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Modification of the Electron Entropy Production in a Plasma

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Abstract: A modified expression of the electron entropy production in a plasma is deduced by means of the Kelly equations of state instead of the ideal gas equations of state. From the Debye–Hückel model which considers the interaction between the charges, such equations of state are derived for a plasma and the entropy is deduced. The technique to obtain the modified entropy production is based on usual developments but including the modified equations of state giving the regular result plus some extra terms. We derive an expression of the modified entropy production in terms of the tensorial Hermitian moments $h_{r_1 \dots r_m}^{(m)}$ by means of the irreducible tensorial Hermite polynomials.

Keywords: plasma; entropy production; Hermite polynomials

1. Introduction

The entropy production was studied since the middle of the XIX century by Clausius [1]. It has to be highlighted the great exhibition on the subject by Prigogine [2] which permits to understand it from a modern point of view. In the case of a Plasma, a classical review has been done by Hinton and Hazeltine [3]. By assuming a local equilibrium, the rate of change of the entropy can be deduced [4]. The moment method developed by Grad [5] when the distribution function is close to the Boltzmann distribution function, see Equation (20), permits to express such a rate, by using the reducible tensorial Hermite polynomials, in terms of the Hermitian moments $\tilde{h}_{r_1 \dots r_m}^{(m)}$. Moreover, an improved expression can be derived in terms of the Hermitian moments $h_{r_1 \dots r_m}^{(m)}$ by means of the irreducible tensorial Hermite polynomials [4]. In order to highlight the importance of the Hermitian moments, it has to be mentioned that they were included recently by Sonnino et al. [6] to compute the heat loss in L-mode, collisional, tokamak plasmas to test the validity of the sophisticated thermodynamical field theory. However, in the deduction of both the entropy production and, in an implicit form, the Hermitian moments, the equations of state for the Plasma are required but nevertheless, the equations of state of the ideal gas are used as an approximation [4].

On the other hand, by giving different equations of state, it is possible to not abandon the Coulombic interactions of charged plasma particles. The historic paper in Electrochemistry and

Plasmas by Debye–Hückel [7] consider such interactions obtaining that the average potential $\langle \varphi_\alpha(r) \rangle_{av}$ surrounding a plasma ion bearing a charge Z_e is

$$\langle \varphi_\alpha(r) \rangle_{av} = \left(\frac{Z_\alpha e}{r} \right) e^{-\frac{r}{D}}, \quad (1)$$

where D represents the Debye length

$$D = \sqrt{\frac{kT}{4\pi e^2 \sum_\alpha \frac{N_\alpha}{V} Z_\alpha^2}}, \quad (2)$$

where N_α represents the number of particles in the volume V and the Boltzmann distribution function is considered for obtaining the average. Notice that the same temperature is considered for the ions and the electrons. This will be analyzed in Section 3. In our times, the theory is still being applied to modeling successfully electrolyte solutions [8]. In Plasma Physics, important articles from the middle of last century have analyzed such theory. Kirwood and Poirier [9] have shown that the Coulomb contribution to the virial is

$$-\frac{1}{2} \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle = e^2 \sum_\alpha \frac{N_\alpha Z_\alpha^2 V}{4D} = \frac{e^3}{2} \left(\frac{\pi}{kT} \right)^{\frac{1}{2}} \left(\sum_\alpha N_\alpha Z_\alpha^2 \right) V. \quad (3)$$

From this, Kelly [10] deduced a new equation of state for the pressure, that is:

$$P = \sum_\alpha \frac{N_\alpha}{V} kT \left(1 - \frac{1}{18N_D} \right), \quad (4)$$

where α denotes the different ions and

$$N_D = 4\pi D^3 \sum_\alpha \frac{N_\alpha}{3}. \quad (5)$$

On the other hand, Wergeland [11] relates the corrected term of the pressure P_{corr}^e for the electrons with the corrected term of the internal energy as

$$P_{corr}^e = \frac{1}{3} \frac{U_e}{V}. \quad (6)$$

It should also be noted that over time there have been many attempts to describe equations of state being some of them very sophisticated where the magnetic and electrical fields are included such as the work of Liboff and Lie [12] and recently experimental works for measuring the equations of state [13]. It must be noticed that, considering a small plasma parameter $g = (8\pi e^2/kT)^{3/2} N/V = (D^3 N/V)^{-1} \ll 1$, by means of the correlation function Krall and Trivelpiece [14] calculated the Gibbs Free Energy and arrived to an expression for the pressure of a plasma similar to the result already obtained by Kelly [10]. Moreover, Krall and Trivelpiece [14] successfully gave a simple expression of the pressure

$$P = 2 \frac{N}{V} kT \left(1 - \frac{g}{48\pi} \right). \quad (7)$$

Furthermore, Wergeland [11] was able to complete the equations of state of a plasma by proposing an expression for the corrected energy (see Equation (6)).

On the other hand, Callen [15] proposed a set of postulates that must be accomplished by the equations of state in order to represent a thermodynamic system (It has to be noticed that in our case the fourth postulate, which represents the third law of thermodynamics, does not have to be satisfied because the analysis of the modified equations of state must be done in a range of temperatures far away from the zero temperature). In this order of ideas, recently, Essex and Andresen [16] exposed

an equivalent method to verify that a set of equations of state represents a thermodynamic system. It consists of calculating the eigenvalues of the Hessian derived from the fundamental equation of state, $U = U(S, V, N)$, in the energy picture. One of the eigenvalue must be zero and the others must be positive real numbers. Therefore, we propose the modified equations of state by following Kelly [10] and Wergeland [11]. Then, we verify that they meet the principles of thermodynamics by using the Essex and Andresen method [16] (see [17]). By means of the modified equations of state, the modified electron entropy production can be calculated.

The purpose of this article is threefold:

A—To obtain the modified electron entropy production by using the set of modified equations of state (Section 4) instead of the ideal gas equations of state. Notice that this does not include any effects due to toroidal configurations, rotating plasmas or electrostatic turbulence [18,19]. It has to be noticed that interesting results about bounds for entropy production and H theorems for rf current drive (1D) have been obtained by Bizarro [20]. However, we are not including such kind of analysis in this work.

B—To derive an expression of the modified electron entropy production in terms of the Hermitian moments $h_{r_1 \dots r_m}^{(m)}$ by means of the irreducible tensorial Hermite polynomials (Section 4).

C—The modified electron entropy production will contain new terms which are composed by new products of fluxes and forces that may not be so small after an evaluation of them under certain circumstances.

The article is organized as follows: in Section 2, the usual entropy production deduction is presented emphasizing where the ideal gas equations of state are included in order to prepare the modified expression of the entropy production. We also deduce the electron entropy production in terms of the Hermitian moments. In Section 3, the equations of state which include the Coulombic interaction between the electron and the ions, are deduced. To verify that such equations of state describe a thermodynamic system, the entropy is expressed. The thermodynamic properties of the plasma as the heat capacities are deduced. Finally in this section, the eigenvalues of the Hessian are calculated finding a zero eigenvalue which reinforces the idea that we are dealing with a thermodynamic system. In the process, the relaxation coefficients are obtained. In Section 4, we incorporate the modified equations of state to calculate the modified electron entropy production in a Plasma giving the usual results plus extra terms. In addition, by means of the modified equations of state, we derive an expression of the modified electron entropy production in terms of the Hermitian moments $h_{r_1 \dots r_m}^{(m)}$ by means of the irreducible tensorial Hermite polynomials. In Section 5, we give some concluding remarks and future work.

2. Usual Deduction of the Entropy Production

In order to understand the thermodynamics of a plasma, the entropy represents a fundamental concept and, consequently, it is necessary to have an expression for the entropy production, in particular the entropy production due to the electrons. We present a usual derivation of it where it is emphasized the role played by the equations of the state of the ideal gas. Then, we express the electron entropy production in terms of the Hermitian moments. This is done in order to understand the change that has to be done when the equations of state are different from those of the ideal gas.

2.1. Standard Deduction of Entropy Production

Let us begin by considering the equations of state and the average entropy per particle of the ideal gas,

$$P_\alpha = n_\alpha T_\alpha, \quad U_\alpha = \frac{3}{2} N T_\alpha, \quad s_\alpha = \ln \left(\frac{T_\alpha^{3/2}}{n_\alpha} \right), \quad (8)$$

where the subscripts α , T_α , U_α , N , n_α and s_α denote the considered species, the temperature, the internal energy, the number of particles, the density of particles and the average entropy per particle of species α where some constants have been withdrawn, respectively. It has to be pointed out that we are putting

the Boltzmann constant $k = 1$. The first identity in Equation (8) represents an equation of state which implies that the pressure tensor may be seen as:

$$P_{ij}^\alpha = \delta_{ij}P_\alpha. \tag{9}$$

Therefore, we can put $s_\alpha = s_\alpha(T_\alpha, n_\alpha)$ and consequently

$$\frac{\partial n_\alpha s_\alpha}{\partial t} = n_\alpha \left(\frac{\partial s_\alpha}{\partial n_\alpha} \frac{\partial n_\alpha}{\partial t} + \frac{\partial s_\alpha}{\partial T_\alpha} \frac{\partial T_\alpha}{\partial t} \right) + s_\alpha \frac{\partial n_\alpha}{\partial t}, \tag{10}$$

where no summation Einstein convention has been applied. If we use the Equation (8) in Equation (10), we obtain

$$\frac{\partial n_\alpha s_\alpha}{\partial t} = n_\alpha \left(\frac{\partial s_\alpha}{\partial n_\alpha} \frac{\partial n_\alpha}{\partial t} + \frac{\partial s_\alpha}{\partial T_\alpha} \frac{\partial (P_\alpha/n_\alpha)}{\partial t} \right) + s_\alpha \frac{\partial n_\alpha}{\partial t}. \tag{11}$$

Therefore, we arrive at

$$\frac{\partial n_\alpha s_\alpha}{\partial t} = \left(s_\alpha - \frac{5}{2} \right) \frac{\partial n_\alpha}{\partial t} + \frac{1}{T_\alpha} \frac{\partial}{\partial t} \left(\frac{3}{2} P_\alpha \right). \tag{12}$$

From the Fokker–Planck Equation [3], which includes the Fokker–Planck collision operator, it can be derived that

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_\alpha + \frac{(m_\alpha n_\alpha \mathbf{u}_\alpha^2)}{2} \right) + \nabla \cdot \mathbf{Q}_\alpha = Q_\alpha + \mathbf{u}_\alpha \cdot (\mathbf{F}_\alpha + e_\alpha n_\alpha \mathbf{E}), \tag{13}$$

where \mathbf{u}_α and \mathbf{E} represent the averaged velocity of the species α and the electric field, respectively; the friction force \mathbf{F}_α is

$$\mathbf{F}_\alpha = \int m_\alpha \mathbf{v} C_\alpha d^3v, \tag{14}$$

with C_α the collision term of the species α ; the collisional rate of heat exchange Q_α is

$$Q_\alpha = \int (\mathbf{v} - \mathbf{u}_\alpha)^2 \frac{m_\alpha}{2} C_\alpha d^3v, \tag{15}$$

(notice that C_α does not depend on the equations of state) and the energy flux \mathbf{Q}_α is

$$\mathbf{Q}_\alpha = \int \frac{m_\alpha \mathbf{v}^2}{2} \mathbf{v} f_\alpha d^3v, \tag{16}$$

where f_α represents the distribution function. By defining the heat flux as

$$\mathbf{q}_\alpha = \int \frac{m_\alpha}{2} (\mathbf{v} - \mathbf{u}_\alpha)^2 (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha d^3v, \tag{17}$$

We know that

$$\int \frac{m_\alpha \mathbf{v}^2}{2} f_\alpha d^3v = \int \frac{m_\alpha (\mathbf{v} - \mathbf{u}_\alpha)^2}{2} f_\alpha d^3v + \frac{m_\alpha n_\alpha \mathbf{u}_\alpha^2}{2} = 3p_\alpha/2 + \frac{m_\alpha n_\alpha \mathbf{u}_\alpha^2}{2}. \tag{18}$$

The thermal velocity v_{Te}^2 is related with the temperature by [3]

$$\frac{v_{Te}^2}{2} = \frac{T_e}{m_e}, \tag{19}$$

and the following approximation for the distribution function,

$$f_\alpha = f_{\alpha 0}(1 + \chi_\alpha), \text{ with } \chi_\alpha \simeq \frac{\mathbf{u}_\alpha}{v_{T\alpha}} \ll 1, \tag{20}$$

where $f_{\alpha 0}$ represents the Boltzmann function or any distribution function which must decrease sufficiently fast at infinity and permits to obtain

$$\frac{m_{\alpha} n_{\alpha} \mathbf{u}_{\alpha}^2}{2} = n_{\alpha} T_{\alpha} \left(\frac{\mathbf{u}_{\alpha}}{v_{T_{\alpha}}} \right)^2 \Rightarrow \frac{m_{\alpha} n_{\alpha} \mathbf{u}_{\alpha}}{2} \simeq 0. \quad (21)$$

Consequently,

$$\int \frac{m_{\alpha} \mathbf{v}^2}{2} f_{\alpha} d\mathbf{v} = 3p_{\alpha}/2. \quad (22)$$

Taking into account Equation (20), we have

$$\int v_r^2 f_{\alpha} d\mathbf{v} \approx \int v_k^2 f_{\alpha 0} d^3v = \frac{p_{\alpha}}{m} \quad (23)$$

by using Equation (8) in Equation (17), we obtain (neglecting \mathbf{u}_{α}^2)

$$\mathbf{q}_{\alpha} = \mathbf{Q}_{\alpha} - \frac{5P_{\alpha} \mathbf{u}_{\alpha}}{2}. \quad (24)$$

By using the approximation described in Equation (20), which implies that the term $\partial(m_{\alpha} n_{\alpha} \mathbf{u}_{\alpha}^2)/\partial t$ can be neglected, and by using Equation (13), Equation (12) can be expressed as

$$\frac{\partial n_{\alpha} s_{\alpha}}{\partial t} = (s_{\alpha} - 5/2) \frac{\partial n_{\alpha}}{\partial t} + \frac{(Q_{\alpha} + \mathbf{u}_{\alpha} \cdot (\mathbf{F}_{\alpha} + e_{\alpha} n_{\alpha} \mathbf{E}))}{T_{\alpha}} - \frac{1}{T_{\alpha}} \nabla \cdot \mathbf{Q}_{\alpha}. \quad (25)$$

On the other hand, by using Equation (8) in Equation (24), we have

$$\mathbf{q}_{\alpha} = \mathbf{Q}_{\alpha} - \frac{5}{2} n_{\alpha} T_{\alpha} \mathbf{u}_{\alpha}. \quad (26)$$

Substituting Equation (26) into Equation (25), we obtain

$$\begin{aligned} \frac{\partial n_{\alpha} s_{\alpha}}{\partial t} &= (s_{\alpha} - 5/2) \frac{\partial n_{\alpha}}{\partial t} + \frac{Q_{\alpha} + \mathbf{u}_{\alpha} \cdot (\mathbf{F}_{\alpha} + e_{\alpha} n_{\alpha} \mathbf{E})}{T_{\alpha}} \\ &\quad - \frac{1}{T_{\alpha}} \nabla \cdot \mathbf{q}_{\alpha} - \frac{5}{2} \nabla \cdot (n_{\alpha} \mathbf{u}_{\alpha}) - \frac{5}{2T_{\alpha}} n_{\alpha} \mathbf{u}_{\alpha} \cdot \nabla T_{\alpha}. \end{aligned} \quad (27)$$

Considering the equation of continuity

$$\frac{\partial n_{\alpha}}{\partial t} + \nabla \cdot (n_{\alpha} \mathbf{u}_{\alpha}) = 0, \quad (28)$$

we arrive at

$$\frac{\partial n_{\alpha} s_{\alpha}}{\partial t} = -s_{\alpha} \nabla \cdot (n_{\alpha} \mathbf{u}_{\alpha}) - \frac{1}{T_{\alpha}} \nabla \cdot \mathbf{q}_{\alpha} + \frac{Q_{\alpha} + \mathbf{u}_{\alpha} \cdot (\mathbf{F}_{\alpha} + e_{\alpha} n_{\alpha} \mathbf{E})}{T_{\alpha}} - \frac{5}{2T_{\alpha}} n_{\alpha} \mathbf{u}_{\alpha} \cdot \nabla T_{\alpha}. \quad (29)$$

Considering the equation of state, Equation (8), we obtain

$$\frac{\partial n_{\alpha} s_{\alpha}}{\partial t} = -s_{\alpha} \nabla \cdot (n_{\alpha} \mathbf{u}_{\alpha}) - \frac{1}{T_{\alpha}} \nabla \cdot \mathbf{q}_{\alpha} + \frac{Q_{\alpha} + \mathbf{u}_{\alpha} \cdot (\mathbf{F}_{\alpha} + e_{\alpha} n_{\alpha} \mathbf{E})}{T_{\alpha}} - \frac{5}{2T_{\alpha}} n_{\alpha} \mathbf{u}_{\alpha} \cdot \nabla \left(\frac{p_{\alpha}}{n_{\alpha}} \right). \quad (30)$$

That is:

$$\begin{aligned} \frac{\partial n_{\alpha} s_{\alpha}}{\partial t} &= -s_{\alpha} \nabla \cdot (n_{\alpha} \mathbf{u}_{\alpha}) - \frac{1}{T_{\alpha}} \nabla \cdot \mathbf{q}_{\alpha} + \frac{Q_{\alpha} + \mathbf{u}_{\alpha} \cdot (\mathbf{F}_{\alpha} + e_{\alpha} n_{\alpha} \mathbf{E})}{T_{\alpha}} \\ &\quad - \frac{5}{2T_{\alpha}} \mathbf{u}_{\alpha} \cdot \nabla (p_{\alpha}) + \frac{5}{2} \mathbf{u}_{\alpha} \cdot \nabla n_{\alpha}. \end{aligned} \quad (31)$$

We obtain

$$\frac{\partial n_\alpha s_\alpha}{\partial t} + \nabla \cdot \left(s_\alpha n_\alpha \mathbf{u}_\alpha + \frac{\mathbf{q}_\alpha}{T_\alpha} \right) = -\frac{\mathbf{q}_\alpha}{T_\alpha^2} \cdot \nabla T_\alpha + \frac{Q_\alpha + \mathbf{u}_\alpha \cdot (\mathbf{F}_\alpha + e_\alpha n_\alpha \mathbf{E})}{T_\alpha} - \frac{\mathbf{u}_\alpha}{T_\alpha} \cdot \nabla p_\alpha. \quad (32)$$

We define the flux vector

$$\mathbf{J}_{\alpha s} = s_\alpha n_\alpha \mathbf{u}_\alpha + \frac{\mathbf{q}_\alpha}{T_\alpha}, \quad (33)$$

and considering that

$$\sigma_\alpha = \partial/\partial t(n_\alpha s_\alpha) + \nabla \cdot \mathbf{J}_{\alpha s}, \quad (34)$$

we arrive at:

$$\sigma_\alpha = -\frac{\mathbf{q}_\alpha}{T_\alpha^2} \cdot \nabla T_\alpha + \frac{Q_\alpha + \mathbf{u}_\alpha \cdot (\mathbf{F}_\alpha + e_\alpha n_\alpha \mathbf{E})}{T_\alpha} - \frac{\mathbf{u}_\alpha}{T_\alpha} \cdot \nabla p_\alpha. \quad (35)$$

This result coincides with the one obtained by Hinton and Hazeltine [3].

2.2. Electron Entropy Production in Terms of the Hermitian Moments

Let us give the Balescu's [4] expression of the electron entropy production:

$$\sigma^e = \frac{n_e}{\tau_e} \left(h_r^{(1)} g_r^{(1)} + h_r^{e(3)} g_r^{e(3)} + h_{rs}^{e(2)} g_{rs}^{e(2)} \right), \quad (36)$$

where some dimensionless Hermitian moments and dimensionless source terms are defined as:

$$\begin{aligned} h_r^{(1)} &= \left(\frac{m_e}{T_e} \right)^{\frac{1}{2}} \frac{j_r}{en_e}, & h_r^{e(3)} &= \sqrt{\frac{2}{5}} \left(\frac{m_e}{T_e} \right)^{\frac{3}{2}} \frac{q_r^e}{m_e n_e}, & h_{rs}^{e(2)} &= \frac{\pi_{rs}}{\sqrt{2} n_e T_e}, \\ g_r^{(1)} &= \tau_e \left(\frac{m_e}{T_e} \right)^{\frac{1}{2}} \left(\frac{eEr}{m_e} - w_e \varepsilon_{rmm} u_m b_n + \frac{\nabla_r P_\alpha}{m_e n_e} \right), & & & & \\ g_r^{e(3)} &= -\sqrt{\frac{5}{2}} \tau_e \left(\frac{T_e}{m_e} \right)^{\frac{1}{2}} \frac{\nabla_r T_e}{T_e}, & g_{rs}^{e(2)} &= -\sqrt{2} \tau_e \nabla_r u_s, & & \end{aligned} \quad (37)$$

with the relaxation electron time τ_e ,

$$\tau_e = \frac{3}{4\sqrt{2}\pi} \frac{m_e^{\frac{1}{2}} T_e^{\frac{3}{2}}}{z^2 e^4 n_e \ln \Lambda} \quad \text{with} \quad \ln \Lambda = \ln \left(\frac{\frac{3}{2} (T_e + T_i) D}{Ze^2} \right), \quad (38)$$

being $\ln \Lambda$ the Coulomb logarithm; and

$$w_e = -\frac{eB}{m_e c}, \quad \text{and} \quad \mathbf{B} = B\mathbf{b}. \quad (39)$$

Let us constrain the Plasma to two species: the electrons and one type of ion. π_{rs} are defined below in Equation (41), \mathbf{j} in Equation (43) and \mathbf{u} in Equation (46). First, some approximations made by Balescu should be set; in order to obtain Equation (36), the collisional rate of heat exchange and the friction force must be considered as negligible, that is:

$$Q_e \simeq 0 \quad \text{and} \quad \mathbf{F}_e \simeq 0. \quad (40)$$

If we want to compare Equation (36) with Equation (35), it is necessary to consider the fundamental approximation done obtaining Equation (35) which is (see Equation (9)), that is:

$$P_{ij}^e = P_e \delta_{ij} + \pi_{ij} \quad \text{with} \quad \pi_{ij} \simeq 0. \quad (41)$$

Applying this last equation, Equation (36) turns to be

$$\sigma_e = \frac{\mathbf{j} \cdot \mathbf{E}}{T_e} - w \frac{\mathbf{j} \cdot (\mathbf{u} \times \mathbf{b})}{T_e m_e} + \frac{\mathbf{j} \cdot \nabla P_e}{en_e} - \frac{\mathbf{q} \cdot \nabla T_e}{T_e^2}. \quad (42)$$

Therefore, from the one fluid model, we have

$$\mathbf{u}_i - \mathbf{u}_e = \frac{\mathbf{j}}{en_e}, \quad (43)$$

where we have considered neutral total charge. Since

$$\mathbf{u}_i \ll \mathbf{u}_e, \quad (44)$$

we have

$$-\mathbf{u}_e = \frac{\mathbf{j}}{en_e}. \quad (45)$$

On the other hand, the average velocity is defined as

$$\mathbf{u} = \frac{m_e n_e \mathbf{u}_e + m_i n_i \mathbf{u}_i}{m_e n_e + m_i n_i} \simeq \mathbf{u}_i, \quad (46)$$

because $(m_e/m_i) \simeq 0$. Therefore, by substituting Equations (45) and (46) in Equation (42), for the electron entropy production, we obtain

$$\sigma_e = -\frac{en_e \mathbf{u}_e \cdot \mathbf{E}}{T_e} + w_e \frac{n_e \mathbf{u}_e \cdot (\mathbf{u}_i \times \mathbf{b})}{T_e m_e} - \frac{\mathbf{u}_e}{T_e} \cdot \nabla P_e - \frac{\mathbf{q}_e}{T_e^2} \cdot \nabla T_e. \quad (47)$$

Since \mathbf{u}_e collinear with \mathbf{u}_i , we arrive at

$$\sigma_e = -\frac{en_e \mathbf{u}_e \cdot \mathbf{E}}{T_e} - \frac{\mathbf{u}_e}{T_e} \cdot \nabla P_e - \frac{\mathbf{q}_e}{T_e^2} \cdot \nabla T_e. \quad (48)$$

Making a comparison between Equations (35), (36) and (48), we can see that they coincide if we consider that Equations (40) and (41) are satisfied. This method is used to obtain the modified electron entropy production when we modify the equations of state in Section 4.

3. Equations of State for the Kelly Plasma

For more than 80 years, there has been much interest in Plasma Physics due to its applications in Tokamaks and Astrophysics. However, under non-relativistic conditions, the equations of state for an ideal gas are used as the first approximation to calculate the equations of balance, the moment equations and thermodynamic flows [3,4]. Nevertheless, among other proposals, Kelly [10] gave an equation of state for a Plasma with different species relating the pressure P , the number of particles for each species N_i , the Volume V and the temperature T . However, one equation of state is not enough to describe a system composed by many species without given other equations of state. This point was partially corrected by Wergeland [11] by including an expression for the corrected internal energy due to Debye–Hückel theory [7] where the energy is obtained by making an average of the Coulomb energy.

Our purpose in this section consists of completing the set of equations of state for a plasma with two species: electrons and one type of ion. Since in many situations in Plasma Physics, it is just necessary to consider only the electrons, we analyze the Kelly equation for one species. We will obtain the electron entropy for a plasma composed of electrons where the effect of the ions are considered just in the equations of state. We obtain the entropy as a function of the volume V , the number of electrons N and the temperature T . However, it is not possible to write explicitly the entropy as a function of the internal energy U , the volume V and the number of electrons N (or the internal energy as a function of

the entropy S , the volume V and the number of electrons N). Accordingly, by using a technique based on Maxwell's relations [17], we obtain the eigenvalues of the Hessian in order to check the viability of the system. Moreover, The nonzero eigenvalues are related with the relaxation times which are very important in Plasma Physics. With these results, we can affirm that the new set of equations of state describes a thermodynamic system and can be used instead of the ideal gas equations of state in the calculation of the electron entropy production.

3.1. Preliminaries

In the case of considering only electrons and ions in a quasi-neutral plasma, the Kelly equation for the partial pressure of the electrons can be written as

$$P_e = \frac{N}{V}kT \left(1 - \frac{1}{18N_D}\right) = \frac{N}{V}kT \left(1 - \frac{1}{24\frac{N}{V} \left(\frac{VkT}{4\pi Ne^2}\right)^{\frac{3}{2}}}\right), \quad (49)$$

where

$$N_D = \frac{4}{3}\pi D^3 \frac{N}{V} \quad D = \left(\frac{k}{4\pi e^2 Z_1^2}\right)^{\frac{1}{2}} \frac{V^{\frac{1}{2}} T^{\frac{1}{2}}}{N^{\frac{1}{2}}} = \left(\frac{kTV}{4\pi Ne^2}\right)^{1/2}. \quad (50)$$

It is necessary to note that D is the Debye's length and N_D represents the number of particles contained in a Debye's sphere. Following Wergeland [11] and Equation (6), the equations of state are:

$$P = \frac{NkT}{V} - \frac{N^{\frac{3}{2}}}{3V^{\frac{3}{2}}T^{\frac{1}{2}}} \left(\frac{\pi^{\frac{1}{2}}e^3}{k^{\frac{1}{2}}}\right), \quad (51)$$

and

$$U = \frac{3}{2}NkT - \frac{N^{\frac{3}{2}}}{V^{\frac{1}{2}}T^{\frac{1}{2}}} \left(\frac{\pi^{\frac{1}{2}}e^3}{k^{\frac{1}{2}}}\right). \quad (52)$$

It is interesting that we can obtain the heat capacity $C_{V,N}$, that is:

$$C_{V,N} = \left[\frac{\partial U}{\partial T}\right]_{V,N} = \frac{3}{2}Nk + \frac{N^{\frac{3}{2}}}{2V^{\frac{1}{2}}T^{\frac{3}{2}}} \left(\frac{\pi^{\frac{1}{2}}e^3}{k^{\frac{1}{2}}}\right). \quad (53)$$

It has to be highlighted that if we consider the expression of the usual entropy production, Equation (48), and under certain circumstances it is considered to cancel, the modified entropy production, Equation (94), may not vanish. Although the term may be considered in first instance as negligible, because the correction is of the order of N_D^{-1} which for plasma is very small, we will see that due to the new terms, the modified entropy production may not be so small.

3.2. Entropy

Due to the similarity with the relationship between energy and pressure in the case of photons, we can propose a correction for the entropy given by the following expression

$$S_{corr} = \frac{U_{corr}}{3T}. \quad (54)$$

Notice that due to the expression of the corrected energy in our case (see Equation (52)), there is a factor 1/3 and not a factor 4/3 as it happens in the photon case. Then, we can propose the total entropy as

$$S = S_{ig} + S_{corr}, \quad (55)$$

where S_{ig} represents the entropy for a null charge, $e = 0$, that is the ideal gas. Therefore,

$$\begin{aligned} S &= S_{ig} + \frac{U_{corr}}{3T} \\ &= S_{ig} - \frac{N^{\frac{3}{2}}}{3V^{\frac{1}{2}}T^{\frac{3}{2}}} \left(\frac{\pi^{\frac{1}{2}}e^3}{k^{\frac{1}{2}}} \right). \end{aligned} \tag{56}$$

If we calculate the Helmholtz free energy $A = A_{ig} + U_{corr} - TS_{corr}$ and deduce the pressure by means of $P = - \left[\frac{\partial A}{\partial V} \right]_{T,N}$, we obtain the equations of state described in Equation (51) which partially verifies that we are dealing with a thermodynamic system. However, this is not enough to assure that the set of equations of state, Equations (51) and (52), represents a thermodynamic system. Indeed, it is necessary to show that entropy as a fundamental equation is a first degree homogeneous function and that the temperature is positive, or simply obtain the eigenvalues of the Hessian showing one of them is null and the others are positive real quantities.

3.3. the Hessian of the Kelly Plasma

Since the electron entropy and the internal energy cannot be written explicitly as fundamental equations ($S = S(U, V, N)$ or $U = U(S, V, N)$), we proceed to calculate the eigenvalues of the Hessian. Since the Hessian is constituted by entrees of this kind $[\partial^2 U / \partial X_i \partial X_j]_{X_k}$, where the X_i represents the extensive variables of the system, we can use thermodynamic relations as the Maxwell's relations in order to obtain the Hessian. For example, if we have an expression of $T = T(S, V, N)$, it will be sufficient in order to calculate

$$\left[\frac{\partial^2 U}{\partial S \partial S} \right]_{V,N} = \left[\frac{\partial T}{\partial S} \right]_{V,N}. \tag{57}$$

Then, the eigenvalues are obtained by following this technique [17] and they are the following: we found an eigenvalue $\lambda = 0$, and the other two eigenvalues are

$$\lambda_{\pm} = \alpha \pm \beta, \tag{58}$$

The eigenvalues α and β have been calculated by Arango-Reyes and Ares de Parga [17] and are positive real quantities for the range of temperatures of the Plasma and they are related with the relaxation times. Therefore, the system described by Equations (51), (52) and (56) represents a thermodynamic system and it can be used to calculate the electron entropy production.

4. Modified Electron Entropy Production

Once we have a new set of equations of state for the electrons in a Plasma, Equations (51), (52) and (56), we can calculate the new expression of the electron entropy production applying the same procedure than in the usual case but using the new set of equations of state. In our treatment, we rewrite the equations of state for the electrons as (in this section we return to consider $k = 1$ as in Section 2):

$$P_e = n_e T_e - \frac{\gamma n_e^{3/2}}{T_e^{1/2}}, \tag{59}$$

where

$$\gamma = \frac{\pi^{1/2} e^3}{3}, \tag{60}$$

and Equation (41) has been considered. In addition,

$$s = \ln\left(\frac{T_e^{3/2}}{n_e}\right) - \frac{\gamma n_e^{1/2}}{T_e^{3/2}}. \tag{61}$$

4.1. Deduction of the Modified Entropy Production

Let us begin by calculating as in Section 2,

$$\frac{\partial P_e}{\partial t} = n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{\partial T_e}{\partial t} + T_e \left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{\partial n_e}{\partial t}. \tag{62}$$

Therefore, we can express $\frac{\partial T_e}{\partial t}$ as

$$\frac{\partial T_e}{\partial t} = \frac{2}{3} \frac{1}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \frac{\partial \left(\frac{3}{2} P_e \right)}{\partial t} - \frac{T_e \left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \frac{\partial n_e}{\partial t} \tag{63}$$

With these results, we can proceed to calculate $\frac{\partial s_e}{\partial n_e}$ and $\frac{\partial s_e}{\partial T_e}$ with the help of Equation (61), we have

$$\frac{\partial s_e}{\partial n_e} = -\frac{1}{n_e} \left(1 + \frac{\gamma n_e^{1/2}}{2} T_e^{-3/2} \right), \tag{64}$$

and

$$\frac{\partial s_e}{\partial T_e} = \frac{3}{2T_e} \left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right) \tag{65}$$

By using Equation (24) and using the new equations of state, we arrive at

$$\mathbf{Q}_e = \mathbf{q}_e + \frac{5}{2} n_e T_e \mathbf{u}_e - \frac{5}{2} \frac{\gamma n_e^{3/2}}{T_e^{1/2}} \mathbf{u}_e. \tag{66}$$

Once we have these identities, we can develop $\frac{\partial n_e s_e}{\partial t}$; that is, substituting Equations (63)–(65) into Equation (10), we have

$$\frac{\partial n_e s_e}{\partial t} = n_e \left(\left(s_e - \frac{1}{n_e} \left(1 + \frac{\gamma}{2 n_e^{1/2}} T_e^{-3/2} \right) - \frac{3}{2} \frac{1}{T_e} \left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{T_e \left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \right) \frac{\partial n_e}{\partial t} + \frac{1}{T_e} \left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{1}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \frac{\partial \left(\frac{3}{2} P_e \right)}{\partial t} \right). \tag{67}$$

By using Equation (13), considering too that the term proportional to \mathbf{u}_e^2 is neglected, we have

$$\frac{\partial n_e s_e}{\partial t} = n_e \left(\left(s_e - \frac{1}{n_e} \left(1 + \frac{\gamma}{2 n_e^{1/2}} T_e^{-3/2} \right) - \frac{3}{2} \frac{1}{T_e} \left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{T_e \left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \right) \frac{\partial n_e}{\partial t} + \frac{\left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e T_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \left(Q_e + \mathbf{u}_e \cdot \left(\mathbf{F}_e - e n_e \mathbf{E} \right) - \nabla \cdot \mathbf{Q}_e \right) \right) \tag{68}$$

Substituting Equation (66) in the last equation, we obtain:

$$\frac{\partial n_e s_e}{\partial t} = n_e \left(\left(s_e - \frac{1}{n_e} \left(1 + \frac{\gamma}{2 n_e^{1/2}} T_e^{-3/2} \right) - \frac{3}{2} \frac{1}{T_e} \left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{T_e \left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \right) \frac{\partial n_e}{\partial t} + \frac{\left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e T_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \left(Q_e + \mathbf{u}_e \cdot \left(\mathbf{F}_e - e n_e \mathbf{E} \right) - \nabla \cdot \left(\mathbf{q}_e + \frac{5}{2} n_e T_e \mathbf{u}_e - \frac{5}{2} \frac{\gamma n_e^{3/2}}{T_e^{1/2}} \mathbf{u}_e \right) \right) \right) \tag{69}$$

Let us calculate first $\nabla \cdot (n_e T_e \mathbf{u}_e)$ and $\nabla \cdot \left(\frac{\gamma n_e^{3/2}}{T_e^{1/2}} \mathbf{u}_e \right)$:

$$\nabla \cdot (n_e T_e \mathbf{u}_e) = n_e \mathbf{u}_e \cdot \nabla \cdot T_e + T_e \nabla \cdot n_e \mathbf{u}_e, \tag{70}$$

and

$$\begin{aligned} \nabla \cdot \left(\frac{\gamma n_e^{3/2}}{T_e^{1/2}} \mathbf{u}_e \right) &= n_e \mathbf{u}_e \cdot \left(\frac{1}{2} n_e^{-1/2} T_e^{-1/2} \nabla n_e - \frac{1}{2} n_e^{1/2} T_e^{-3/2} \nabla T_e \right) \\ &\quad + \frac{n_e^{1/2}}{T_e^{1/2}} \nabla \cdot (n_e \mathbf{u}_e). \end{aligned} \tag{71}$$

Then, let us calculate ∇P_e ,

$$\nabla P_e = n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \nabla T_e + T_e \left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \nabla n_e. \tag{72}$$

We can write explicitly ∇T_e ,

$$\nabla T_e = \frac{1}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \nabla P_e - \frac{T_e \left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \nabla n_e. \tag{73}$$

By using Equations (70) and (72), we can calculate

$$\begin{aligned} -\nabla \cdot \left(\frac{5}{2} n_e T_e \mathbf{u}_e - \frac{5}{2} \frac{\gamma n_e^{3/2}}{T_e^{1/2}} \mathbf{u}_e \right) &= -\frac{5}{2} \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) n_e \mathbf{u}_e \cdot \nabla T_e \\ &\quad - \frac{5}{2} \left(1 - \gamma n_e^{1/2} T_e^{-3/2} \right) \nabla \cdot (n_e \mathbf{u}_e) \\ &\quad + \frac{5}{4} \gamma n_e^{1/2} T_e^{-1/2} n_e \mathbf{u}_e \cdot \nabla n_e \end{aligned} \tag{74}$$

By substituting Equation (73) into Equation (74), we obtain

$$\begin{aligned} -\nabla \cdot \left(\frac{5}{2} n_e T_e \mathbf{u}_e - \frac{5}{2} \frac{\gamma n_e^{3/2}}{T_e^{1/2}} \mathbf{u}_e \right) &= \frac{5}{2} T_e \left(1 - \gamma n_e^{1/2} T_e^{-3/2} \right) (\mathbf{u}_e \cdot \nabla n_e - \nabla \cdot (n_e \mathbf{u}_e)) \\ &\quad - \frac{5}{2} \mathbf{u}_e \cdot \nabla P_e \end{aligned} \tag{75}$$

By using this last equation in Equation (69), we arrive at

$$\frac{\partial n_e s_e}{\partial t} = \left(\begin{aligned} &\left(s_e - \left(1 + \frac{\gamma}{2 n_e^{1/2}} T_e^{-3/2} \right) - \frac{3}{2} \left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{\left(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)}{1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}} \right) \frac{\partial n_e}{\partial t} \\ &+ \frac{\left(1 + \gamma n_e^{1/2} T_e^{-3/2} \right)}{n_e \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right)} \left(\begin{aligned} &\frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - e n_e \mathbf{E})}{T_e} \\ &\frac{5}{2} \left(1 - \gamma n_e^{1/2} T_e^{-3/2} \right) (\mathbf{u}_e \cdot \nabla n_e - \nabla \cdot (n_e \mathbf{u}_e)) \\ &- \frac{5}{2 T_e} \mathbf{u}_e \cdot \nabla P_e \end{aligned} \right) \end{aligned} \right). \tag{76}$$

Let us now make an approximation

$$\gamma n_e^{1/2} T_e^{-3/2} \ll 1, \tag{77}$$

which is a natural approximation in a Tokamak [21], such that

$$\frac{\left(1 + \gamma n^{1/2} T^{-3/2} \right)}{\left(1 + \frac{1}{2} \gamma n^{1/2} T^{-3/2} \right)} \approx 1 + \frac{1}{2} \gamma n^{1/2} T^{-3/2}, \tag{78}$$

Therefore,

$$\frac{\partial n_e s_e}{\partial t} = \left(\begin{array}{l} \left(s_e - \left(1 + \frac{\gamma}{2n_e^{1/2}} T_e^{-3/2} \right) - \frac{3}{2} \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{(1 - \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2})}{(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2})} \right) \frac{\partial n_e}{\partial t} \\ + \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \left(\begin{array}{l} \frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E})}{T_e} \\ + \frac{5}{2} \left(1 - \gamma n_e^{1/2} T_e^{-3/2} \right) (\mathbf{u}_e \cdot \nabla n_e - \nabla \cdot (n_e \mathbf{u}_e)) \\ - \frac{5}{2T_e} \mathbf{u}_e \cdot \nabla P_e \end{array} \right) \end{array} \right) \quad (79)$$

Simplifying, we arrive at

$$\frac{\partial n_e s_e}{\partial t} = \left(\begin{array}{l} \left(s_e - \frac{5}{2} + \frac{\gamma}{n_e^{1/2}} T_e^{-3/2} \right) \frac{\partial n_e}{\partial t} + \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \left(\frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E}) - \nabla \cdot \mathbf{q}_e}{T_e} \right) \\ - \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{5}{2} \mathbf{u}_e \cdot \nabla P_e + \frac{5}{2} \left(1 - \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) (\mathbf{u}_e \cdot \nabla n_e - \nabla \cdot (n_e \mathbf{u}_e)) \end{array} \right) \quad (80)$$

If we use the equation of continuity in Equation (28), we obtain

$$\begin{aligned} \frac{\partial n_e s_e}{\partial t} &= -s_e \nabla \cdot (n_e \mathbf{u}_e) + \frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E})}{T_e} \\ &\quad - \frac{\nabla \cdot \mathbf{q}_e}{T_e} - \frac{5}{2T_e} \mathbf{u}_e \cdot \nabla P_e + \frac{5}{2} \mathbf{u}_e \cdot \nabla n_e \\ &\quad + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \left(\begin{array}{l} \frac{1}{2} \nabla \cdot (n_e \mathbf{u}_e) + \frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E})}{T_e} \\ - \frac{\nabla \cdot \mathbf{q}_e}{T_e} - \frac{5}{2T_e} \mathbf{u}_e \cdot \nabla P_e - \frac{5}{2} \mathbf{u}_e \cdot \nabla n_e \end{array} \right). \end{aligned} \quad (81)$$

On the other hand, from Equation (59), we can obtain

$$\nabla n_e = \frac{1}{\left(T_e - \frac{3}{2} \frac{\gamma n_e^{1/2}}{T_e^{1/2}} \right)} \nabla P_e - \frac{\left(n_e + \frac{1}{2} \frac{\gamma n_e^{1/2}}{T_e^{3/2}} \right)}{\left(T_e - \frac{3}{2} \frac{\gamma n_e^{1/2}}{T_e^{1/2}} \right)} \nabla T_e. \quad (82)$$

If in Equation (82), we use the approximation of Equation (77), we have

$$\nabla n_e = \frac{1}{T_e} \left(T_e + \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \nabla P_e - \frac{n_e}{T_e} \left(1 + 2 \gamma n_e^{1/2} T_e^{-3/2} \right) \nabla T_e. \quad (83)$$

By making the scalar product of the last equation with $(5/2)(T_e - \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}) \mathbf{u}_e$, we arrive at

$$\begin{aligned} \frac{5}{2} \left(T_e - \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \mathbf{u}_e \cdot \nabla n_e &= \frac{5}{2} \left(T_e + \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{1}{T_e} \mathbf{u}_e \cdot \nabla P_e \\ &\quad - \frac{5}{2} \left(1 + \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e. \end{aligned} \quad (84)$$

By using Equation (84) in Equation (81), we obtain

$$\begin{aligned} \frac{\partial n_e s_e}{\partial t} &= \left(\left(s_e - \frac{1}{4} \gamma n_e^{1/2} T_e^{-3/2} \right) \nabla \cdot (n_e \mathbf{u}_e) \right) \\ &\quad + \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \left(\frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E}) - \nabla \cdot \mathbf{q}_e}{T_e} \right) \\ &\quad + \frac{5}{4} \left(\gamma n_e^{1/2} T_e^{-3/2} \right) \frac{\mathbf{u}_e}{T_e} \cdot \nabla P_e - \frac{5}{2} \left(1 + \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e. \end{aligned} \quad (85)$$

In order to put our attention in the equation of balance, we must consider the following identity:

$$\begin{aligned}
 -s_e \nabla \cdot n_e \mathbf{u}_e &= -\nabla \cdot (s_e n_e \mathbf{u}_e) - \left(1 + 2\gamma n_e^{1/2} T_e^{-3/2}\right) \frac{1}{T_e} \mathbf{u}_e \cdot \nabla P_e \\
 &\quad + \left(\frac{5}{2} + 4\gamma n_e^{1/2} T_e^{-3/2}\right) \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e
 \end{aligned} \tag{86}$$

If we substitute Equation (86) into Equation (85), we have

$$\begin{aligned}
 \frac{\partial n_e s_e}{\partial t} + \nabla \cdot \left(s_e n_e \mathbf{u}_e + \frac{\mathbf{q}_e}{T_e}\right) &= \frac{1}{T_e} (\mathbf{u}_e \cdot \nabla P_e) \left(1 + \frac{3}{4} \gamma n_e^{1/2} T_e^{-3/2}\right) \\
 &\quad - \left(\frac{\mathbf{q}_e}{T_e} \cdot \nabla T_e\right) \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \\
 &\quad + \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \left(\frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E})}{T_e}\right) \\
 &\quad + \frac{1}{4} \gamma n_e^{1/2} T_e^{-3/2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e + \frac{1}{4} \gamma n_e^{1/2} T_e^{-3/2} \nabla \cdot (n_e \mathbf{u}_e) \\
 &\quad - \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \nabla \cdot \left(\frac{\mathbf{q}_e}{T_e}\right).
 \end{aligned} \tag{87}$$

Simplifying, we arrive at

$$\begin{aligned}
 &\frac{\partial n_e s_e}{\partial t} + \nabla \cdot \left(n_e \mathbf{u}_e \left(s_e - \frac{1}{4} \gamma n_e^{1/2} T_e^{-3/2}\right) + \frac{\mathbf{q}_e}{T_e} \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right)\right) \\
 = &-\frac{\mathbf{u}_e}{T_e} \cdot (\nabla P_e) \left(1 + \frac{7}{8} \gamma n_e^{1/2} T_e^{-3/2}\right) - \frac{\mathbf{q}_e}{T_e^2} \cdot (\nabla T_e) \left(1 + \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \\
 &+ \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \left(\frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E})}{T_e}\right) \\
 &+ \frac{1}{4} \gamma n_e^{1/2} T_e^{-3/2} \frac{\mathbf{q}_e}{n_e T_e^2} \cdot \nabla P_e + \frac{3}{4} \gamma n_e^{1/2} T_e^{-3/2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e.
 \end{aligned} \tag{88}$$

We can define now

$$\mathbf{J}_{esm} = n_e \mathbf{u}_e \left(s_e - \frac{1}{4} \gamma n_e^{1/2} T_e^{-3/2}\right) + \frac{\mathbf{q}_e}{T_e} \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right). \tag{89}$$

Since

$$\sigma_{em} = \frac{\partial}{\partial t} (n_e s_e) + \nabla \cdot \mathbf{J}_{esm}, \tag{90}$$

we have

$$\begin{aligned}
 \sigma_{em} &= -\frac{\mathbf{u}_e}{T_e} \cdot (\nabla P_e) \left(1 + \frac{7}{8} \gamma n_e^{1/2} T_e^{-3/2}\right) - \frac{\mathbf{q}_e}{T_e^2} \cdot (\nabla T_e) \left(1 + \frac{3}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \\
 &\quad + \left(1 + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \left(\frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E})}{T_e}\right) \\
 &\quad + \frac{1}{4} \gamma n_e^{1/2} T_e^{-3/2} \frac{\mathbf{q}_e}{n_e T_e^2} \cdot \nabla P_e + \frac{3}{4} \gamma n_e^{1/2} T_e^{-3/2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e.
 \end{aligned} \tag{91}$$

Notice that by using Equations (33) and (35), we arrive at

$$\mathbf{J}_{esm} = \mathbf{J}_{es} + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \left(\frac{\mathbf{q}_e}{T_e} - \frac{n_e \mathbf{u}_e}{2}\right) \tag{92}$$

and

$$\sigma_{em} = \sigma_e + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \left(\begin{array}{l} -3 \frac{\mathbf{q}_e}{T_e^2} \cdot (\nabla T_e) + \left(\frac{Q_e + \mathbf{u}_e \cdot (\mathbf{F}_e - en_e \mathbf{E})}{T_e} \right) \\ -\frac{7}{4} \frac{\mathbf{u}_e}{T_e} \cdot (\nabla P_e) + \frac{1}{2} \frac{\mathbf{q}_e}{n_e T_e^2} \cdot \nabla P_e + \frac{3}{2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e \end{array} \right). \quad (93)$$

4.2. Deduction of the Modified Electron Entropy Production in Terms of The Hermitian Moments

Let us consider the modified entropy production, Equations (91) or (93) but with the approximations described in Equations (40), (41) and (77), that is:

$$\sigma_{em} = \sigma_e + \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \left(\begin{array}{l} -3 \frac{\mathbf{q}_e}{T_e^2} \cdot (\nabla T_e) - en_e \frac{\mathbf{u}_e \cdot \mathbf{E}}{T_e} \\ -\frac{7}{4} \frac{\mathbf{u}_e}{T_e} \cdot (\nabla P_e) + \frac{1}{2} \frac{\mathbf{q}_e}{n_e T_e^2} \cdot \nabla P_e + \frac{3}{2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e \end{array} \right) \quad (94)$$

If we compare the usual entropy production σ_e , Equation (48), and the modified entropy production σ_{em} , Equation (94), we can see that in the correction term there are the same three products of flows by forces except for the factor $\left(\frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right)$ and their respective different coefficients as -3 and $-7/4$ unless for two new terms $\frac{1}{2} \frac{\mathbf{q}_e}{n_e T_e^2} \cdot \nabla P_e$ and $\frac{3}{2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e$. Under certain circumstances, the usual energy production may vanish as for example: $\mathbf{q}_e = 0$, $\mathbf{E} = \mathbf{0}$ and $\nabla P_e = 0$, which implies that $\sigma_e = 0$. In such a case due to the resemblance of three of the terms in the correction term of the entropy production, the modified entropy production, for typical conditions of a Tokamak [21], can be approximated to

$$\sigma_{em} \simeq \left(\frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \right) \frac{3}{2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e \simeq 4 \times 10^{-10} 6 \frac{10^{14}}{10^{-8}} \simeq 24 \times 10^{12} \mathbf{u}_e \cdot \nabla T_e. \quad (95)$$

Therefore, even if $\mathbf{u}_e \cdot \nabla T_e$ is very small, σ_{em} cannot be neglected and consequently the corrections done to the equations of state lead us to results that in certain circumstances are not negligible.

Furthermore, these last expressions show that the change in the expression of entropy production can be significant when equations of state very different from the ideal gas equations are used. Moreover, new products of fluxes and forces, $\left(\frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \frac{1}{2} \frac{\mathbf{q}_e}{n_e T_e^2} \cdot \nabla P_e$ and $\left(\frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2}\right) \frac{3}{2} \frac{n_e}{T_e} \mathbf{u}_e \cdot \nabla T_e$, appear in the modified entropy production which represent an important result if we want to analyze the system from the point of view of the Thermodynamics of Irreversible Processes [2,22].

By using Equations (38) and (45) and reminding that \mathbf{u}^e is collinear with \mathbf{u}^i , we have

$$\begin{aligned} u_r^e &= - \left(\frac{T_e}{m_e} \right)^{\frac{1}{2}} h_r^{(1)}, & q_r^e &= m_e n_e \sqrt{\frac{5}{2}} \left(\frac{T_e}{m_e} \right)^{\frac{3}{2}} h_r^{e(3)}, \\ g_r^{(1)} &= \tau_e \left(\frac{m_e}{T_e} \right)^{\frac{1}{2}} \left(\frac{eE_r}{m_e} + \frac{1}{m_e n_e} \frac{\partial P_e}{\partial x_r} \right), \\ g_r^{e(3)} &= -\sqrt{\frac{5}{2}} \tau_e \left(\frac{T_e}{m_e} \right)^{\frac{1}{2}} \frac{1}{T_e} \frac{\partial T_e}{\partial x_r}. \end{aligned} \quad (96)$$

Therefore, in this case, we can express the regular electron entropy production σ_e as

$$\sigma_e = \frac{n_e}{\tau_e} \left(h_r^{(1)} g_r^{(1)} + h_r^{e(3)} g_r^{e(3)} \right). \quad (97)$$

For the modified electron entropy production we need to redefine the g 's terms, that is

$$g_{mr}^{(1)} = g_r^{(1)} - \tau_e \left(\frac{m_e}{T_e} \right)^{\frac{1}{2}} \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \left(\frac{eE_r}{m_e} - \frac{7}{4} \frac{1}{m_e n_e} \frac{\partial P_e}{\partial x_r} + \frac{3}{2} \frac{1}{m_e} \frac{\partial T_e}{\partial x_r} \right), \quad (98)$$

and

$$g_{mr}^{e(3)} = g_r^{e(3)} + \tau_e \sqrt{\frac{5}{2}} \left(\frac{m_e}{T_e} \right)^{\frac{1}{2}} \frac{1}{2} \gamma n_e^{1/2} T_e^{-3/2} \left(3 \frac{1}{T_e} \frac{\partial T_e}{\partial x_r} + \frac{1}{2} \frac{1}{n_e T_e} \frac{\partial P_e}{\partial x_r} \right). \quad (99)$$

We arrive at

$$\sigma_{em} = \frac{n_e}{\tau_e} \left(h_r^{(1)} g_{mr}^{(1)} + h_r^{e(3)} g_{mr}^{e(3)} \right). \quad (100)$$

5. Concluding Remarks

We obtain an expression for the modified electron entropy production by means of the Kelly equations of state within the approximation described by Equations (40) and (41) but keeping the constraint in Equation (77). It has to be noticed that within our approximations the ion entropy production does not suffer any modification.

The new terms in the modified entropy production represent new products of fluxes and forces. Such products may not be so negligible in some circumstances which implies that some analysis can be done with respect the Linear Irreversible Thermodynamics [22]. The modifications done to the expression of the energy production in terms of dimensionless Hermitian moments and dimensionless source terms, Equation (100), may suggest that when distribution functions are different from the Boltzmann distribution function, the irreducible tensorial Hermite polynomials will not be useful as Izacard [23] proposed by making a generalization of the Hermite polynomial representation.

The next step consists of obtaining the modified electron entropy production without using such approximations. Such expression will include the moment $h_r^{e(2)}$ by considering that the collisional rate of heat exchange Q_e , the friction force \mathbf{F}_e and the traceless pressure tensor π_{ij}^e do not have to be neglected.

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References

1. Clausius, R. Ueber eine veränderte Form des zweiten Hauptsatzes der mechanischen Wärmetheoriein. *Ann. Phys. Chem.* **1854**, *93*, 481–506. [[CrossRef](#)]
2. Prigogine, I. *Introduction to Thermodynamics of Irreversible Processes*, 2nd ed.; Interscience Publishers: New York, NY, USA, 1961.
3. Hinton, F.; Hazeltine, R. Theory of plasma transport in toroidal confinement systems. *Rev. Mod. Phys.* **1976**, *48*, 239–308. [[CrossRef](#)]
4. Balescu, R. *Transport Processes in Plasmas*; North Holland: Amsterdam, The Netherlands, 1988; Volume 1.
5. Grad, H. On the kinetic theory of rarefied gases. *Commun. Pure Appl. Math.* **1949**, *2*, 311. [[CrossRef](#)]
6. Sonnino, G.; Evslin, J.; Sonnino, A.; Steinbrecher, G.; Tirapegui, E. Symmetry group and group representations associated with the thermodynamic covariance principle. *Phys. Rev. E* **2016**, *95*, 042103. [[CrossRef](#)] [[PubMed](#)]
7. Debye, P.; Hückel, E. Zur Theorie der Elektrolyte. I. Gefrierpunktserniedrigung und verwandte Erscheinungen. *Phys. Z.* **1923**, *24*, 185–206.
8. Kontogeorgis, G.; Maribo-Mogensen, B.; Thomsen, K. The Debye-Hückel theory and its importance in modeling electrolyte solutions. *Fluid Phase Equilib.* **2018**, *462*, 130–152. [[CrossRef](#)]
9. Kirwood, G.; Poirier, J. The Statistical Mechanical Basis of the Debye-Hückel Theory of Strong Electrolytes. *J. Phys. Chem.* **1954**, *58*, 591. [[CrossRef](#)]
10. Kelly, D. Plasma Equation of State. *Am. J. Phys.* **1963**, *31*, 827. [[CrossRef](#)]
11. Wergeland, H. Remark on the Plasma Equation of State. *Am. J. Phys.* **1964**, *32*, 566. [[CrossRef](#)]

12. Liboff, R.; Lie, T. Plasma Virial and Equations of State for a Plasma. *Phys. Fluids* **1968**, *11*, 1943. [[CrossRef](#)]
13. Kaur, M.; Barbano, L.; Suen-Lewis, E.; Shrock, J.; Light, A.; Brown, M.; Schaffner, D. Measuring the equations of state in a relaxed magnetohydrodynamic plasma. *Phys. Rev. E* **2018**, *97*, 011202. [[CrossRef](#)] [[PubMed](#)]
14. Krall, N.A.; Trivelpiece, A.W. *Principles of Plasma Physics*; McGraw-Hill: New York, NY, USA, 1973.
15. Callen, H. *Thermodynamics and an Introduction to Thermostatistics*; John Wiley and Sons: New York, NY, USA, 1985.
16. Essex, C.; Andresen, B. The principal equations of state for classical particles, photons, and neutrinos. *J. Non-Equilib. Thermodyn.* **2013**, *38*, 293–312. [[CrossRef](#)]
17. Arango-Reyes, K.; Ares de Parga, G. Completeness of Classical Thermodynamics: The Ideal Gas, the Unconventional Systems, the Rubber Band, the Paramagnetic Solid and the Kelly Plasma. *Entropy* **2020**, *22*, 398–437. [[CrossRef](#)]
18. Sugama, M.; Horton, W. Transport processes and entropy production in toroidally rotating plasmas with electrostatic turbulence. *Phys. Plasmas* **1997**, *4*, 405. [[CrossRef](#)]
19. Abel, I.G.; Plunk, G.G.; Wang, E.; Barnes, M.; Cowley, S.C.; Dorland, W.; Schekochihin, A.A. Multiscale gyrokinetics for rotating tokamak plasmas: fluctuations, transport and energy flows. *Rep. Prog. Phys.* **2013**, *76*, 116201. [[CrossRef](#)] [[PubMed](#)]
20. Bizarro, J.P.S. A thermodynamical analysis of rf current drive with fast electrons. *Phys. Plasmas* **2015**, *22*, 082510. [[CrossRef](#)]
21. Riviere, A.C. Penetration of fast hydrogen atoms into a fusion reactor. *Nucl. Fusion* **1971**, *11*, 363–369. [[CrossRef](#)]
22. Onsager, L. Reciprocal Relations in Irreversible Processes. I. *Phys. Rev.* **1931**, *37*, 405. [[CrossRef](#)]
23. Izacard, O. Kinetic corrections from analytic non-Maxwellian distribution functions in magnetized plasmas. *Phys. Plasmas* **2016**, *23*, 082504. [[CrossRef](#)]



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