}{\int_B w_2 d\nu_1} \iint_{A \times B} w_1(\omega) w_2(s) f_1^{(1)}(\omega) f_1^{(2)}(s) \log_k^T \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right) d\lambda_1(\omega) d\lambda_2(s) + k \cdot \\ &\quad \frac{1}{\int_A w_1 d\mu_1} \cdot \frac{1}{\int_B w_2 d\nu_1} \cdot \\ &\quad \iint_{A \times B} w_1(\omega) w_2(s) f_1^{(1)}(\omega) f_1^{(2)}(s) \left( \log_k^T \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right) \right) \left( \log_k^T \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right) \right) d\lambda_1(\omega) d\lambda_2(s) = \\ &= \frac{1}{\int_A w_1 d\mu_1} \left( \int_A w_1(\omega) f_1^{(1)}(\omega) \log_k^T \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right) d\lambda_1(\omega) \right) \frac{1}{\int_B w_2 d\nu_1} \cdot \\ &\quad \left( \int_B w_2(s) f_1^{(2)}(s) d\lambda_2(s) \right) + \end{aligned}$$

$$\begin{aligned} & \frac{1}{\int_A w_1 d\mu_1} \left( \int_A w_1(\omega) f_1^{(1)}(\omega) d\lambda_1(\omega) \right) \frac{1}{\int_B w_2 d\nu_1} \left( \int_B w_2(s) f_1^{(2)}(s) \log_k^T \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right) d\lambda_2(s) \right) + \\ & k \cdot \frac{1}{\int_A w_1 d\mu_1} \left( \int_A w_1(\omega) f_1^{(1)}(\omega) \log_k^T \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right) d\lambda_1(\omega) \right) \frac{1}{\int_B w_2 d\nu_1} \cdot \\ & \left( \int_B w_2(s) f_1^{(2)}(s) \log_k^T \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right) d\lambda_2(s) \right) = \\ & D_k^{w_1, T}(\mu_1 | \mu_2, A) + D_k^{w_2, T}(\nu_1 | \nu_2, B) + k D_k^{w_1, T}(\mu_1 | \mu_2, A) D_k^{w_2, T}(\nu_1 | \nu_2, B). \end{aligned}$$

(b) Because  $\log_k^K x = \frac{1}{2} (\log_k^T x + \log_{-k}^T x)$ , we have

$$\begin{aligned} & D_k^{w_1 w_2, K}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = \\ & \frac{1}{\int_A w_1 d\mu_1 \int_B w_2 d\nu_1} \iint_{A \times B} w_1(\omega) w_2(s) f_1(\omega, s) \log_k^K \left( \frac{f_1(\omega, s)}{f_2(\omega, s)} \right) d\lambda(\omega, s) = \\ & \frac{1}{\int_A w_1 d\mu_1} \cdot \frac{1}{\int_B w_2 d\nu_1} \cdot \iint_{A \times B} w_1(\omega) w_2(s) f_1^{(1)}(\omega) f_1^{(2)}(s) \cdot \\ & \frac{1}{2} \left( \log_k^T \left( \frac{f_1^{(1)}(\omega) f_1^{(2)}(s)}{f_2^{(1)}(\omega) f_2^{(2)}(s)} \right) + \log_{-k}^T \left( \frac{f_1^{(1)}(\omega) f_1^{(2)}(s)}{f_2^{(1)}(\omega) f_2^{(2)}(s)} \right) \right) d\lambda_1(\omega) d\lambda_2(s) = \\ & \frac{1}{2} \left( D_k^{w_1 w_2, T}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) + D_{-k}^{w_1 w_2, T}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) \right). \end{aligned}$$

Using (a), we get

$$\begin{aligned} & D_k^{w_1 w_2, K}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = \\ & \frac{1}{2} \left( D_k^{w_1, T}(\mu_1 | \mu_2, A) + D_k^{w_2, T}(\nu_1 | \nu_2, B) + k D_k^{w_1, T}(\mu_1 | \mu_2, A) D_k^{w_2, T}(\nu_1 | \nu_2, B) \right) + \\ & \frac{1}{2} \left( D_{-k}^{w_1, T}(\mu_1 | \mu_2, A) + D_{-k}^{w_2, T}(\nu_1 | \nu_2, B) - k D_{-k}^{w_1, T}(\mu_1 | \mu_2, A) D_{-k}^{w_2, T}(\nu_1 | \nu_2, B) \right) = \\ & D_k^{w_1, K}(\mu_1 | \mu_2, A) + D_k^{w_2, K}(\nu_1 | \nu_2, B) + \\ & \frac{k}{2} \left( D_k^{w_1, T}(\mu_1 | \mu_2, A) D_k^{w_2, T}(\nu_1 | \nu_2, B) - D_{-k}^{w_1, T}(\mu_1 | \mu_2, A) D_{-k}^{w_2, T}(\nu_1 | \nu_2, B) \right). \end{aligned}$$

(c) It is easy to prove that

$$\log_k^K(xy) = y^k \cdot \frac{x^k - x^{-k}}{2k} + x^{-k} \cdot \frac{y^k - y^{-k}}{2k} = y^k \log_k^K x + x^{-k} \log_k^K y \text{ for any } x, y \in (0, \infty).$$

Hence,

$$\begin{aligned} & D_k^{w_1 w_2, K}(\mu_1 \times \nu_1 | \mu_2 \times \nu_2, A \times B) = \\ & \frac{1}{\int_A w_1 d\mu_1 \int_B w_2 d\nu_1} \iint_{A \times B} w_1(\omega) w_2(s) f_1(\omega, s) \log_k^K \left( \frac{f_1(\omega, s)}{f_2(\omega, s)} \right) d\lambda(\omega, s) = \\ & \frac{1}{\int_A w_1 d\mu_1} \cdot \frac{1}{\int_B w_2 d\nu_1} \iint_{A \times B} w_1(\omega) w_2(s) f_1^{(1)}(\omega) f_1^{(2)}(s) \left[ \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right)^k \log_k^K \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right) + \right. \\ & \left. \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right)^{-k} \log_k^K \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right) \right] d\lambda_1(\omega) d\lambda_2(s) = \\ & \frac{1}{\int_A w_1 d\mu_1} \left( \int_A w_1(\omega) f_1^{(1)}(\omega) \log_k^K \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right) d\lambda_1(\omega) \right) \cdot \end{aligned}$$

$$\begin{aligned}
& \frac{1}{\int_B w_2 d\nu_1} \left( \int_B w_2(s) f_1^{(2)}(s) \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right)^k d\lambda_2(s) \right) + \\
& \frac{1}{\int_A w_1 d\mu_1} \left( \int_A w_1(\omega) f_1^{(1)}(\omega) \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right)^{-k} d\lambda_1(\omega) \right) \cdot \\
& \frac{1}{\int_B w_2 d\nu_1} \left( \int_B w_2(s) f_1^{(2)}(s) \log_k \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right) d\lambda_2(s) \right) = \\
& \frac{1}{\int_B w_2 d\nu_1} \left( \int_B w_2(s) f_1^{(2)}(s) \left( \frac{f_1^{(2)}(s)}{f_2^{(2)}(s)} \right)^k d\lambda_2(s) \right) D_k^{w_1, K}(\mu_1 | \mu_2, A) + \\
& \frac{1}{\int_A w_1 d\mu_1} \left( \int_A w_1(\omega) f_1^{(1)}(\omega) \left( \frac{f_1^{(1)}(\omega)}{f_2^{(1)}(\omega)} \right)^{-k} d\lambda_1(\omega) \right) D_k^{w_2, K}(\nu_1 | \nu_2, B).
\end{aligned}$$

□

## 5. Conclusions

With the help of some inequalities concerning Tsallis logarithm, we obtained inequalities between the weighted Tsallis and Kaniadakis divergences on an arbitrary measurable non-negligible set and Tsallis logarithm, respectively, Kaniadakis logarithm (Theorem 1). We showed that the aforementioned divergences are limited by similar bounds with those that limit Kullback–Leibler divergence (Theorem 2) and proved that are pseudo-additive (Theorem 3).

**Author Contributions:** Conceptualization, R.-C.S., S.-C.S. and V.P.; Formal analysis, V.P.; Investigation, R.-C.S., S.-C.S. and V.P.; Methodology, R.-C.S.; Validation, S.-C.S.; Writing—original draft, R.-C.S.; Writing—review and editing, S.-C.S. and V.P. All authors contributed equally to the paper. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors are very much indebted to the anonymous referees and to the editors for their most valuable comments and suggestions which improved the quality of the paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Tsallis, C. Possible generalization of Boltzmann–Gibbs statistics. *J. Stat. Phys.* **1988**, *52*, 479–487. [\[CrossRef\]](#)
2. Preda, V.; Dedu, S.; Gheorghe, C. New classes of Lorenz curves by maximizing Tsallis entropy under mean and Gini equality and inequality constraints. *Physics A* **2015**, *436*, 925–932. [\[CrossRef\]](#)
3. Soares, A.D.; Moura, N.J., Jr.; Ribeiro, M.B. Tsallis statistics in the income distribution of Brazil. *Chaos Solitons Fractals* **2016**, *88*, 158–171. [\[CrossRef\]](#)
4. Abe, S.; Suzuki, N. Iteration of the Internet over nonequilibrium stationary states in Tsallis statistics. *Phys. Rev. E* **2003**, *67*, 016106. [\[CrossRef\]](#) [\[PubMed\]](#)
5. Oikonomou, N.; Provata, A.; Tirnakli, U. Nonextensive statistical approach to non-coding human DNA. *Physics A* **2008**, *387*, 2653–2659. [\[CrossRef\]](#)
6. Lima, J.; Silva, R., Jr.; Santos, J. Plasma oscillations and nonextensive statistics. *Phys. Rev. E* **2000**, *61*, 3260. [\[CrossRef\]](#)
7. Jiang, Z.Q.; Chen, W.; Zhou, W.X. Scaling in the distribution of intertrade durations of Chinese stocks. *Physics A* **2008**, *387*, 5818–5825. [\[CrossRef\]](#)
8. Kaizoji, T. An interacting-agent model of financial markets from the viewpoint of nonextensive statistical mechanics. *Physics A* **2006**, *370*, 109–113. [\[CrossRef\]](#)

9. Kaniadakis, G. Non-linear kinetics underlying generalized statistics. *Physics A* **2001**, *296*, 405–425. [\[CrossRef\]](#)
10. Preda, V.; Dedu, S.; Sheraz, M. New measure selection for Hunt-Devolder semi-Markov regime switching interest rate models. *Physics A* **2014**, *407*, 350–359. [\[CrossRef\]](#)
11. Trivellato, B. The minimal  $k$ -entropy martingale measure. *Int. J. Theor. Appl. Financ.* **2012**, *15*, 1250038. [\[CrossRef\]](#)
12. Trivellato, B. Deformed exponentials and applications to finance. *Entropy* **2013**, *15*, 3471–3489. [\[CrossRef\]](#)
13. Abreul, E.M.C.; Neto, J.A.; Barboza, E.M., Jr.; Nunes, R.C. Jeans instability criterion from the viewpoint of Kaniadakis' statistics. *EPL* **2016**, *114*, 55001. [\[CrossRef\]](#)
14. Cure M.; Rial, D.F.; Christen, A.; Cassetti, J. A method to deconvolve stellar rotational velocities. *Astron. Astrophys.* **2014**, *564*, A85.
15. Macedo-Filho, A.; Moreira, D.A.; Silva, R.; da Silva, L.R. Maximum entropy principle for Kaniadakis statistics and networks. *Phys. Lett. A* **2013**, *377*, 842–846. [\[CrossRef\]](#)
16. Stella, M.; Brede, M. A kappa-deformed model of growing complex networks with fitness. *Physics A* **2014**, *407*, 360–368. [\[CrossRef\]](#)
17. Clementi, F.; Gallegati, M.; Kaniadakis, G. A  $k$ -generalized statistical mechanics approach to income analysis. *J. Stat. Mech.* **2009**, 2009, P02037. [\[CrossRef\]](#)
18. Modanese, G. Common origin of power-law tails in income distributions and relativistic gases. *Phys. Lett. A* **2016**, *380*, 29–32. [\[CrossRef\]](#)
19. Kaniadakis, G. Statistical mechanics in the context of special relativity. *Phys. Rev. E* **2002**, *66*, 056125. [\[CrossRef\]](#)
20. Kaniadakis, G. Statistical mechanics in the context of special relativity. II. *Phys. Rev. E* **2005**, *72*, 036108. [\[CrossRef\]](#)
21. Kaniadakis, G.; Scarfone, A.M.; Sparavigna, A.; Wada, T. Composition law of  $k$ -entropy for statistically independent systems. *Phys. Rev. E* **2017**, *95*, 052112. [\[CrossRef\]](#)
22. Kullback, S.; Leibler, R.A. On information and sufficiency. *Ann. Math. Stat.* **1951**, *22*, 79–86. [\[CrossRef\]](#)
23. Amari, S.; Nagaoka, H. *Methods of Information Geometry (Translated from the Japanese by Daishi Harada)*; American Mathematical Society: Providence, RI, USA, 2000. [\[CrossRef\]](#)
24. Gkelsinis, T.; Karagrigoriou, A.; Barbu, V.S. Statistical inference based on weighted divergence measures with simulations and applications. *Stat. Pap.* **2022**, *63*, 1511–1536. [\[CrossRef\]](#)
25. Nawrocki, D.N.; Harding, W.H. State-value weighted entropy as a measure of investment risk. *Appl. Econ.* **1986**, *18*, 411–419. [\[CrossRef\]](#)
26. Guiaşu, S. Grouping data by using the weighted entropy. *J. Stat. Plan. Inference* **1990**, *15*, 63–69. [\[CrossRef\]](#)
27. Di Crescenzo, A.; Longobardi, M. On weighted residual and past entropies. *arXiv* **2007**, arXiv:math/0703489. [\[CrossRef\]](#)
28. Suhov, Y.; Zohren, S. Quantum weighted entropy and its properties. *arXiv* **2014**, arXiv:11411.0892. [\[CrossRef\]](#)
29. Furuichi, S.; Yanagi, K.; Kuriyama, K. Fundamental properties of Tsallis relative entropy. *J. Math. Phys.* **2004**, *45*, 4868–4877. [\[CrossRef\]](#) [\[PubMed\]](#)
30. Huang, J.; Yong, W.-A.; Hong, L. Generalization of the Kullback-Leibler divergence in the Tsallis statistics. *J. Math. Anal. Appl.* **2016**, *436*, 501–512. [\[CrossRef\]](#)
31. Sfetcu, R.-C. Tsallis and Rényi divergences of generalized Jacobi polynomials. *Physics A* **2016**, *460*, 131–138.
32. Sfetcu, R.-C.; Sfetcu, S.-C.; Preda, V. On Tsallis and Kaniadakis divergences. *Math. Phys. Anal. Geom.* **2022**, *25*, 23. [\[CrossRef\]](#)
33. Barbu, V.S.; Karagrigoriou, A.; Preda, V. Entropy and divergence rates for Markov chains. II: The weighted case. *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.* **2018**, *19*, 3–10. [\[CrossRef\]](#)
34. Beliş M.; Guiaşu S. A quantitative-qualitative measure of information in cybernetic systems. *IEEE Trans. Inf. Theory* **1968**, *14*, 593–594. [\[CrossRef\]](#)
35. Guiaşu, S. Weighted entropy. *Rep. Math. Phys.* **1971**, *2*, 165–179.
36. Barbu, V.S.; Karagrigoriou, A.; Makrides, A. Semi-Markov modelling for multi-state systems. *Methodol. Comput. Appl. Probab.* **2017**, *19*, 1011–1028.
37. Bulinski, A.; Dimitrov, D. Statistical estimation of the Kullback-Leibler divergence. *Mathematics* **2021**, *9*, 544. [\[CrossRef\]](#)
38. Gkelsinis, T.; Karagrigoriou, A. Theoretical aspects on measures of directed information with simulations. *Mathematics* **2020**, *8*, 587. [\[CrossRef\]](#)
39. Toma, A. Optimal robust M-estimators using divergences. *Stat. Probab. Lett.* **2009**, *79*, 1–5. [\[CrossRef\]](#)
40. Toma, A. Model selection criteria using divergences. *Entropy* **2014**, *16*, 2686–2698. [\[CrossRef\]](#)
41. Tsallis, C. *Introduction to Nonextensive Statistical Mechanics*; Springer Science Business Media LLC: Berlin/Heidelberg, Germany, 2009.