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Dual Loomis-Whitney Inequalities via Information Theory

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Abstract: We establish lower bounds on the volume and the surface area of a geometric body using the size of its slices along different directions. In the first part of the paper, we derive volume bounds for convex bodies using generalized subadditivity properties of entropy combined with entropy bounds for log-concave random variables. In the second part, we investigate a new notion of Fisher information which we call the L_1 -Fisher information and show that certain superadditivity properties of the L_1 -Fisher information lead to lower bounds for the surface areas of polyconvex sets in terms of its slices.

Keywords: Loomis-Whitney inequality; fisher information; volume; surface area; log-concave distributions

1. Introduction

Tomography concerns reconstructing a probability density by synthesizing data collected along sections (or slices) of that density and is a problem of great significance in applied mathematics. Some popular applications of tomography in the field of medical imaging are computed tomography (CT), magnetic resonance imaging (MRI) and positron emission tomography (PET). In each of these, sectional data is obtained in a non-invasive manner using penetrating waves and images are generated using tomographic reconstruction algorithms. *Geometric tomography* is a term coined by Gardner [1] to describe an area of mathematics that deals with the retrieval of information about a geometric object from data about its sections, projections or both. Gardner notes that the term *geometric* is deliberately vague, since it may be used to study convex sets or polytopes as well as more general shapes such as star-shaped bodies, compact sets or even Borel sets.

An important problem in geometric tomography is estimating the size of a set using lower dimensional sections or projections. Here, projection of a geometric object refers to its shadow or orthogonal projection, as opposed to the marginal of a probability density. As detailed in Campi and Gronchi [2], this problem is relevant in a variety of settings ranging from the microscopic study of biological tissues [3,4], to the study of fluid inclusions in minerals [5,6] and to reconstructing the shapes of celestial bodies [7,8]. Various geometric inequalities provide bounds on the sizes of sets using lower dimensional data pertaining to their projections and slices of sets. The “size” of a set often refers to its volume but it may also refer to other geometric properties such as its surface area or its mean width. A canonical example of an inequality that bounds the volume of set using its orthogonal projections is the Loomis-Whitney inequality [9]. This inequality states that for any Borel measurable set $K \subseteq \mathbb{R}^n$,

$$V_n(K) \leq \left(\prod_{i=1}^n V_{n-1}(P_{e_i^\perp} K) \right)^{\frac{1}{n-1}}. \quad (1)$$

Equality holds in (1) if and only if K is a box with sides parallel to the coordinate axes. The Loomis-Whitney inequality has been generalized and strengthened in a number of ways. Burago and Zalgaller [10] proved a version of (1) that considers projections of K on to all m -dimensional spaces spanned by $\{e_1, \dots, e_n\}$. Bollobas and Thomason [11] proved the Box Theorem which states that for every Borel set $K \subseteq \mathbb{R}^n$, there exists a box B such that $V_n(B) = V_n(K)$ and $V_m(P_S B) \leq V_m(P_S K)$ for every m -dimensional coordinate subspace S . Ball [12] showed that the Loomis-Whitney inequality is closely related to the Brascamp-Lieb inequality [13,14] from functional analysis and generalized it to projections along subspaces that satisfy a certain geometric condition. Inequality (1) also has deep connections to additive combinatorics and information theory. Some of these connections have been explored in Balister and Bollobas [15], Gyarmati et al. [16] and Madiman and Tetali [17].

A number of geometric inequalities also provide upper bounds for the surface area of a set using projections. Naturally, it is necessary to make some assumptions for such results, since one can easily conjure sets that have small projections while having a large surface area. Betke and McMullen [2,18] proved that, for compact convex bodies,

$$V_{n-1}(\partial K) \leq 2 \sum_{i=1}^n V_{n-1}(P_{e_i^\perp} K). \quad (2)$$

Motivated by inequalities (1) and (2), Campi and Gronchi [2] investigated upper bounds for *intrinsic volumes* [19] of compact convex sets.

Inequalities (1) and (2) provide upper bounds and a natural question of interest is developing analogous lower bounds. Lower bounds are obtained via *reverse* Loomis-Whitney inequalities or *dual* Loomis-Whitney inequalities. The former uses projection information whereas the latter uses slice information, often along the coordinate axes. A canonical example of a dual Loomis-Whitney inequality is Meyer's inequality [20], which states that for a compact convex set $K \subseteq \mathbb{R}^n$, the following lower bound holds:

$$V_n(K) \geq \left(\frac{n!}{n^n} \prod_{i=1}^n V_{n-1}(K \cap e_i^\perp) \right)^{\frac{1}{n-1}}, \quad (3)$$

with equality if and only if K is a regular crosspolytope. Generalizations of Meyer's inequality have also been recently obtained in Li and Huang [21] and Liakopoulos [22]. Betke and McMullen [2,18] established a reverse Loomis-Whitney type inequality for surface areas of compact convex sets:

$$V_{n-1}(\partial K)^2 \geq 4 \sum_{i=1}^n V_{n-1}(P_{e_i^\perp} K)^2. \quad (4)$$

Campi et al. [23] extended inequalities (3) and (4) for intrinsic volumes of certain convex sets.

Our goal in this paper is to develop lower bounds on volumes and surface areas of geometric bodies that are most closely related to dual Loomis-Whitney inequalities; that is, inequalities that use slice-based information. The primary mathematical tools we use are entropy and information inequalities; namely, the Brascamp-Lieb inequality, entropy bounds for log-concave random variables and superadditivity properties of a suitable notion of Fisher information. Using information theoretic tools allows our results to be quite general. For example, our volume bounds rely on maximal slices parallel to a set of subspaces and are valid for very general choices of subspaces. Our surface area bounds are valid for polyconvex sets, which are finite unions of compact convex sets. The drawback of using information theoretic strategies is that the resulting bounds are not always tight; that is, equality may not be achieved by any geometric body. However, we show that in some cases our bounds are asymptotically tight as the dimension n tends to infinity, thus partly mitigating the drawbacks. Our main contributions are as follows:

- *Volume lower bounds:* In Theorem 3, we establish a new lower bound on the volume of a compact convex set in terms of the size of its slices. Just as Ball [12] extended the Loomis-Whitney

inequality to projections in more general subspaces, our inequality also allows for slices parallel to subspaces that are not necessarily e_i^\perp . Another distinguishing feature of this bound is that unlike classical dual Loomis-Whitney inequalities, the lower bound is in terms of *maximal slices*; that is, the largest slice parallel to a given subspace. The key ideas we use are the Brascamp-Lieb inequality and entropy bounds for log-concave random variables.

- **Surface area lower bounds:** Theorem 7 contains our main result that provides lower bounds for surface areas. Unlike the volume bounds, the surface area bounds are valid for the larger class of polyconvex sets, which consists of finite unions of compact, convex sets. Moreover, the surface area lower bound is not simply in terms of the maximal slice; instead, this bound uses all available slices along a particular hyperplane. As in the volume bounds, the slices used may be parallel to general $(n - 1)$ -dimensional subspaces and not just e_i^\perp . The key idea is motivated by a superadditivity property of Fisher information established in Carlen [24]. Instead of classical Fisher information, we develop superadditivity properties for a new notion of Fisher information which we call the L_1 -Fisher information. This superadditivity property when restricted to uniform distributions over convex bodies yields the lower bound in Theorem 7.

In Section 2 we state and prove our volume lower bound and in Section 3 we state and prove our surface area bound. We conclude with some open problems and discussions in Section 4.

Notation: For $n \geq 1$, let $[n]$ denote the set $\{1, 2, \dots, n\}$. For $K \subseteq \mathbb{R}^n$ and any subspace $E \subseteq \mathbb{R}^n$, the orthogonal projection of K on E is denoted by $P_E K$. The standard basis vectors in \mathbb{R}^n are denoted by $\{e_1, e_2, \dots, e_n\}$. We use the notation V_r to denote the volume functional in \mathbb{R}^r . The boundary of K is denoted by ∂K and its surface area is denoted by $V_{n-1}(\partial K)$. For a random variable X taking values in \mathbb{R}^n , the marginal of X along a subspace E is denoted by $P_E X$. In this paper, we shall consider random variables with bounded variances and whose densities lie in the convex set $\{f \mid \int_{\mathbb{R}^n} f(x) \log(1 + f(x)) < \infty\}$. The differential entropy of such random variables is well-defined and is given by

$$h(X) = - \int_{\mathbb{R}^n} p_X(x) \log p_X(x) dx,$$

where $X \sim p_X$ is an \mathbb{R}^n -valued random variable. The Fisher information of a random variable X with a differentiable density p_X is given by

$$I(X) = \int_{\mathbb{R}^n} \|\nabla \log p_X(x)\|_2^2 p_X(x) dx.$$

2. Volume Bounds

The connection between functional/information theoretic inequalities and geometric inequalities is well-known. In particular, the Brascamp-Lieb inequality has found several applications in geometry as detailed in Ball [13]. In the following section we briefly discuss the Brascamp-Lieb inequality and its relation to volume inequalities.

2.1. Background on the Brascamp-Lieb Inequality

We shall use the information theoretic form of the Brascamp-Lieb inequality, as found in Carlen et al. [25]:

Theorem 1. [Brascamp-Lieb inequality] Let X be a random variable taking values in \mathbb{R}^n . Let $E_1, E_2, \dots, E_m \subseteq \mathbb{R}^n$ be subspaces with dimensions r_1, r_2, \dots, r_m respectively and $c_1, c_2, \dots, c_m > 0$ be constants. Define

$$M = \sup_X \left(h(X) - \sum_{j=1}^m c_j h(P_{E_j} X) \right), \tag{5}$$

and

$$M_g = \sup_{X \in \mathcal{G}} \left(h(X) - \sum_{j=1}^m c_j h(P_{E_j} X) \right), \tag{6}$$

where \mathcal{G} is the set of all Gaussian random variables taking values in \mathbb{R}^n . Then $M = M_g$ and M_g (and therefore M) is finite if and only if $\sum_{i=1}^m r_i c_i = n$ and for all subspaces $V \subseteq \mathbb{R}^n$, we have $\dim(V) \leq \sum_{i=1}^m \dim(P_{E_i} V) c_i$.

Throughout this paper, we assume that E_i and c_i are such that $M < \infty$. As detailed in Bennett et al. [14], the Brascamp-Lieb inequality generalizes many popular inequalities such as Holder’s inequality, Young’s convolution inequality and the Loomis-Whitney inequality. In particular, Ball [12] showed that the standard Loomis-Whitney inequality in (1) could be extended to settings where projections are obtained on more general subspaces:

Theorem 2 (Ball [12]). *Let K be a closed and bounded set in \mathbb{R}^n . Let E_i and c_i for $i \in [m]$ and M_g be as in Theorem 1. Let $P_{E_i} K$ be the projection of K on to the subspace E_i , for $i \in [m]$. Let the dimension of E_i be r_i for $i \in [m]$. Then the volume of K may be upper-bounded as follows:*

$$V_n(K) \leq e^{M_g} \prod_{i=1}^m V_{r_i}(P_{E_i} K)^{c_i}. \tag{7}$$

Since we shall be using a similar idea in Section 2, we include a proof for completeness.

Proof. Consider a random variable X that is uniformly distributed on K ; that is $X \sim p_X = \text{Unif}(K)$. Let $P_{E_i} X$ denote the random variable obtained by projecting X on E_i or equivalently the marginal of X in subspace E_i . Naturally, $\text{supp}(P_{E_i} X) \subseteq P_{E_i} K$ and thus

$$h(P_{E_i} X) \leq \log V_{r_i}(P_{E_i} K), \quad \text{for } i \in [m]. \tag{8}$$

Substituting these inequalities in the Brascamp-Lieb inequality for X , we obtain

$$h(X) = \log V_n(K) \leq \sum_{i=1}^m c_i \log V_{r_i}(P_{E_i} K) + M_g. \tag{9}$$

Exponentiating both sides concludes the proof. \square

To show that the Loomis-Whitney inequality is implied by Theorem 2, we set $E_i = e_i^\perp$, $c_i = 1/(n - 1)$ for $i \in [n]$ and use Szasz’s inequality or other tools from linear algebra [26] to show that the supremum below evaluates to 1:

$$e^{M_g} = \sup_{K \geq 0} \frac{\det K}{\prod_{i=1}^n \det K_i^{\frac{1}{n}}}.$$

In general, Ball [12] showed that if the E_i and c_i satisfy what is called John’s condition; that is $\sum_{i=1}^m c_i P_{E_i} x = x$ for all $x \in \mathbb{R}^n$, then $M_g = 0$.

2.2. Volume Bounds Using Slices

Providing lower bounds for volumes in terms of projections requires making additional assumptions on the set K . A simple counterexample is the $(n - 1)$ dimensional sphere (shell), which can have arbitrarily large projections in lower dimensional subspaces but has 0 volume. Even for convex K , providing lower bounds using a finite number of projections fails. For example, given a finite collection of subspaces, we may consider any convex set supported on a random $(n - 1)$ dimensional subspace of \mathbb{R}^n which will have (with high probability) non-zero projections on all subspaces in the

collection. Clearly, such a set has volume 0. Therefore, it makes sense to obtain lower bounds on volumes using slices instead of projections, as in Meyer’s inequality (3).

Given a subspace E_i , the slice parallel to E_i^\perp is not unambiguously defined as it depends on translations of E_i^\perp . For this reason we consider the maximal slice; that is, the largest slice parallel to a given subspace. Note that although Meyer’s inequality (3) is not stated in terms of maximal slices, it remains valid even if the right hand side of inequality (3) is replaced by maximal slices parallel to e_i^\perp . This is because one can always choose the origin of the coordinate system such that the largest slice parallel to e_i^\perp is $K \cap e_i^\perp$. However, when subspaces are in a more general orientation, it is not always possible to select the origin that simultaneously maximizes the slices along all subspaces. Our main result is the following:

Theorem 3. *Let K be a compact convex body in \mathbb{R}^n . For $j \in [m]$, let $E_j \subseteq \mathbb{R}^n$ be subspaces with dimensions r_j and $c_j > 0$ be constants. Let $S_{\max}(j)$ be the largest slice of K by a subspace orthogonal to E_j ; that is,*

$$S_{\max}(j) = \sup_{t \in E_j} V_{n-r_j}(K \cap (E_j^\perp + t)). \tag{10}$$

Then the following inequality holds:

$$V_n(K) \geq \left(\frac{\prod_{j=1}^m S_{\max}(j)^{c_j}}{e^{n+M_g}} \right)^{1/(C-1)}, \tag{11}$$

where $C = \sum_{j=1}^m c_j$ and M_g is the Brascamp-Lieb constant corresponding to $\{E_j, c_j\}_{j \in [m]}$.

Proof. There are two main components in the proof. First, let X be a random variable that is uniformly distributed on K . The Brascamp-Lieb inequality yields the bound

$$h(X) \leq \sum_{j=1}^m c_j h(P_{E_j} X) + M_g.$$

When deriving upper bounds on volume, we employ the upper bound $h(P_{E_i} X) \leq \log V_{r_i}(P_{E_i} K)$. Here, we employ a slightly different strategy. Note that X , being a uniform distribution on a convex set, is a log-concave random variable. Thus, any lower dimensional marginal of X is also log-concave [27]. Furthermore, the entropy of a log-concave random variable is tightly controlled by the maximum value of its density. For a log-concave random variable Z taking values in \mathbb{R}^n and distributed as p_Z , it was shown in Bobkov and Madiman [28] that

$$\frac{1}{n} \log \frac{1}{\|p_Z\|_\infty} \leq \frac{h(Z)}{n} \leq \frac{1}{n} \log \frac{1}{\|p_Z\|_\infty} + 1,$$

where $\|p_Z\|_\infty$ is the largest value of the probability density p_Z . Define $Z_i := P_{E_i} X$. The key point to note is that $\|p_{Z_i}\|_\infty$ is given by the size of the largest slice parallel to E_i^\perp , normalized by $V_n(K)$; that is, $\|p_{Z_i}\|_\infty = \frac{S_{\max}(i)}{V_n(K)}$. Thus, for $i \in [m]$,

$$h(Z_i) \leq r_i + \log \frac{1}{\|p_{Z_i}\|_\infty} = r_i + \log \frac{V_n(K)}{S_{\max}(i)}.$$

Substituting this in the Brascamp-Lieb bound, we obtain

$$\begin{aligned} \log V_n(K) &\leq \sum_{j=1}^m \left(c_j r_j + c_j \log \frac{V_n(K)}{S_{\max}(j)} \right) + M_g \\ &= n + C \log V_n(K) - \sum_{j=1}^m c_j \log S_{\max}(j) + M_g. \end{aligned}$$

Note that $\sum_{j=1}^m c_j > \sum_{j=1}^m c_j (r_j/n) = 1$, and thus we may rearrange and exponentiate to obtain

$$V_n(K) \geq \left(\frac{\prod_{j=1}^m S_{\max}(j)^{c_j}}{e^{n+M_g}} \right)^{\frac{1}{c-1}}.$$

□

It is instructive to compare Meyer’s inequality to the bound obtained using Theorem 3 for the same choice of parameters. Substituting $M_g = 0$, $E_i = e_i$ and $c_i = 1$, Theorem 3 gives the bound

$$V_n(K) \geq \left(\frac{\prod_{j=1}^n S_{\max}(j)}{e^n} \right)^{\frac{1}{n-1}}.$$

To compare with Meyer’s inequality (3), first assume that the origin of the coordinate planes is selected such that the intersection with e_i^\perp corresponds to the maximal slice along e_i^\perp . With such a choice, we may simply compare the constants in the two inequalities. Observe that

$$\left(\frac{n!}{n^n} \right)^{\frac{1}{n-1}} \leq \left(\frac{1}{e^n} \right)^{\frac{1}{n-1}},$$

and thus Meyer’s inequality (3) yields a tighter bound. However, Sterling’s approximation implies that for large enough n the two constants are approximately the same. Thus, Theorem 3 yields an asymptotically tight result.

Note that if the slices are not aligned along the coordinate axes or if the slices are in larger dimensions, then Meyer’s inequality (3) is not applicable but Theorem 3 continues to yield valid inequalities. An important special case is when there are more than n directions along which slices are available. If u_1, u_2, \dots, u_m are unit vectors and constants c_1, c_2, \dots, c_m satisfy John’s condition [12]; that is, $\sum_{j=1}^m c_j P_{u_j}(x) = x$ for all $x \in \mathbb{R}^n$, then Theorem 3 yields the bound

$$V_n(K) \geq \left(\frac{\prod_{j=1}^m S_{\max}(j)^{c_j}}{e^n} \right)^{\frac{1}{n-1}}, \tag{12}$$

where $S_{\max}(j)$ is the size of the largest slice by a hyperplane perpendicular to u_j . Note that if $\{u_j, c_j\}$ satisfy John’s condition, then so do $\{u_j^\perp, c_j/(n-1)\}$. Applying the bound from Theorem 2 in this case yields

$$V_n(K) \leq \prod_{j=1}^m V_{n-1}(P_{u_j^\perp} K)^{c_j/(n-1)},$$

which may be compared with inequality (12) by observing $S_{\max}(j) \leq V_{n-1}(P_{u_j^\perp} K)$.

3. Surface Area Bounds

The information theoretic quantities of entropy and Fisher information are closely connected to the geometric quantities of volume and surface area, respectively. Surface area of $K \subseteq \mathbb{R}^n$ is defined as

$$V_{n-1}(\partial K) = \lim_{\epsilon \rightarrow 0^+} \frac{V_n(K \oplus \epsilon B_n) - V_n(K)}{\epsilon}, \quad (13)$$

where B_n is the Euclidean ball in \mathbb{R}^n with unit radius and \oplus refers to the Minkowski sum. The Fisher information of a random variable X satisfies a similar relation,

$$\frac{I(X)}{2} = \lim_{\epsilon \rightarrow 0} \frac{h(X + \sqrt{\epsilon}Z) - h(Z)}{\epsilon}, \quad (14)$$

where Z is a standard Gaussian random variable that is independent of X . Other well-known connections include the relation between the entropy of a random variable and the volume of its typical set [29,30], isoperimetric inequalities concerning Euclidean balls and Gaussian distributions and the observed similarity between the Brunn-Minkowski inequality and the entropy power inequality [31]. In Section 2, we used subadditivity of entropy as given by the Brascamp-Lieb inequality to develop volume bounds. To develop surface area bounds, it seems natural to use Fisher information inequalities and adapt them to geometric problems. In the following subsection, we discuss relevant Fisher-information inequalities.

3.1. Superadditivity of Fisher Information

The Brascamp-Lieb subadditivity of entropy has a direct analog noted in Reference [25]. We focus on the case when $\{u_j\}$ and constants $\{c_j\}$ for $j \in [m]$ satisfy John's condition. The authors of Reference [25] provide an alternate proof to the Brascamp-Lieb inequality in this case by first showing a superadditive property of Fisher information, which states that

$$I(X) \geq \sum_{j=1}^m c_j I(P_{u_j} X). \quad (15)$$

The Brascamp-Lieb inequality follows by integrating inequality (15) using the following identity that holds for all random variables X taking values in \mathbb{R}^n [32]:

$$h(X) = \frac{n}{2} \log 2\pi e - \int_{t=0}^{\infty} \left(I(X_t) - \frac{n}{1+t} \right) dt, \quad (16)$$

where $X_t = X + \sqrt{t}Z$ for a standard normal random variable Z that is independent of X . In particular, using this formula for inequality (15) yields the geometric Brascamp-Lieb inequality of Ball [33]:

$$h(X) \leq \sum_{j=1}^m c_j h(P_{u_j} X).$$

If $u_i = e_i$ and $c_i = 1$ for $i \in [n]$, then inequality (15) reduces to the standard superadditivity of Fisher information:

$$I(X) \geq \sum_{i=1}^n I(X_i), \quad (17)$$

where $X = (X_1, \dots, X_n)$.

In Section 2, we directly used the entropic Brascamp-Lieb inequality on random variables uniformly distributed over suitable sets $K \subseteq \mathbb{R}^n$. It is tempting to use inequality (15) to derive surface area bounds for geometric bodies. Unfortunately, directly substituting X to be uniform over $K \subseteq \mathbb{R}^n$ in inequality (15) does not lead to any useful bounds. This is because the left hand side, namely

$I(X)$, is $+\infty$ since the density of X is not differentiable. Thus, it is necessary to modify inequality (15) before we can apply it to geometric problems. A classical result concerning superadditivity of Fisher information-like quantities is provided in Carlen [24]:

Theorem 4 (Theorem 2, [24]). *For $p \in [1, \infty)$, let $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function in $L^p(\mathbb{R}^m) \otimes W^{1,p}(\mathbb{R}^n)$. Define the marginal map M as*

$$G(y) = \left(\int_{\mathbb{R}^m} |f(x, y)|^p dx \right)^{1/p}, \tag{18}$$

denoted by $Mf = G$. Then the following inequality holds:

$$\int_{\mathbb{R}^n} |\nabla_y G(y)|^p dy \leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |\nabla_y f(x, y)|^p dx dy. \tag{19}$$

Carlen [24] also established the (weak) differentiability of G and the continuity of M prior to proving Theorem 4, so the derivatives in its statement are well-defined. The notion of Fisher information we wish to use is essentially identical to the case of $p = 1$ in Theorem 4. However, since our goal is to use this result for uniform densities over compact sets, we cannot directly use Theorem 4, since such densities do not satisfy the required assumptions. In particular, the (weak) partial derivatives of conditional densities are defined in terms of Dirac delta distributions and so the densities do not lie in the Sobolev space $W^{1,1}(\mathbb{R}^n)$. To get around this, we redefine the $p = 1$ case as follows:

Definition 1. *Let $X = (X_1, \dots, X_n)$ be a random vector on \mathbb{R}^n and $f_X(\cdot)$ be its density function. For any unit vector $u \in \mathbb{R}^n$, define*

$$I_1(X)_u := \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{|f_X(x) - f_X(x - \epsilon u)|}{\epsilon} dx,$$

given that the limit exists. Define the L_1 -Fisher information of X as

$$I_1(X) := \sum_{i=1}^n I_1(X)_{e_i},$$

given that the right hand side is well-defined. In particular, when X is a real-valued random variable,

$$I_1(X) = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx.$$

Our new definition is motivated by observing that Theorem 4 is essentially a data processing result for ϕ -divergences and specializing it to the total variation divergence yields our definition. To see this, consider real-valued random variables X and Y with a joint density $\tilde{f}(x, y)$. Let the marginal of Y on \mathbb{R} be $\tilde{G}(\cdot)$. For $\epsilon > 0$, consider the perturbed random variable $(X_\epsilon, Y_\epsilon) = (X, Y + \epsilon)$. Let the joint density of this perturbed random variable be \tilde{f}_ϵ and the marginal of Y_ϵ by \tilde{G}_ϵ . Recall that for every convex function ϕ satisfying $\phi(1) = 0$, it is possible to define the divergence $D_\phi(p||q) = \int \phi\left(\frac{p(x)}{q(x)}\right) q(x) dx$ for two probability densities p and q . It is well-known that such divergences satisfy the data-processing inequality [34]; that is, if $X' \rightarrow X \rightarrow Y \rightarrow Y'$ is a Markov chain, then $D_\phi(X||Y) \geq D_\phi(X'||Y')$. Using this fact, we obtain

$$D_\phi(\tilde{f}_\epsilon || \tilde{f}) \geq D_\phi(\tilde{G}_\epsilon || \tilde{G}). \tag{20}$$

Choosing $\phi(t) = (t - 1)^p$ and using Taylor’s expansion, it is easy to see that

$$D_\phi(\tilde{f}_\epsilon || \tilde{f}) = \int_{\mathbb{R}^2} \frac{|\tilde{f}(x, y) - \tilde{f}(x, y - \epsilon)|^p}{\tilde{f}(x, y)^{p-1}} dx dy = \epsilon^p \left(\int_{\mathbb{R}^2} \frac{|\partial \tilde{f}(x, y) / \partial y|^p}{\tilde{f}(x, y)^{p-1}} dx dy \right) + o(\epsilon^p).$$

And similarly,

$$D_\phi(\tilde{G}_\epsilon || \tilde{G}) = \int_{\mathbb{R}} \frac{|\tilde{G}(y) - \tilde{G}(y - \epsilon)|^p}{\tilde{G}(y)^{p-1}} dy = \epsilon^p \left(\int_{\mathbb{R}} \frac{|d\tilde{G}(y)/dy|^p}{\tilde{G}(y)^{p-1}} dy \right) + o(\epsilon^p).$$

Substituting in inequality (20), dividing by ϵ^p and taking the limit as $\epsilon \rightarrow 0$ yields

$$\int_{\mathbb{R}^2} \frac{|\partial \tilde{f}(x, y) / \partial y|^p}{\tilde{f}(x, y)^{p-1}} dx dy \geq \int_{\mathbb{R}} \frac{|d\tilde{G}/dy|^p}{\tilde{G}(y)^{p-1}} dy. \tag{21}$$

The above inequality is exactly equivalent to that in Theorem 4 using the substitution $\tilde{G} = G^p$ and $\tilde{f} = f^p$. Although we focused on joint densities over $\mathbb{R} \times \mathbb{R}$, the same argument also goes through for random variables on $\mathbb{R}^m \times \mathbb{R}^n$.

Recall that Definition 1 redefines the case of $p = 1$ in Theorem 4. Such redefinitions could indeed be done for $p > 1$ as well. However, the perturbation argument presented above makes it clear that if $p > 1$, the ϕ -divergence between a random variable (taking uniform values on some compact set) and its perturbation will be $+\infty$, since their respective supports are mismatched. Thus, analogous definitions for $p > 1$ will not yield useful bounds for such distributions. Using Definition 1, we now establish superadditivity results for the L_1 -Fisher information.

Lemma 1. *Let X be an \mathbb{R}^n -valued random variable with a smooth density $f_X(\cdot)$. Let $u \in \mathbb{R}^n$ be any unit vector. Define $X \cdot u$ to be the projection of X along u . Then the following inequality holds when both sides are well-defined:*

$$I_1(X \cdot u) \leq I_1(X)_u. \tag{22}$$

Proof. Define the random variable $X_\epsilon := X + \epsilon u$. Then the distribution of X_ϵ satisfies

$$f_{X_\epsilon}(x) = f_X(x - \epsilon u), \tag{23}$$

and is therefore a translation of f_X along the direction u by a distance ϵ . Using the data-processing inequality for total-variation distance, we obtain

$$d_{TV}(X \cdot u, X_\epsilon \cdot u) \leq d_{TV}(X_\epsilon, X), \tag{24}$$

where d_{TV} is the total variation divergence. Notice that $X_\epsilon \cdot u = X \cdot u + \epsilon$ and thus $f_{X_\epsilon \cdot u}(x) = f_{X \cdot u}(x - \epsilon)$. Dividing the left hand side of inequality (24) by ϵ and taking the limit as $\epsilon \rightarrow 0$, we obtain

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \frac{d_{TV}(X \cdot u, X_\epsilon \cdot u)}{\epsilon} &= \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{|f_{X \cdot u}(x) - f_{X \cdot u}(x - \epsilon)|}{\epsilon} dx \\ &\stackrel{(a)}{=} \frac{1}{2} I_1(X \cdot u). \end{aligned}$$

Here, equality (a) follows by the definition of $I_1(X \cdot u)$ and the assumption that it is well-defined. Doing a similar calculation for the right hand side of inequality (24) leads to

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \frac{d_{TV}(X, X_\epsilon)}{\epsilon} &= \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} \frac{|f_X(x) - f_X(x - \epsilon u)|}{\epsilon} dx \\ &\stackrel{(a)}{=} \frac{1}{2} I_1(X)_u. \end{aligned}$$

The equality in (a) follows from the definition of $I_1(X)_u$ and the assumption that it is well-defined. \square

Our next result is a counterpart to the superadditivity property of Fisher information as in inequality (17).

Theorem 5. Let $X = (X_1, \dots, X_n)$ be an \mathbb{R}^n -valued random variable. Then the following superadditivity property holds:

$$\sum_{i=1}^n I_1(X_i) \leq I_1(X).$$

Proof. Applying Lemma 1 for the unit vectors e_1, \dots, e_n , we obtain

$$\sum_{i=1}^n I_1(X_i) \leq \sum_{i=1}^n I_1(X)_{e_i} = I_1(X).$$

□

3.2. Surface Integral Form of the L_1 -Fisher Information

If we consider a random variable X that takes values uniformly over a set $K \subseteq \mathbb{R}^n$, then the L_1 -Fisher information superadditivity from Theorem 5 allows us to derive surface area inequalities once we observe two facts:

- The L_1 -Fisher information $I_1(X)$ is well-defined for X and is given by a surface integral over ∂K and
- The quantity $I_1(X)_{e_i}$ may be calculated exactly given the sizes of all slices parallel to e_i^\perp or may be lower-bounded by using any finite number of slices parallel to e_i^\perp .

Establishing the surface integral result in part (a) requires making some assumptions on the shape of the geometric body. We focus on the class of polyconvex sets [19,35], which are defined as follows:

Definition 2. A set $K \subseteq \mathbb{R}^n$ is called a polyconvex set if it can be written as $K = \cup_{i=1}^m C_i$, where $m < \infty$ and each C_i is a compact, convex set in \mathbb{R}^n that has positive volume. Denote the set of polyconvex sets in \mathbb{R}^n by \mathcal{K} .

In order to make our analysis tractable and rigorous, we first focus on polytopes and prove the polyconvex case by taking a limiting sequence of polytopes. Recall that convex polytope is the convex hull of a set of points. A precise definition of a polytope is as follows:

Definition 3. Define the set of polytopes, denoted by \mathcal{P} to be all subsets of \mathbb{R}^n such that every $K \in \mathcal{P}$ admits a representation $K = \cup_{j=1}^m P_j$, where $m > 0$ and P_j is a compact, convex polytope in \mathbb{R}^n with positive volume for each $1 \leq j \leq m$.

In what follows, we make observations (a) and (b) precise.

Theorem 6. Let X be uniformly distributed over a polytope K . Then the following equality holds:

$$I_1(X) = \frac{1}{V_n(K)} \int_{\partial K} \|n(x)\|_1 dS. \quad (25)$$

where $n(x)$ is a unit normal vector at x on ∂K .

Proof of Theorem 6. The equality in (25) is not hard to see intuitively. Consider the set K and its perturbed version K_ϵ that is obtained by translating K in the direction of e_i by ϵ . The L_1 distance between the uniform distributions on K and K_ϵ is easily seen to be

$$\frac{1}{V_n(K)} (V_n(K \cup K_\epsilon) - V_n(K \cap K_\epsilon)).$$

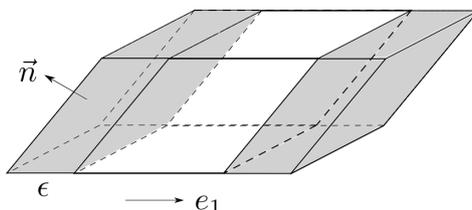


Figure 1. Perturbing a set by ϵ .

As shown in Figure 1, each small patch dS contributes $|n(x) \cdot e_i|dS$ volume to $(K \cup K_\epsilon) \setminus (K \cap K_\epsilon)$, where $n(x)$ is the normal to the surface at dS . Summing up over all such patches dS yields the desired conclusion. We make this proof rigorous with the aid of two lemmas:

Lemma 2 (Proof in Appendix A). *Let X be uniformly distributed over a compact measurable set $K \subseteq \mathbb{R}^n$. If there exists an integer L such that the intersection between K and any straight line can be divided into at most L disjoint closed intervals, then*

$$I_1(X)_{e_i} = \int_{\mathbb{R}^{n-1}} \frac{2N_i(\dots, \widehat{x}_i, \dots)}{V_n(K)} dx_1 \dots \widehat{dx}_i \dots dx_n. \tag{26}$$

Here \widehat{x}_i stands for removing x_i from the expression. The function $N_i(\dots, \widehat{x}_i, \dots)$ is the number of disjoint closed intervals of the intersection of K and line $\{X_j = x_j, 1 \leq j \leq n, j \neq i\}$.

The above lemma does not require K to be a polytope. However, the surface integral Lemma 3 below uses this assumption.

Lemma 3 (Proof in Appendix B). *Let X be uniform over a polytope $K \in \mathcal{P}$. Then*

$$\int_{\mathbb{R}^{n-1}} \frac{2N_i(\dots, \widehat{x}_i, \dots)}{V_n(K)} dx_1 \dots \widehat{dx}_i \dots dx_n = \frac{1}{V_n(K)} \int_{\partial K} |n(x) \cdot e_i| dS.$$

Here $n(x)$ is the normal vector at point $x \in \partial K$ and dS is the element for surface area.

Lemmas 2 and 3 immediately yield the desired conclusion, since $I_1(X) = \sum_{i=1}^n I_1(X)_{e_i}$ and $\|n(x)\|_1 = \sum_{i=1}^n |n(x) \cdot e_i|$. \square

Our goal now is to connect $I_1(X_i)$ to the size of the slices of K along e_i^\perp .

3.3. L_1 -Fisher Information via Slices

Consider the marginal density of X_1 , which we denote by f_{X_1} . It is easy to see that for each $x_1 \in \text{supp } f_{X_1}$, we have

$$f_{X_1}(x_1) = \frac{V_{n-1}(K \cap (e_1^\perp + x_1))}{V_n(K)}. \tag{27}$$

Thus, the distribution of X_1 is determined by the slices of K by hyperplanes parallel to e_1^\perp . Since Theorem 5 is expressed in terms of $I_1(X_i)$, where each X_i is a real-valued random variable, we establish a closed form expression for real-valued random variables in terms of their densities as follows:

Lemma 4 (Proof in Appendix C). Let X be a continuous real-valued random variable with density f_X . If we can find $-\infty = a_0 < a_1 < \dots < a_{M+1} = \infty$ such that (a) f_X is continuous and monotonic on each open interval (a_i, a_{i+1}) ; (b) For $i = 0, \dots, M$, the limits

$$f(a_i^+) = \lim_{x \rightarrow a_i^+} f_X(x) \text{ for } i = 1, \dots, M, \quad \text{and}$$

$$f(a_i^-) = \lim_{x \rightarrow a_i^-} f_X(x) \text{ for } i = 1, \dots, M$$

exist and are finite. Then

$$I_1(X) = \sum_{i=0}^M |f(a_{i+1}^-) - f(a_i^+)| + \sum_{i=1}^M |f(a_i^+) - f(a_i^-)|. \tag{28}$$

We can see that the first sum in (28) captures the change of function values on each monotonic interval and the second term captures the difference of the one-sided limits at end points. The following two corollaries are immediate.

Corollary 1. Let X be uniformly distributed on finitely many disjoint closed intervals; that is, there exist disjoint intervals $[a_i, b_i] \subseteq \mathbb{R}$ for $i \in [N]$ and $\tau \in \mathbb{R}$ such that

$$f_X(x) = \begin{cases} \tau & x \in \cup_{i=1}^N [a_i, b_i], \quad \text{and} \\ 0 & \text{otherwise,} \end{cases}$$

then $I_1(X) = 2N\tau$.

Corollary 2. Let X be a real-valued random variable with unimodal piecewise continuous density function f_X . Then the following equality holds:

$$I_1(X) = 2\|f\|_\infty. \tag{29}$$

Lemma 4 gives an explicit expression to compute I_1 when we know the whole profile of f_X . When f_X is only known for certain values x , we are able to establish a lower bound for $I_1(X)$. Note that knowing f_X for only certain values corresponds to knowing the sizes of slices along a certain directions.

Corollary 3. Let $X \sim f_X$ where f_X is as in Lemma 4. If there exists a set

$$S = \{-\infty = \theta_0 < \theta_1 < \dots < \theta_N < \theta_{N+1} = \infty\}$$

such that f_X is continuous at each θ_i for $i \in [N]$, then

$$I_1(X) \geq \sum_{i=0}^N |f(\theta_{i+1}) - f(\theta_i)|.$$

Proof. We can find $T = \{a_i | i = 0, \dots, M + 1\}$ where $a_0 = \theta_0 = -\infty, a_{M+1} = \theta_{N+1} = +\infty$ such that they satisfy the conditions in Lemma 4 and

$$I_1(X) = \sum_{i=0}^M |f_X(a_{i+1}^-) - f_X(a_i^+)| + \sum_{i=1}^M |f_X(a_i^+) - f_X(a_i^-)|. \tag{30}$$

Consider the set $S \cup T = \{c_0, \dots, c_{L+1}\}$, which divides \mathbb{R} into subintervals

$$(c_i, c_{i+1}) \quad \text{for } 0 \leq i \leq L + 1.$$

We claim that

$$I_1(X) = \sum_{i=0}^L |f_X(c_{i+1}^-) - f_X(c_i^+)| + \sum_{i=1}^L |f_X(c_i^+) - f_X(c_i^-)|. \tag{31}$$

For the second term, note that

$$\sum_{i=1}^L |f_X(c_i^+) - f_X(c_i^-)| = \sum_{i=1}^M |f_X(a_i^+) - f_X(a_i^-)|, \tag{32}$$

since f_X is assumed to be continuous at θ_i for $i \in [N]$. The points in $S \setminus T$ subdivide each of the intervals (a_i, a_{i+1}) ; that is, for each interval (a_i, a_{i+1}) we can find an index j_0 such that $a_i = c_{j_0} < c_{j_0+1} < \dots < c_{j_0+r} < c_{j_0+r+1} = a_{i+1}$ and the monotonicity of the function over (a_i, a_{i+1}) gives

$$|f_X(a_i^+) - f_X(a_{i+1}^-)| = \sum_{j=0}^r |f_X(c_{j_0+j+1}^-) - f_X(c_{j_0+j}^+)|. \tag{33}$$

Summing up over all intervals yields equality (31). To conclude the proof, note that f_X is not necessarily monotonic in the interval (θ_i, θ_{i+1}) . Thus, if we have indices $\theta_i = c_{k_0} < \dots < c_{k_0+s+1} = \theta_{i+1}$, the triangle inequality yields

$$\begin{aligned} |f_X(\theta_{i+1}) - f_X(\theta_i)| &\stackrel{(a)}{=} |f_X(\theta_{i+1}^-) - f_X(\theta_i^+)| \\ &= \left| \sum_{u=0}^s f_X(c_{k_0+u}^+) - f_X(c_{k_0+u+1}^-) + \sum_{u=1}^s f_X(c_{k_0+u}^-) - f_X(c_{k_0+u}^+) \right| \\ &\leq \sum_{u=0}^s |f_X(c_{k_0+u}^+) - f_X(c_{k_0+u+1}^-)| + \sum_{u=1}^s |f_X(c_{k_0+u}^-) - f_X(c_{k_0+u}^+)|. \end{aligned}$$

Here, equality (a) follows from the continuity of f_X at the points in S . Performing the above summation over all intervals (θ_i, θ_{i+1}) for $0 \leq i \leq N$ and using equality (31), we may conclude the inequality

$$I_1(X) \geq \sum_{i=0}^N |f(\theta_i) - f(\theta_{i+1})|.$$

□

Remark 1. Suppose K is the union of two squares joined at the corner as shown in Figure 2. Let X be uniformly distributed on K . Suppose also that the slice of K is known only at θ_1 . By direct calculation, we have $I_1(X \cdot e_1) = 2$, since $X \cdot e_1$ is uniform over $[0, 1]$. Notice that $f_{X \cdot e_1}(\theta_1) = 2$ and thus the bound from Corollary 3 is 4, which is larger than $I_1(X \cdot e_1)$. This reversal is due to the discontinuity of $f_{X \cdot e_1}$ at the sampled location θ_1 — $f_{X \cdot e_1}(\theta_1)$ equals neither the left limit or the right limit at θ_1 . To avoid such scenarios, we require continuity of the density at sampled points.

Corollary 3 shows that under mild conditions, we can estimate $I_1(X)$ when only limited information is known about its density function.

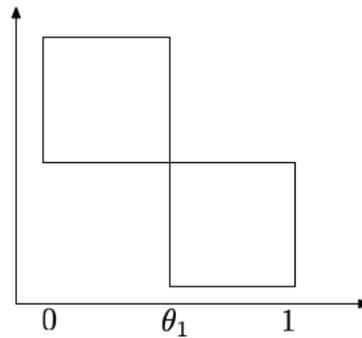


Figure 2. Uniform distribution over a union of squares.

3.4. Procedure to Obtain Lower Bounds on the Surface Area

We first verify that the assumptions required by Lemma 4 are satisfied by the marginals of uniform densities over polytopes.

Lemma 5 (Proof in Appendix D). *Suppose $X = (X_1, \dots, X_n)$ is uniformly distributed over a polytope $K \in \mathcal{P}$. Let u be any unit vector and let $f_{X \cdot u}$ be the marginal density of $X \cdot u$. Then $f_{X \cdot u}(\cdot)$ satisfies the conditions in Lemma 4.*

Now suppose $X = (X_1, \dots, X_n)$ is uniformly distributed over a polytope K . Since K is a polytope, we may write $K = \cup_{i=1}^m P_i$ where each P_i is a compact, convex polytope. Theorem 5 provides the lower bound:

$$\frac{1}{V_n(K)} \int_{\partial K} \|n(x)\|_1 dS \geq \sum_{i=1}^n I_1(X_i).$$

To derive surface area bounds, notice that

$$\sqrt{n} = \sqrt{n} \|n(x)\|_2 \geq \|n(x)\|_1,$$

and thus

$$\frac{V_{n-1}(\partial K)}{V_n(K)} \geq \frac{1}{\sqrt{n}} \sum_{i=1}^n I_1(X_i). \tag{34}$$

Suppose we know the sizes of some finite number of slices by hyperplanes parallel to e_i^\perp for $i \in [n]$. We may use Corollary 3 to obtain lower bounds $\frac{B_i}{V_n(K)}$ on $I_1(X_i)$ for each $i \in [n]$ using the available slice information. This leads to the lower bound

$$\frac{V_{n-1}(\partial K)}{V_n(K)} \geq \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{B_i}{V_n(K)},$$

and thereby we may conclude the lower bound

$$V_{n-1}(\partial K) \geq \frac{1}{\sqrt{n}} \sum_{i=1}^n B_i.$$

This is made rigorous in the following result, which may be considered to be our main result concerning surface areas.

Theorem 7. Let K be a polyconvex set. For $i \in [n]$, suppose that we have $M_i \geq 0$ slices of K obtained by hyperplanes $e_i^\perp + t_1^i, \dots, e_i^\perp + t_{M_i}^i$ ($t_1^i < \dots < t_{M_i}^i$), with sizes $\alpha_1^i, \dots, \alpha_{M_i}^i$. Then the surface area of K is lower-bounded by

$$V_{n-1}(\partial K) \geq \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\sum_{j=0}^{M_i} |\alpha_j^i - \alpha_{j+1}^i| \right), \tag{35}$$

where $\alpha_0^i, \alpha_{M_i+1}^i = 0$ for all $i \in [n]$.

Proof. Let K be a polyconvex set with a representation $K = \cup_{i=1}^m C_i$ where C_i are compact, convex sets. For each C_i , we construct a sequence of convex polytopes $\{P_i^k\}$ which approximate C_i from the outside. This means that $C_i \subseteq P_i^k$ for all $k \geq 1$ and $\lim_{k \rightarrow \infty} d(P_i^k, C_i) \rightarrow 0$, where d is the Hausdorff metric. (This is easily achieved, for instance by sampling the support function of C_i uniformly at random and constructing the corresponding polytope.) Consider the sequence of polytopes $P^k = \cup_{i=1}^m P_i^k$. For each k , we would like to assert that inequality (35) holds for the polytope P^k ; that is, we would like to lower bound $V_{n-1}(\partial P^k)$ using the slices of P^k at $(e_i^\perp + t_j^i)$ for $i \in [n]$ and $j \in [M_i]$. The only difficulty in applying Corollary 3 to obtain such a lower bound on $V_{n-1}(\partial P^k)$ is the continuity assumption, which states that the marginal of the uniform density of P_k on e_i , denoted by f_{P^k, e_i} , should be continuous at t_j^i for all $i \in [n]$ and all $j \in [M_i]$. However, this is easily ensured by choosing an outer approximating polytope for C_i that has no face parallel to e_i^\perp for all $i \in [n]$.

To complete the proof for K , we need to show that $\lim_{k \rightarrow \infty} V_{n-1}(\partial P^k) = V_{n-1}(\partial K)$ and $\lim_{k \rightarrow \infty} V_{n-1}((e_i^\perp + t_j^i) \cap P^k) = V_{n-1}((e_i^\perp + t_j^i) \cap K)$ for any $i \in [n]$ and any $j \in [M_i]$. To show this, we use the following lemma [36]:

Lemma 6 (Lemma 1 [36]). Let $K_1, \dots, K_m \subseteq \mathbb{R}^n$ be compact sets. Let $\{K_i^k\}, k \geq 1$ be a sequence of compact approximations converging to K_i in Hausdorff distance, such that $K_i \subseteq K_i^k$ for all $n \geq 1$ and for $i \in [m]$. Then it holds that

$$\lim_{k \rightarrow \infty} d \left(\cap_{i=1}^m K_i, \cap_{i=1}^m K_i^k \right) = 0. \tag{36}$$

Using Lemma 6, we observe that for any collection of indices $1 \leq i_1 < \dots < i_l \leq m$, we must have $d(P_{i_1}^k \cap \dots \cap P_{i_l}^k, C_{i_1} \cap \dots \cap C_{i_l}) \rightarrow 0$ as $k \rightarrow \infty$. Since surface area is convex continuous with respect to the Hausdorff measure [19,35], we have the limit

$$\lim_{n \rightarrow \infty} V_{n-1}(\partial(P_{i_1}^k \cap \dots \cap P_{i_l}^k)) = V_{n-1}(\partial(C_{i_1} \cap \dots \cap C_{i_l})). \tag{37}$$

Moreover, surface area is a valuation on polyconvex sets [19,35] and thus the surface area of a union of convex sets is obtained using the inclusion exclusion principle. In particular, the surface area of K is

$$V_{n-1}(\partial K) = \sum_{i=1}^n V_{n-1}(\partial C_i) - \sum_{i_1 < i_2} V_{n-1}(\partial(C_{i_1} \cap C_{i_2})) + \dots + (-1)^{m+1} V_{n-1}(\partial(\cap_{i=1}^m C_i)), \tag{38}$$

and the surface area of P^k is given by

$$V_{n-1}(\partial P^k) = \sum_{i=1}^n V_{n-1}(\partial P_i^k) - \sum_{i_1 < i_2} V_{n-1}(\partial(P_{i_1}^k \cap P_{i_2}^k)) + \dots + (-1)^{m+1} V_{n-1}(\partial(\cap_{i=1}^m P_i^k)). \tag{39}$$

Using the limit in equation (37), we may conclude that every single term in (39) converges to the corresponding term in (38) and so

$$\lim_{k \rightarrow \infty} V_{n-1}(\partial P^k) = V_{n-1}(\partial K). \tag{40}$$

We now show that each slice of P^k converges in size to the corresponding slice of K . Let H be some fixed hyperplane that is orthogonal to one of the coordinate axes. Since each C_i can be replaced by a polytope $\cap_{k=1}^n P_i^k$, we can assume without loss of generality that for each $i \in [m]$, the sequence of polytopes that approximate C_i from outside is monotonically decreasing; that is, $P_i^k \supseteq P_i^{k+1}$ for all $k \geq 1$. For any fixed compact convex set $L \subseteq H$, Lemma 6 yields

$$d(L \cap P_i^k, L \cap C_i) \rightarrow 0, \tag{41}$$

and thus the $(n - 1)$ -dimensional volume of the two sets also converges. Picking L to be $P_i^1 \cap H$, we see that $L \cap P_i^k = H \cap P_i^k$ and $L \cap C_i$ is $H \cap C_i$ and thus equation (41) yields

$$d(H \cap P_i^k, H \cap C_i) \rightarrow 0. \tag{42}$$

The sequence of set $H \cap P_i^k$ for $k \geq 1$ is an outer approximation to $H \cap C_i$ that converges in the Hausdorff metric. Therefore, using Lemma 6,

$$d((H \cap P_{i_1}^k) \cap \dots \cap (H \cap P_{i_l}^k), (H \cap C_{i_1}) \cap \dots \cap (H \cap C_{i_l})) \rightarrow 0. \tag{43}$$

Using the continuity of the volume functional,

$$V_{n-1}((H \cap P_{i_1}^k) \cap \dots \cap (H \cap P_{i_l}^k)) \rightarrow V_{n-1}((H \cap C_{i_1}) \cap \dots \cap (H \cap C_{i_l})). \tag{44}$$

Now an identical argument as above says that the $(n - 1)$ -dimensional volume of $H \cap K$ is obtained via an inclusion exclusion principle applied to the convex sets $H \cap C_i$ for $i \in [m]$. Applying Equation (44) to all the terms in the inclusion exclusion expression, we conclude that

$$V_{n-1}(H \cap P^k) \rightarrow V_{n-1}(H \cap K). \tag{45}$$

This concludes the proof. \square

Note that there is nothing restricting us to hyperplanes parallel to e_i^\perp . For example, suppose we have slice information available via hyperplanes parallel to $\{u_1^\perp, \dots, u_m^\perp\}$ for some unit vectors u_i for $i \in [m]$. In this case, we have the inequality

$$\frac{1}{V_n(K)} \int_{\partial K} \left(\sum_{j=1}^m |n(x) \cdot u_j| \right) dS \geq \sum_{j=1}^m I_1(X \cdot u_j).$$

Using the slice information, we may lower bound $I_1(X \cdot u_i)$ via Corollary 3. Suppose this bound is $\frac{1}{V_n(K)} \sum_{j=1}^m B_j$. To arrive at a lower bound for the surface area, all we need is the best possible constant C_n such that

$$C_n \geq \sum_{j=1}^m |n(x) \cdot u_j|$$

for all unit vectors $n(x)$. (This constant happened to be \sqrt{n} when u_j 's were the coordinate vectors.) With such a constant, we may conclude

$$V_{n-1}(\partial K) \geq \frac{\sum_{j=1}^m B_j}{C_n}.$$

In Appendix E, we work out the surface area lower bound from Theorem 7 for a particular example of a nonconvex (yet polyconvex) set.

4. Conclusions

In this paper, we provided two different families of geometric inequalities to provide (a) Lower bounds on the volumes of convex sets using their slices and (b) Lower bounds on the surface areas of polyconvex sets using their slices. These inequalities were derived using information theoretic tools. The volume bounds were obtained by using the Brascamp-Lieb subadditivity of entropy in conjunction with entropy bounds for log-concave random variables. Our main innovation in the surface area bounds is interpreting superadditivity of Fisher information as a consequence of the data-processing inequality applied to perturbed random variables. With this interpretation, we show that using the total variation distance for data-processing allows us to derive superadditivity results for the L_1 -Fisher information. Crucially, the L_1 -Fisher information is well-defined even for non-smooth densities and thus we are able to calculate it for uniform distributions over compact sets.

There are a number of future directions worth pursuing. One interesting question is whether the volume bounds can be tightened further using entropy bounds for log-concave random variables that depend not just on the maximum value of the density but also on the size of the support. Note that this means knowing the largest slices as well as the sizes of the projections of a convex set. Another interesting question is characterizing the equality cases of the superadditivity of Fisher information in Theorem 5 and thereby get a better understanding of when the resulting bounds provide meaningful estimates on the surface area of geometric body.

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Appendix A. Proof of Lemma 2

If $K \in \mathcal{P}$, the assumption in Lemma 2 may be verified. Clearly, K has finitely many faces F_1, F_2, \dots, F_M . For a line ℓ intersecting K in some closed intervals, one of two events can happen. Either $\ell \cap F_j$ is one of the intervals or the interval has endpoints that are marked by $\ell \cap F_{i_1}$ and $\ell \cap F_{i_2}$ for some $i_1, i_2 \in [M]$. The maximum number of intervals may be loosely bounded by $L := M + \binom{M}{2}$, which is finite.

We show (26) for $i = 1$ and the others can be proved in the same way. Since X is uniformly distributed over K ,

$$f_X(x) = \begin{cases} \frac{1}{V_n(K)} & \forall x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$F(x_2, \dots, x_n, \epsilon) = \int_{\mathbb{R}} \frac{|f_X(x_1, \dots, x_n) - f_X(x_1 - \epsilon, \dots, x_n)|}{\epsilon} dx_1.$$

We claim that there exists $g(x_2, \dots, x_n) \in L^1$, such that

$$F(x_2, \dots, x_n, \epsilon) \leq g(x_2, \dots, x_n).$$

This would allow us to use the dominated convergence theorem to conclude

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^{n-1}} F(x_2, \dots, x_n, \epsilon) dx_2 \dots dx_n = \int_{\mathbb{R}^{n-1}} \lim_{\epsilon \rightarrow 0} F(x_2, \dots, x_n, \epsilon) dx_2 \dots dx_n$$

Fix the coordinates x_2, \dots, x_n . If $(x_2, \dots, x_n) \notin P_{e_1^\perp}(K)$, we clearly have $F(x_2, \dots, x_n, \epsilon) = 0$. Let $(x_2, \dots, x_n) \in P_{e_1^\perp}(K)$. Since K intersects any straight line at most L times, $f_X(x_1, \dots, x_n)$ is a constant function on at most L line segments and 0 else where. We can write it as $\sum_i f_i(x_1)$ where each

f_i is a constant function with value $\frac{1}{V_n(K)}$ on a small interval of x_1 and 0 elsewhere. Let f_k be a function in this sum. We consider $F(x_2, \dots, x_n, \epsilon)$ in the following situations.

- Support of f_k is larger than or equal to ϵ , then

$$\int_{\mathbb{R}} \frac{|f_k(x_1) - f_k(x_1 - \epsilon)|}{\epsilon} dx = 2 \cdot \frac{1}{V_n(K)} \epsilon / \epsilon = \frac{2}{V_n(K)}.$$

See Figure A1.

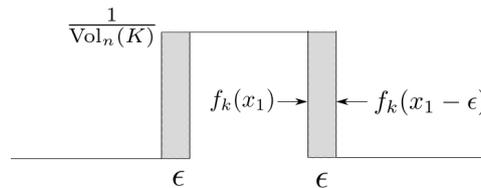


Figure A1. Support $> \epsilon$.

- Support of f_k is $\epsilon' < \epsilon$, then

$$\int_{\mathbb{R}} \frac{|f_k(x_1) - f_k(x_1 - \epsilon)|}{\epsilon} dx_1 = 2 \cdot \frac{1}{V_n(K)} \epsilon' / \epsilon \leq \frac{2}{V_n(K)}.$$

See Figure A2.

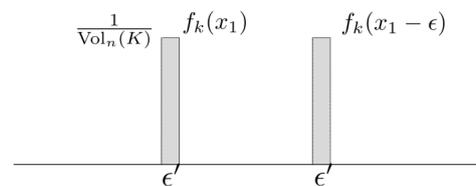


Figure A2. Support $< \epsilon$.

In both cases, we have

$$\int_{\mathbb{R}} \frac{|f_k(x_1) - f_k(x_1 - \epsilon)|}{\epsilon} dx_1 \leq \frac{2}{V_n(K)}.$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}} \frac{|f_X(x_1, \dots, x_n) - f_X(x_1 - \epsilon, \dots, x_n)|}{\epsilon} dx_1 &= \int_{\mathbb{R}} \frac{|\sum_i f_i(x_1) - \sum_i f_i(x_1 - \epsilon)|}{\epsilon} dx_1 \\ &\leq \int_{\mathbb{R}} \frac{\sum_i |f_i(x_1) - f_i(x_1 - \epsilon)|}{\epsilon} dx_1 \\ &\leq \sum_i \frac{2}{V_n(K)} \\ &\leq \frac{2L}{V_n(K)}. \end{aligned}$$

Let

$$g(x_2, \dots, x_n) = \begin{cases} \frac{2L}{V_n(K)} & (x_2, \dots, x_n) \in P_{e_1^\perp}(K), \\ 0 & \text{otherwise.} \end{cases}$$

Then $F(x_2, \dots, x_n, \epsilon) \leq g(x_2, \dots, x_n)$ and

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} g(x_2, \dots, x_n) dx_2 \dots dx_n &= \int_{P_{e_1^\perp}(K)} \frac{2L}{V_n(K)} dx_2 \dots dx_n \\ &= \frac{2LV_{n-1}(P_{e_1^\perp}(K))}{V_n(K)}, \end{aligned}$$

which shows that g is integrable. Using the dominated convergence theorem, we know that

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} \frac{|f_X(x_1, \dots, x_n) - f_X(x_1 - \epsilon, \dots, x_n)|}{\epsilon} dx_1 dx_2 \dots dx_n \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^{n-1}} F(x_2, \dots, x_n, \epsilon) dx_2 \dots dx_n \\ &= \int_{\mathbb{R}^{n-1}} \lim_{\epsilon \rightarrow 0^+} F(x_2, \dots, x_n, \epsilon) dx_2 \dots dx_n \\ &= \int_{\mathbb{R}^{n-1}} dx_2 \dots dx_n \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{|f_X(x_1, \dots, x_n) - f_X(x_1 - \epsilon, \dots, x_n)|}{\epsilon} dx_1. \end{aligned}$$

Lastly, by Corollary 1,

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{|f_X(x_1, \dots, x_n) - f_X(x_1 - \epsilon, \dots, x_n)|}{\epsilon} dx_1 = \frac{2N(x_2, \dots, x_n)}{V_n(K)}.$$

This concludes the proof.

Appendix B. Proof of Lemma 3

Recall that we need to show

$$\frac{1}{V_n(K)} \int_{\partial K} |n(x) \cdot e_i| dS = \int_{\mathbb{R}^{n-1}} \frac{2N_i(\dots, \widehat{x}_i, \dots)}{V_n(K)} dx_1 \dots \widehat{dx}_i \dots dx_n \tag{A1}$$

Without loss of generality, let $i = 1$. Denote ∂K as $\cup_{j=1}^M F_j$ where F_j are the faces of K . Let n_j be the outward normal to F_j for $j \in [m]$. We have the equality

$$V_{n-1}(P_{e_1^\perp}(F_j)) = \int_{P_{e_1^\perp}(F_j)} dx_2 dx_3 \dots dx_n = |n_j \cdot e_1| V_{n-1}(F_j).$$

Summing up for all F_j , the left hand side of (A1) is given by

$$\begin{aligned} \frac{1}{V_n(K)} \int_{\partial K} |n(x) \cdot e_1| dS &= \frac{1}{V_n(K)} \sum_{i=1}^M |n_j \cdot e_1| V_{n-1}(F_j) \\ &= \sum_{i=1}^M \frac{1}{V_n(K)} \int_{P_{e_1^\perp}(F_j)} dx_2 dx_3 \dots dx_n. \end{aligned}$$

If for some n_j , the equality $n_j \cdot e_1 = 0$ holds, then $V_{n-1}(P_{e_1^\perp}(F_j)) = 0$. Clearly,

$$\int_{P_{e_1^\perp}(F_j)} dx_2 dx_3 \dots dx_n = \int_{P_{e_1^\perp}(F_j)} 2N(x_2, \dots, x_n) = 0.$$

Without loss of generality, we assume $n_j \cdot e_1 \neq 0$. Let $\delta_{P_{e_1^\perp}(F_j)}(x_2, \dots, x_n)$ be the indicator function on $P_{e_1^\perp}(F_j)$; that is,

$$\delta_{P_{e_1^\perp}(F_j)}(x_2, \dots, x_n) = \begin{cases} 1 & (x_2, \dots, x_n) \in P_{e_1^\perp}(F_j), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{i=1}^M \int_{P_{e_1^\perp}(F_j)} dx_2 \dots dx_n &= \sum_{i=1}^M \int_{\mathbb{R}^{n-1}} \delta_{P_{e_1^\perp}(F_j)}(x_2, \dots, x_n) dx_2 \dots dx_n \\ &= \int_{\mathbb{R}^{n-1}} \sum_{i=1}^M \delta_{P_{e_1^\perp}(F_j)}(x_2, \dots, x_n) dx_2 \dots dx_n. \end{aligned}$$

For every (x_2, \dots, x_n) , there will be $2N(x_2, \dots, x_n)$ many F_j 's such that $(x_2, \dots, x_n) \in F_j$. Therefore

$$\int_{\mathbb{R}^{n-1}} \sum_{i=1}^k \delta_{P_{e_1^\perp}(U_i)}(x_2, \dots, x_n) dx_2 \dots dx_n = \int_{\mathbb{R}^{n-1}} 2N(x_2, \dots, x_n) dx_2 \dots dx_n,$$

which completes the proof.

Appendix C. Proof of Lemma 4

We claim that

$$\lim_{\epsilon \rightarrow 0^+} \int_{a_i+\epsilon}^{a_{i+1}} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx = |f_X(a_{i+1}^-) - f_X(a_i^+)|, \quad (i = 0, \dots, M), \text{ and} \tag{A2}$$

$$\lim_{\epsilon \rightarrow 0^+} \int_{a_i}^{a_i+\epsilon} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx = |f_X(a_i^+) - f_X(a_i^-)|, \quad (i = 1, \dots, M). \tag{A3}$$

If f_X is increasing on (a_i, a_{i+1}) , then

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_{a_i+\epsilon}^{a_{i+1}} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx &= \lim_{\epsilon \rightarrow 0^+} \int_{a_i+\epsilon}^{a_{i+1}} \frac{f_X(x) - f_X(x - \epsilon)}{\epsilon} dx \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{a_i+\epsilon}^{a_{i+1}} \frac{f_X(x)}{\epsilon} dx - \int_{a_i+\epsilon}^{a_{i+1}} \frac{f_X(x - \epsilon)}{\epsilon} dx \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{a_i+\epsilon}^{a_{i+1}} \frac{f_X(x)}{\epsilon} dx - \int_{a_i}^{a_{i+1}-\epsilon} \frac{f_X(x)}{\epsilon} dx \\ &= \lim_{\epsilon \rightarrow 0^+} - \int_{a_i}^{a_i+\epsilon} \frac{f_X(x)}{\epsilon} dx + \int_{a_{i+1}-\epsilon}^{a_{i+1}} \frac{f_X(x)}{\epsilon} dx \\ &= -f_X(a_i^+) + f_X(a_{i+1}^-). \end{aligned}$$

The last equality is true since $\int_{a_i}^{a_i+\epsilon} \frac{f_X(x)}{\epsilon} dx = f_X(\theta)$ for $a_i < \theta < a_i + \epsilon$ due to mean value theorem. This value approaches $f_X(a_i^+)$ when $\epsilon \rightarrow 0^+$. Using the same argument, we can show $\lim_{\epsilon \rightarrow 0^+} \int_{a_{i+1}-\epsilon}^{a_{i+1}} \frac{f_X(x)}{\epsilon} dx = f_X(a_{i+1}^-)$. Similarly, when f_X is decreasing on (a_i, a_{i+1}) , we have

$$\lim_{\epsilon \rightarrow 0^+} \int_{a_i+\epsilon}^{a_{i+1}} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx = |f_X(a_i^+) - f_X(a_{i+1}^-)|$$

Therefore we have established (A2). Similarly, $\int_{a_i}^{a_i+\epsilon} \frac{|f_X(x)-f_X(x-\epsilon)|}{\epsilon} = |f_X(\theta') - f_X(\theta' - \epsilon)|$ for $a_i < \theta' < a_i + \epsilon$. Since $a_i - \epsilon < \theta' - \epsilon < a_i$, this approaches $|f_X(a_i^-) - f_X(a_i^+)|$ as $\epsilon \rightarrow 0^+$. So we have also established (A3). Lastly,

$$\begin{aligned} I_1(X) &= \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx \\ &= \lim_{\epsilon \rightarrow 0^+} \sum_{i=0}^M \int_{a_i+\epsilon}^{a_{i+1}} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx + \sum_{i=1}^M \int_{a_i}^{a_i+\epsilon} \frac{|f_X(x) - f_X(x - \epsilon)|}{\epsilon} dx \\ &= \sum_{i=0}^M |f_X(a_{i+1}^-) - f_X(a_i^+)| + \sum_{i=1}^M |f_X(a_i^+) - f_X(a_i^-)|. \end{aligned}$$

Appendix D. Proof of Lemma 5

Without loss of generality, assume $u = e_1$. Let $K = \cup_{i=1}^m P_i$ where P_i are compact, convex polytopes. Denote the projection of $f_X(\cdot)$ restricted to some set $C \subseteq K$ on the e_1 axis by $f_{C \cdot e_1}(\cdot)$. If C is a convex polytope, we may verify that $f_{C \cdot e_1}$ is log concave and therefore a continuous function on some closed interval. Furthermore, if C is a convex and compact polytope, then we may triangulate C ; i.e., express $C = \cup_{i=1}^r T_i$ where T_i are n -dimensional compact simplices for $i \in [r]$ such that their interiors partition the interior of C . Then $f_{C \cdot e_1} = \sum_{i=1}^r f_{T_i \cdot e_1}$. Each function in the summation is a degree $(n - 1)$ polynomial with a compact interval as its support in \mathbb{R} [37]. Thus, $f_{C \cdot e_1}$ is a continuous function consisting of finitely many pieces such that $f_{C \cdot e_1}$ restricted to each piece is a polynomial of degree $(n - 1)$. Note that the overall density $f_{K \cdot e_1} := f_{X \cdot e_1}(\cdot)$ is given via the inclusion exclusion principle by

$$f_{K \cdot e_1} = \sum_{i=1}^m f_{P_i} - \sum_{i_1 < i_2} f_{P_{i_1} \cap P_{i_2}} + \sum_{i_1 < i_2 < i_3} f_{P_{i_1} \cap P_{i_2} \cap P_{i_3}} + \dots$$

For each collection of indices i_1, \dots, i_k , we have that $\cap_{j=1}^k P_{i_j}$ is a compact, convex polytope, possibly with 0 volume but such sets do not contribute to the above sum so we only consider cases where the intersection has a positive volume. The sum (or difference) of finitely many bounded continuous functions on closed intervals is easily seen to satisfy the following property: We may find finitely many points $-\infty = \gamma_0 < \gamma_1 < \dots < \gamma_R < \gamma_{R+1} = +\infty$ such that the function is continuous on each open interval (γ_i, γ_{i+1}) and the left and right limits at the endpoints in each interval are finite. To verify the assumptions in Lemma 4, we simply check that on each interval (γ_i, γ_{i+1}) , the function does not have infinitely many local optima. This is clearly true since restricted to (γ_i, γ_{i+1}) , the function is a piecewise polynomial of degree $(n - 1)$. This proves the claim.

Remark A1. Note that in general, it is possible for the difference of log concave functions to have infinitely many local optima. For example, if $f_1, f_2 : [-1, 1] \rightarrow \mathbb{R}$ such that $f_1(x) = 2 - x^2$ and $f_2(x) = (2 - x^2) + \frac{e^{-1/x^2} \sin(1/x)}{2}$, then both functions are concave and positive and therefore log-concave. However, $f_2 - f_1 = \frac{e^{-1/x^2} \sin(1/x)}{2}$ has infinitely many local optima close to 0. The observation that the marginals in our case are piecewise polynomials is therefore necessary in the above argument.

Appendix E. Example

To illustrate (34), we consider the example of a cube with a hole in it as in Figure A3. Note that this is a nonconvex set but is easily seen to be polyconvex. The density $f_{X_1}(x_1)$ can be computed using area of slices along x_1 -axis and

$$\begin{aligned} I_1(X_1) &= |f_{X_1}(0) - f_{X_1}(-\infty)| + |f_{X_1}(1) - f_{X_1}(0)| + |f_{X_1}(3) - f_{X_1}(2)| + |f_{X_1}(\infty) - f_{X_1}(3)| \\ &= \left| \frac{9}{26} - 0 \right| + \left| \frac{8}{26} - \frac{9}{26} \right| + \left| \frac{9}{26} - \frac{8}{26} \right| + \left| 0 - \frac{9}{26} \right| \\ &= \frac{20}{26}. \end{aligned}$$

By symmetry,

$$\frac{1}{\sqrt{3}}(I_1(X_1) + I_1(X_2) + I_1(X_3)) = \frac{60}{26\sqrt{3}} \approx 1.33,$$

By direct calculation,

$$\frac{V_{n-1}(\partial K)}{V_n(K)} = \frac{48}{26} \approx 1.85.$$

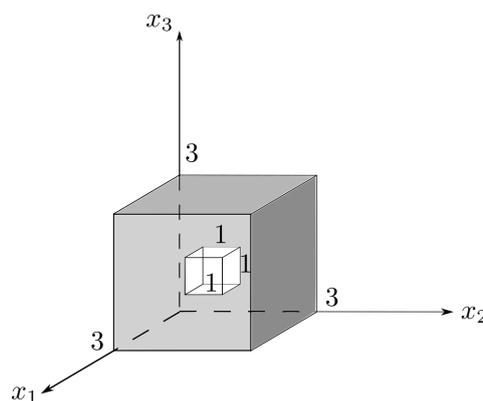


Figure A3. Cube with a hole.

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