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A New Proposal to Create a Valid Quantization of Einstein's Gravity

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Abstract: Canonical quantization has created many valid quantizations that require infinite-line coordinate variables. However, the half-harmonic oscillator, which is limited to the positive coordinate half, cannot receive a valid canonical quantization because of the reduced coordinate space. Instead, affine quantization, which is a new quantization procedure, has been deliberately designed to handle the quantization of problems with reduced coordinate spaces. Following examples of what affine quantization is, and what it can offer, a remarkably straightforward quantization of Einstein's gravity is attained, in which a proper treatment of the positive definite metric field of gravity has been secured.

Keywords: affine quantization; field theories; Einstein's gravity

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1. Introduction

1.1. Elementary Examples of Canonical Quantization

The quantization of a full-harmonic oscillator can start with a simple classical Hamiltonian, $H(p,q) = (p^2+q^2)/2$, along with $-\infty < p$ & $q < \infty$, and a Poisson bracket, $\{q,p\} = 1$. These favored classical variables are promoted to self-adjoint operators, i.e., $p \to P = P^\dagger$ & $q \to Q = Q^\dagger$, and, following Dirac [1], and others, the quantum Hamiltonian for the full-harmonic oscillator is then given by $\mathcal{H}(P,Q) = (P^2+Q^2)/2$. Schrödinger's representation, where $P = -i\hbar\,\partial/\partial x$, $Q = x$, and $-\infty < x < \infty$, along wih Schrödinger's equation for the full-harmonic oscillator, is given by

$$i\hbar\,\partial\,\psi(x,t)/\partial t = [-\hbar^2\,\partial^2/\partial x^2 + x^2]/2\;\psi(x,t)\;, \qquad (1)$$

which, with solutions that obey $\int |\psi(x,t)|^2\,dx < \infty$, provides the foundation that leads to valid results of the quantization of the full-harmonic oscillator.

Our next example is that of the half-harmonic oscillator, which has the same classical Hamiltonian, i.e., $H(p,q) = (p^2+q^2)/2$, but now $q > 0$. In that case, while $Q = Q^\dagger > 0$, it follows that $P^\dagger \neq P$, which means that this example can not be properly quantized by canonical quantization, either because there are infinitely many distinct, self-adjoint, candidates for the Hamiltonian operator, or by forcing the negative portion of any wave function to zero, this act can only lead to half of the needed eigenfunctions that would become part of the eigenfunctions of the full-harmonic oscillator due to the requirement that all of the acceptable eigenfunctions must vanish at $x = 0$. Briefly stated, canonical quantization fails to provide a proper quantization of the half-harmonic oscillator.

1.2. Elementary Examples of Affine Quantization

Now we introduce affine quantization and see what it has to say about the half-harmonic oscillator. We start with the classical variables $d \equiv pq$ and q as our principal variables (instead of p and q). However, we need to insist that $q \neq 0$, meaning we must discard the point $q = 0$. This is required because if $q = 0$ then $d = 0$ and p can not help; however, we keep $p = 0$ so $d = 0$ is still available. With $q \neq 0$, it follows, as already noticed, that $P^\dagger \neq P$. However, the dilation operator, $d = pq \to D = (P^\dagger Q + QP)/2\,(= D^\dagger)$, is self-adjoint. Therefore, we choose as our principal operators, D and $Q \neq 0$. Indeed, we can go further and accept only $q > 0$ or $q < 0$, for which then $Q > 0$ or $Q < 0$. Since we now focus on the half-harmonic oscillator, for which we require $q > 0$, we choose D and $Q > 0$ (Note: $e^{iqP/\hbar}\, Q\, e^{-iqP/\hbar} = Q + q$, the dilation operator D gets its name (for a dimensionless $q > 0$) from the fact that $e^{i\ln(q)D/\hbar}\, Q\, e^{-i\ln(q)D/\hbar} = qQ$. Moreover, while $P^\dagger \neq P$, it is important to note that $P^\dagger Q = PQ$ and thus $D = (PQ + QP)/2$ as well. This property holds because P requires to face a wave function that is zero, while $P^\dagger$ can face any value of an acceptable wave function. Having Q times a wave function brings it to zero so $P^\dagger$ then acts just like P).

Once again the classical Hamiltonian for the half-harmonic oscillator is $H = (p^2 + q^2)/2 = (d^2/q^2 + q^2)/2)$, with $q > 0$, and for that we suggest that the quantum Hamiltonian operator can be given by $\mathcal{H}'(D, Q) = (DQ^{-2}D + Q^2)/2$, an expression that directly imitates the classical Hamiltonian with the classical variables d and $q > 0$. This procedure is exactly like the process for canonical quantization using preferred classical variables that are promoted to operators and then placed in the same position that they have in the classical Hamiltonian to build the quantum Hamiltonian.

Now, the first discovery is that

$$\mathcal{H}' = (DQ^{-2}D + Q^2)/2 = [P^2 + (3/4)\hbar^2/Q^2 + Q^2]/2\,, \tag{2}$$

and the second discovery is that, within this equation, $P^\dagger$ & $(P^\dagger)^2$ *act* like P & P^2 thanks to the '3/4' term [2]. It follows that Schrödinger's representation and equation for the half-harmonic oscillator becomes

$$i\hbar\,\partial\,\psi(x,t)/\partial t = [-\hbar^2\partial^2/\partial x^2 + (3/4)\hbar^2/x^2 + x^2]/2]\ \psi(x,t)\,, \tag{3}$$

and, once again, provided that $\int |\psi(x,t)|^2\ dx < \infty$, we are led to the foundations of a complete affine quantization story.

If the reader finds it difficult to accept that the expression in Equation (3) is as valid as the expression in (1), we shall compare properties that will link them together. For example, the eigenvalues of the full-harmonic oscillator in (1) are $E_n = \hbar(n + 1/2)$, with $n = 0, 1, 2, \ldots$, and it has an equally spaced spectrum. Likewise, the eigenvalues of the half-harmonic oscillator in (3) are $E'_n = 2\hbar(n+1)$, with again $n = 0, 1, 2, \ldots$, in which they are also equally spaced by exactly twice that of the full-harmonic oscillator [2] (Each eigenfunction in this case has the unusual form of $x^{3/2}$(polynomial)$e^{-x^2/2\hbar}$). Moreover, if we change the cutoff limit from $x = 0$ to $x = -b$, where $b > 0$, a change which affects only the '3/4' term that becomes $(3/4)\hbar^2/(x+b)^2$, the b-dependent eigenvalues are still equally spaced and, as $b \to \infty$, it follows that the eigenvalues become those of the full-harmonic oscillator along with the eigenvectors being the usual ones of the traditional full-harmonic oscillator [3] (A beautiful illustration of a set of several eigenvalues changing as a function of b appears as Figure 1 in [3] (page 15). That author chose to shift their eigenfunctions, e.g., $\psi(x) \to \psi(x-b)$, but the eigenvalues remain the same).

Such a close connection between the half- and full-harmonic oscillators serves to ensure that an affine quantization of the half-harmonic oscillator is as valid a quantization as that of the canonical quantization for the full-harmonic oscillator. In other words, by using favored classical affine variables to promote to basic operators, affine quantization has been found to be as valid as it is for using favored classical canonical variables to promote to basic operators for canonical quantization.

Now we examine a few complex problems that have encountered difficulties when being quantized by canonical procedures and see what happens if instead we choose to quantize such problems using affine quantization procedures. The following story will involve a field theory model, used as an introductory field theory example, followed by a careful examination of Einstein's gravity.

2. A Brief Overview of Quantum Field Theory

To begin with, we focus on a feature of mathematics that impacts physics. As an example, consider the function $f(x) = 1/|x|^{1/3}$ in the interval $-1 < x < 1$, with $f(0) = \infty$. It follows that $\int_{-1}^{1} f(x)^2 \, dx < \infty$, while $\int_{-1}^{1} f(x)^4 \, dx = \infty$. Later, we will refer to this situation as an 'f-issue' which involves a field that reaches infinity, and also is part of an integration that is finite.

Physics is frequently engaged in studying Nature, and then having any field with an f-issue, as in the last paragraph, seems impossible. Stated bluntly, can the strength of any field of Nature reach infinity? The author believes that no field of Nature reaches infinity, and we now endeavor to make that happen in our analysis. For example, a classical Hamiltonian density, $H(x)$, that describes a part of Nature, should not reach infinity for any x, and that fact needs to be part of the mathematics involved. Presently, the mathematical focus only requires that $\int H(x) \, d^s x < \infty$, and this approach leads to a nonrenormalizable behavior if the interaction power of a term such as $\varphi(x)^p$, when $p \geq 2n/(n-2)$, and where $n = s+1$ is the number of spacetime dimensions. Let us examine a procedure in which we can actually favor Nature.

2.1. A Typical Model of a Covariant Scalar Field

Our model of interest has the classical Hamiltonian given by

$$H(\pi, \varphi) = \int \{ \tfrac{1}{2} [\pi(x)^2 + (\overrightarrow{\nabla} \varphi(x))^2 + m^2 \varphi(x)^2] + g \, \varphi(x)^p \} \, d^s x \, . \tag{4}$$

When such a model is quantized, say by a path integration procedure, the classical Hamiltonian is evaluated by a vast number of functions for both $\pi(x)$ and $\varphi(x)$. In so doing, fields that can diverge but still offer finite integrations—like the example of an f-issue in a previous paragraph—are conventionally introduced along with fields without any divergencies.

How can we limit the classical fields so that f-issues do not arise? The answer to that question appears in the next section, and it is much easier than could have been expected.

2.2. An Affine Quantization of Classical Field Theories

Following the simple rules of Section 1.2, we introduce the dilation field $\kappa(x) = \pi(x) \, \varphi(x)$, along with $\varphi(x)^2 > 0$, because if $\varphi(x) = 0$, then $\kappa(x) = 0$ and $\pi(x)$ cannot help. With these new classical variables, the classical Hamiltonian of (4) becomes

$$H'(\kappa, \varphi) = \int \{ \tfrac{1}{2} [\kappa(x)^2 / \varphi(x)^2 + (\overrightarrow{\nabla} \varphi(x))^2 + m^2 \varphi(x)^2] + g \, \varphi(x)^p \} \, d^s x \, , \tag{5}$$

Already we see that $0 < |\varphi(x)| < \infty$ for if $\varphi(x)^2 = 0$ or $1/\varphi(x)^2 = 0$, it means that $\kappa(x)$ fails to offer any value for $\pi(x)$. It follows then that $0 < |\varphi(x)|^p < \infty$, and therefore *nonrenormalizability vanishes!* It is noteworthy, that for this field model, we keep both sides of $\varphi(x) \neq 0$, i.e., both $\varphi(x) > 0$ and $\varphi(x) < 0$, and since there is the gradient term, the field will appear to be continuous and integrations should not be affected.

To offer an affine quantization for this example, we first introduce the dilation operator $\hat{\kappa}(x) = [\hat{\pi}(x)^\dagger \hat{\varphi}(x) + \hat{\varphi}(x) \hat{\kappa}(x)]/2$ and $\hat{\varphi}(x) \neq 0$. Next, adopting a Schrödinger representation for the quantum Hamiltonian, we are led to

$$\begin{aligned} \mathcal{H}'(\hat{\kappa}, \varphi) = \int \{ \tfrac{1}{2} [\hat{\kappa}(x) \, (\varphi(x)^{-2}) \, \hat{\kappa}(x) + (\overrightarrow{\nabla} \varphi(x))^2 + m^2 \varphi(x)^2] \\ + g \, \varphi(x)^p \} \, d^s x \, , \end{aligned} \tag{6}$$

where it is evident that $1/\varphi(x)^2 > 0$, and therefore $0 < |\varphi(x)|^p < \infty$ as desired. Finally, we offer Schrödinger's equation as

$$i\hbar \, \partial \, \Psi(\varphi, t) / \partial t = \mathcal{H}'(\hat{\kappa}, \varphi) \, \Psi(\varphi, t) \, . \tag{7}$$

As usual, it may be necessary to introduce some version of a regularization for these equations, but these same equations should point the way to proceed. To offer some support, we note that although $\hat{\pi}(x)^\dagger \neq \hat{\pi}(x)$ it can be helpful to know that $\hat{\pi}(x)^\dagger \varphi(x) = \hat{\pi}(x)\varphi(x)$.

If the reader has already accepted the expressions for the half-harmonic oscillator in (3), they may be willing to accept (6) and (7) for this quantized covariant scalar model as well (We note that Monte Carlo studies of the scalar fields φ_4^4 and φ_3^{12} using canonical quantization have led to "free-results", as if the interaction term was absent when it was not. However, using affine quantization has led to "non-free-results", in which the interaction term leads to different results when the coupling constant changes [4–8]).

3. Applying Affine Quantization to Einstein's Gravity

In order to quantize gravity it is important to render a valid quantization of the ADM classical Hamiltonian [9]. We first choose our new classical variables which include what we call the dilation field $\pi_b^a(x) \equiv \pi^{ac}(x)\, g_{bc}(x)$ along with the metric field $g_{ab}(x)$. We do not need to impose any restriction on the metric field because physics already requires that $ds(x)^2 = g_{ab}(x)\; dx^a\; dx^b > 0$ provided that $\{dx^a\} \neq 0$. The metric can also be diagonalized by non-physical, orthogonal matrices, and then it includes only $g_{11}(x), g_{22}(x),\ \&\ g_{33}(x)$, each of which must be strictly positive as required by physics (The reader should compare the three diagonalized positive metric variables with $q > 0$, which then requires an affine quantization for the half-harmonic oscillator, and also then appreciate the need for such a quantization that led to Equation (3)).

Next we present the ADM classical Hamiltonian in our chosen affine variables, which, introducing $g(x) \equiv \det[g_{ab}(x)] > 0$, leads to

$$\begin{aligned} H(\pi, g) = \textstyle\int \{ g(x)^{-1/2} [\pi_b^a(x)\pi_a^b(x) - \tfrac{1}{2}\pi_a^a(x)\pi_b^b(x)] \\ + g(x)^{1/2}\, {}^{(3)}\!R(x)\}\; d^3\!x\,, \end{aligned} \tag{8}$$

where ${}^{(3)}\!R(x)$ is the Ricci scalar for three spatial coordinates and which contains all of the derivatives of the metric field. Already this version of the classical Hamiltonian contains reasons that restrict $g(x)$ to $0 < g(x) < \infty$, $|\pi_b^a(x)| < \infty$, and $|{}^{(3)}\!R(x)| < \infty$, which, like the field theory example of Section 2, leads to no f-issues for the gravity story.

Finally, we introduce the dilation gravity operator $\hat{\pi}_b^a(x) = [\hat{\pi}^{ac}(x)^\dagger\, \hat{g}_{bc}(x) + \hat{g}_{bc}(x)\, \hat{\pi}^{ac}(x)]/2$ along with $\hat{g}_{ab}(x) > 0$, and adopting Schrödinger's representation and equation, we are led to

$$\begin{aligned} \mathcal{H}'(\hat{\pi}, g) = \textstyle\int \{\, [\, \hat{\pi}_b^a(x)\, g(x)^{-1/2}\, \hat{\pi}_a^b(x) - \tfrac{1}{2}\hat{\pi}_a^a(x)\, g(x)^{-1/2}\, \hat{\pi}_b^b(x)\,] \\ + g(x)^{1/2}\; {}^{(3)}\!R(x)\; \}\; d^3\!x\,. \end{aligned} \tag{9}$$

And now, as before, we close with Schrödinger's equation

$$i\hbar\, \partial\, \Psi(g,t)/\partial t = \mathcal{H}'(\hat{\pi}, g)\; \Psi(g,t)\,, \tag{10}$$

which offers the necessary ingredients for the foundation of a valid quantization of the classical Hamiltonian, which is an important part of the full story.

As before, it may be necessary to introduce some version of regularization for these equations, but these same equations point the way to proceed. In that effort, note that although $\hat{\pi}^{ac}(x)^\dagger \neq \hat{\pi}^{ac}(x)$ it can be helpful to know that $\hat{\pi}^{ac}(x)^\dagger\, \hat{g}_{bc}(x) = \hat{\pi}^{ac}(x)\, \hat{g}_{bc}(x)$.

A full quantization of gravity must deal with first and likely second order constraints, which are designed to reduce the overall Hilbert space to secure a final quantization. This paper is not the proper place to finalize a quantization of gravity, but several of the author's articles have been designed to go further toward the final steps, Refs. [9–17], and even an earlier paper that saw the future [18].

For those who like path integration, a recent paper has found that affine quantization provides a royal path to a valid path integration of Einstein's gravity [19].

4. Summary

We have stressed the definition, procedures, and advantages of affine quantization in offering to secure valid quantizations of several different examples. We first used simple models with different coordinate spaces to quantize models that fit their coordinate space in order to insure a valid result. Following that path we examined field models by letting affine variables remove all f-issues as unphysical for any of Nature's fields. Finally, we were able to put affine procedures to work on an f-issue free version of an affine quantization of gravity, a contribution that has been needed for a long tine. It is hoped that readers can use affine quantization procedures to help solve their problems for which they may be well suited.

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