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Stochastic Comparisons of Weighted Distributions and Their Mixtures

Abdulahkim A. Albabtain ¹, Mansour Shrahili ¹, M. A. Al-Shehri ¹ and M. Kayid ^{1,2,*} 

¹ Department of Statistics and Operations Research College of Science, King Saud University, 11451 Riyadh, Saudi Arabia; hakim@ksu.edu.sa (A.A.A.); msharahili@ksu.edu.sa (M.S.); mashaeh@ksu.edu.sa (M.A.A.-S.)

² Department of Mathematics and Computer Science, Faculty of Science, Suez University, 14028 Suez, Egypt

* Correspondence: drkayid@ksu.edu.sa

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Abstract: In this paper, various stochastic ordering properties of a parametric family of weighted distributions and the associated mixture model are developed. The effect of stochastic variation of the output random variable with respect to the parameter and/or the underlying random variable is specifically investigated. Special weighted distributions are considered to scrutinize the consistency as well as the usefulness of the results. Stochastic comparisons of coherent systems made of identical but dependent components are made and also a result for comparison of Shannon entropies of weighted distributions is developed.

Keywords: stochastic order; mixture distribution; hazard rate; preservation; coherent systems; shannon entropy

1. Introduction

In the literature, weighted distributions have been exhaustively applied and put to use to model data in nature, as they provide more insights to provide more adequacy in modelling as a result of variety of sampling surveys (cf. Rao [1], Patil and Rao [2] and Patil [3]). Let X be a random variable with cumulative distribution function (cdf) F and probability density function (pdf) f and let $w(\cdot, \theta)$ is a non-negative function such that $E(w(X, \theta))$ exists and is finite for all $\theta \in \chi$, where χ is an arbitrary subset of \mathbb{R} . Then X_w is taken to be a random variable with weighted distribution associated with f , having pdf

$$f_w(x, \theta) = \frac{w(x, \theta)f(x)}{E[w(X, \theta)]}. \quad (1)$$

Many families of statistical distributions hold at the disposal of the family of weighted distributions in (1) (see, e.g., the typical weighted distributions in Sections 3.1 and 3.2). Suppose that the hazard rate function h corresponds to the pdf f so that $h(x) = f(x)/\bar{F}(x)$ where $\bar{F} \equiv 1 - F$ is the survival function of X . In spirit of Jain et al. [4], the hazard rate of X_w is characterized by

$$h_w(x, \theta) = \frac{w(x, \theta)}{B(x, \theta)}h(x), \quad (2)$$

in which $B(x, \theta) = E[w(X, \theta) | X > x]$. As for the reversed hazard rate of X_w we have from Sunoj and Maya [5]

$$\tilde{h}_w(x, \theta) = \frac{w(x, \theta)}{A(x, \theta)}\tilde{h}(x), \quad (3)$$

where $A(x, \theta) = E[w(X, \theta) | X \leq x]$ and $\tilde{h}(x) = f(x)/F(x)$ is the reversed hazard rate function of X .

The density in (1) may be used to model data randomly drawn from population at a certain level θ of some quantity of interest. For example, θ could be a particular age for an individual, a certain time point or a given threshold with a specific amount. In many realistic circumstances it is acknowledged that the parameter θ may not be constant so that the occurrence of heterogeneity is sometimes incalculable and unexplained. In addition, it often takes place in practical situations where data from several populations is mixed. To model such data sets mixture models are used. For example, the measurements on the life lengths of a device may be gathered regardless of the manufacturer, or data may be gathered on humans without regard, say, to blood type. If the ignored variable has a bearing on the characteristic which is being measured, then the data follow a mixture model. To all intents and purposes, it is hard to find data that are not some kind of a mixture, because there is almost always some relevant covariate that is not observed.

The study of reliability properties of various mixture models has recently received much attention in the literature. When a mixture model is fitted to survival data, the mixing operation can change the pattern of aging for the lifetime unit under consideration in some favorite way (see, for example, Finkelstein and Esaulova [6], Alves and Dias [7], Arbel et al. [8], Cole and Bauer [9], Bordes and Chauveau [10], Li and Liu [11], Amini-Seresht and Zhang [12], Misra and Naqvi [13] and Badía and Lee [14]).

Mixture models capture heterogeneity in data by decomposing the population into latent subgroups, each of which is governed by its own subgroup-specific set of parameters. To represent a general formulation of the mixture model in the case of our study, consider the density

$$f^*(x) = \int_{\theta \in \mathcal{X}} f_w(x, \theta) dG(\theta) = f(x) \int_{\theta \in \mathcal{X}} \frac{w(x, \theta)}{\mu(\theta)} dG(\theta), \quad (4)$$

associated with (1), where $\mu(\theta) = E(w(X, \theta))$ and G is the cdf of the random variable Θ . It is known that $f^*(x) = f(x)v(x)$ with

$$v(x) = E\left(\frac{w(x, \Theta)}{\mu(\Theta)}\right), \quad (5)$$

playing the role of the weight function through which f is altered to f^* . This signifies that the mixture density in (4) can be thought as the density of a weighted distribution with weight function v for which $E(v(X)) = 1$. In situations where Θ is designated by a discrete random variable, a finite mixture model is considered. To this end, the model (4) is developed as

$$f^*(x) = f(x) \sum_{i=1}^{\infty} \frac{w(x, \theta_i)}{\mu(\theta_i)} g(\theta_i), \quad (6)$$

where $g(\theta_i)$ represents the value of the probability mass function (pmf) of Θ at θ_i for $i = 1, 2, \dots$. Throughout the paper, it is assumed that the output random variables following the mixture weighted distribution (4) have absolutely continuous distribution functions.

To the best of our knowledge, there has not been a work on the literature to argue different stochastic properties of the parametric weighted distributions as well as their mixtures in general to be attractive for broader audiences. There is a need for an effective study in this direction. The main objective of this paper is to initiate such a study to investigate the impact of the association of the model to a parameter on some general stochastic aspects of the model.

The rest of the paper is organized as follows. In Section 2, some useful notions of stochastic orders and some further stochastic properties are presented. In Section 3, some special applied weighted distributions are introduced. In Section 4, preservation of several ordinary as well as relative stochastic orderings is studied in Section 4.1. In Section 4.2, preservation properties of some stochastic orders in the extended mixture model of weighted distributions are secured and in the long run in Section 4.3, a link to information theory is provided.

2. Preliminaries

Assume that the random variables X and Y have distribution functions F and G , survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, density functions f and g , hazard rate functions $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$ and reversed hazard rate functions $\tilde{h}_X = f/F$ and $\tilde{h}_Y = g/G$, respectively. To compare the magnitude of random variables some notions of stochastic orders are introduced below.

Definition 1. The random variable X is said to be smaller than the random variable Y in the

- (i) usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x ;
- (ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is non-decreasing in x , or equivalently, if $h_X(x) \geq h_Y(x)$ for all x ;
- (iii) reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ is non-decreasing in x , or equivalently, if $\tilde{h}_X(x) \leq \tilde{h}_Y(x)$ for all x ;
- (iv) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is non-decreasing in x .
- (v) relative hazard rate order (denoted by $X \leq_{rhr} Y$) if $h_Y(x)/h_X(x)$ is non-increasing in x .
- (vi) relative reversed hazard rate order (denoted by $X \leq_{rrh} Y$) whenever $\tilde{h}_Y(x)/\tilde{h}_X(x)$ is non-decreasing in x .

It is known that the following implications hold:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr[rh]} Y \Rightarrow X \leq_{st} Y.$$

The notions of the totally positive of order 2 (TP2) and the reverse regular of order 2 (RR2) are defined as follows.

Definition 2 (Karlin [15]). A function $h(x, y)$ is said to be Sign-Regular of order 2 (SR₂) if $\epsilon_1 h(x, y) \geq 0$ and

$$\epsilon_2 \begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geq 0,$$

for all $x_1 \leq x_2$ and for all $y_1 \leq y_2$ for ϵ_1 and ϵ_2 equaling to +1 or -1.

If $\epsilon_1 = +1$ and $\epsilon_2 = +1$, then h is said to be TP₂. If $\epsilon_1 = +1$ and $\epsilon_2 = -1$, then h is said to be RR₂. It is readily pointed out that the TP₂ [RR₂] property of $h(t, x)$ is equivalent to saying that $h(t, x_2)/h(t, x_1)$ is non-decreasing [non-increasing] in t whenever $x_1 \leq x_2$ after making the conventions that $a/0 = +\infty$ when $a > 0$ and $a/0 = 0$ if $a = 0$. In view of the foregoing statements and by assuming $h_Y = h_2$ and $h_X = h_1$ and also $\tilde{h}_Y = \tilde{h}_2$ and $\tilde{h}_X = \tilde{h}_1$, one observes $X_1 \leq_{rhr} X_2$ holds if, and only if, $h_i(x)$ is RR₂ as a function of $(i, x) \in \{1, 2\} \times \zeta$, where ζ is the common support of X and Y . In a similar manner we can establish that $X_1 \leq_{rrh} X_2$ is equivalent to $\tilde{h}_i(x)$ being TP₂ in $(i, x) \in \{1, 2\} \times \zeta$.

3. Special Weighted Distributions

In this section, several special parametric weight functions are presented making the investigation of the main model of (4) more developed. First, some general formations of the weight function are considered by which many important families of weighted distributions are included. In all of the cases we assume that the weight function has a finite mean with respect to the underlying distribution.

3.1. Distribution-Free Weight Functions

Here, several weight functions which do not depend on the underlying distribution are given. Suppose that $w_i, i = 1, 2$ are two non-negative functions of x and that $k_i, i = 1, 2$ are two proper functions of θ so that the following weight functions satisfy the requirement that $\mu(\theta) < \infty$. Substituting any of these weight functions in the density (4) leads to a particular model that might be of some interest in a context.

- (i) (weighted power) $w(x, \theta) = w_1(x)(w_2(x))^{k_1(\theta)}$
- (ii) (weighted exponentiated) $w(x, \theta) = w_1(x)(k_1(\theta))^{w_2(x)}$
- (iii) (multiplicative) $w(x, \theta) = w_1(x)k_1(\theta)$
- (iv) (additive-multiplicative) $w(x, \theta) = w_1(x) + k_1(\theta)w_2(x)$
- (v) (weighted left-truncated) $w(x, \theta) = w_1(x)I[w_2(x) > k_1(\theta)]$
- (vi) (weighted right-truncated) $w(x, \theta) = w_1(x)I[w_2(x) \leq k_1(\theta)]$

3.2. Semiparametric Models

Models where the parameters of interest are finite-dimensional and the nuisance parameters are infinite-dimensional are called semiparametric models. There are some choices for the weight function $w(x, \theta)$ that are functional of the underlying distribution function F , including the parameter θ within. Below, we list some kinds of those choices whose associated weight function depend on the underlying distribution.

- (i) (Proportional hazards) $w(x, \theta) = \bar{F}^{\theta-1}(x)$, where $\theta > 0$.
- (ii) (Proportional reversed hazards) $w(x, \theta) = F^{\theta-1}(x)$ where $\theta > 0$.
- (iii) (Proportional odds ratio) $w(x, \theta) = 1/((1 - \bar{\theta}\bar{F}(x))^2)$ where $\theta > 0$, and $\bar{\theta} = 1 - \theta$.
- (iv) (Upper records) $w(x, \theta) = -\ln^\theta(\bar{F}(x))$, in which $\theta \in \mathbb{N}$.
- (v) (Lower records) $w(x, \theta) = -\ln^\theta(F(x))$, in which $\theta \in \mathbb{N}$.
- (vi) (Residual life) $w(x, \theta) = f(\theta + x)/f(x)$, where $\theta > 0$ is the guaranteed survival time.
- (viii) (Inactivity time) $w(x, \theta) = f(\theta - x)/f(x)$, in which $\theta > 0$ is the time of observation of failure.
- (viii) (Scale) $w(x, \theta) = f(\theta x)/f(x)$, in which $\theta > 0$.

4. Stochastic Orderings

In this section, preservation properties of some stochastic orders under the formation of the weighted model in the fixed as well as the random levels of the parameter θ are studied.

4.1. Weighted Distribution with Specific Parameter

Here, in the same vein as Misra et al. [16] several preservation properties on likelihood ratio, hazard rate and reversed hazard rates orders can be established in the sense of the model (1). Suppose that X_i is a random variable with pdf f_i and cdf F_i , for $i = 1, 2$, and assume that X_{iw_i} follows the weighted distribution of X_i with weight function $w_i(x) = w(x, \theta_i)$ having pdf

$$f_{iw_i}(x, \theta_i) = \frac{w(x, \theta_i)f_i(x)}{E[w(X_i, \theta_i)]}, \quad i = 1, 2, \quad (7)$$

where θ_1 and θ_2 are two fixed numbers in χ . In the next round, as will be presented, conditions for stochastic orders made of X_{1w_1} and X_{2w_2} to emulate the same type of stochastic orders between X_1 and X_2 are obtained.

The following Proposition is a direct conclusion of Theorem 3.2 in Misra et al. [16].

Proposition 1. Let $X_i, i = 1, 2$ have support ζ and let $w(x, \theta)$ be TP_2 in $(x, \theta) \in \zeta \times \chi$. Then

- (i) If $\theta_1 \leq \theta_2$, then $X_1 \leq_{lr} X_2$ implies that $X_{1w_1} \leq_{lr} X_{2w_2}$.
- (ii) If $\theta_1 \leq \theta_2$ and $w(x, \theta)$ is non-decreasing in x , then $X_1 \leq_{hr} X_2$ implies that $X_{1w_1} \leq_{hr} X_{2w_2}$.
- (iii) If $\theta_1 \leq \theta_2$ and $w(x, \theta)$ is non-increasing in x , then $X_1 \leq_{rh} X_2$ implies that $X_{1w_1} \leq_{rh} X_{2w_2}$.

Preservation properties of the stochastic orders considered in Proposition 1 have been procured for some special weighted distributions by Izadkhah et al. [17] including the models of proportional (reversed) hazard rates, upper (lower) records, right (left) truncation, moment generating and

size-biased distributions. Izadkhah et al. [18] obtained sufficient conditions for preservation of reversed mean residual life order and Izadkhah et al. [19] presented some conditions under which the mean residual life order is preserved under weighting. For the sake of completeness, the preservation properties of the likelihood ratio, the hazard rate and the reversed hazard rates orders are studied for some of the parametric weighted distributions considered in Sections 3.1 and 3.2. Suppose that X_1 and X_2 are two non-negative random variables with distribution functions F_1 and F_2 , survival functions $\bar{F}_1 = 1 - F_1$ and $\bar{F}_2 = 1 - F_2$ and density functions f_1 and f_2 , respectively.

Example 1 (Weighted power distribution). Assume that

$$w(x, \theta_i) = w_1(x)(w_2(x))^{k_1(\theta_i)}, i = 1, 2,$$

in which $\theta_1 \leq \theta_2 \in \chi$. Suppose that w_2 and k_1 are both non-decreasing (or non-increasing) functions. By Proposition 1(i), if $X_1 \leq_{lr} X_2$ then $X_{1w_1} \leq_{lr} X_{2w_2}$. Let us further assume that $k_1(\theta_i) > 0$ for $i = 1, 2$. Then, by Proposition 1(ii) $X_1 \leq_{hr} X_2$ implies $X_{1w_1} \leq_{hr} X_{2w_2}$ provided that w_1 and w_2 are both non-decreasing. In parallel, if w_1 and w_2 are both non-increasing then using Proposition 1(iii), $X_1 \leq_{rh} X_2$ concludes that $X_{1w_1} \leq_{rh} X_{2w_2}$.

Example 2 (Weighted exponentiated distribution). Consider the weight function

$$w(x, \theta_i) = w_1(x)(k_1(\theta_i))^{w_2(x)}, i = 1, 2$$

such that $\theta_1 \leq \theta_2 \in \chi$ in which $k_1(\theta_i) > 0$ for $i = 1, 2$. Presume that w_2 and k_1 are both non-decreasing (or non-increasing) functions. From Proposition 1(i), $X_1 \leq_{lr} X_2$ yields $X_{1w_1} \leq_{lr} X_{2w_2}$. If w_1 is non-decreasing, w_2 is non-decreasing and $k_1(\theta_i) > 1$ (resp. w_2 is non-increasing and $k_1(\theta_i) < 1$) for $i = 1, 2$ then by Proposition 1(ii) $X_1 \leq_{hr} X_2$ gives $X_{1w_1} \leq_{hr} X_{2w_2}$. Besides, if w_1 is non-increasing, w_2 is non-increasing and $k_1(\theta_i) > 1$ (resp. w_2 is non-decreasing and $k_1(\theta_i) < 1$) for $i = 1, 2$ then on applying by Proposition 1(iii) $X_1 \leq_{rh} X_2$ implies $X_{1w_1} \leq_{rh} X_{2w_2}$.

Example 3 (Additive-Multiplicative weighted distribution). Let

$$w(x, \theta_i) = w_1(x) + k_1(\theta_i)w_2(x), i = 1, 2,$$

where $\theta_1 \leq \theta_2 \in \chi$. Suppose that w_1/w_2 is non-increasing (resp. non-decreasing) and k_1 is non-decreasing (resp. non-increasing). Proposition 1(i) provides that $X_1 \leq_{lr} X_2$ implies $X_{1w_1} \leq_{lr} X_{2w_2}$. If, moreover, we assume that $k_1(\theta_i) > 0, i = 1, 2$ then when w_1 and w_2 are both non-decreasing by Proposition 1(ii) we deduce that $X_1 \leq_{hr} X_2$ gives $X_{1w_1} \leq_{hr} X_{2w_2}$ and once w_1 and w_2 are both non-increasing by Proposition 1(iii) $X_1 \leq_{rh} X_2$ entails that $X_{1w_1} \leq_{rh} X_{2w_2}$. In a suchlike manner, whenever $k_1(\theta_i) \leq 0, i = 1, 2$ then if w_1 is non-decreasing and w_2 is non-increasing, on applying Proposition 1(ii) $X_1 \leq_{hr} X_2$ assures that $X_{1w_1} \leq_{hr} X_{2w_2}$ and similarly when w_1 is non-increasing and w_2 is non-decreasing by Proposition 1(iii) it is deducible that $X_1 \leq_{rh} X_2$ gives $X_{1w_1} \leq_{rh} X_{2w_2}$.

Example 4 (Weighted left-truncated distribution). Let

$$w(x, \theta_i) = w_1(x)I[w_2(x) > k_1(\theta_i)], i = 1, 2$$

with $\theta_1 \leq \theta_2 \in \chi$. Let us suppose that w_2 and k_1 are both non-decreasing (or non-increasing) functions. By Proposition 1(i), $X_1 \leq_{lr} X_2$ implies $X_{1w_1} \leq_{lr} X_{2w_2}$. If w_1, w_2 and k_1 are all non-decreasing functions then Proposition 1(ii) establishes that $X_1 \leq_{hr} X_2$ implicate $X_{1w_1} \leq_{hr} X_{2w_2}$. In a similar manner, if w_1, w_2 and k_1 are all non-increasing functions then from Proposition 1(iii) it is deduced that $X_1 \leq_{rh} X_2$ implies $X_{1w_1} \leq_{rh} X_{2w_2}$.

Example 5 (Weighted right-truncated distribution). *Let*

$$w(x, \theta_i) = w_1(x)I[w_2(x) \leq k_1(\theta_i)], i = 1, 2,$$

in which $\theta_1 \leq \theta_2 \in \chi$. We assume that w_2 and k_1 are both non-decreasing (or non-increasing) functions. Proposition 1(i) guarantees that $X_1 \leq_{lr} X_2$ implies $X_{1w_1} \leq_{lr} X_{2w_2}$. If w_1 is non-decreasing and w_2 and k_1 are non-increasing functions then by Proposition 1(ii) $X_1 \leq_{hr} X_2$ yields $X_{1w_1} \leq_{hr} X_{2w_2}$. In the dual case, if w_1 is non-increasing and further w_2 and k_1 are both non-decreasing functions then Proposition 1(iii) concludes that $X_1 \leq_{rh} X_2$ gives $X_{1w_1} \leq_{rh} X_{2w_2}$.

Some relative stochastic orders including the relative (reversed) hazard rate and relative mean residual life orders have attracted the attention of researchers in the last decade (cf. Di-Crescenzo and Longobardi [20], Kayid et al. [21], Misra and Francis [22], Misra et al. [23], Ding et al. [24], Ding and Zhang [25], Misra and Francis [26] and Misra and Francis [27]). We reminisce about the definition of these orders from Rezaei et al. [28] and Kayid et al. [21] [see, for example, Definition 1(v) and (vi)]. In the next theorem, the study of preservation of the relative hazard rate and the relative reversed hazard rate orders are initiated for a well-known class of semiparametric distributions. For $i = 1, 2$, denote by $h_{iw_i}(t, \theta_i)$ ($\tilde{h}_{iw_i}(t, \theta_i)$) the hazard rate (resp. the reversed hazard rate) of X_{iw_i} , where w_i and is supposed to be valid as a weight function. Before stating the result, we introduce some notations. Let $w_i(t, \theta_i) = v(\bar{F}_i(t), \theta_i), i = 1, 2$ be two appropriate weight functions and set

$$\zeta(x, \theta) = \frac{\int_0^x v(u, \theta) du}{xv(x, \theta)}, x \in [0, 1], \theta \in \chi.$$

Denote by $\zeta'(x, \theta)$ partial derivative of $\zeta(x, \theta)$ with respect to x , that is, $\zeta'(x, \theta) = \frac{\partial}{\partial x} \zeta(x, \theta)$. The symbol $\stackrel{sign}{=}$ is used to denote the similar sign.

Theorem 1. *Suppose that $w_i(t, \theta_i) = v(\bar{F}_i(t), \theta_i), i = 1, 2$ so that $\theta_1 \leq \theta_2$ and $X_1 \leq_{hr} X_2$. Let $x\zeta'(x, \theta)/\zeta(x, \theta)$ is non-increasing (resp. non-decreasing) in x , for all θ and non-increasing (resp. non-decreasing) in θ , for all x where $\zeta(x, \theta)$ is non-decreasing (resp. non-increasing) in x , for all $\theta \in \chi$. Then, $X_1 \leq_{rhr} X_2$ implies that $X_{1w_1} \leq_{rhr} X_{2w_2}$.*

Proof. From (2), one has $h_{iw_i}(t, \theta_i) = w_i(t, \theta_i)h_i(t)/B_i(t, \theta_i)$, where

$$\begin{aligned} B_i(t, \theta_i) &= E[v(\bar{F}_i(X_i), \theta_i) | X_i > t] \\ &= \frac{1}{\bar{F}_i(t)} \int_t^\infty v(\bar{F}_i(x), \theta_i) dF_i(x) \\ &= \frac{1}{\bar{F}_i(t)} \int_0^{\bar{F}_i(t)} v(u, \theta_i) du, i = 1, 2. \end{aligned}$$

Thus, for all $t > 0$, we have:

$$\begin{aligned} \frac{h_{2w_2}(t, \theta_2)}{h_{1w_1}(t, \theta_1)} &= \frac{h_2(t) w_2(t, \theta_2) B_1(t, \theta_1)}{h_1(t) w_1(t, \theta_1) B_2(t, \theta_2)} \\ &= \frac{h_2(t) \int_0^{\bar{F}_1(t)} v(u, \theta_1) du / (\bar{F}_1(t)v(\bar{F}_1(t), \theta_1))}{h_1(t) \int_0^{\bar{F}_2(t)} v(u, \theta_2) du / (\bar{F}_2(t)v(\bar{F}_2(t), \theta_2))} \\ &= \frac{h_2(t) \zeta(\bar{F}_1(t), \theta_1)}{h_1(t) \zeta(\bar{F}_2(t), \theta_2)}. \end{aligned}$$

By assumption, $h_2(t)/h_1(t)$ is non-increasing in $t > 0$. It suffices only to prove that:

$\zeta(\bar{F}_1(t), \theta_1) / \zeta(\bar{F}_2(t), \theta_2)$ is non-increasing in $t > 0$.

The assumption $X_1 \leq_{hr} X_2$ yields $h_1(t) \geq h_2(t)$, for all $t > 0$, which further concludes that $\bar{F}_1(t) \leq \bar{F}_2(t)$, for all $t > 0$. Therefore,

$$\begin{aligned} \frac{d}{dt} \frac{\zeta(\bar{F}_1(t), \theta_1)}{\zeta(\bar{F}_2(t), \theta_2)} &\stackrel{sign}{=} f_2(t)\zeta'(\bar{F}_2(t), \theta_2)\zeta(\bar{F}_1(t), \theta_1) - f_1(t)\zeta'(\bar{F}_1(t), \theta_1)\zeta(\bar{F}_2(t), \theta_2) \\ &= \frac{f_2(t)}{\bar{F}_2(t)}\bar{F}_2(t)\zeta'(\bar{F}_2(t), \theta_2)\zeta(\bar{F}_1(t), \theta_1) - \frac{f_1(t)}{\bar{F}_1(t)}\bar{F}_1(t)\zeta'(\bar{F}_1(t), \theta_1)\zeta(\bar{F}_2(t), \theta_2) \\ &= h_2(t)\bar{F}_2(t)\zeta'(\bar{F}_2(t), \theta_2)\zeta(\bar{F}_1(t), \theta_1) - h_1(t)\bar{F}_1(t)\zeta'(\bar{F}_1(t), \theta_1)\zeta(\bar{F}_2(t), \theta_2), \end{aligned}$$

is non-positive (resp. non-negative) for all $t > 0$, if, and only if, for all $x_1 \leq x_2$ and for all $\theta_1 \leq \theta_2$ it holds that

$$\frac{x_1\zeta'(x_1, \theta_1)}{\zeta(x_1, \theta_1)} \geq (\text{resp. } \leq) \frac{x_2\zeta'(x_2, \theta_2)}{\zeta(x_2, \theta_2)},$$

which is validated by assumption. \square

To present the result about the preservation of the relative reversed hazard rate order we introduce some other notation. Let us define for $x \in [0, 1]$,

$$\zeta^*(x, \theta) = \frac{\int_x^1 v(u, \theta) du}{(1-x)v(x, \theta)}, \theta \in \chi.$$

Suppose $\zeta^{*/'}(x, \theta)$ stands for the partial derivative of $\zeta^*(x, \theta)$ with respect to x , that is $\zeta^{*/'}(x, \theta) = \frac{\partial}{\partial x} \zeta^*(x, \theta)$.

Theorem 2. Let $w_i(t, \theta_i) = v(\bar{F}_i(t), \theta_i), i = 1, 2$ so that $\theta_1 \leq \theta_2$ and $X_1 \leq_{rh} X_2$. If $(1-x)\zeta^{*/'}(x, \theta) / \zeta^*(x, \theta)$ is non-decreasing (resp. non-increasing) in x , for all θ and non-decreasing (resp. non-increasing) in θ , for all x , in which $\zeta^*(x, \theta)$ is non-decreasing (non-increasing) in x for all $\theta \in \chi$, then $X_1 \leq_{rrh} X_2$ implies that $X_{1w_1} \leq_{rrh} X_{2w_2}$.

Proof. In spirit of (3), we can write $\tilde{h}_{iw_i}(i, \theta_i) = w_i(t, \theta_i)\tilde{h}_i(t) / A_i(t, \theta_i)$, in which

$$\begin{aligned} A_i(t, \theta_i) &= E[v(\bar{F}_i(X_i), \theta_i) \mid X_i \leq t] \\ &= \frac{1}{1 - \bar{F}_i(t)} \int_0^t v(\bar{F}_i(x), \theta_i) dF_i(x) \\ &= \frac{1}{1 - \bar{F}_i(t)} \int_{\bar{F}_i(t)}^1 v(u, \theta_i) du, \text{ for any } i = 1, 2. \end{aligned}$$

For all $t > 0$, one obtains

$$\begin{aligned} \frac{\tilde{h}_{2w_2}(t, \theta_2)}{\tilde{h}_{1w_1}(t, \theta_1)} &= \frac{\tilde{h}_2(t) w_2(t, \theta_2) A_1(t, \theta_1)}{\tilde{h}_1(t) w_1(t, \theta_1) A_2(t, \theta_2)} \\ &= \frac{\tilde{h}_2(t) \int_{\bar{F}_1(t)}^1 v(u, \theta_1) du / ((1 - \bar{F}_1(t))v(\bar{F}_1(t), \theta_1))}{\tilde{h}_1(t) \int_{\bar{F}_2(t)}^1 v(u, \theta_2) du / ((1 - \bar{F}_2(t))v(\bar{F}_2(t), \theta_2))}. \\ &= \frac{\tilde{h}_2(t) \zeta^*(\bar{F}_1(t), \theta_1)}{\tilde{h}_1(t) \zeta^*(\bar{F}_2(t), \theta_2)}. \end{aligned}$$

Since $X_1 \leq_{rrh} X_2$, thus $\tilde{h}_2(t) / \tilde{h}_1(t)$ is non-decreasing in $t > 0$. It remains to demonstrate that $\zeta^*(\bar{F}_2(t), \theta_1) / \zeta^*(\bar{F}_1(t), \theta_2)$ is non-increasing in $t > 0$. It is known that $X_1 \leq_{hr} X_2$ implies $\tilde{h}_1(t) \leq \tilde{h}_2(t)$, for all $t > 0$, which in turn yields $\bar{F}_1(t) \leq \bar{F}_2(t)$, for all $t > 0$. For that reason,

$$\begin{aligned} \frac{d}{dt} \frac{\zeta^*(\bar{F}_2(t), \theta_2)}{\zeta^*(\bar{F}_1(t), \theta_1)} &\stackrel{\text{sign}}{=} -f_2(t)\zeta^{*'}(\bar{F}_2(t), \theta_2)\zeta^*(\bar{F}_1(t), \theta_1) + f_1(t)\zeta^{*'}(\bar{F}_1(t), \theta_1)\zeta^*(\bar{F}_2(t), \theta_2) \\ &= \frac{f_1(t)}{F_1(t)}F_1(t)\zeta^{*'}(\bar{F}_1(t), \theta_1)\zeta^*(\bar{F}_2(t), \theta_2) - \frac{f_2(t)}{F_2(t)}F_2(t)\zeta^{*'}(\bar{F}_2(t), \theta_2)\zeta^*(\bar{F}_1(t), \theta_1) \\ &= \tilde{h}_1(t)F_1(t)\zeta^{*'}(\bar{F}_1(t), \theta_1)\zeta^*(\bar{F}_2(t), \theta_2) - \tilde{h}_2(t)F_2(t)\zeta^{*'}(\bar{F}_2(t), \theta_2)\zeta^*(\bar{F}_1(t), \theta_1), \end{aligned}$$

is non-positive (resp. non-negative) for all $t > 0$, if, and only if, for all $x_1 \leq x_2$ and for all $\theta_1 \leq \theta_2$:

$$\frac{(1-x_1)\zeta^{*'}(x_1, \theta_1)}{\zeta^*(x_1, \theta_1)} \leq (\text{resp. } \geq) \frac{(1-x_2)\zeta^{*'}(x_2, \theta_2)}{\zeta^*(x_2, \theta_2)},$$

which holds by assumption. \square

The weight functions considered in Theorems 1 and 2 encompass some particular cases which may be of independent interest. In that regard, the following corollary is resulted.

Corollary 1. Let $w_i(t) = v(\bar{F}_i(t)), i = 1, 2$ such that $X_1 \leq_{hr} X_2$. Set

$$\zeta(x) = \frac{1}{xv(x)} \int_0^x v(u)du \text{ and } \zeta^*(x) = \frac{1}{(1-x)v(x)} \int_x^1 v(u)du.$$

- (i) If $x\zeta'(x)/\zeta(x)$ is non-increasing (resp. non-decreasing) in $x \in [0, 1]$, in which $\zeta(x)$ is non-decreasing (non-increasing) in x , then $X_1 \leq_{rhr} X_2$ implies that $X_{1w_1} \leq_{rhr} X_{2w_2}$.
- (ii) If $(1-x)\zeta^{*'}(x)/\zeta^*(x)$ is non-decreasing (resp. non-increasing) in $x \in [0, 1]$, in which $\zeta^*(x)$ is non-decreasing (non-increasing) in x , then $X_1 \leq_{rrh} X_2$ implies that $X_{1w_1} \leq_{rrh} X_{2w_2}$.

Theorem 3. Let $w_i(t, \theta_i) = v_i(\bar{F}(t), \theta_i)$ and set $\zeta_i(x, \theta_i) = (1/v_i(x, \theta_i)) \int_0^x v_i(u, \theta_i) du$ and $\zeta_i^*(x, \theta_i) = (1/v_i(x, \theta_i)) \int_x^1 v_i(u, \theta_i) du$ for $i = 1, 2$. Then

- (i) $X_{1w_1} \leq_{rhr} X_{2w_2}$ if, and only if, $\frac{\zeta_2(x, \theta_2)}{\zeta_1(x, \theta_1)}$ is non-increasing in x .
- (ii) $X_{1w_1} \leq_{rrh} X_{2w_2}$ if, and only if, $\frac{\zeta_2^*(x, \theta_2)}{\zeta_1^*(x, \theta_1)}$ is non-decreasing in x .

Proof. We only prove the assertion (i) as the proof of (ii) is similarly accomplished. Note that analogously as in the proof of Theorem 1, we can get

$$\begin{aligned} \frac{h_{2w_2}(t, \theta_2)}{h_{1w_1}(t, \theta_1)} &= \frac{w_2(t, \theta_2)}{w_1(t, \theta_1)} \frac{\int_t^\infty v_1(\bar{F}(x), \theta_1)f(x) dx}{\int_t^\infty v_2(\bar{F}(x), \theta_2)f(x) dx} \\ &= \frac{v_2(\bar{F}(t), \theta_2)}{v_1(\bar{F}(t), \theta_1)} \frac{\int_0^{\bar{F}(t)} v_1(u, \theta_1) du}{\int_0^{\bar{F}(t)} v_2(u, \theta_2) du} \\ &= \frac{\zeta_1(\bar{F}(t), \theta_1)}{\zeta_2(\bar{F}(t), \theta_2)}, \text{ for all } t > 0. \end{aligned}$$

It can be seen that, for all $t \geq 0$,

$$\frac{d}{dt} \frac{\zeta_1(\bar{F}(t), \theta_1)}{\zeta_2(\bar{F}(t), \theta_2)} \stackrel{\text{sign}}{=} f(t)[\zeta_2'(\bar{F}(t), \theta_2)\zeta_1(\bar{F}(t), \theta_1) - \zeta_1'(\bar{F}(t), \theta_1)\zeta_2(\bar{F}(t), \theta_2)]$$

which is non-positive if, and only if,

$$\frac{\zeta_2'(x, \theta_2)}{\zeta_2(x, \theta_2)} \leq \frac{\zeta_1'(x, \theta_1)}{\zeta_1(x, \theta_1)}, \text{ for all } x \in [0, 1],$$

or equivalently if $\zeta_2(x, \theta_2)/\zeta_1(x, \theta_1)$ is non-increasing in $x \in [0, 1]$ according which the ratio $h_{2w_2}(t, \theta_2)/h_{1w_1}(t, \theta_2)$ is also non-increasing in $t > 0$, that is, $X_{1w_1} \leq_{rhr} X_{2w_2}$. The proof is complete. \square

The following corollary is a useful observation in the context of Theorem 3 as it illustrates that a typical parametric family of weighted distributions enjoys the relative hazard rate and the relative reversed hazard rate ordering properties in some cases.

Corollary 2. *Suppose that the random variable $X(\theta_1)$ and $X(\theta_2)$ for $\theta_1, \theta_2 \in \chi$ have density functions*

$$f_{\theta_1}(t) = \frac{v(\bar{F}(t), \theta_1)f(t)}{\int_0^\infty v(\bar{F}(t), \theta_1)f(t) dt} \text{ and } f_{\theta_2}(t) = \frac{v(\bar{F}(t), \theta_2)f(t)}{\int_0^\infty v(\bar{F}(t), \theta_2)f(t) dt}.$$

For $\zeta(x, \theta) = (1/v(x, \theta)) \int_0^x v(u, \theta) du$ and $\zeta^*(x, \theta) = (1/v(x, \theta)) \int_x^1 v(u, \theta) du$, we have:

- (i) $X(\theta_1) \leq_{rhr} X(\theta_2)$, for all $\theta_1 \leq \theta_2 \in \chi$ if, and only if, $\zeta(x, \theta)$ is RR_2 in (x, θ) .
- (ii) $X(\theta_1) \leq_{rrh} X(\theta_2)$, for all $\theta_1 \leq \theta_2 \in \chi$ if, and only if, $\zeta^*(x, \theta)$ TP_2 in (x, θ) .

In reliability and survival theories, feature of ordering for lifetime of coherent systems is a relevant subject to be studied. To this end, Navarro et al. [29] obtained a representation of the system reliability \bar{F}_{Sys} as a distorted function of the common component reliability \bar{F} such that $\bar{F}_{Sys}(t) = \bar{F}(t)$, where h is an non-decreasing function depending on the structure of the underlying system and the survival copula of the joint distribution of the component lifetimes. In this context, they have shown that the reliability function of a coherent system with dependent identically distributed (DID) components can be written as a distorted function of the common component reliability function. The following lemma is due to Navarro et al. [29].

Lemma 1. *Let $\tau(\mathbf{X})$ be the lifetime of a coherent system formed by n DID components with the vector of random lifetimes $\mathbf{X} = (X_1, X_2, \dots, X_n)$ with common survival function \bar{F} . Then the reliability function of $\tau(\mathbf{X})$ can be written as*

$$\bar{F}_{\tau(\mathbf{X})}(x) = h(\bar{F}(x)),$$

where $h(\cdot) : [0, 1] \rightarrow [0, 1]$ is an non-decreasing continuous function such that $h(0) = 0$ and $h(1) = 1$. The function h is called the domination (or distortion) function which is characterized through the structure function $\phi(\cdot)$ of the system (see, e.g., Barlow and Proschan [30]) and on the survival copula \hat{C} of X_1, X_2, \dots, X_n .

In the set up of the particular weighted distributions given in Section 3.2, the survival function of the arisen weighted distribution can be commuted to a distorted survival function, as specified earlier in Lemma 1, for which the domination function is characterized by the associated weight function. To this purpose, consider the weight function $w(x, \theta) = v(\bar{F}(x), \theta)$ and notice that in this case X_w has the survival function

$$\bar{F}_w(t, \theta) = \frac{\int_t^\infty v(\bar{F}(x), \theta) dF(x)}{\int_{-\infty}^\infty v(\bar{F}(x), \theta) dF(x)} = \frac{\int_0^{\bar{F}(t)} v(u, \theta) du}{\int_0^1 v(u, \theta) du} = h_\theta(\bar{F}(t)),$$

where

$$h_\theta(x) = \frac{\int_0^x v(u, \theta) du}{\int_0^1 v(u, \theta) du}, \quad x \in [0, 1]$$

plays the role of a parametric domination function. Note that $h_\theta(\cdot) : [0, 1] \rightarrow [0, 1]$ is a non-decreasing continuous function with $h_\theta(0) = 0$ and $h_\theta(1) = 1$. In the reversed direction, if h_θ is a distortion (domination) function and $v(x, \theta) = h'_\theta(x)$, for any $x \in [0, 1]$ then $\int_0^1 v(x, \theta) dx = 1$ and thus $\bar{F}_w(t, \theta) = h_\theta(\bar{F}(t))$. Therefore, there is a unique relationship between $v(\cdot, \theta)$ and $h_\theta(\cdot)$ that is the studies of weighted distributions in the context of semiparametric models entertain the studies of distorted

survival functions and vice versa. The parameter θ may be an appropriate quantity that affects the magnitude of system's lifetime. In the case when DID components construct the system, θ may be related to the dependency of the component lifetimes in a way that the survival copula in Lemma 1 depends on θ . For instance, in the case where the Archimedean copula or the FGM copula is adopted to model the association of lifetime of components in a coherent system. The following results are useful to analysis of relative ordering properties of coherent systems as to the best of our knowledge such a study has not been developed in the literature thus far. The following proposition is a direct consequence of Theorem 3.

Proposition 2. Let $\zeta_i(x) = h_i(x)/h'_i(x)$ and $\zeta_i^*(x) = (1 - h_i(x))/h'_i(x)$ for $i = 1, 2$ where h_1 and h_2 are two domination functions. Let T_1 and T_2 have respective survival functions $h_1(\bar{F})$ and $h_2(\bar{F})$. Then,

- (i) $T_1 \leq_{rhr} T_2$, if, and only if, $\zeta_i(x)$ is RR_2 in $(i, x) \in \{1, 2\} \times [0, 1]$.
- (ii) $T_1 \leq_{rrh} T_2$, if, and only if, $\zeta_i^*(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times [0, 1]$.

The following example illustrates an application of Proposition 2.

Example 6. Suppose that $X_i, i = 1, 2, 3, 4$ denote the lifetime of four components having survival function \bar{F} . Let

$$T_1 = \max\{\min\{X_1, X_2\}, \min\{X_2, X_3\}, \min\{X_3, X_4\}\} \text{ and } T_2 = \max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}\}$$

denote the lifetime of two coherent systems. According to Table I in Navarro et al. [29], when X_i 's are independent, T_1 and T_2 have survival functions $h_1(\bar{F})$ and $h_2(\bar{F})$, respectively, in which $h_1(u) = 3u^2 - 2u^3$ and $h_2(u) = 2u^2 - u^4$. For all $x \in [0, 1]$, we have

$$\frac{d}{dx} \frac{\zeta_2(x)}{\zeta_1(x)} = \frac{d}{dx} \frac{2 - x^2}{(1 + x)(3 - 2x)} = -\frac{1 + (1 - x)^2}{((1 + x)(3 - 2x))^2} \leq 0.$$

Hence, $\zeta_i(x)$ is RR_2 in $(i, x) \in \{1, 2\} \times [0, 1]$. That is, Proposition 2(i) concludes that $T_1 \leq_{rhr} T_2$.

In the following example, we show that Proposition 2 can also be applied to systems with DID components.

Example 7. Suppose that (X_1, X_2, X_3) are dependent with a Farlie-Gumbel-Morgenstern (FGM) joint reliability function given by

$$\bar{F}(x_1, x_2, x_3) = \bar{F}(x_1)\bar{F}(x_2)\bar{F}(x_3)(1 - \alpha F(x_1)F(x_2)F(x_3)),$$

where $\alpha \in [-1, 1]$, F the marginal distribution of X_i 's and $\bar{F} = 1 - F$. Consider the parallel system with lifetime $T_\alpha = \max\{X_1, X_2, X_3\}$ for which

$$\begin{aligned} \bar{F}_\alpha(x) &= \bar{F}(x, x, x) - 3\bar{F}(x, x, 0) + 3\bar{F}(x) \\ &= \bar{F}^3(x) - \alpha(\bar{F}(x) - \bar{F}^2(x))^3 - 3\bar{F}^2(x) + 3\bar{F}(x). \end{aligned}$$

Thus, $\bar{F}_\alpha(x) = h_\alpha(\bar{F}(x))$, where $h_\alpha(u) = u^3 - \alpha(u - u^2)^3 - 3u^2 + 3u$. It can be shown that if $h_1(x) = x^3 - 0.5(x - x^2)^3 - 3x^2 + 3x$ and $h_2(x) = x^3 - 0.75(x - x^2)^3 - 3x^2 + 3x$ then $\zeta_i^*(x) = (1 - h_i(x))/h'_i(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times [0, 1]$ because the function $\zeta_2^*(x)/\zeta_1^*(x)$ is non-decreasing in $x \in [0, 1]$ as the Figure 1 indicates. Hence, according to Proposition 2(ii) one obtains $T_{0.5} \leq_{rrh} T_{0.75}$.

The following example reveals a relative ordering property in the Marshall-Olkin family of distributions.

Example 8. Suppose that the incorporated weight function is

$$w(x, \theta) = v(\bar{F}(x), \theta) = \frac{1}{(1 - \bar{\theta}\bar{F}(x))^2},$$

where $\bar{\theta} = 1 - \theta$ and $\theta > 0$. The random variable X_w has survival function

$$\bar{F}_w(x, \theta) = \frac{\theta\bar{F}(x)}{(1 - \bar{\theta}\bar{F}(x))} = h_\theta(\bar{F}(x)), \tag{8}$$

so that $h_\theta(u) = \theta u / (1 - \bar{\theta}u)$ which is considered to be the relevant domination function. Note that the family of distributions characterized via (8) is called the proportional odds family of distributions which is due to Marshall and Olkin [31]. Let T_1 and T_2 be two random variables with respective survival functions $h_{\theta_1}(\bar{F})$ and $h_{\theta_2}(\bar{F})$ such that $\theta_1 > \theta_2$. It can be seen that

$$\xi_i(x) = \frac{h_{\theta_i}(x)}{h'_{\theta_i}(x)} = x(1 - \bar{\theta}_i x), \quad i = 1, 2.$$

It follows that

$$\frac{d}{dx} \frac{\xi_2(x)}{\xi_1(x)} = \frac{\theta_2 - \theta_1}{(1 - \bar{\theta}_1 x)^2} < 0,$$

that is $\xi_i(x)$ is RR_2 in $i = 1, 2$ and $x > 0$. Thus, according to Proposition 2(i) we deduce that $T_1 \leq_{rhr} T_2$.

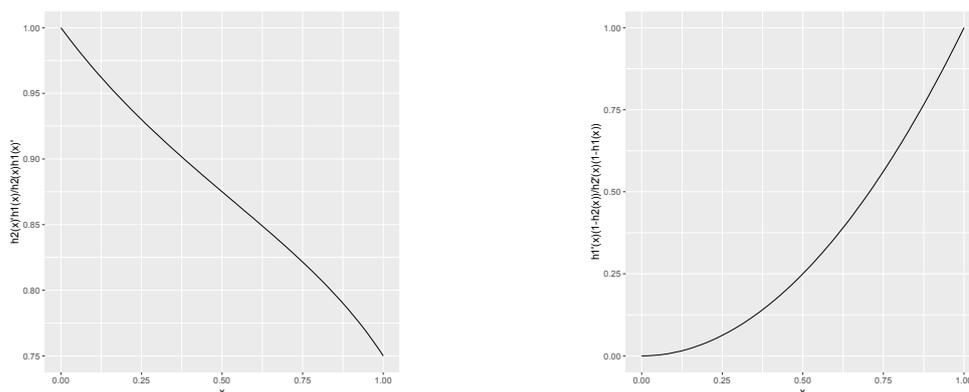


Figure 1. Plot of $\xi_2(x)/\xi_1(x)$ (left); Plot of $\xi_2^*(x)/\xi_1^*(x)$ (right).

4.2. Comparisons of Mixture Weighted Distribution

In this segment, the problem of preservation of a number of stochastic orderings in the mixture weighted model is investigated. The study is carried out in two different settings, where firstly the random parameter varies in distribution while the underlying distribution remains unchanged and secondly the underlying distribution is changed in the case when the random parameter is fixed in distribution. The results obtained by Kayid et al. [32] are developed to entertain more dynamic weighted distributions.

It is followed up that some stochastic orders of random parameters as well as the underlying random variables are transmitted to the random variables with the associated mixture weighted distribution. Give thought to Θ_i as a random variable with the pdf g_i , the cdf G_i and the sf $\bar{G}_i = 1 - G_i$, for $i = 1, 2$. Contemplate the random variable $X_i^*, i = 1, 2$ having pdf

$$f_i^*(x) = f(x) \int_{\theta \in \mathcal{X}} \frac{w(x, \theta)}{\mu(\theta)} g_i(\theta) d\theta = f(x) E \left[\frac{w(x, \Theta_i)}{\mu(\Theta_i)} \right] \tag{9}$$

from which the cdf F_i^* and the sf $\bar{F}_i^* = 1 - F_i^*$ of X_i^* are procured after somewhat plain algebraic calculations, respectively, by

$$F_i^*(x) = F(x)E\left(\frac{A(x, \Theta_i)}{\mu(\Theta_i)}\right) \text{ and } \bar{F}_i^*(x) = \bar{F}(x)E\left(\frac{B(x, \Theta_i)}{\mu(\Theta_i)}\right), \quad i = 1, 2, \tag{10}$$

where the bivariate functions A and B and the function μ are all determined as earlier in Section 1. In the rest of the paper, it is taken for granted that the random variables Θ_1 and Θ_2 are independent. Denote by h_i^* and \tilde{h}_i^* the hazard rate and the reversed hazard rate of X_i^* , respectively. It can be seen, after some integral calculation, that

$$h_i^*(x) = \int_{\theta \in \chi} \frac{f_w(x, \theta)}{\bar{F}_i^*(x)} dG_i(\theta) = h(x) E\left[\frac{w(x, \Theta_i)}{B(x, \Theta_i)} \mid X_i^* > x\right] \tag{11}$$

and

$$\tilde{h}_i^*(x) = \int_{\theta \in \chi} \frac{f_w(x, \theta)}{F_i^*(x)} dG_i(\theta) = \tilde{h}(x) E\left[\frac{w(x, \Theta_i)}{A(x, \Theta_i)} \mid X_i^* \leq x\right] \tag{12}$$

The following result demonstrates the likelihood ratio order preservation in the model (9).

Theorem 4. Let $w(x, \theta)$ be TP_2 (resp. RR_2) in $(x, \theta) \in \zeta \times \chi$. Then, $\Theta_1 \leq_{lr} \Theta_2$ implies $X_1^* \leq_{lr}$ (resp. \geq_{lr}) X_2^* .

Proof. It is not impenetrable to realize that $X_1^* \leq_{lr} X_2^*$ if, and only if, $f_i^*(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times \zeta$. In spirit of (9), one gets

$$f_i^*(x) = \int_{\theta \in \chi} f_w(x, \theta) g_i(\theta) d\theta = \int_{\theta \in \chi} \frac{w(x, \theta) f(x)}{\mu(\theta)} g_i(\theta) d\theta, \quad i = 1, 2.$$

By the assumption of $\Theta_1 \leq_{lr} \Theta_2$ we can rely on the fact that $g_i(\theta)$ is TP_2 (resp. RR_2) in $(i, \theta) \in \{1, 2\} \times \chi$. It is also obvious that $\frac{f(x)w(x, \theta)}{\mu(\theta)}$ is TP_2 in $(x, \theta) \in \zeta \times \chi$. The general composition theorem of Karlin [15] concludes the desired result. \square

Theorem 5. Let $w(x, \theta)$ be TP_2 (resp. RR_2) in $(x, \theta) \in \zeta \times \chi$. Then, $\Theta_1 \leq_{hr} \Theta_2$ implies $X_1^* \leq_{hr}$ (resp. \geq_{hr}) X_2^* .

Proof. We prove the non-parenthetical part. The parenthetical part of the theorem can be similarly proved. In consideration of the second identity in (10) one observes

$$\bar{F}_i^*(x) = \int_{\theta \in \chi} \Psi(\theta, x) dG_i(\theta), \quad i = 1, 2;$$

where

$$\Psi(\theta, x) = \frac{\bar{F}(x) B(x, \theta)}{\mu(\theta)}.$$

Take into account that

$$\begin{aligned} B(x, \theta) &= \int_x^\infty w(x', \theta) \frac{f(x')}{\bar{F}(x)} dx' \\ &= \int_{-\infty}^\infty \Phi(x, x') w(x', \theta) dx' \end{aligned}$$

such that

$$\Phi(x, x') = \begin{cases} 0, & \text{for } x' \leq x \\ \frac{f(x')}{\bar{F}(x)}, & \text{for } x' > x. \end{cases}$$

By assumption $w(x', \theta)$ is TP_2 in $(x', \theta) \in \zeta \times \chi$ and $\Phi(x, x')$ is TP_2 in $(x, x') \in \zeta \times \zeta$. Hence, the general composition theorem of Karlin [15] concludes that $B(x, \theta)$ is TP_2 in $(x, \theta) \in \zeta \times \chi$. Since $\mu(\theta) = B(-\infty, \theta)$ and since $B(x, \theta)$ is TP_2 in (x, θ) , then $B(x, \theta) / \mu(\theta)$ is non-decreasing in $\theta \in \chi$ thus $\Psi(\theta, x)$ is

TP₂ in $(\theta, x) \in \chi \times \zeta$ and further it is non-decreasing in $\theta \in \chi$. It can be readily claimed by assumption that $\bar{G}_i(\theta)$ is TP₂ in $(i, \theta) \in \{1, 2\} \times \chi$. On account of Lemma 4.2 in Li and Xu [33] we deduce that $\bar{F}_i^*(x)$ is TP₂ in $(i, x) \in \{1, 2\} \times \zeta$ and the result follows. \square

In the setup of the model (9), the reversed hazard rate order of the random parameters is relocated into the overall random variables.

Theorem 6. Let $w(x, \theta)$ be TP₂ (resp. RR₂) in $(x, \theta) \in \zeta \times \chi$. Then, $\Theta_1 \leq_{rh} \Theta_2$ implies $X_1^* \leq_{rh}$ (resp. \geq_{rh}) X_2^* .

Proof. The non-parenthetical part is only proved since the proof for the parenthetical part is analogously carried out. In view of the former identity in (10), it is inferred that $X_1^* \leq_{rh} X_2^*$ if, and only if,

$$E \left[\frac{A(x_1, \Theta_1)}{\mu(\Theta_1)} \frac{A(x_2, \Theta_2)}{\mu(\Theta_2)} \right] \geq E \left[\frac{A(x_1, \Theta_2)}{\mu(\Theta_2)} \frac{A(x_2, \Theta_1)}{\mu(\Theta_1)} \right]$$

for all $x_1 \leq x_2 \in \zeta$. Let us explicate that

$$\phi_1(\theta_1, \theta_2) = \frac{A(x_1, \theta_2)}{\mu(\theta_2)} \times \frac{A(x_2, \theta_1)}{\mu(\theta_1)},$$

and

$$\phi_2(\theta_1, \theta_2) = \frac{A(x_1, \theta_1)}{\mu(\theta_1)} \times \frac{A(x_2, \theta_2)}{\mu(\theta_2)}.$$

It can be seen that

$$\begin{aligned} A(x, \theta) &= \int_0^x w(x', \theta) \frac{f(x')}{F(x)} dx' \\ &= \int_0^\infty \phi(x, x') w(x', \theta) dx' \end{aligned}$$

so that

$$\phi(x, x') = \begin{cases} \frac{f(x')}{F(x)}, & \text{for } x' \leq x \\ 0, & \text{for } x' > x. \end{cases}$$

From assumption $w(x', \theta)$ is TP₂ in $(x', \theta) \in \zeta \times \chi$ and $\phi(x, x')$ is TP₂ in $(x, x') \in \zeta \times \zeta$. The general composition theorem of Karlin [15] therefore is applied to draw the inference that $A(x, \theta)$ is TP₂ in $(x, \theta) \in \zeta \times \chi$. For that reason, for all $\theta_1 \leq \theta_2 \in \chi$ and $x_1 \leq x_2 \in \zeta$

$$\begin{aligned} \Delta\phi_{21}(\theta_1, \theta_2) &= \phi_2(\theta_1, \theta_2) - \phi_1(\theta_1, \theta_2) \\ &= \frac{1}{\mu(\theta_1)\mu(\theta_2)} [A(x_1, \theta_1)A(x_2, \theta_2) - A(x_1, \theta_2)A(x_2, \theta_1)] \\ &\geq 0. \end{aligned}$$

It is clearly seen that $\mu(\theta_i) = A(\infty, \theta_i)$, for $i = 1, 2$. Since $A(x, \theta)$ is TP₂ in $(x, \theta) \in \zeta \times \chi$ thus $A(x_1, \theta_1)/\mu(\theta_1)$ is non-increasing in $\theta_1 \in \chi$ and in addition $A(x_2, \theta_1)/\mu(x_1, \theta_1)$ is non-decreasing in $\theta_1 \chi$. On that account,

$$\Delta\phi_{21}(\theta_1, \theta_2) = \frac{A(x_1, \theta_1)}{\mu(\theta_1)\mu(\theta_2)} \left[A(x_2, \theta_2) - A(x_1, \theta_2) \frac{A(x_2, \theta_1)}{A(x_1, \theta_1)} \right]$$

is non-negative and also non-increasing in $\theta_1 \in \chi$, for all $\theta_1 \leq \theta_2 \in \chi$ and for all $x_1 \leq x_2 \in \zeta$. The proof is completed by Theorem 1.B.48 of Shaked and Shanthikumar [34]. \square

The weight functions brought in Sections 3.1 and 3.2 are all TP₂ (or RR₂) at least under some (mild) condition. That is, they are applicable to develop the \leq_{lr} , the \leq_{hr} and the \leq_{rh} orders from the random parameter into the mixture (average) variable in the model of (9) according to the result of Theorems 4–6, respectively. It is remarkable that

- (i) $w(x, \theta) = w_1(x)(w_2(x))^{k_1(\theta)}$ is TP₂ in $(x, \theta) \in \zeta \times \chi$ whenever $w_2(x)$ is non-decreasing (resp. non-increasing) in $x \in \zeta$ and $k_1(\theta)$ is non-decreasing (resp. non-increasing) in $\theta \in \chi$.
- (ii) $w(x, \theta) = w_1(x)(k_1(\theta))^{w_2(x)}$ is TP₂ in $(x, \theta) \in \zeta \times \chi$ whenever $w_2(x)$ is non-decreasing (resp. non-increasing) in $x \in \zeta$ and $k_1(\theta)$ is non-decreasing (resp. non-increasing) in $\theta \in \chi$.
- (iii) $w(x, \theta) = w_1(x)k_1(\theta)$ is TP₂ in $(x, \theta) \in \zeta \times \chi$.
- (iv) $w(x, \theta) = w_1(x) + k_1(\theta)w_2(x)$ is TP₂ in $(x, \theta) \in \zeta \times \chi$ whenever $w_1(x)/w_2(x)$ is non-increasing (resp. non-decreasing) in $x \in \zeta$ and $k_1(\theta)$ is non-decreasing (resp. non-increasing) in $\theta \in \chi$.
- (v) $w(x, \theta) = w_1(x)I[w_2(x) > k_1(\theta)]$ is TP₂ in $(x, \theta) \in \zeta \times \chi$ whenever $w_2(x)$ is non-decreasing (resp. non-increasing) in $x \in \zeta$ and $k_1(\theta)$ is non-decreasing (resp. non-increasing) in $\theta \in \chi$.
- (vi) $w(x, \theta) = w_1(x)I[w_2(x) \leq k_1(\theta)]$ is TP₂ in $(x, \theta) \in \zeta \times \chi$ whenever $w_2(x)$ is non-decreasing (resp. non-increasing) in $x \in \zeta$ and $k_1(\theta)$ is non-decreasing (resp. non-increasing) in $\theta \in \chi$.
- (vii) $w(x, \theta) = \bar{F}^{\theta-1}(x)$ is RR₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$.
- (viii) $w(x, \theta) = F^{\theta-1}(x)$ is TP₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$.
- (ix) $w(x, \theta) = 1/((1 - \theta\bar{F}(x))^2)$ is TP₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$.
- (x) $w(x, \theta) = -\ln^\theta(\bar{F}(x))$ is TP₂ in $(x, \theta) \in \zeta \times \mathbb{N}$.
- (xi) $w(x, \theta) = -\ln^\theta(F(x))$ is RR₂ in $(x, \theta) \in \zeta \times \mathbb{N}$.
- (xii) $w(x, \theta) = f(\theta + x)/f(x)$ is TP₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$ whenever f , as the density function of X , is log-convex on ζ and it is RR₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$ provided that f is log-concave on ζ .
- (xiii) $w(x, \theta) = f(\theta - x)/f(x)$ is RR₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$ whenever f is log-convex on ζ and it is TP₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$ when f is log-concave on ζ .
- (xiv) $w(x, \theta) = f(\theta x)/f(x)$ is TP₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$ whenever $\ln(X)$ has a log-convex density function while it is RR₂ in $(x, \theta) \in \zeta \times \mathbb{R}^+$ provided that $\ln(X)$ has a log-concave density function.

In a modified setup of the mixture weighted model, we consider the case where X_i is a lifespan with pdf f_i and cdf F_i , for $i = 1, 2$. Then X_i^{**} is taken as the random variable with average density

$$f_i^{**}(x) = f_i(x) \int_{\theta \in \chi} \frac{w(x, \theta)}{\mu_i(\theta)} g(\theta) d\theta = f_i(x) E \left[\frac{w(x, \Theta)}{\mu_i(\Theta)} \right], \quad (13)$$

where g stands for the pdf of Θ and $\mu_i(\theta) = E[w(X_i, \theta)]$, for $i = 1, 2$. In the new setting with the mixture model given in (13) two baseline variables X_1 and X_2 are implicated. The random lifetimes X_1^{**} and X_2^{**} are assumed to have density functions f_1^{**} and f_2^{**} , respectively. The mixing variable Θ shares an equal impact upon the construction of the mixture densities.

Theorem 7. Let $(\Theta | X_1^{**} > x) \leq_{st} (\Theta | X_2^{**} > x)$, for all $x > 0$. Let $w(x, \theta)$ be TP₂ in $(x, \theta) \in \zeta \times \chi$ and that it is non-decreasing in $x \in \zeta$, for all $\theta \in \chi$. Then $X_1 \leq_{hr} X_2$ implies $X_1^{**} \leq_{hr} X_2^{**}$.

Proof. For $i = 1, 2$, suppose that h_i and h_i^{**} represent the hazard rate functions of X_i and X_i^{**} , respectively. In a same manner as in (12),

$$h_i^{**}(x) = h_i(x) E \left[\frac{w(x, \Theta)}{B_i(x, \Theta)} | X_i^{**} > x \right],$$

where $B_i(x, \theta) = E(w(X_i, \theta) | X_i > x)$, for $i = 1, 2$. It is known that $X_1 \leq_{hr} X_2$ if, and only if, $(X_1 | X_1 > x) \leq_{st} (X_2 | X_2 > x)$, for all $x > 0$. Therefore,

$$\begin{aligned}
 B_1(x, \theta) &= E(w(X_1, \theta) \mid X_1 > x) \\
 &\leq B_2(x, \theta) = E(w(X_2, \theta) \mid X_2 > x),
 \end{aligned}$$

for all $x > 0$ and $\theta \in \chi$. By assumption, $h_1(x) \geq h_2(x)$, for all $x > 0$. Hence,

$$\begin{aligned}
 h_1^{**}(x) - h_2^{**}(x) &= \int_0^\infty \frac{w(x, \theta)h_1(x)}{B_1(x, \theta)} d\Pi(\theta \mid X_1^{**} > x) \\
 &\quad - \int_0^\infty \frac{w(x, \theta)h_2(x)}{B_2(x, \theta)} d\Pi(\theta \mid X_2^{**} > x) \\
 &\geq h_2(x) \int_0^\infty \frac{w(x, \theta)}{B_2(x, \theta)} d[\Pi(\theta \mid X_1^{**} > x) - \Pi(\theta \mid X_2^{**} > x)]. \tag{14}
 \end{aligned}$$

Since $w(x, \theta)/B_2(x, \theta)$ is non-increasing in $\theta \in \chi$, thus by assumption,

$$\int_0^T d[\Pi(\theta \mid X_1^{**} > x) - \Pi(\theta \mid X_2^{**} > x)] \geq 0,$$

for all $T > 0$. By Lemma 7.1(b) of Barlow and Proschan [30] to (14) we attain the proof. \square

The last result establishes the reversed hazard rate ordering preservation in the baseline-varied mixture weighted model of (13).

Theorem 8. Let $(\Theta \mid X_1^{**} \leq x) \leq_{st} (\Theta \mid X_2^{**} \leq x)$, for all $x > 0$, and let $(X_1^{**} \mid \Theta = \theta_1) \leq_{rh} (X_2^{**} \mid \Theta = \theta_2)$, for all $\theta_1 \leq \theta_2 \in \chi$. Furthermore, let $w(x, \theta)$ be TP₂ in $(x, \theta) \in \zeta \times \chi$. Then $X_1 \leq_{rh} X_2$ implies $X_1^{**} \leq_{rh} X_2^{**}$.

Proof. First, we denote by \tilde{h}_i and \tilde{h}_i^{**} the reversed hazard rate functions of X_i and X_i^{**} , respectively, for $i = 1, 2$. For all $x > 0$,

$$\tilde{h}_i^{**}(x) = \tilde{h}_i(x) E \left[\frac{w(x, \Theta)}{A_i(x, \Theta)} \mid X_i^{**} \leq x \right],$$

where $A_i(x, \theta) = E[w(X_i, \theta) \mid X_i \leq x]$, for $i = 1, 2$. The order relation $(X_1^{**} \mid \Theta = \theta_1) \leq_{rh} (X_2^{**} \mid \Theta = \theta_2)$, for all $\theta_1 \leq \theta_2 \in \chi$ yields

$$\begin{aligned}
 \frac{w(x, \theta)\tilde{h}_2(x)}{A_2(x, \theta)} &= \frac{w(x, \theta)f_2(x)}{\int_0^x w(x', \theta)f_2(x') dx'} \\
 &\geq \frac{w(x, \theta)f_1(x)}{\int_0^x w(x', \theta)f_1(x') dx'} \\
 &= \frac{w(x, \theta)\tilde{h}_1(x)}{A_1(x, \theta)},
 \end{aligned}$$

for all $x > 0$, and $\theta \in \chi$. Thus

$$\begin{aligned}
 \tilde{h}_2^{**}(x) - \tilde{h}_1^{**}(x) &= \int_0^\infty \frac{w(x, \theta)\tilde{h}_2(x)}{A_2(x, \theta)} d\Pi(\theta \mid X_2^{**} \leq x) - \int_0^\infty \frac{w(x, \theta)\tilde{h}_1(x)}{A_1(x, \theta)} d\Pi(\theta \mid X_1^{**} \leq x) \\
 &\geq \tilde{h}_1(x) \int_0^\infty \frac{w(x, \theta)}{A_1(x, \theta)} d[\Pi(\theta \mid X_2^{**} \leq x) - \Pi(\theta \mid X_1^{**} \leq x)]. \tag{15}
 \end{aligned}$$

It can be seen that $w(x, \theta)/A_1(x, \theta)$ is non-decreasing in $\theta \in \chi$. From assumption,

$$\int_T^\infty d[\Pi(\theta | X_2^{**} \leq x) - \Pi(\theta | X_1^{**} \leq x)] \geq 0,$$

for all $T \geq 0$. Lemma 7.1(a) of Barlow and Proschan [30] is applicable in (15) and provides the proof. \square

4.3. A Link to Information Theory

The concept of entropy in information theory has played a prominent role in a broad area of science including statistical thermodynamics, urban and regional planning, business, economics, finance, operations research, queueing theory, spectral analysis, image reconstruction, biology and manufacturing (see, for example, El Gamal and Kim [35], Brillouin [36], Khinchin [37] and Grant [38]).

Stochastic comparisons of distributions have found a link to information theory in the literature (see, for instance, Ebrahimi et al. [39], Belzunce et al. [40], Nanda and Prasanta [41], Qiu [42], Toomaj and Di Crescenzo [43] and Toomaj and Di Crescenzo [44]).

Here before closing the paper, we impose a stochastic ordering property that leads to ordering of entropies of weighted distributions with weight functions given in Section 3.2. The extension of the Shannon entropy from the discrete case to the absolutely continuous case when dealing with lifetime random variables is defined by

$$H(X) = - \int_0^\infty f(x) \log(f(x)) dx,$$

where f is the pdf of non-negative random variable X with an absolutely continuous distribution function. Note that \log , with convention $0 \log(0) = 0$ stands for the natural logarithm.

However, it is found that the entropy is related to the concept of dispersion of (random) variables. Being aware of this certitude, it is useful to concentrate on dispersion measures of probability distributions as well as their related stochastic dispersion orderings.

Let us recall from Shaked and Shanthikumar [34] that X with the pdf f and the cdf F is less (or equal) than (with) Y with the pdf g and the cdf G in dispersive order (denoted by $X \leq_{disp} Y$) whenever

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha), \quad \text{for all } 0 < \alpha \leq \beta < 1.$$

It follows from (3.B.25) in Shaked and Shanthikumar [34] that

$$X \leq_{disp} Y \implies \text{Var}(X) \leq \text{Var}(Y).$$

In spirit of Theorem 3.B.20(a) and Theorem 3.B.20(b) in Shaked and Shanthikumar [34] if X or Y has an increasing hazard rate function, then

$$X \leq_{disp} Y \implies X \leq_{hr} Y,$$

and if X or Y has a decreasing hazard rate function, then

$$X \leq_{hr} Y \implies X \leq_{disp} Y.$$

In accordance with Corollary 4.4 in Bartoszewicz [45], if X or Y has a decreasing reversed hazard rate function, then

$$X \leq_{disp} Y \implies X \leq_{rh} Y,$$

and if X or Y has an increasing reversed hazard rate function, then

$$X \leq_{rh} Y \implies X \leq_{disp} Y.$$

If X and Y are two random variables with supports $S_X = (l_X, u_X)$ and $S_Y = (l_Y, u_Y)$, respectively, then according to Theorem 3.B.13(a) in Shaked and Shanthikumar [34] when $l_X = l_Y > -\infty$,

$$X \leq_{disp} Y \implies X \leq_{st} Y,$$

and also according to Theorem 3.B.13(b) in Shaked and Shanthikumar [34] when $u_X = u_Y < \infty$,

$$X \leq_{disp} Y \implies X \geq_{st} Y.$$

The weight functions $w_\theta(x)$ and $v_\theta(x)$ considered in the following theorem depend on x only through $F(x)$ and $G(x)$, respectively.

Theorem 9. Let X_{w_θ} and Y_{v_θ} be the weighted versions of X and Y with weight functions $w_\theta(x) = d_\theta(F(x))$ and $v_\theta(x) = d_\theta(G(x))$, respectively, where $\theta \in \chi$ is a parameter and d_θ is a non-negative function so that $\int_0^1 d_\theta(u)du < \infty$ for any $\theta \in \chi$. Then, $X \leq_{disp} Y$ implies $H(X_{w_\theta}) \leq H(Y_{v_\theta})$ for all $\theta \in \chi$.

Proof. Take $b(\theta) = \int_0^1 d_\theta(u)du$ and observe that, for all $\theta \in \chi$,

$$E(w_\theta(X)) = \int_0^\infty d_\theta(F(x))f(x) dx = b(\theta)$$

and also

$$E(v_\theta(Y)) = \int_0^\infty d_\theta(G(y))g(y) dy = b(\theta).$$

Note that

$$\begin{aligned} H(X_{w_\theta}) &= - \int_0^\infty \log\left(\frac{d_\theta(F(x))f(x)}{b(\theta)}\right) \frac{d_\theta(F(x))f(x)}{b(\theta)} dx \\ &= - \int_0^\infty \log\left(\frac{d_\theta(F(x))}{b(\theta)}\right) \frac{d_\theta(F(x))}{b(\theta)} f(x) dx - \frac{1}{b(\theta)} \int_0^\infty d_\theta(F(x))f(x) \log(f(x)) dx \\ &= c(\theta) - \frac{1}{b(\theta)} \int_0^1 d_\theta(u) \log(f(F^{-1}(u))) du, \end{aligned}$$

where $c(\theta) = - \int_0^1 (d_\theta(u)/b(\theta)) \log(d_\theta(u)/b(\theta)) du$ is obviously independent of the underlying distribution. In a similar manner,

$$H(Y_{v_\theta}) = c(\theta) - \frac{1}{b(\theta)} \int_0^1 d_\theta(u) \log(g(G^{-1}(u))) du.$$

From (3.B.23) in Shaked and Shanthikumar [34], $X \leq_{disp} Y$ implies that $f(F^{-1}(u)) \geq g(G^{-1}(u))$, for all $u \in (0, 1)$. Thus, we conclude that

$$\log\left(\frac{f(F^{-1}(u))}{g(G^{-1}(u))}\right) \geq 0, \text{ for all } u \in (0, 1).$$

As a result, for all $\theta \in \chi$, we deduce $H(Y_{v_\theta}) - H(X_{w_\theta}) = \frac{1}{b(\theta)} \left(\int_0^1 d_\theta(u) \log\left(\frac{f(F^{-1}(u))}{g(G^{-1}(u))}\right) du \right) \geq 0. \quad \square$

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