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Hausdorff Dimension and Topological Entropies of a Dynamical Solenoid

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Abstract: The purpose of this paper is to elucidate the interrelations between three essentially different concepts: dynamical solenoids, topological entropy, and Hausdorff dimension, where by a dynamical solenoid we mean a sequence of continuous epimorphisms of a compact metric space. For this purpose, we describe a dynamical solenoid by topological entropy-like quantities and investigate the relations between them. For L-Lipschitz dynamical solenoids and locally $\lambda-$expanding dynamical solenoids, we show that the topological entropy and fractal dimensions are closely related. For a locally $\lambda-$expanding dynamical solenoid, we prove that its topological entropy is lower estimated by the Hausdorff dimension of X multiplied by the logarithm of λ.

Keywords: entropy; Hausdorff measure; Hausdorff dimension; box dimension; dynamical solenoid; locally expanding map

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1. Introduction

A solenoid, which was introduced to mathematics by Vietoris [1] as the topological object, can be presented either in an abstract way as an inverse limit or in a geometric way as a nested intersection of a sequence of tori. The classical construction of Vietoris was modified by McCord [2], Williams [3], and others. Since the publication of William's paper on expanding attractors [3], inverse limit spaces have played a key role in dynamical systems and in continuum theory. Smale [4] introduced the solenoid to dynamical systems as a hyperbolic attractor.

In the paper, a sequence $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ of continuous epimorphisms of a compact metric space (X, d) is called a dynamical solenoid while the inverse limit

$$X_\infty = \varprojlim(X, f_k) := \{(x_k)_{k=0}^{\infty} : x_{k-1} = f_k(x_k)\}.$$

is called a classical solenoid. Since the paper is not about classical solenoids, the term dynamical solenoid is sometimes abbreviated as solenoid.

In mathematical literature, one can also find a more restrictive definition of the solenoid as a finite-dimensional, connected, compact abelian group. These solenoids generalize torus groups, and their entropic properties have been studied by Berg [5], Lind and Ward [6], Einsiedler and Lindenstrauss [7], and others. A less restrictive definition of the solenoid was considered in [8–10].

A dynamical solenoid is a natural generalization of a classical dynamical system. In contrast with the classical dynamical systems, the properties of solenoid entropies have not been fully investigated. In the paper, we consider several different definitions of entropy-like quantities for a dynamical solenoid f_∞:

topological entropy $h_{top}(f_\infty)$, topological cover entropy $h_{top-cov}(f_\infty)$, and topological dimensional entropy $h_{top-dim}(f_\infty)$.

Both nonautonomous dynamical systems and dynamical solenoids are determined by compositions of continuous self-maps; therefore, in both cases, the entropy-like quantities that capture complexities of these generalized dynamical systems can be similar. For example, the topological entropy of a dynamical solenoid coincides with the topological entropy of a nonautonomous dynamical system defined in [11]. In this paper, we derive the following relations between the entropies of a dynamical solenoid which were previously known for continuous maps on compact metric spaces, and we obtained the following results.

Theorem 1. $h_{top-dim}(f_\infty) \leq h_{top-cov}(f_\infty)$.

Theorem 2. $h_{top}(f_\infty) = h_{top-cov}(f_\infty)$.

In 2002, Milnor [12] stated two questions related to the classical dynamical system: "Is entropy of it effectively computable?" "Given an explicit dynamical system and given $\epsilon > 0$, is it possible to compute the entropy with maximal error of ϵ?" In most cases the answer is negative. For the recent results on computability of topological entropy, we recommend [13,14].

Therefore, in mathematical literature, there were many attempts to estimate entropy of dynamical systems by Lyapunov exponents, volume growth, Hausdorff dimension, or fractal dimensions.

The theory of Carathéodory structures, introduced by Pesin [15] for a single map, has been applied in [11] to get some estimations of the topological entropy of a nonautonomous dynamical system. To show a comprehensive picture and beauty of dynamics of dynamical solenoids, we rewrite the Theorem 3 in [11] to express complexity of so called L-Lipschitz dynamical solenoid. A dynamical solenoid $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ is called L-Lipschitz if it consists of L-Lipschitz epimorphisms; the following inequality holds.

Theorem 3. *Assume that $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ is a L-Lipschitz dynamical solenoid with $L > 1$. Then, for any $Y \subset X$, we obtain*

$$HD(Y) \geq \frac{h_{top-dim}((f_\infty), Y)}{\log(L)},$$

where $HD(Y)$ is the Hausdorff dimension of Y.

Finally, we investigate so called locally $\lambda-$expanding dynamical solenoids, in the sense of Ruelle [16] (see Definition 6). We prove that the topological entropy of a $\lambda-$expanding dynamical solenoid, defined on the space X, is related to the upper box dimension of X multiplied by the logarithm of λ. We obtained the following inequalities.

Theorem 4. *Given a locally $\lambda-$expanding dynamical solenoid $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$. Then,*

$$h_{top}(f_\infty) \geq (\log \lambda) \cdot \overline{\dim_B(X)} \geq (\log \lambda) \cdot HD(X),$$

where $\overline{\dim_B(X)}$ is the upper box dimension of X.

The paper is organized as follows. In Section 2, we introduce several definitions of entropy-like quantities for a dynamical solenoid: topological entropy, topological cover entropy, and topological dimensional entropy. In Section 3, we prove the relations between them (Theorems 1 and 2). Section 4 is devoted to L-Lipschitz dynamical solenoids; we present Theorem 3. Finally, in Section 5, we investigate locally $\lambda-$expanding dynamical solenoids and prove Theorem 4.

2. Topological Entropies of a Dynamical Solenoid

In 1965, Adler, Konheim, and McAndrew [17] introduced a definition of topological entropy for the classical dynamical system (i.e., a pair (X, f), where X is a topological space and $f : X \to X$ is a continuous map) as a non-negative number assigned to an open cover of X. A different definition of entropy of a continuous self-map defined on a compact metric space was introduced by Bowen [18] and independently by Dinaburg [19]. In [20], Bowen proved that the definitions are equivalent. Nowadays, topological entropy is a main notion in topological dynamics. In the paper, we present a few generalizations of the classical topological entropy of a single map to dynamical solenoids.

In the paper, we consider a dynamical solenoid determined by a sequence $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ of continuous epimorphisms of a compact metric spaces (X, d). Thus, the dynamical solenoid is a generalized dynamical system. Its complexity and chaos can be measured by several entropy-like quantities. First, we introduce topological entropy via $(n, \epsilon)-$separated sets.

2.1. Topological Entropy of a Dynamical Solenoid via $(n, \epsilon)-$Separated or $(n, \epsilon)-$Spanning Sets

Let $B(x, r) = \{y \in X : d(x, y) \leq r\}$ denote a closed ball in the metric space (X, d) centered at $x \in X$ and with radius r.

Definition 1. *Fix $\epsilon > 0$, $n \in \mathbb{N}$. A subset $F \subset X$ is called (n, ϵ)-spanning if for any $x \in X$ there exists $y \in F$ such that*

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(x), f_i \circ f_{i+1} \circ ... \circ f_n(y)) : i \in \{1, ..., n\}\} \leq \epsilon.$$

Let $r(n, \epsilon) := \min\{card(F) : F$ is (n, ϵ)-spanning set$\}$.

A set $E \subset X$ is called (n, ϵ)-separated if for any pair of distinct points $x, y \in E$ we have

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(x), f_i \circ f_{i+1} \circ ... \circ f_n(y)) : i \in \{1, ..., n\}\} > \epsilon.$$

Let $s(n, \epsilon) := \max\{card(E) : E$ is (n, ϵ)-separated$\}$.

The following two lemmas are a reformulation of Definition 1.

Lemma 1. *A set $F \subset X$ is $(n, \epsilon)-$spanning if and only if*

$$X = \bigcup_{y \in F} \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(y), \epsilon].$$

Proof. $(\Rightarrow)$ Assume that a subset $F \subset X$ is $(n, \epsilon)-$spanning. Then for any point $x \in X$ there exists a point $y \in F$ such that

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(x), f_i \circ f_{i+1} \circ ... \circ f_n(y)) : i \in \{1, ..., n\}\} \leq \epsilon.$$

For any $i \in \{1, ..., n\}$ we obtain

$$f_i \circ f_{i+1} \circ ... \circ f_n(x) \in B[f_i \circ f_{i+1} \circ ... \circ f_n(y), \epsilon]$$

and

$$x \in (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(y), \epsilon].$$

So

$$x \in \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(y), \epsilon].$$

Since x is an arbitrary point of X we conclude

$$X \subset \bigcup_{y \in F} \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(y), \epsilon] \subset X.$$

($\Leftarrow$) Now assume that the following equality

$$X = \bigcup_{y \in F} \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(y), \epsilon]$$

holds for a subset $F \subset X$. Then, for any $x \in X$ there exists $y \in F$ such that

$$x \in \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(y), \epsilon]$$

which is equivalent to

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(x), f_i \circ f_{i+1} \circ ... \circ f_n(y)) : i \in \{1, ..., n\}\} \leq \epsilon.$$

Thus the set F is $(n, \epsilon)-$spanning and the proof is finished. $\square$

Lemma 2. *A set $E \subset X$ is (n, ϵ)-separated if and only if for any $x \in E$ the set $\bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x), \epsilon]$ contains no other points of E.*

Proof. ($\Rightarrow$) Assume that a set $E \subset X$ is $(n, \epsilon)-$separated and choose two distinct points $x_1, x_2 \in E$. For any $i \in \{1, ..., n\}$ we get

$$x_1 \in (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), \epsilon]$$

so

$$x_1 \in \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), \epsilon].$$

Assume that

$$x_2 \in \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), \epsilon]$$

then we obtain the following inequality

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(x_1), f_i \circ f_{i+1} \circ ... \circ f_n(x_2)) : i \in \{1, ..., n\}\} \leq \epsilon$$

which gives a contradiction with the assumption that x_1, x_2 are $(n, \epsilon)-$separated. Thus the intersection

$$(*) \quad E \cap \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), \epsilon] = \{x_1\}.$$

($\Leftarrow$) Now assume that for a given subset $E \subset X$ the condition $(*)$ holds. For two distinct points $x_1, x_2 \in E$ we have

$$x_2 \notin \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), \epsilon].$$

Therefore, there exists $i \in \{1, ..., n\}$ such that

$$d(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), f_i \circ f_{i+1} \circ ... \circ f_n)(x_2)) > \epsilon.$$

We have proved that the set $E \subset X$ is $(n, \epsilon)-$separated. $\square$

Modifying slightly the classical Bowen's definition [18] of the topological entropy of a single map (for details see also Chapter 7 in [21]), we present the definition of topological entropy of a dynamical solenoid as follows.

Definition 2. *The quantity*

$$h_{top}(f_\infty) := \lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon)$$

is called the topological entropy of f_∞.

Remark 1. *The topological entropy of a dynamical solenoid can also be expressed in the language of (n, ϵ)-spannings sets. Using arguments similar to remarks on page 169 in [21], we get estimations*

$$r(n, \epsilon) \leq s(n, \epsilon) \leq r(n, \epsilon/2).$$

Indeed, for any two distinct points x_1, x_2 of an $(n, \epsilon)-$separated set E with cardinality $card(E) = s(n, \epsilon)$ we have

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(x_1), f_i \circ f_{i+1} \circ ... \circ f_n(x_2)) : i \in \{1, ..., n\}\} > \epsilon.$$

Since E is $(n, \epsilon)-$separated set with maximal cardinality, for any $y \in X \setminus E$ there exists $x_3 \in E$ such that

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(y), f_i \circ f_{i+1} \circ ... \circ f_n(x_3)) : i \in \{1, ..., n\}\} \leq \epsilon.$$

It means that E is $(n, \epsilon)-$spanning and

$$r(n, \epsilon) \leq card(E) = s(n, \epsilon).$$

To show the other inequality for the set E and an $(n, \frac{\epsilon}{2})-$ spanning set $F \subset X$ with cardinality $card(F) = r(n, \epsilon/2)$ define $\phi : E \to F$ by choosing, for each point $x \in E$ some point $\phi(x) \in E$ with

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n(x), f_i \circ f_{i+1} \circ ... \circ f_n(\phi(x))) : i \in \{1, ..., n\}\} \leq \frac{\epsilon}{2}.$$

The map $\phi : E \to F$ is injective and therefore $card(E) \leq card(F)$. Hence $s(n, \epsilon) \leq r(n, \epsilon/2)$.

Applying the inequalities

$$r(n, \epsilon) \leq s(n, \epsilon) \leq r(n, \epsilon/2)$$

and passing to the suitable limits, we obtain the equality

$$h_{top}(f_\infty) = \lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon).$$

Remark 2. *Assume that all maps of the sequence $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ coincide with a fixed continuous map $f : X \to X$ of a compact metric space (X, d). Then, the topological entropy of f_∞ is equal to the topological entropy of f. For example, the topological entropy of a dynamical solenoid coincides with the topological entropy of a nonautonomous dynamical system defined in [11].*

2.2. Topological Entropy of a Dynamical Solenoid via Open Covers

It is a well-known fact that topological entropy of a single continuous map $f : X \to X$ can be defined by open covers of the compact metric space (X, d). We intend to show that similar approach can be applied to a dynamical solenoid. For this purpose, notice that for two open covers $\mathcal{A}$, $\mathcal{B}$ of X, the family

$$\mathcal{A} \vee \mathcal{B} := \{A \cap B : A \in \mathcal{A},\ B \in \mathcal{B}\}$$

is an open cover of X. Moreover, for a continuous map $f_i \circ f_{i+1} \circ ... \circ f_n : X \to X$ and an open cover $\mathcal{A}$ of X the family

$$(f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}\mathcal{A} := \{(f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}A : A \in \mathcal{A}\}$$

is an open cover of X. Thus, for the open cover $\mathcal{A}$ of X, the family

$$\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}\mathcal{A} :=$$

$$(f_1 \circ f_2 \circ ... \circ f_n)^{-1}(\mathcal{A}) \vee (f_2 \circ f_3 \circ ... \circ f_n)^{-1}(\mathcal{A}) \vee ... \vee (f_n)^{-1}(\mathcal{A})$$

is an open cover of X.

For an open cover $\mathcal{B}$ of X let us denote by $N(\mathcal{B})$ the number of sets in a finite subcover of $\mathcal{B}$ covering X, with the smallest cardinality.

Definition 3. *The topological cover entropy of f_∞, relative to an open cover $\mathcal{A}$ of X, is defined as*

$$h_{top-cov}(f_\infty, \mathcal{A}) := \limsup_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}\mathcal{A}\right),$$

whereas the topological cover entropy of f_∞ is the quantity

$$h_{top-cov}(f_\infty) := \sup_{\mathcal{A}} h(f_\infty, \mathcal{A}),$$

where $\mathcal{A}$ ranges over all open covers of X.

2.3. Topological Entropy as a Dimension Theory Quantity

Here, we modify the Bowen's definition [20] of the topological entropy of a continuous single map, which is similar to the construction of the Hausdorff measure, to obtain the topological dimensional entropy of f_∞.

2.3.1. The Hausforff Measure and the Hausdorff Dimension

For the convenience of the reader, we recall briefly the classical construction of the Hausdorff measure and the Hausdorff dimension.

For a metric space (X, d) and a subset $Y \subset X$, let us denote by $Cov_\epsilon(Y)$ the family of open covers $\mathcal{B}$ of Y with $diam(B) < \epsilon$, for any $B \in \mathcal{B}$. Here, $diam(B)$ denotes the diameter of B.

For any $\lambda > 0$ the classical Hausdorff $\lambda-$measure $\mu_\lambda(Y)$ of a subset $Y \subset X$ is defined as follows,

$$\mu_\lambda(Y) := \lim_{\epsilon \to 0} \inf\{\sum_{B \in \mathcal{B}} [diam(B)]^\lambda : \mathcal{B} \in Cov_\epsilon(Y)\}.$$

The function $\lambda \to \mu_\lambda(Y)$ has a unique critical point, where it jumps from ∞ to 0. The Hausdorff dimension $HD(Y)$ of Y is defined as the critical point of the function $\lambda \to \mu_\lambda(Y)$, i.e.,

$$HD(Y) = \sup\{\lambda : \mu_\lambda(Y) = \infty\} = \inf\{\lambda : \mu_\lambda(Y) = 0\}.$$

2.3.2. Generalized Hausdorff Measure and Generalized Hausdoff Dimension

Arguments similar to the construction of the classical Hausdorff λ-measure and the Hausdorff dimension lead to another entropy-like quantity for $f_\infty = (f_n : X \to X)_{n=1}^\infty$. Denote by $\mathcal{A}$ a finite open cover of X. For a subset $B \subset X$, we write $B \prec \mathcal{A}$ if there exists $A_i \in \mathcal{A}$ such that $B \subset A_i$. Denote by $n_\mathcal{A}(B)$ the largest non-negative integer n such that $f_k \circ f_{k+1} \circ ... \circ f_n(B) \prec \mathcal{A}$ for $k = 1, ..., n$. If there is no element $A_i \in \mathcal{A}$ such that $B \subset A_i$, then we write $n_\mathcal{A}(B) = 0$. Let us introduce the following notations.

$$diam_\mathcal{A}(B) := exp(-n_\mathcal{A}(B)),$$

$$diam_\mathcal{A}(\mathcal{B}) := \sup\{diam_\mathcal{A}(B) : B \in \mathcal{B}\}$$

and

$$\mathcal{D}_\mathcal{A}(\mathcal{B}, \lambda) := \sum_{B \in \mathcal{B}} [diam_\mathcal{A}(B)]^\lambda$$

for a family $\mathcal{B}$ of subsets of X and a real number, $\lambda > 0$. For a subset $Y \subset X$ and $\epsilon > 0$, let $Cov_\epsilon^\mathcal{A}(Y)$ denote the family of open covers $\mathcal{B}$ of Y with $diam_\mathcal{A}(\mathcal{B}) < \epsilon$. Now we set

$$\mu_{\mathcal{A},\lambda}(Y) := \lim_{\epsilon \to 0} \inf\{\mathcal{D}_\mathcal{A}(\mathcal{B}, \lambda) : \mathcal{B} \in Cov_\epsilon^\mathcal{A}(Y)\}.$$

The behavior of the function $\lambda \to \mu_{\mathcal{A},\lambda}(Y)$ is very similar to the behavior of $\lambda \to \mu_\lambda(Y)$: it has a unique critical point, where it jumps from ∞ to 0. More precisely.

Lemma 3. *For the function $\lambda \to \mu_{\mathcal{A},\lambda}(Y)$, there exists a unique critical number λ_0 such that $\mu_{\mathcal{A},\lambda}(Y) = \infty,\ for\ 0 \leq \lambda < \lambda_0$ and $\mu_{\mathcal{A},\lambda}(Y) = 0,\ for\ \lambda_0 < \lambda$.*

Proof. For any $\epsilon \in (0, 1)$ there exists a cover $\mathcal{B}$ of Y with $\exp(-n_\mathcal{A}(B)) < 1$, for any $B \in \mathcal{B}$. Therefore, the inequality $0 < \beta < \alpha$ implies

$$\sum_{B \in \mathcal{B}} \exp(-n_\mathcal{A}(B) \cdot \alpha) \leq \sum_{B \in \mathcal{B}} \exp(-n_\mathcal{A}(B) \cdot \beta),$$

so

$$\mu_{\mathcal{A},\alpha}(Y) = \lim_{\epsilon \to 0} \inf\{\sum_{B \in \mathcal{B}} \exp(-n_\mathcal{A}(B) \cdot \alpha) : \mathcal{B} \in Cov_\epsilon^\mathcal{A}(Y)\} \leq$$

$$\lim_{\epsilon \to 0} \inf\{\sum_{B \in \mathcal{B}} \exp(-n_\mathcal{A}(B) \cdot \beta) : \mathcal{B} \in Cov_\epsilon^\mathcal{A}(Y)\} = \mu_{\mathcal{A},\beta}(Y).$$

Therefore,

$$(**)\ \ 0 < \beta < \alpha \Rightarrow \mu_{\mathcal{A},\alpha}(Y) \leq \mu_{\mathcal{A},\beta}(Y).$$

First assume that $\mu_{\mathcal{A},\delta}(Y) = \infty$ for some $\delta > 0$ and that $\beta < \delta$. By $(**)$ we conclude that

$$\infty = \mu_{\mathcal{A},\delta}(Y) \leq \mu_{\mathcal{A},\beta}(Y).$$

In a similar way, we prove that if $\mu_{\mathcal{A},\lambda}(Y) = 0$, then for $\lambda_1 > \lambda$ we obtain the equality $\mu_{\mathcal{A},\lambda_1}(Y) = 0$. □

Definition 4. *Denote by λ_0 the critical point of the function $\lambda \to \mu_{\mathcal{A},\lambda}(Y)$. Let $\lambda_0 = h_{top-dim}((f_\infty), Y, \mathcal{A})$. In other words, let*

$$h_{top-dim}((f_\infty), Y, \mathcal{A}) := \sup\{\lambda : \mu_{\mathcal{A},\lambda}(Y) = \infty\} = \inf\{\lambda : \mu_{\mathcal{A},\lambda}(Y) = 0\}.$$

The number

$$h_{top-dim}(f_\infty, Y) := \sup\{h_{top-dim}((f_\infty), Y, \mathcal{A}) : \mathcal{A}\ finite\ open\ cover\ of\ Y\}$$

is called the topological dimensional entropy of f_∞ restricted to Y. If $Y = X$, we write $h_{top-dim}(f_\infty, X) = h_{top-dim}(f_\infty)$.

Remark 3. *Our definition of topological dimension entropy of a dynamical solenoid is an extension of Bowen's entropy [20]. Moreover, the topological dimensional entropy of a dynamical solenoid is similar to Bowen's topological entropy of nonautonomous dynamical systems in [22].*

3. Relations between Topological Entropies of a Dynamical Solenoid

In the previous section, we introduced three entropy-like quantities for a dynamical solenoid. Now, we relate the topological dimensional entropy of a dynamical solenoid to its topological covering entropy. We obtain the following result.

Theorem 1. $h_{top-dim}(f_\infty) \leq h_{top-cov}(f_\infty)$.

Proof. Choose a finite open cover $\mathcal{A}$ of X and let

$$\mathcal{A}_n = \{\bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}(A_i) : A_i \in \mathcal{A}\}.$$

Denote by $\mathcal{B}_n$ a finite subcover of $\mathcal{A}_n$ with cardinality $|\mathcal{B}_n| = N(\mathcal{A}_n)$. Then, for any $B \in \mathcal{B}_n$, we obtain that $n_{\mathcal{A}}(B) \geq n$, so

$$diam_{\mathcal{A}}(B) \leq \exp(-n)$$

and for any $\lambda > 0$ we get

$$\mathcal{D}_{\mathcal{A}}(\mathcal{B}_n, \lambda) = \sum_{B \in \mathcal{B}_n} [diam_{\mathcal{A}}(B)]^\lambda = \sum_{B \in \mathcal{B}_n} exp(-\lambda \cdot n_{\mathcal{A}}(B)) \leq |\mathcal{B}_n| \cdot \exp(-\lambda \cdot n).$$

As $|\mathcal{B}_n| = N(\mathcal{A}_n)$, we have

$$|\mathcal{B}_n| \cdot \exp(-\lambda \cdot n) = \exp(-\lambda \cdot n + \log|\mathcal{B}_n|) = \exp(-n(\lambda - \frac{1}{n}\log N(\mathcal{A}_n))).$$

Consequently,

$$\mathcal{D}_{\mathcal{A}}(\mathcal{B}_n, \lambda) \leq \exp(-n \cdot (\lambda - \frac{1}{n}\log N(\mathcal{A}_n))).$$

Fix $\epsilon > 0$ and an arbitrary small $\gamma > 0$. Choose λ^* such that $\lambda^* > h_{top-cov}(f_\infty, \mathcal{A}) > \lambda^* - \gamma$. For sufficiently large $n \in \mathbb{N}$, we obtain the inequalities

$$\lambda^* - \frac{1}{n}\log N(\mathcal{A}_n)) > 0,$$

$$diam_{\mathcal{A}}(B) < \exp(-n) < \epsilon,\ for\ B \in \mathcal{B}, and$$

$$\mathcal{D}_{\mathcal{A}}(\mathcal{B}_n, \lambda^*) \leq \exp(-n \cdot (\lambda^* - \frac{1}{n}\log N(\mathcal{A}_n))) < \epsilon.$$

As $\epsilon > 0$ is arbitrarily small, the above two inequalities yield $\mu_{\mathcal{A},\lambda^*}(X) = 0$. Therefore,

$$h_{top-dim}(f_\infty, Y, \mathcal{A}) \leq \lambda^* \leq h_{top-cov}(f_\infty, \mathcal{A}) + \gamma.$$

As $\mathcal{A}$ is an arbitrary finite open cover of X, we obtain

$$h_{top-dim}(f_\infty) = \sup\{h_{top-dim}((f_\infty), X, \mathcal{A}) : \mathcal{A} - finite\ open\ cover\ of\ X\}$$

$$\leq \sup\{h_{top-cov}(f_\infty, \mathcal{A}) : \mathcal{A} - finite\ open\ cover\ of\ X\} + \gamma = h_{top-cov}(f_\infty) + \gamma.$$

Finally, passing with γ to zero, we get

$$h_{top-dim}(f_\infty) \leq h_{top-cov}(f_\infty).$$

□

Lemma 4. *For an open cover $\mathcal{A}$ of X with the Lebegue number $Leb(\mathcal{A}) = \delta$, we get*

$$N\left(\bigvee_{i=1}^{n}(f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}\mathcal{A}\right) \leq r(n, \frac{\delta}{2}).$$

Proof. Fix $n \in \mathbb{N}$ and $\delta > 0$. Choose an $(n, \frac{\delta}{2})$-spanning set F with cardinality $card(F) = r(n, \frac{\delta}{2})$. As $Leb(\mathcal{A}) = \delta$, we obtain that any ball $B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x), \frac{\delta}{2}]$ of radius $\delta/2$, where $x \in F$ and $i \in \{1, ..., n\}$, is included in some set $A_i \in \mathcal{A}$, so

$$\bigcap_{i=1}^{n}(f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x), \frac{\delta}{2}] \subset \bigcap_{i=1}^{n}(f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}A_i,$$

for some $A_1, A_2, ...A_n \in \mathcal{A}$. It means that the set

$$\bigcap_{i=1}^{n}(f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}B[(f_i \circ f_{i+1} \circ ... \circ f_n)(x), \frac{\delta}{2}]$$

is a subset of some member of the covering

$$\bigvee_{i=1}^{n}(f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}\mathcal{A}.$$

On the other hand, applying Lemma 1, we get

$$X = \bigcup_{x \in F} \bigcap_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} B((f_i \circ f_{i+1} \circ ... \circ f_n)(x), \frac{\delta}{2}),$$

so

$$N(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} \mathcal{A}) \leq card(F) = r(n, \frac{\delta}{2}).$$

□

Lemma 5. *Assume that $\epsilon > 0$ and $\mathcal{B}$ is an open cover of X, with $diam(\mathcal{B}) \leq \epsilon$. Then,*

$$s(n, \epsilon) \leq N(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} \mathcal{B}).$$

Proof. Choose an (n, ϵ)-separated set E with cardinality $card(E) = s(n, \epsilon)$. Assume that two distinct points $x_1, x_2 \in E$ belong to the same member of the cover $\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} \mathcal{B}$. Therefore, there exist sets $B_i \in \mathcal{B}$ such that $(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), (f_i \circ f_{i+1} \circ ... \circ f_n)(x_2) \in B_i$ for any $i \in \{1, .., n\}$. On the other hand, as the set E is (n, ϵ)-separated, there exists $j \in \{1, ..., n\}$ such that

$$d((f_j \circ f_{j+1} \circ ... \circ f_n)(x_1), (f_j \circ f_{j+1} \circ ... \circ f_n)(x_2)) =$$

$$\max\{d(f_i \circ f_{i+1} \circ ... \circ f_n)(x_1), (f_i \circ f_{i+1} \circ ... \circ f_n)(x_2) : i \in \{1, .., n\}\} > \epsilon.$$

Thus, we get a contradiction with $diam(B_j) \leq \epsilon$. Therefore,

$$s(n, \epsilon) \leq N(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} \mathcal{B}).$$

□

Now, we are ready to prove that the topological entropy of a dynamical solenoid is equivalent to its topological covering entropy.

Theorem 2. $h_{top}(f_\infty) = h_{top-cov}(f_\infty)$.

Proof. Fix $\epsilon > 0$. Let $\mathcal{A}_\epsilon$ be the cover of X by all open balls of radius $2 \cdot \epsilon$ and denote by $\mathcal{B}_\epsilon$ the cover of X by all open balls of radius $\frac{\epsilon}{2}$. Due to Lemma 4, we obtain

$$N(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} \mathcal{A}_\epsilon) \leq r(n, \epsilon),$$

so

$$\limsup_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1} \mathcal{A}_\epsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon)$$

and

$$h_{top-cov}(f_\infty) \leq h_{top}(f_\infty).$$

Applying Lemma 5, we get

$$s(n,\epsilon) \leq N(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}\mathcal{B}_\epsilon),$$

so

$$\limsup_{n\to\infty} \frac{1}{n} \log s(n,\epsilon) \leq \limsup_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n} (f_i \circ f_{i+1} \circ ... \circ f_n)^{-1}\mathcal{B}_\epsilon)$$

and finally we get the second inequality

$$h_{top}(f_\infty) \leq h_{top-cov}(f_\infty).$$

The theorem is proved. □

4. Topological Entropy of L-Lipschitz Dynamical Solenoids

Dai, Zhou, and Geng [23] proved the following result. If X is a metric compact space and $f : X \to X$ a Lipschitz continuous map, then the Hausdorff dimension of X is lower estimated by the topological entropy of f divided by the logarithm of its Lipschitz constant. In 2004, Misiurewicz [24] provided a new definition of topological entropy of a single transformation, which was a kind of hybrid between the Bowen's definition and the original definition of Adler, Konheim, and McAndrew [17]. The main theorem in [24] is similar to the result in [23]. In this section, we consider a special class of dynamical solenoids called *L-Lipschitz dynamical solenoids*. We say that a dynamical solenoid $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ is a *L-Lipschitz* if there exists $L > 0$ such that each map $f_n : X \to X$ is an Lipschitz epimorphism with Lipschitz constant L, i.e., for any $x, y \in X$ and arbitrary $n \in \mathbb{N}$

$$d(f_n(x), f_n(y)) \leq L \cdot d(x,y).$$

Let us start with the following example.

Example 1. *Consider the dynamical solenoid $f_\infty = (f_n : \mathbb{T}^2 \to \mathbb{T}^2)_{n=1}^{\infty}$, where $\mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathbb{Z}^2}$ is two-dimensional torus and each $f_n : \mathbb{T}^2 \to \mathbb{T}^2$ is the doubling map, i.e., $f_n(x_1, x_2) = 2 \cdot (x_1, x_2)$, for any $(x_1, x_2) \in \mathbb{T}^2$. Then,*

$$\frac{h_{top}(f_\infty, \mathbb{T}^2)}{\log(2)} = HD(\mathbb{T}^2) = \frac{h_{top-dim}(f_\infty, \mathbb{T}^2)}{\log(2)}.$$

Indeed, the Hausdorff dimension of the two dimensional torus is equal to two (see page 23 in [25]). Due to Remarks 2 and 3, we get $h_{top}(f_\infty, \mathbb{T}^2) = h_{top}(f_2) = h_{top-dim}(f_\infty, \mathbb{T}^2)$. On the other hand, the doubling map $f_2 : \mathbb{T}^2 \to \mathbb{T}^2$ can be considered as the Cartesian product of two doubling maps $g : \frac{\mathbb{R}}{\mathbb{Z}} \to \frac{\mathbb{R}}{\mathbb{Z}}$ defined by $g(x) = 2 \cdot x \ mod\ 1$, for $x \in \frac{\mathbb{R}}{\mathbb{Z}}$. Moreover, $h_{top}(g) = \log(2)$ (see Example on page 29 in [26]). Consequently, $h_{top}(f_\infty, \mathbb{T}^2) = 2 \cdot \log(2) = h_{top-dim}(f_\infty, \mathbb{T}^2)$.

To show the comprehensive picture of dynamics of L-Lipschitz dynamical solenoids, we rewrite the Theorem 3 published in [11], written for nonautonomous dynamical systems, in the set up of dynamical solenoids as follows.

Theorem 3. *Assume that $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ is a L-Lipschitz dynamical solenoid with $L > 1$. Then, for any $Y \subset X$, we obtain*

$$HD(Y) \geq \frac{h_{top-dim}((f_\infty), Y)}{\log(L)}.$$

For the convenience of the reader and to make the paper self-contained, we write the proof of Theorem 3 which is essentially the same as the proof of Theorem 3 in [11].

Proof. Choose a finite open cover $\mathcal{A}$ of Y and denote by $\delta = Leb(\mathcal{A})$ its Lebesgue number. It means that for an open subset $C \subset Y$ with diameter $diam(C) < \delta$, there exists $A \in \mathcal{A}$ such that $C \subset A$. Choose an open set B with $\frac{\delta}{L^n} \leq diam(B) < \frac{\delta}{L^{n-1}}$, for some $n \in \mathbb{N}$. We obtain that

$$diam(f_k \circ f_{k+1} \circ .... \circ f_{n-1}(B)) < \delta$$

for any $k = 1, 2, ...n-1$, so $n_{\mathcal{A}}(B) \geq n-1$. From the inequality

$$\frac{\delta}{L^n} \leq diam(B)$$

we conclude that

$$n \geq \frac{\log(\delta) - \log(diam(B))}{\log(L)}.$$

Consequently,

$$\frac{\log(\delta) - \log(diam(B))}{\log(L)} \leq n_{\mathcal{A}}(B) + 1$$

and

$$diam_{\mathcal{A}}(B) = \exp(-n_{\mathcal{A}}(B)) \leq \exp(1 - \frac{\log(\delta) - \log(diam(B))}{\log(L)}) =$$

$$\exp[1 - (\frac{\log(\delta)}{\log(L)})] \cdot (diam(B))^{\frac{1}{\log(L)}}.$$

Therefore, for an open cover $\mathcal{B}$ of Y consisting of open sets B with $\frac{\delta}{L^n} \leq diam(B) < \frac{\delta}{L^{n-1}}$ and $\lambda > 0$, we get

$$\mathcal{D}_{\mathcal{A}}(\mathcal{B}, \lambda) \leq \exp[\lambda - \lambda \cdot (\frac{\log(\delta)}{\log(L)})] \cdot \sum_{B \in \mathcal{B}} (diam(B))^{\frac{\lambda}{\log(L)}}.$$

Fix $\gamma > 0$ and choose λ_1 such that

$$\frac{\lambda_1}{\log(L)} > HD(Y) \geq \frac{\lambda_1}{\log(L)} - \gamma.$$

By definition of the Hausdorff measure, the equality $\mu_{\frac{\lambda_1}{\log(L)}}(Y) = 0$ holds. Therefore, for any $\epsilon > 0$ there exists and an open cover $\mathcal{B}_\epsilon$ of Y such that for any $B \in \mathcal{B}_\epsilon$

$$\epsilon > \exp[1 - (\frac{\log(\delta)}{\log(L)})] \cdot (diam(B))^{\frac{1}{\log(L)}} > diam_{\mathcal{A}}(B)$$

and

$$\epsilon > \exp[\lambda_1 - \lambda_1 \cdot (\frac{\log(\delta)}{\log(L)})] \cdot \sum_{B \in \mathcal{B}_\epsilon} (diam(B))^{\frac{\lambda_1}{\log(L)}} > D_{\mathcal{A}}(\mathcal{B}_\epsilon, \lambda_1).$$

The inequalities

$$\mu_{\mathcal{A}, \lambda_1}(Y) \leq \mathcal{D}_{\mathcal{A}}(\mathcal{B}_\epsilon, \lambda_1) < \epsilon$$

yield $\mu_{\mathcal{A}, \lambda_1}(Y) = 0$. According to Definition 4, we get

$$h_{top-dim}((f_\infty), Y, \mathcal{A}) = \inf\{\lambda : \mu_{\mathcal{A},\lambda}(Y) = 0\} \leq \lambda_1.$$

Taking supremum over all open finite covers of Y, we obtain

$$h_{top-dim}((f_\infty), Y) =$$

$$\sup\{h_{top-dim}((f_\infty), Y, \mathcal{A}) : \mathcal{A} - finite\ open\ cover\ of\ Y\} \leq$$

$$\lambda_1 \leq \log(L) \cdot (HD(Y) + \gamma).$$

Finally,

$$h_{top-dim}((f_\infty), Y) \leq \log(L) \cdot HD(Y),$$

as γ is an arbitrarily small positive number. $\square$

In particular, taking $Y = X$, we obtain the following corollary.

Corollary 1. *Assume that* $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ *is a L-Lipschitz dynamical solenoid. Then, the inequality*

$$HD(X) \geq \frac{h_{top-dim}((f_\infty))}{\log(L)}$$

holds.

In the special case, for $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$ being a L-Lipschitz dynamical solenoid such that all maps $f_n : X \to X$ coincide with a continuous map $f : X \to X$, we get that

$$h_{top}(f_\infty) = h_{top}(f),$$

where $h_{top}(f)$ is the classical topological entropy of $f: X \to X$. Bowen proved (Proposition 1 in [20]) that $h_{top-dim}(f) = h_{top}(f)$. Consequently, as a corollary of Theorem 3, we get the result of Misiurewicz [24].

Corollary 2 (Theorem 2.1 in [24])**.** *If* $f : X \to X$ *is a continuous L-Lipschitz map of a compact metric space* (X, d), *then*

$$HD(X) \geq \frac{h_{top}(f)}{\log(L)}.$$

5. Topological Entropy of Locally Expanding Dynamical Solenoids

In this section, we investigate locally expanding dynamical solenoids. Ruelle [16] introduced the notion of a locally expanding map in the following way.

Definition 5. *Let* (X, d) *be a compact metric space and* $f : X \to X$ *a continuous selfmap. If for* $\lambda > 1$ *there exists* $\epsilon > 0$ *such that for every pair of distinct points* $x, y \in X$

$$d(x, y) < \epsilon \Rightarrow\ d(f(x), f(y)) \geq \lambda \cdot d(x, y),$$

then we say that f *is a locally* (ϵ, λ)*-expanding map and* λ *is an* expanding coefficient *of* f.

Notice that any finite composition of locally (ϵ_i, λ_i)-expanding maps is an (ϵ, λ)-locally expanding map for some $\epsilon > 0$ and $\lambda > 1$. We extend the notion of locally expanding map to a dynamical solenoid as follows.

Definition 6. *Given a dynamical solenoid $f_\infty = (f_n : X \to X)_{n=1}^\infty$. If there exists $\epsilon > 0$ such that all maps $f_n : X \to X$ are locally (ϵ, λ_n)-expanding and $\lambda := \inf\{\lambda_n : n \in \mathbb{N}\} > 1$, then we say that f_∞ is locally λ-expanding.*

Lemma 6. *Given a locally λ-expanding dynamical solenoid $f_\infty = (f_n : X \to X)_{n=1}^\infty$. Then, there exists $\epsilon > 0$ such that for any $x \in X$, $k \in \mathbb{N}$, and $\gamma \in (0, \epsilon)$ we get*

$$\bigcap_{i=1}^{k} (f_i \circ f_{i+1} \circ \ldots \circ f_k)^{-1} B(f_i \circ f_{i+1} \circ \ldots \circ f_k(x), \gamma) \subset B\left(x, \frac{\gamma}{\lambda^k}\right).$$

Proof. Choose $\epsilon > 0$ such that for any $k \in \mathbb{N}$ and for every pair of distinct points $x, y \in X$, we get

$$d(x,y) < \epsilon \Rightarrow\ d(f_k(x), f_k(y)) \geq \lambda_k \cdot d(x,y).$$

Fix $\gamma \in (0, \epsilon)$ and let

$$y \in \bigcap_{i=1}^{k} (f_i \circ f_{i+1} \circ \ldots \circ f_k)^{-1} B(f_i \circ f_{i+1} \circ \ldots \circ f_k(x), \gamma).$$

Then, we get inequalities

$$\epsilon > \gamma > d(f_1 \circ f_2 \circ \ldots \circ f_k(x), f_1 \circ f_2 \circ \ldots \circ f_k(y)) \geq$$
$$\lambda_1 \cdot d(f_2 \circ \ldots \circ f_k(x), f_2 \circ \ldots \circ f_k(y)) \geq \lambda_1 \cdot \ldots \cdot \lambda_k \cdot d(x,y) \geq$$
$$\lambda^k \cdot d(x,y).$$

Therefore, $d(x,y) < \frac{\gamma}{\lambda^k}$ and $y \in B(x, \frac{\gamma}{\lambda^k})$. The lemma is proved. □

The notion of the box dimension is an example of fractal dimension which belongs to fractal geometry. It was defined independently by Minkowski and Bouligard for a subset of Euclidean space. For modern presentation of fractal dimensions see the classical books of Falconer [25,27] or the monograph written by Przytycki and Urbański [28].

Definition 7 (Chapter 2 in [25])**.** *Recall that the upper box dimension of a closed subset Z of a compact metric space X is*

$$\overline{\dim_B(Z)} := \limsup_{\gamma \to 0} \frac{\log N(Z, \gamma)}{-\log \gamma},$$

where $N(Z, \gamma)$ denotes the smallest number of balls $B(x, \gamma)$ of radius $\gamma > 0$ needed to cover Z.

Lemma 7 ([28])**.** *For a compact metric space X, the Hausdorff dimension $HD(X)$ of X and the upper box dimension $\overline{\dim_B(X)}$ of X are interrelated*

$$HD(X) \leq \overline{\dim_B(X)}.$$

In the proof of Theorem 4 we need the following lemma.

Lemma 8 (Lemma 6.2 in [29])**.** *Let $\phi : R \to R_+$ be a decreasing function. If $\delta \in (0,1)$ and $\gamma > 0$, then*

$$\limsup_{r \to 0} \frac{\log \phi(r)}{\log r} = \limsup_{n \to \infty} \frac{\log \phi(\delta^n \gamma)}{\log(\delta^n \gamma)}.$$

Theorem 4. *Given a locally λ-expanding dynamical solenoid $f_\infty = (f_n : X \to X)_{n=1}^{\infty}$. Then,*

$$h_{top}(f_\infty) \geq (\log \lambda) \cdot \overline{\dim_B(X)} \geq (\log \lambda) \cdot HD(X).$$

Proof. In the first part of the proof we intend to show that

$$h_{top}(f_\infty) \geq (\log \lambda) \cdot \overline{\dim_B(X)}. \tag{1}$$

Fix $\epsilon > 0$ such that for every pair of distinct points $x, y \in X$ and for every $n \in \mathbb{N}$,

$$d(x,y) < \epsilon \Rightarrow d(f_n(x), f_n(y)) \geq \lambda \cdot d(x,y).$$

By Lemma 6 and Lemma 1, for any $\gamma \in (0, \epsilon)$ and an arbitrary $n \in \mathbb{N}$, we have

$$N\left(X, \frac{\gamma}{\lambda^n}\right) \leq r(n, \gamma), \tag{2}$$

consequently, applying Lemma 8 for the first equality and (2) for the subsequent inequality, we get

$$\overline{\dim_B(X)} = \limsup_{n\to\infty} \frac{\log N(X, \frac{\gamma}{\lambda^n})}{-\log \frac{\gamma}{\lambda^n}} \leq \limsup_{n\to\infty} \frac{\log r(n,\gamma)}{-\log \frac{r}{\lambda^n}} =$$

$$\frac{1}{\log \lambda} \cdot \limsup_{n\to\infty} \frac{\log r(n,\gamma)}{n}.$$

Therefore,

$$h_{top}(f_\infty) = \lim_{\gamma\to 0} \limsup_{n\to\infty} \frac{\log r(n,\gamma)}{n} \geq (\log \lambda) \cdot \overline{\dim_B(X)}.$$

According to the Lemma 7, we finally get

$$h_{top}(f_\infty) \geq (\log \lambda) \cdot \overline{\dim_B(X)} \geq (\log \lambda) \cdot HD(X).$$

□

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