Analytic Expression of Quantum Discords in Werner States under LQCC

Chuanmei Xie [1], Zhanjun Zhang [2],*, Jianlan Chen [1] and Xiaofeng Yin [1]

[1] School of Physics & Material Science, Anhui University, Hefei 230039, China; cmxie@ahu.edu.cn (C.X.); jianlanchen@ahu.edu.cn (J.C.); xiaofengyin@ahu.edu.cn (X.Y.)

[2] School of Information and Electronic Engineering, Zhejiang Gongshang University, Hangzhou 310018, China

* Correspondence: zhangzhanjun@zjgsu.edu.cn

Abstract: In this paper, quantum discords in a special kind of states, i.e., Werner states by local quantum operations and classical communication (LQCC) protocols (WLQCC states), are studied. Nineteen parameters to quantify the quantum discords are reduced to four parameters in terms of properties of Werner states and quantum discord. In the case of orthogonal projective measures, analytic expression of quantum discords in WLQCC states is analytically worked out. Some properties of the quantum discord in the WLQCC states are obtained, especially the variation relations between the quantum discords and the parameters characterizing the WLQCC states. By virtue of numerical computations, quantum discords in a Werner state before and after LQCC protocols are compared. It is found that quantum discord in any WLQCC state cannot exceed that in the original Werner state.

1. Introduction

In quantum information processing, particularly in the theoretical and experimental investigation of quantum correlations, the local quantum operations, and classical communication (LQCC) protocols play an essential role. The importance of LQCC in quantum information processing has long been recognized, playing a key role in teleportation [1], entanglement distillation [2,3], one-way [4] and distributed [5] quantum computing, local cloning [6], quantum secret sharing [7,8], and beyond. While many important results have been obtained concerning LQCC [9–17], it has, nonetheless, proven difficult to characterize in simple terms.

Up to now, the form of quantum correlation related to LQCC operations is focused on quantum entanglement. A distinct feature has been found that quantum entanglement cannot be increased through the LQCC channel. It is well known that quantum correlations can be classified into two forms. One form is quantum entanglement, another is quantum correlation different from quantum entanglement (QCDE). The first kind of QCDE, namely quantum discord (QD), was first revealed by Ollivier and Zurek [18] in 2001. Distinctly, it exists even in some separable states, where quantum entanglement does not emerge obviously. This distinct feature attracted many researchers further investigations. Later, other kinds of QCDEs were proposed [19–31] and their role in quantum information theory is standing out gradually. Moreover,

the potential applications of these discovered QCDEs in quantum information processing are gradually exploited, such as quantum computation [32,33], state merging [34], remote state preparation [35], assisted state discrimination [36], quantum correlation swapping [37–39], etc. By far, it has been well admitted that QCDE is another form of important quantum resources. Hence, it is interesting to study the behavior of QCDEs in quantum states through LQCC operations. Particularly, similar to the case of quantum entanglement, we wonder whether LQCC can increase QCDEs in quantum states or not.

In this paper, we will study the influences of LQCC on QD in a given quantum state. Considering the difficulties in characterizing the LQCC operation and quantifying QD in a quantum state, we will originally study a comparatively simple case. To be concrete, we will study whether QDs involved in two-qubit Werner states can be increased by the single-state LQCC protocol or not. As for this specific topic, our main motivations are fourfold: (1) study the QD in the concerned states through LQCC operations quantitatively; (2) discover whether LQCC can increase QD or not similar to the quantum entanglement case; (3) if yes or no, seek out the similarities and differences among them qualitatively; (4) try to expose the quantitative features of QDs in the LQCC process. It is worth stressing that so far quantum discord in a generic state is extremely difficult to evaluate. Because of the definition of discord requires the optimization to be executed over all measurements on one subsystem. Until now, the explicit analytical results on QD are known only for a few classes of quantum states [40–44].

The rest of this paper is outlined as follows. In Section 2, WLQCC states are originally introduced and pointedly simplified. In Section 3, the analytical expression of QD in the WLQCC states is derived. In Section 4, some analyses and discussions about the QDs are made and some distinct features are revealed. Finally, a concise summary is given in Section 5.

2. WLQCC State

Usually, a two-qubit Werner state is written as [45]

$$\rho_{ab}^W = \frac{1-F}{3}\mathbb{I} + \frac{4F-1}{3}|\Psi^-\rangle\langle\Psi^-|, \tag{1}$$

where $\mathbb{I}$ denotes the 4×4 identity matrix and $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state. The Werner state ρ_{ab}^W is characterized by a single real parameter F called fidelity with $1/4 \leq F \leq$. The single-state LQCC protocol maps a Werner state ρ_{ab}^W to WLQCC state ϱ_{ab} in the form of [46]

$$\varrho_{ab} = \frac{A \otimes B \rho_{ab}^W A^\dagger \otimes B^\dagger}{\text{Tr}(A \otimes B \rho_{ab}^W A^\dagger \otimes B^\dagger)}, \tag{2}$$

where A and B are operators act on subsystems a and b, respectively. The success probability of this protocol is $\text{Tr}(A \otimes B \rho_{ab}^W A^\dagger \otimes B^\dagger)$. The only restrictions for operators A and B are $A^\dagger A \leq I_2$ and $B^\dagger B \leq I_2$. They can be written as $A = U_a f_a U_a'$ and $B = V_b f_b V_b'$, where U_a, U_a', V_b and V_b' are arbitrary unitary operators, f_a and f_b are filtering operators taking the forms of $f_a = \sum_{i=0}^1 \alpha_a^i |i\rangle\langle i|$ and $f_b = \sum_{j=0}^1 \alpha_b^j |j\rangle\langle j|$, respectively. Considering physical realizations of filtering operators f_a and f_b, their eigenvalues must be between zero and one, i.e., $0 \leq \alpha_a^i, \alpha_b^j \leq 1$. Thus any nontrivial LQCC (i.e., any LQCC which is not the zero map) for two-qubit states can be written in the form of [46]

$$A \otimes B = \gamma U_a f_a' U_a' \otimes V_b f_b' V_b', \tag{3}$$

where γ is a scale factor in the range $0 < \gamma \leq 1$, $f'_a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha_a \end{smallmatrix}\right)$ and $f'_b = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha_b \end{smallmatrix}\right)$ with $0 \leq \alpha_a, \alpha_b \leq 1$. Actually, the scale factor γ can be ignored for that it will vanish in the normalizing process. In spite of the trivial parameter γ, there are still nineteen parameters required to determine a WLQCC state.

According to [47], there exists an equivalence relation $V'_b \rho^W_{ab} V'^\dagger_b = V^T_a \rho^W_{ab} V^*_a$ by virtue of the inherent permutation symmetry of two-qubit Werner states ρ^W_{ab}. Making use of this property, one can further reexpress the state ϱ_{ab} as

$$\varrho_{ab} = \frac{U_a V_b \rho^k_{ab} V^\dagger_b U^\dagger_a}{\text{Tr}(U_a V_b \rho^k_{ab} V^\dagger_b U^\dagger_a)} = U_a V_b \varrho^k_{ab} V^\dagger_b U^\dagger_a, \tag{4}$$

where

$$\varrho^k_{ab} = f'_a W_a \otimes f'_b \rho^W_{ab} W^\dagger_a f'_a \otimes f'_b \tag{5}$$

with $W_a = U'_a V'^T_a$. Obviously, W_a can be seen as an arbitrary unitary operator on subsystem a in the matrix form of $\left(\begin{smallmatrix} \sqrt{\zeta}e^{i(\theta+\mu)} & \sqrt{1-\zeta}e^{i\nu} \\ -\sqrt{1-\zeta}e^{i(\theta-\nu)} & \sqrt{\zeta}e^{-i\mu} \end{smallmatrix}\right)$, here ζ is in the range $0 \leq \zeta \leq 1$ and $0 \leq \mu, \nu, \theta \leq 2\pi$. After substituting the matrix forms of ρ^W_{ab}, W_a, and $W^\dagger_a$ into Equation (5), one can achieve the matrix form of ϱ^k_{ab} in computational bases:

$$\varrho^k_{ab}$$
$$= \frac{(4F-1)}{6} \begin{pmatrix} (1-\zeta)+\frac{2(1-F)}{(4F-1)} & -\alpha_b\sqrt{\zeta(1-\zeta)}e^{-i(\mu-\nu+\theta)} & \alpha_a\sqrt{\zeta(1-\zeta)}e^{i(\mu+\nu)} & \alpha_a\alpha_b(1-\zeta)e^{i(2\nu-\theta)} \\ -\alpha_b\sqrt{\zeta(1-\zeta)}e^{i(\mu-\nu+\theta)} & \alpha^2_b\zeta+\frac{2(1-F)}{(4F-1)}\alpha^2_b & -\alpha_a\alpha_b\zeta e^{i(2\mu+\theta)} & -\alpha_a\alpha^2_b\sqrt{\zeta(1-\zeta)}e^{i(\mu+\nu)} \\ \alpha_a\sqrt{\zeta(1-\zeta)}e^{-i(\mu+\nu)} & -\alpha_a\alpha_b\zeta e^{-i(2\mu+\theta)} & \alpha^2_a\zeta+\frac{2(1-F)}{(4F-1)}\alpha^2_a & \alpha^2_a\alpha_b\sqrt{\zeta(1-\zeta)}e^{-i(\mu-\nu+\theta)} \\ \alpha_a\alpha_b(1-\zeta)e^{-i(2\nu-\theta)} & -\alpha_a\alpha^2_b\sqrt{\zeta(1-\zeta)}e^{-i(\mu+\nu)} & \alpha^2_a\alpha_b\sqrt{\zeta(1-\zeta)}e^{i(\mu-\nu+\theta)} & \alpha^2_a\alpha^2_b(1-\zeta)+\frac{2(1-F)}{(4F-1)}\alpha^2_a\alpha^2_b \end{pmatrix}. \tag{6}$$

Furthermore, after the normalizing process, one can get

$$\varrho^k_{ab} = \frac{\xi_{ab}}{\Omega}, \tag{7}$$

where $\xi_{ab} = 6\varrho^k_{ab}$ and

$$\Omega = \text{Tr}\xi_{ab} = 3(\alpha^2_a + \alpha^2_b) + (1 - \alpha^2_a)(1 - \alpha^2_b)[2(1 - F) + (4F - 1)(1 - \zeta)]. \tag{8}$$

Obviously, one can find that the number of parameters required to determine the state ϱ^k_{ab} is seven, i.e., $F, \alpha_a, \alpha_b, \zeta, \mu, \nu$, and θ.

3. Quantum Discord in the WLQCC State

Quantum discord (QD) is a kind of quantum correlation quantifier proposed in 2001 [18]. In terms of definition, it is easy to prove that QD is invariant under local unitary operations on host qubits. Using this property, according to Equation (4), one can get

$$Q(\varrho_{ab}) = Q(\varrho^k_{ab}), \tag{9}$$

Hence, to calculate QD one can directly consider the kernel state ϱ^k_{ab} instead of the state ϱ_{ab}.

QD is defined as the discrepancy between two kinds of mutual information expressions which are identical in classical information theory. One kind of mutual information is generalized directly from classical mutual information. It is defined as $\mathcal{I}(\rho_{ab}) = S(\rho_a) + S(\rho_b) - S(\rho_{ab})$, where $S(\cdot)$ denotes the von Neumann entropy and $\rho_{a(b)}$ is a marginal state of ρ_{ab}. This kind of mutual information is called quantum mutual information, which is utilized to characterize and quantify the total correlation in state ρ_{ab}. For a WLQCC state ϱ_{ab} in Equation (7), its total correlation can be obtained as

$$
\begin{aligned}
\mathcal{I}(\varrho_{ab}^k) &= S(\varrho_a^k) + S(\varrho_b^k) - S(\varrho_{ab}^k) \\
&= 2 - \big[(1 - \sqrt{l_a})\log_2(1 - \sqrt{l_a}) + (1 + \sqrt{l_a})\log_2(1 + \sqrt{l_a}) \\
&\quad + (1 - \sqrt{l_b})\log_2(1 - \sqrt{l_b}) + (1 + \sqrt{l_b})\log_2(1 + \sqrt{l_b})\big]/2 \\
&\quad + \log_2 \Omega - \frac{1}{\Omega}\sum_{n=1}^{4} r_n \log_2 r_n,
\end{aligned}
\tag{10}
$$

where

$$
l_{a(b)} = 1 - \big[(4F - 1)^2 \alpha_{a(b)}^2 + 2(1 - F)(2F + 1)(1 + \alpha_{a(b)}^2)^2\big]\frac{4\alpha_{b(a)}^2}{\Omega^2},
\tag{11}
$$

$$
\begin{cases}
r_1 = \big[\Omega - 2s_1 - \sqrt{4s_2 - (\Omega^3 - 4c_1\Omega + 8c_2)/s_1}\big]/4, \\
r_2 = \big[\Omega + 2s_1 - \sqrt{4s_2 + (\Omega^3 - 4c_1\Omega + 8c_2)/s_1}\big]/4, \\
r_3 = \Omega/2 - s_1 - r_1, \\
r_4 = \Omega - r_1 - r_2 - r_3,
\end{cases}
\tag{12}
$$

and

$$
\begin{cases}
s_1 &= \sqrt{\Omega^2/4 + \{s_4/s_3 + s_3 - 2c_1\}/3}, \\
s_2 &= \Omega^2/2 - \{s_4/s_3 + s_3 + 4c_1\}/3, \\
s_3 &= \big[(s_5 + \sqrt{s_5^2 - 4s_4^3})/2\big]^{\frac{1}{3}}, \\
s_4 &= c_1^2 + 12c_3 - 3c_2\Omega, \\
s_5 &= 27(c_2^2 + \Omega^2 c_3) + c_1(2c_1^2 - 72c_3 - 9c_2\Omega),
\end{cases}
\tag{13}
$$

$$
\begin{cases}
c_1 &= 2(1 - F)(2F + 1)[(\alpha_a^2 + \alpha_b^2)(\alpha_a^2\alpha_b^2 + 1) + 2\alpha_a^2\alpha_b^2], \\
c_2 &= 4(1 - F)^2\alpha_a^2\alpha_b^2\{6F(1 + \alpha_a^2)(1 + \alpha_b^2) \\
&\quad - (4F - 1)[(1 + \alpha_a^2\alpha_b^2)(1 - \zeta) + (\alpha_a^2 + \alpha_b^2)\zeta]\}, \\
c_3 &= 48F(1 - F)^3\alpha_a^4\alpha_b^4.
\end{cases}
\tag{14}
$$

Actually, r_ks are the solutions of the quartic equation $r^4 - \Omega r^3 + c_1 r^2 - c_2 r + c_3 = 0$ with r unknown. Three special cases should be clarified here: (1) when $F = 1$, $r_1 = \Omega$ and $r_2 = r_3 = r_4 = 0$; (2) when $\alpha_a = 0$ and $\alpha_b = 0$, $r_1 = \Omega$ and $r_2 = r_3 = r_4 = 0$; (3) when $\alpha_a = 1$ and $\alpha_b = 1$, Equation (11) is not suitable anymore, and this case will be discussed in the analyses part of this paper. Easily, one can see that $\mathcal{I}(\varrho_{ab})$ is a function of four parameters F, α_a, α_b and ζ.

Another kind of the mutual information is generalized from the classical analogy which is related to conditional entropy. It is defined as the maximal classical correlation $\mathcal{C}_a(\rho_{ab}) = S(\rho_b) - \min_{\{\hat{E}_a^i\}} \sum_i p_a^i S(\rho_{b|i})$ by optimizing over all possible measurements on part a (or b). Here $\rho_{b|i} = \mathrm{Tr}_a(\hat{E}_a^i \otimes I_b \rho_{ab})/\mathrm{Tr}_{ab}(\hat{E}_a^i \otimes I_b \rho_{ab})$ is the state of part b conditioned on outcome i in part a with probability $p_a^i = \mathrm{Tr}_{ab}(\hat{E}_a^i \otimes I_b \rho_{ab})$,

where $\{\hat{E}^i\}$ represents the set of positive-operator-valued measure (POVM) elements such that $\sum_i \hat{E}^i = I$. To get access to the maximal classical correlation, one must consider all possible POVM for optimization. As a matter of fact, the optimization is quite difficult for a generic state. In this study, von Neumann measurements are employed as an approximation. The measure element performed on subsystem a (or b) of the state ρ_{ab} can be generally expressed as $\hat{E}^i = \hat{\Gamma}\hat{P}^i\hat{\Gamma}^\dagger$, $i = 0, 1$, where $\hat{P}^i = |i\rangle\langle i|$ is the projector for the subsystem a (or b) along the computational bases $|i\rangle$ and $\hat{\Gamma} \in SU(2)$ is a unitary operator. $\hat{\Gamma} = \sqrt{\tau}e^{i\alpha}|0\rangle\langle 0| + \sqrt{1-\tau}e^{i\beta}|0\rangle\langle 1| - \sqrt{1-\tau}e^{-i\beta}|1\rangle\langle 0| + \sqrt{\tau}e^{-i\alpha}|1\rangle\langle 1|$ with parameters τ in the range $0 \leq \tau \leq 1$ and $0 \leq \alpha, \beta \leq \pi$. Note that classical correlation is generally not a symmetric quantity, i.e., $\mathcal{C}_a(\rho_{ab}) \neq \mathcal{C}_b(\rho_{ab})$. $\mathcal{C}_a(\rho_{ab})$ is usually referred to as the "left" classical correlation, while $\mathcal{C}_b(\rho_{ab})$ the 'right' classical correlation.

For our concerned state ϱ_{ab}^k, it's "left" classical correlation is a function of nine parameters $F, \alpha_a, \alpha_b, \zeta, \mu, \nu, \tau, \alpha,$ and β before optimizing processes:

$$\begin{aligned}
\mathcal{C}_a(\varrho_{ab}^k) &= S(\varrho_b^k) - \min_{\{\hat{E}_a^i\}} \sum_{i=0}^{1} p_a^i S(\varrho_{b|i}^k) \\
&= 1 - \frac{1}{2}(1 - \sqrt{l_a})\log_2(1 - \sqrt{l_a}) - \frac{1}{2}(1 + \sqrt{l_a})\log_2(1 + \sqrt{l_a}) \\
&\quad - \min_{\{\tau,\alpha,\beta\}} \sum_{i=0}^{1} p_a^i \Big[1 - \frac{1}{2}(1 - \sqrt{1 - \kappa_a^i})\log_2(1 - \sqrt{1 - \kappa_a^i}) \\
&\quad - \frac{1}{2}(1 + \sqrt{1 - \kappa_a^i})\log_2(1 + \sqrt{1 - \kappa_a^i}) \Big].
\end{aligned} \tag{15}$$

where

$$\begin{cases}
p_a^0 &= \{3[\alpha_b^2\tau + \alpha_a^2(1-\tau)] + (1 - \alpha_b^2)[\tau(1 + \alpha_a^2) - \alpha_a^2][2(1-F) + (4F-1)(1-\zeta)] \\
&\quad - 2\alpha_a(4F-1)(1 - \alpha_b^2)\cos(\mu + \nu - \alpha - \beta)\sqrt{\zeta\tau(1-\zeta)(1-\tau)}\}/\Omega, \\
p_a^1 &= 1 - p_a^0, \\
\kappa_a^0 &= 8\alpha_b^2(1-F)(2F+1)[\alpha_a^2 + \tau(1 - \alpha_a^2)]^2/(p_a^0\Omega)^2, \\
\kappa_a^1 &= 8\alpha_b^2(1-F)(2F+1)[1 - \tau(1 - \alpha_a^2)]^2/(p_a^1\Omega)^2.
\end{cases} \tag{16}$$

To get the value of "left" classical correlation for a given Werner derivative ($F, \alpha_a, \alpha_b, \zeta, \mu, \nu,$ and θ are given), one has to work out the extreme points of $\sum_{i=0}^{1} p_a^i S(\varrho_{b|i})$ at first by solving the following three partial derivative equations:

$$\frac{\partial \sum_{i=0}^{1} p_a^i S(\varrho_{b|i})}{\partial \tau} = 0, \quad \frac{\partial \sum_{i=0}^{1} p_a^i S(\varrho_{b|i})}{\partial \alpha} = 0, \quad \frac{\partial \sum_{i=0}^{1} p_a^i S(\varrho_{b|i})}{\partial \beta} = 0. \tag{17}$$

After inspecting the form of $\sum_{i=0}^{1} p_a^i S(\varrho_{b|i})$, we find that parameters α and β induced by measurement are only present in p_a^0 in the form of $\cos(\mu + \nu - \alpha - \beta)$. To simplify partial derivative procedures, we regard the function $\cos(\mu + \nu - \alpha - \beta)$ as a new parameter X in the scope of $[-1, 1]$. Thus the latter two equations in Equation (14) are transformed into the following two judgments:

$$\frac{\partial \sum_{i=0}^{1} p_a^i S(\varrho_{b|i})}{\partial X} = 0 \text{ or } \frac{\partial X}{\partial \alpha} = \frac{\partial X}{\partial \beta} = \sin(\mu + \nu - \alpha - \beta) = 0. \tag{18}$$

Actually, it is still quite difficult to directly solve them. Fortunately, we find that the partial derivative equations hold if $\tau = 0.5$ and $X = 0$. This implies that the point $(\tau, X) = (0.5, 0)$ is an extreme point.

Compare values of this extreme point with other special points, one easily finds that the former value is smaller than the latter ones. This means that this extreme point is at least a local minimal point. We conjecture that it is actually the global minimal point. To verify this conjecture, we have further investigated this feature via numerical calculations and conclusively confirmed that it is exactly the global minimal point. This global minimal point means that to achieve the "left" classical correlation for each state ϱ_{ab}, the optimal set of measurement $\{\hat{E}_a^i\}$ is determined by parameters $\tau = 0.5$ and α, β, where α, β are related to μ, ν by the equation $\cos(\mu + \nu - \alpha - \beta) = 0$.

After substituting $\tau = 0.5$ and $X = 0$ into Equation (12), we achieve the analytic expression for "left" classical correlation in state ϱ_{ab}:

$$
\begin{aligned}
\mathcal{C}_a(\varrho_{ab}^k) &= -\frac{1}{2}(1 - \sqrt{l_a})\log_2(1 - \sqrt{l_a}) - \frac{1}{2}(1 + \sqrt{l_a})\log_2(1 + \sqrt{l_a}) \\
&\quad + \frac{1}{2}(1 - \Delta_a)\log_2(1 - \Delta_a) + \frac{1}{2}(1 + \Delta_a)\log_2(1 + \Delta_a),
\end{aligned}
\tag{19}
$$

where $\Delta_a = \sqrt{l_a + 4\alpha_a^2\alpha_b^2(4F - 1)^2/\Omega^2}$.

Now let us turn to the "right" classical correlation of ϱ_{ab}, it is a function of $F, \alpha_a, \alpha_b, \zeta, \mu, \nu,$ and θ after optimized on parameters $\tau, \alpha,$ and β which induced by the set of measurement elements $\{\hat{E}_b^i\}$:

$$
\begin{aligned}
\mathcal{C}_b(\varrho_{ab}^k) &= S(\varrho_a^k) - \min_{\{\hat{E}_b^i\}} \sum_{i=0}^{1} p_b^i S(\varrho_{a|i}^k) \\
&= 1 - \frac{1}{2}(1 - \sqrt{l_b})\log_2(1 - \sqrt{l_b}) - \frac{1}{2}(1 + \sqrt{l_b})\log_2(1 + \sqrt{l_b}) \\
&\quad - \min_{\{\tau,\alpha,\beta\}} \sum_{i=0}^{1} p_b^i \left[1 - \frac{1}{2}(1 - \sqrt{1 - \kappa_b^i})\log_2(1 - \sqrt{1 - \kappa_b^i}) \right. \\
&\quad \left. - \frac{1}{2}(1 + \sqrt{1 - \kappa_b^i})\log_2(1 + \sqrt{1 - \kappa_b^i}) \right],
\end{aligned}
\tag{20}
$$

where

$$
\left\{
\begin{aligned}
p_b^0 &= \{3[\alpha_a^2\tau + \alpha_b^2(1 - \tau)] + (1 - \alpha_a^2)[\tau(1 + \alpha_b^2) - \alpha_b^2][2(1 - F) + (4F - 1)(1 - \zeta)] \\
&\quad + 2\alpha_b(4F - 1)(1 - \alpha_a^2)\cos(\mu - \nu + \alpha + \beta + \theta)\sqrt{\zeta\tau(1 - \zeta)(1 - \tau)}\}/\Omega, \\
p_b^1 &= 1 - p_b^0, \\
\kappa_b^0 &= 8\alpha_a^2(1 - F)(2F + 1)[\alpha_b^2 + \tau(1 - \alpha_b^2)]^2/(p_b^0\Omega)^2, \\
\kappa_b^1 &= 8\alpha_a^2(1 - F)(2F + 1)[1 - \tau(1 - \alpha_b^2)]^2/(p_b^1\Omega)^2.
\end{aligned}
\right.
\tag{21}
$$

To get the value of "right" classical correlation for a given Werner derivative, one has to work out the extreme points at first by solving the following three partial derivative equations:

$$
\frac{\partial \sum_{i=0}^{1} p_b^i S(\varrho_{a|i}^k)}{\partial \tau} = 0, \quad \frac{\partial \sum_{i=0}^{1} p_b^i S(\varrho_{a|i}^k)}{\partial \alpha} = 0, \quad \frac{\partial \sum_{i=0}^{1} p_b^i S(\varrho_{a|i}^k)}{\partial \beta} = 0
\tag{22}
$$

Similarly to the solution employed for the optimal procedure in dealing with "left" classical correlation, we regard the function $\cos(\mu - \nu + \alpha + \beta + \theta)$ as a new parameter Y in the scope of $[-1, 1]$. Thus the latter two equations in Equation (19) are transformed into the following two judgments:

$$
\frac{\partial \sum_{i=0}^{1} p_b^i S(\varrho_{a|i}^k)}{\partial Y} = 0 \text{ or } \frac{\partial Y}{\partial \alpha} = \frac{\partial Y}{\partial \beta} = -\sin(\mu - \nu + \alpha + \beta + \theta) = 0.
\tag{23}
$$

With the assistance of numerical calculations, we confirm that the point $(\tau, Y) = (0.5, 0)$ is the optimal point for function $\sum_{i=0}^{1} p_b^i S(\varrho_{a|i}^k)$ for every state ϱ_{ab}^k. The optimal set of measurement elements $\{\hat{E}_b^i\}$ is determined by τ, α, and β, where $\tau = 0.5$ and α, β are determined by the values of μ, ν, and θ.

After substituting values of $\tau = 0.5$ and $Y = 0$ into Equation (17), we achieve the analytic expression for "right" classical correlation in state ϱ_{ab}^k:

$$
\begin{aligned}
\mathcal{C}_b(\varrho_{ab}^k) \;=\; & -\frac{1}{2}(1 - \sqrt{l_b}) \log_2(1 - \sqrt{l_b}) - \frac{1}{2}(1 + \sqrt{l_b}) \log_2(1 + \sqrt{l_b}) \\
& + \frac{1}{2}(1 - \Delta_b) \log_2(1 - \Delta_b) + \frac{1}{2}(1 + \Delta_b) \log_2(1 + \Delta_b),
\end{aligned}
\tag{24}
$$

where $\Delta_b = \sqrt{l_b + 4\alpha_a^2 \alpha_b^2 (4F-1)^2 / \Omega^2}$.

The quantum discord of ϱ_{ab}^k is defined as the discrepancy between the total correlation $\mathcal{I}(\varrho_{ab}^k)$ and the classical correlation $\mathcal{C}_{a(b)}(\varrho_{ab}^k)$, i.e., $\mathcal{D}_{a(b)}(\varrho_{ab}^k) = \mathcal{I}(\varrho_{ab}^k) - \mathcal{C}_{a(b)}(\varrho_{ab}^k)$. With the analytic expression of "left" and "right" classical correlations, we can achieve the analytical expression for "left" and "right" QD in state ϱ_{ab}:

$$
\begin{aligned}
Q_{a(b)}(\varrho_{ab}^k) \;=\; & -\frac{1}{2}(1 - \sqrt{l_{a(b)}}) \log_2(1 - \sqrt{l_{a(b)}}) - \frac{1}{2}(1 + \sqrt{l_{a(b)}}) \log_2(1 + \sqrt{l_{a(b)}}) \\
& -\frac{1}{2}(1 - \Delta_{a(b)}) \log_2(1 - \Delta_{a(b)}) - \frac{1}{2}(1 + \Delta_{a(b)}) \log_2(1 + \Delta_{a(b)}) \\
& + 2 - \log_2 \Omega + \frac{1}{\Omega} \sum_{k=1}^{4} r_k \log_2 r_k.
\end{aligned}
\tag{25}
$$

4. Analyses and Discussions

In the previous section, we have obtained the analytic expressions of the "left" and "right" QDs in the WLQCC states. According to the analytical expression in Equation (25), in this section, we will make some analyses and discussions on these obtained QDs.

Firstly we want to emphasize that, we have developed and used some skills to obtain the analytic expressions of the QD, as is a distinct feature of our present study. In this paper, the concerned state is the WLQCC state ϱ_{ab} (see in Equation (2)). Due to the complicated form of LQCC protocol, the number of parameters needed to define the state ϱ_{ab} is great. The corresponding number of independent parameters is 19. Hence, it is very difficult to consider the state ϱ_{ab} directly. Fortunately, after some analyses, we find one can use some skills to degrade this difficulty. To be concrete, noticing that our main purpose in this paper is to calculate the QD in the WLQCC state ϱ_{ab}. Moreover, ϱ_{ab} is related to ϱ_{ab}^k (see in Equation (5)) with local unitary transformations and QD is invariant under the transformations. Hence, in calculating QD in ϱ_{ab} one can consider the kernel state ϱ_{ab}^k instead. The parameters needed in ϱ_{ab}^k is 7. Hence, the quantity and complexity in calculating QD are reduced to a great extent. See Table 1.

Table 1. The number of parameters of Werner states by local quantum operations and classical communication protocol (WLQCC) states, kernel states, and their quantum discord. N denotes the number of parameters.

	ϱ_{ab} (Equation (2))	ϱ_{ab}^k (Equation (7))	$Q(\varrho_{ab}) = Q(\varrho_{ab}^k)$
N	19	7	8

Furthermore, through complicated calculation and derivation, one can find that the final number of the independent parameters defining QD in the concerned state ϱ_{ab}^k is 4. That is to say, the final analytical expression obtained from the QD of the WLQCC states is only a function of 4 independent parameters, i.e., F, α_a, α_b, and ζ. Using the above skills and derivations, the difficulty is evidently degraded and the analytic expression is obtained.

The second, let us move to investigate the features in the obtained QD. According to the symmetry of the 'left' quantum discord and the "right" one, in the following, we will make some analyses about the variation of the "left" quantum discord $\mathcal{D}_a$ with F, α_a, α_b, and ζ. Through analyses, one can get the following distinct features.

(i) $\mathcal{D}_a$ is an increasing function of F for any given set of $(\alpha_a, \alpha_b, \zeta)$ and reaches its maxima at $F = 1$ (See Figure 1). This phenomenon can be understood as follows. Inspect the original Werner state in Equation (1) and the WLQCC state in Equation (2). Originally, state ρ_{ab}^W consists of two terms, which are mixed with the weights characterized by F. The former represented by I is the maximally mixed state with no quantum correlation. The latter is the maximally entangled state. In this case, a bigger F corresponds to a larger weight of the latter term and accordingly a larger quantum correlation is induced by the mixture of the two terms. As for the WLQCC state ϱ_{ab} which are mapped from the original state ρ_{ab}^W, quantum correlation in it still increases with the parameter F. This means that the WLQCC operation do not change the dependency relation about QC with F. However, according to [40], one can find that for a given F the quantity of QC in ϱ_{ab} is less than that in ρ_{ab}^W. This means that the nonzero set of $(\alpha_a, \alpha_b, \zeta)$ can decrease the quantum discord for a given F. That is to say, the LQCC protocol cannot increase the quantum discord in the Werner state ρ_{ab}^W.
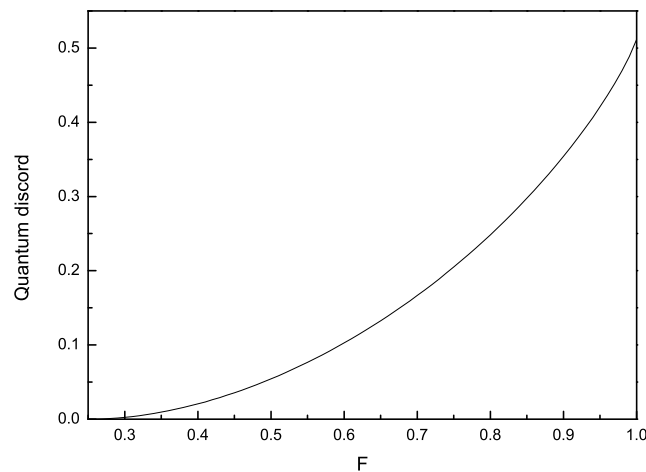

Figure 1. The variation of quantum discord Q_a with F for α_a = 0.3, α_b = 0.6, and ζ = 0.8.

(ii) For given $\zeta \in [0, 0.5]$, F, and α_b, $\mathcal{D}_a$ monotonically increases with α_a and reaches its maxima at $\alpha_a = 1$ (See Figure 2). However, for given $\zeta \in [0.5, 1.0]$, F, and α_b, $\mathcal{D}_a$ first increases then decreases with α_a, and the transition point is determined by the exact value of ζ. As for the case of α_b, one can find that the dependent relation of $\mathcal{D}_a$ on α_b for given ζ, F and α_a is similar to that on α_a. This feature is related to the LQCC operation and its underlying physics can be explained as follows. The LQCC operation is concerned with two parties, Alice and Bob, who each control one subsystem, and who are restricted to carrying out local quantum operations and classical communication (i.e., LQCC). Specifically, the quantum operations Alice and Bob are allowed to perform are local unitary transformations and local filtrations. As analyzed in the second section in this paper, to calculate the quantum discord in the WLQCC state ϱ_{ab}

in Equation (2), we have considered the kernel state ϱ_{ab}^k in Equation (5) instead. From Equation (5), one can see that α_a is the parameter characterizing the filtration operator f_a', α_b is the one for f_b', and ζ is the one for the unitary operator W_a.
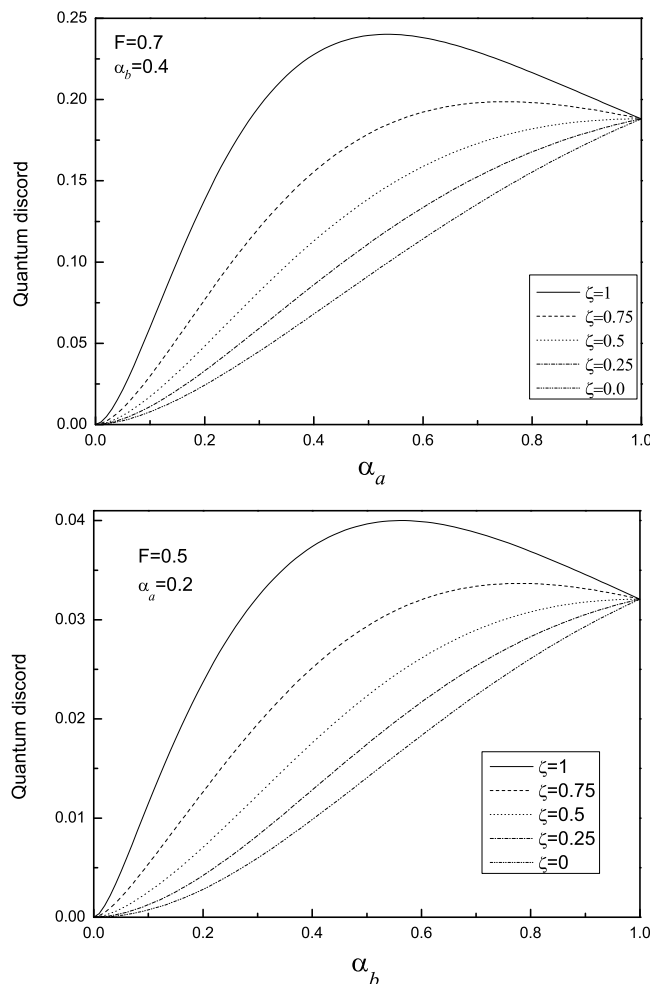

Figure 2. The variation of quantum discord Q_a with α_a or α_b.

(iii) Through analysis, one can find that quantum discord $\mathcal{D}_a$ is a monotonically increasing function of ζ for given α_a, α_b, and F. See Figure 3 for example. ζ is the parameter used to characterize the unitary operation W_a. In addition, according to the expression of quantum discords in WLQCC states, the parameter ζ vanishes when $\alpha_a = 1$ or $\alpha_b = 1$. Especially, in the situation of $\alpha_a = \alpha_b = 1$, LQCC is turned into local unitary operation, which does not change the quantum discords in Werner states. When $\alpha_a = 0$ or $\alpha_b = 0$, the total correlations of WLQCC states are zero. The special case $\alpha_a = \alpha_b = 0$ is meaningless for the matrix form of WLQCC states is zero.

Finally, we want to stress that, although the result, i.e., LQCC cannot increase the QD (similar to the behavior of quantum entanglement), has been obtained, it is still an open question whether this conclusion is applicable for other states or in the framework of other QCDEs. In fact, it is easy to verify that this conclusion is not general. In other words, for other general states and for other quantum correlation measures, the property of LQCC cannot increase and the quantum correlation may not exist at

all. Nonetheless, for other states with high inherent property of symmetry, such as the higher-dimensional (qutrit) Werner states, we conjecture that the coincidence may remain. We will pay attention to it in the near future.
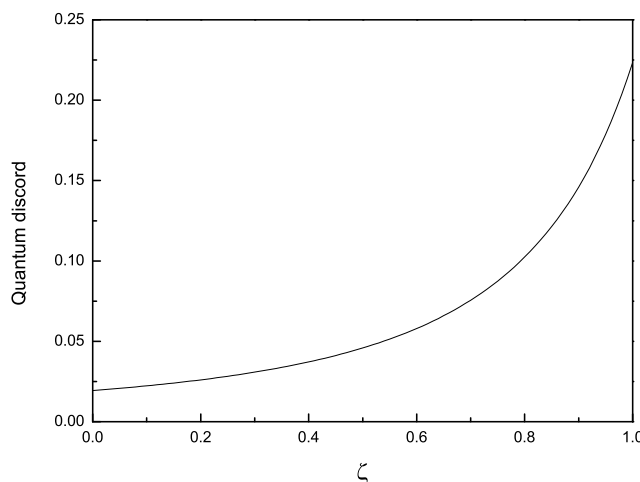

Figure 3. The variation of quantum discord $\mathcal{D}_a$ with ζ for given F = 0.8, α_a = 0.2, α_b = 0.3.

5. Summary

To summarize, in this paper, we have derived the analytical expression of QDs in WLOCC states. With the assistance of numerical computations, we find that the QD in a WLQCC state cannot exceed that of the original Werner state. The research in [PRL 81, 3279] exhibits a similar result in the scenario of quantum entanglement: QDs in two-qubit Werner states cannot be increased by single-state LQCC protocols.

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