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Planck-Scale Soccer-Ball Problem: A Case of Mistaken Identity

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Received: 23 April 2017; Accepted: 18 July 2017; Published: 2 August 2017

Abstract: Over the last decade, it has been found that nonlinear laws of composition of momenta are predicted by some alternative approaches to "real" 4D quantum gravity, and by all formulations of dimensionally-reduced (3D) quantum gravity coupled to matter. The possible relevance for rather different quantum-gravity models has motivated several studies, but this interest is being tempered by concerns that a nonlinear law of addition of momenta might inevitably produce a pathological description of the total momentum of a macroscopic body. I here show that such concerns are unjustified, finding that they are rooted in failure to appreciate the differences between two roles for laws composition of momentum in physics. Previous results relied exclusively on the role of a law of momentum composition in the description of spacetime locality. However, the notion of total momentum of a multi-particle system is not a manifestation of locality, but rather reflects translational invariance. By working within an illustrative example of quantum spacetime, I show explicitly that spacetime locality is indeed reflected in a nonlinear law of composition of momenta, but translational invariance still results in an undeformed linear law of addition of momenta building up the total momentum of a multi-particle system.

Keywords: quantum foundations; relativity; quantum gravity

1. Introduction

An emerging characteristic of quantum-gravity research over the last decade has been a gradual shift of focus toward manifestations of the Planck scale on momentum space, particularly pronounced in some approaches to quantum gravity. For some research lines based on spacetime noncommutativity, several momentum-space structures have been in focus, including the possibility of deformed laws of composition of momenta, which shall be here of interest. While deformed laws of composition of momenta are found to be inevitable in some approaches based on spacetime noncommutativity (e.g., [1–6]), the situation is less certain in the loop-quantum-gravity approach. For "real" 4D loop quantum gravity, the relevant issues are partly obscured by our present limited understanding of the semiclassical limit of that theory [7], but some indirect arguments suggest that a nonlinear law of composition of momenta might arise [8,9]. These arguments find further strength in results on 3D loop quantum gravity, where the simplifications afforded by that dimensionally-reduced model allow one to rigorously show that indeed the nonlinearities on momentum space are present (e.g., [10]). Actually, evidence is growing that in all alternative formulations of 3D quantum gravity coupled to matter there are nonlinearities in momentum space, including nonlinear laws of composition of momenta (e.g., [11]). The role played by nonlinearities on momentum space is also noteworthy in two recently-proposed approaches to the quantum-gravity problem: the one based on group field theory [12] and the one based on the relative-locality framework [13].

Due to the lack of experimental guidance, a variety of approaches to quantum gravity are being developed, and in most cases the different approaches have very little in common. This of course endows with additional reasons of interest any result which is found to apply to more than one approach. Indeed, there has been growing interest in the conceptual implications and possible phenomenological implications [14] of nonlinear laws on momentum space and particularly nonlinear laws of composition of momenta. However, this interest is being tempered by concerns that a nonlinear law of addition of momenta might inevitably produce a pathological description of the total momentum of a macroscopic body [15–23] (also see References [24–26] for a related discussion focused within the novel relative-locality framework). This issue has often been labelled as the "soccer-ball problem" [17]: the quantum-gravity pictures lead one to expect nonlinearities of the law of composition of momenta which are suppressed by the Planck scale ($\sim 10^{28}$ eV) and would be unobservably small for particles at energies we presently can access, but in the analysis of a macroscopic body (e.g., a soccer ball), one might have to add up very many of such minute nonlinearities, ultimately obtaining results in conflict with observations [15–23].

If this so-called "soccer-ball problem" really was a scientific problem (a case of actual conflict with experimental data), we could draw rather sharp conclusions about several areas of quantum-gravity research. Perhaps most notably we should consider as ruled out large branches of research on quantum-gravity based on spacetime noncommutativity and we should consider the whole effort of research on dimensionally-reduced 3D quantum gravity as completely unreliable in forming an intuition for "real" 4D quantum gravity. However, I here show that previous discussions of this soccer-ball problem [15–26] failed to appreciate the differences between two roles for laws of composition of momentum in physics. Previous results supporting a nonlinear law of addition of momenta relied exclusively on the role of a law of momentum composition in the description of spacetime locality. The notion of total momentum of a multi-particle system is not a manifestation of locality, but rather reflects translational invariance in interacting theories. After being myself confused about these issues for quite some time [17] I feel I am now in a position to articulate the needed discussion at a completely general level. However, considering the tone and content of the bulk of literature that precedes this contribution of mine I find it is best to opt here instead for a very explicit discussion based on illustrative examples of calculations performed within a specific simple model affected by nonlinearities for a law of composition of momenta. The model I focus on has $2+1$-dimensional pure-spatial κ-Minkowski noncommutativity [1–6], with the time coordinate left unaffected by the deformation and the two spatial coordinates, x_1 and x_2, governed by

$$[x_1, x_2] = i\ell x_1 \tag{1}$$

(with the deformation scale ℓ expected to be of the order of the inverse of the Planck scale).

In the next section I briefly review within this example of quantum spacetime previous arguments showing that spacetime locality is reflected in a nonlinear law of composition of momenta. Then, Section 3 takes off from known results on translational invariance for κ-Minkowski noncommutative spacetimes and builds on those to achieve the first ever example of translationally-invariant interacting two-particle system in κ-Minkowski. This allows me to explicitly verify that the conserved charge associated with that translational invariance (the total momentum of the two-particle system) adds linearly the momenta of the two particles involved. Section 4 offers some closing remarks.

2. Soccer-Ball Problem and Sum of Momenta from Locality

The ingredients needed for seeing a nonlinear law of composition of momenta emerging from noncommutativity of type (1) are very simple. Essentially, one needs only to rely on results establishing that functions of coordinates governed by (1) still admit a rather standard Fourier expansion (e.g., [1,2])

$$\Phi(x) = \int d^4k\ \tilde{\Phi}(k)\ e^{ik_\mu x^\mu}$$

and that the notion of integration on such a noncommutative space preserves many of the standard properties including [1,3]

$$\int d^4x \; e^{ik_\mu x^\mu} = (2\pi)^4 \delta^{(4)}(k)\,. \tag{2}$$

It is a rather standard exercise for practitioners of spacetime noncommutativity to use these tools in order to enforce locality within actions describing classical fields. For example, one might want to introduce in the action the product of three (possibly identical, but in general different) fields, Φ, Ψ, Υ, insisting on locality in the sense that the three fields be evaluated "at the same quantum point x"; i.e., $\Phi(x)\,\Psi(x)\,\Upsilon(x)$. There is still no consensus on how one should formulate the more interesting quantum-field version of such theories, and it remains unclear to which extent and in which way our ordinary notion of locality is generalized by the requirement of evaluating "at the same quantum point x" fields intervening in a product such as $\Phi(x)\,\Psi(x)\,\Upsilon(x)$. Nonetheless, for the classical-field case there is a sizable body of literature consistently adopting this prescription for locality. Important for my purposes here is the fact that with such a prescription, locality inevitably leads to a nonlinear law of composition of momenta, as I show explicitly in the following example:

$$\begin{aligned} &\int d^4x \; \Phi(x)\,\Psi(x)\,\Upsilon(x) = \\ &= \int d^4x \int d^4k \int d^4p \int d^4q \; \tilde{\Phi}(k)\,\tilde{\Psi}(p)\,\tilde{\Upsilon}(q)\, e^{ik_\mu x^\mu} e^{ip_\nu x^\nu} e^{iq_\rho x^\rho} \\ &= \int d^4x \int d^4k \int d^4p \int d^4q \; \tilde{\Phi}(k)\,\tilde{\Psi}(p)\,\tilde{\Upsilon}(q) e^{i(k\oplus p\oplus q)_\mu x^\mu} \\ &= (2\pi)^4 \int d^4k \int d^4p \int d^4q \; \tilde{\Phi}(k)\,\tilde{\Psi}(p)\,\tilde{\Upsilon}(q)\, \delta^{(4)}(k\oplus p\oplus q) \end{aligned} \tag{3}$$

where $\oplus$ is such that

$$(k\oplus p)_0 = k_0 + p_0 \tag{4}$$

$$(k\oplus p)_2 = k_2 + p_2 \tag{5}$$

$$(k\oplus p)_1 = \frac{k_2+p_2}{1-e^{\ell(k_2+p_2)}}\left[\frac{1-e^{\ell k_2}}{k_2 e^{\ell p_2}}k_1 + \frac{1-e^{\ell p_2}}{p_2}p_1\right] \tag{6}$$

This result is rooted in one of the most studied aspects of such noncommutative spacetimes, which is their "generalized star product" [1–3]. This is essentially a characterization of the properties of products of exponentials induced by rules of noncommutativity of type (1). Specifically, one easily arrives at (3) (with $\oplus$ such that, in particular, (6) holds) by just observing that from the defining commutator (1) it follows that (Equation (7) is a particular example of application of the Baker-Campbell-Hausdorff formula for products of exponentials of noncommuting variables. In general, the Baker–Campbell–Hausdorff formula involves an infinite series of nested commutators, but the case of noncommutativity (1) is one of the cases for which the series of nested commutators can be resummed explicitly [2,3]) [2,3]:

$$\begin{aligned} &\log\left[\exp\left(ik_2x_2 + ik_1x_1\right)\exp\left(ip_2x_2 + ip_1x_1\right)\right] = \\ &= ix_2(p_2+k_2) + ix_1\frac{k_2+p_2}{1-e^{\ell(k_2+p_2)}}\left(\frac{1-e^{\ell k_2}}{k_2 e^{\ell p_2}}k_1 + \frac{1-e^{\ell p_2}}{p_2}p_1\right) \end{aligned} \tag{7}$$

The so-called soccer-ball problem concerns the acceptability of laws of composition of type (6). Since one assumes that the deformation scale ℓ is on the order of the inverse of the Planck scale, applying (6) to microscopic/fundamental particles has no sizable consequences: of course (6) gives us back to good approximation $(k\oplus p)_1 \simeq k_1 + p_1$ whenever $|\ell k_2| \ll 1$ and $|\ell p_2| \ll 1$. However, if a law of composition such as (6) should be used also when we add very many microparticle momenta in obtaining the total momentum of a multiparticle system (such as a soccer ball), then the final result could be pathological [15–26] even when each microparticle in the system has momentum much smaller than $1/\ell$.

3. Sum of Momenta from Translational Invariance

As clarified in the brief review of known results given in the previous section, a nonlinear law of composition of momenta arises in characterizations of locality, as a direct consequence of the form of some star products. My main point here is that a different law of composition of momenta is produced by the analysis of translational invariance, and it is this other law of composition of momenta which is relevant for the characterization of the total momentum of a multi-particle system. Here too I shall use only known facts about the peculiarities of translation transformations in certain noncommutative spacetimes, but exploit them to obtain results that had not been derived before—indeed, results relevant for the description of the total momentum of a multi-particle system.

A first hint that translation transformations should be modified [4–6] in certain noncommutative spacetimes comes from noticing that (1) is incompatible with the standard Heisenberg relations $[p_j, x_k] = i\delta_{jk}$. Indeed, if one adopts (1) and $[p_j, x_k] = i\delta_{jk}$, one then easily finds that some Jacobi identities are not satisfied. The relevant Jacobi identities are satisfied if one allows for a modification of the Heisenberg relations which balances for the noncommutativity of the coordinates:

$$[p_1, x_1] = i\ , \quad [p_2, x_1] = 0\ , \quad [p_2, x_2] = i\ , \tag{8}$$
$$[p_1, x_2] = -i\ell p_1\ , \tag{9}$$

One easily finds that by combining (1), (8), and (9), all Jacobi identities are satisfied [4–6].

Additional intuition for these nonstandard properties of the momenta p_j comes from actually looking at which formulation of translation transformations preserves the form of the noncommutativity of coordinates (1). Evidently, the standard description

$$x_2 \to x_2' = x_2 + a_2\ , \quad x_1 \to x_1' = x_1 + a_1$$

is not a symmetry of (1):

$$[x_1', x_2'] = [x_1 + a_1, x_2 + a_2] = i\ell x_1 = i\ell(x_1' - a_1) \tag{10}$$

Unsurprisingly, what does work is the description of translation transformations using as generators the p_j of (8) and (9), which as stressed above satisfy the Jacobi-identity criterion. These deformed translation transformations take the form

$$\begin{aligned} x_1' &= x_1 - ia_1[p_1, x_1] - ia_2[p_2, x_1] = x_1 + a_1\ , \\ x_2' &= x_2 - ia_1[p_1, x_2] - ia_2[p_2, x_2] = x_2 + a_2 - \ell a_1 p_1 \end{aligned} \tag{11}$$

and indeed are symmetries of the commutation rules (1):

$$\begin{aligned} [x_1', x_2'] &= [x_1 + a_1, x_2 + a_2 - \ell a_1 p_1] = \\ &= i\ell x_1 - \ell a_1 [x_1, p_1] = i\ell(x_1 + a_1) = i\ell x_1' \end{aligned} \tag{12}$$

All this about translation transformations in certain noncommutative spacetimes is well known (e.g., [4–6]). The part which I am here going to contribute is to show how this is relevant for the mentioned much-debated issue about the total momentum of a multi-particle system. My starting point is that in order for us to be able to even contemplate the total momentum of a multiparticle system, we must be dealing with a case where translational invariance is ensured: total momentum is the conserved charge for a translationally invariant multi-particle system. Surely the introduction of translationally invariant multi-particle systems must involve some subtleties due to the noncommutativity of coordinates, and these subtleties are directly connected to the new properties of translation transformations (9), but they are not directly connected to the properties of the star product (7) and the associated law of composition of momenta (6). For my purposes, also considering the heated debate that precedes this contribution of mine, it is best to show the implications of this point very simply and explicitly, focusing on a system of two particles interacting via a harmonic potential.

I start by noticing that evidently one does not achieve translational invariance through a description of the form

$$\begin{aligned}\mathcal{H}_{non-transl} = \quad & \frac{(p_1^A)^2}{2m} + \frac{(p_2^A)^2}{2m} + \frac{(p_1^B)^2}{2m} + \frac{(p_2^B)^2}{2m} + \\ + \quad & \frac{1}{2}\rho[(x_1^A - x_1^B)^2 + (x_2^A - x_2^B)^2] \end{aligned} \tag{13}$$

where indices A and B label the two particles involved in the interaction via the harmonic potential. As stressed above, translation transformations consistent with the coordinate noncommutativity (1) must be such that (see (11)) $x_1 \to x_1 + a_1$ and $x_2 \to x_2 + a_2 - \ell a_1 p_1$, and as a result by writing the harmonic potential with $(x_1^A - x_1^B)^2 + (x_2^A - x_2^B)^2$, one does not achieve translational invariance.

One does get translational invariance by adopting instead

$$\begin{aligned}\mathcal{H} = & \frac{(p_1^A)^2}{2m} + \frac{(p_2^A)^2}{2m} + \frac{(p_1^B)^2}{2m} + \frac{(p_2^B)^2}{2m} + \\ & + \frac{1}{2}\rho[(x_1^A - x_1^B)^2 + (x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B)^2]\end{aligned} \tag{14}$$

This is trivially invariant under translations generated by p_2, which simply produce $x_1 \to x_1$ and $x_2 \to x_2 + a_2$. It is also invariant under translations generated by p_1, since they produce $x_1 \to x_1 + a_1$ and $x_2 \to x_2 - \ell a_1 p_1$, so that $x_2 + \ell x_1 p_1$ is left unchanged:

$$x_2 + \ell x_1 p_1 \to x_2 - \ell a_1 p_1 + \ell(x_1 + a_1)p_1 = x_2 + \ell x_1 p_1$$

It is interesting for my purposes to see which conserved charge is associated with this invariance under translations of the hamiltonian $\mathcal{H}$. This conserved charge will describe the total momentum of the two-particle system governed by $\mathcal{H}$ (i.e., the center-of-mass momentum). It is easy to see that this conserved charge is just the standard $\vec{p}^A + \vec{p}^B$. For the second component, one trivially finds that indeed

$$[p_2^A + p_2^B, \mathcal{H}] = 0$$

and the same result also applies to the first component:

$$\begin{aligned}&[p_1^A + p_1^B, \mathcal{H}] \propto [p_1^A + p_1^B, (x_1^A - x_1^B)^2] + \\ &\quad +[p_1^A + p_1^B, (x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B)^2] = \\ &= [p_1^A + p_1^B, (x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B)^2] \propto \\ &\propto [p_1^A + p_1^B, x_2^A + \ell x_1^A p_1^A - x_2^B - \ell x_1^B p_1^B] \\ &= -i\ell p_1^A + i\ell p_1^A + i\ell p_1^B - i\ell p_1^B = 0\end{aligned} \tag{15}$$

where the only non-trivial observation I have used is that (1) leads to $[p_1, x_2 + \ell x_1 p_1] = -i\ell p_1 + i\ell p_1 = 0$.

The result (15) shows that indeed $\vec{p}^A + \vec{p}^B$ is the momentum of the center of mass of my translationally-invariant two-particle system; i.e., it is the total momentum of the system.

The concerns about total momentum that had been voiced in discussions of the Planck-scale soccer-ball problem were rooted in the different sum of momenta relevant for locality, the $\oplus$ sum discussed in the previous section. It was feared that one should obtain the total momentum by combining single-particle momenta with the nonlinear $\oplus$ sum. The result (15) shows that this expectation was incorrect. One can also directly verify that indeed $\vec{p}^A \oplus \vec{p}^B$ is not a conserved charge for my translationally-invariant two-particle system, and specifically, taking into account (6), one finds that

$$[(\vec{p}^A \oplus \vec{p}^B)_1, \mathcal{H}] \neq 0$$

This completes my thesis, but in closing this section I should warn readers of the fact that while the picture emerging from my analysis is rather compelling, one should not forget that the interpretation of the notion of total momentum in a noncommutative spacetime remains affected by some open issues (see Reference [14] and references therein). Even the physical meaning of having noncommutative spacetime coordinates is still being debated. In the shadow of these interpretational issues, we cannot even be sure that the Hamiltonian of Equation (14) has physical (observable) consequences different from an ordinary harmonic-oscillator theory. Nonetheless, my analysis contributes to this ongoing debate by exposing two notions of momentum conservation: one connected to locality, and one connected with translational invariance. Evidently, if interpreted in standard way, these two notions could be mutually incompatible: in the analysis of a chain of events one might naturally want to insist on overall total-momentum conservation, but in some parts of the chain of events the conserved quantity might be the one coming from locality, while in other parts of the chain of events the conserved quantity might the one coming from translational invariance. Addressing this apparent puzzle might require a totally new interpretation of the notion of momentum of a particle in a quantum spacetime, while failing to address it might be a mortal blow to the whole research area. While in part my results are sub judice because of these interpretational issues, my analysis nonetheless firmly establishes the main conceptual point I am making, which concerns the differences between "composition of momentum appearing in locality analyses" and "composition of momentum appearing in translational-invariance analyses"—two notions which are usually confused with each other due to the fact that in a classical spacetime they coincide.

4. Implications and Outlook

The results here reported suggest that—at least within the framework of κ-Minkowski spacetime noncommutativity, there might be no "soccer-ball problem". I am confident that analogous results will emerge in other similar formalisms, but of course dedicated analyses are needed. A case of particular interest might be that of the Snyder model of spacetime noncommutativity [27], which is already known to have a complicated interplay with translational invariance: the original model of Reference [27] is not invariant under translations, but a variant with an extra dimension recovers translational invariance [28].

The Hamiltonian of Equation (14) is the only one I managed to find which is invariant under the translation transformations (11), but I do not have any proof of uniqueness. It would be interesting to consider other Hamiltonians that are invariant under (11) and give the ordinary harmonic-oscillator Hamiltonian in the $\ell \to 0$ limit.

As usual in physics, attempts to generalize a theory also help us understand the theory itself: the analysis I here reported makes us appreciate how our current theories are built on a non-trivial correspondence between the momentum-space manifestations of locality and translational invariance. This can be viewed from a different perspective by reconsidering the fact that in Galilean relativity all laws of composition of momenta and velocities are linear, and there is a linear relationship between velocity and momentum. Within Galilean-relativistic theories, one could choose to never speak of momentum and work exclusively in terms of velocities, with apparently a single linear law of composition of velocities. In our current post-Galilean theories, the relationship between momentum and velocity is non-linear, and we then manage to appreciate differences between composition laws (in our current theories all laws of composition of momenta remain linear, but velocities are composed non-linearly).

I must also comment on the fact that aspects of my analysis pertaining to translational invariance were confined to a first-quantized system. This came out of necessity since several grey areas remain for the formulation of second quantization with κ-Minkowski noncommutativity. As a matter of fact, I here provided the first ever translationally-invariant formulation of an interacting theory in κ-Minkowski. All previous attempts had been made within quantum field theory, and led to unsatisfactory results, particularly concerning global translational invariance. Perhaps the results I here reported could provide guidance for improving upon previous attempts at formulating interacting quantum field theories in κ-Minkowski. In particular, it might be appropriate to make room for some novel notion of "coincidence of points"—a possibility which had not been considered in previous attempts. I see a hint pointing in this direction in the structure of my translationally-invariant harmonic potential: unlike standard Harmonic potentials, the potential in my Equation (14) does not vanish when the coordinates of the particles coincide: the potential in Equation (14) vanishes for $x_1^A = x_1^B$ and $x_2^A = x_2^B$ only if the momenta also coincide ($p_1^A = p_1^B$). This is reminiscent of some results obtained within the recently-proposed relative-locality framework [13], where the only meaningful notion of "coincidence" is a phase-space notion (not a notion that could be formulated exclusively in spacetime). This suggests that one could perhaps improve upon previous attempts to formulate interacting quantum field theories in κ-Minkowski by exploiting quantum-field-theory results being developed [29] for the relative-locality framework.

Another direction for future studies which might bring some enlightenment concerns building interacting theories with full relativistic covariance. Herein I focused on translation transformations because it was sufficient for the purposes of my study, but it would be interesting to ask what additional constraints would arise if one insists on full relativistic covariance (including boosts and spatial rotations) rather than just translational invariance. For the law of composition of momenta based on locality, a fully consistent relativistic picture is already known [13,14,29], and its consistency with κ-Minkowski noncommutativity is well established. Important insight might be gained by establishing whether or not analogous results are available for the law of composition of momenta based on translational invariance of my interacting Hamiltonian.

Conflicts of Interest: The author declares no conflict of interest.

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