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Two Approaches to Obtaining the Space-Time Fractional Advection-Diffusion Equation

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Abstract: Two approaches resulting in two different generalizations of the space-time-fractional advection-diffusion equation are discussed. The Caputo time-fractional derivative and Riesz fractional Laplacian are used. The fundamental solutions to the corresponding Cauchy and source problems in the case of one spatial variable are studied using the Laplace transform with respect to time and the Fourier transform with respect to the spatial coordinate. The numerical results are illustrated graphically.

Keywords: fractional calculus; advection-diffusion equation; Caputo derivative; Riesz derivative; Laplace transform; Fourier transform

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1. Introduction

The Fokker–Planck equation with constant coefficients has the form [1–5]:

$$\frac{\partial w}{\partial t} = a\Delta w - \mathbf{v} \cdot \operatorname{grad} w. \tag{1}$$

Here, $a > 0$ is the diffusion coefficient, and $\mathbf{v}$ is the drift vector.

The Fokker–Planck equation has many different physical applications: statistics of laser light [1], superionic conductors under the influence of an additional external field [1], diffusion in an external field [3], transport processes in porous media [6,7], groundwater hydrology [8], quantum optics [9], etc. The terms "diffusion coefficient" and "drift vector" do not predetermine their various physical interpretations [3].

Equation (1) can be written in the form of the continuity equation for probability density:

$$\frac{\partial w}{\partial t} = -\operatorname{div} \mathbf{J} \tag{2}$$

and the constitutive equation for the probability current [1,3,10,11]:

$$\mathbf{J} = -a \operatorname{grad} w + \mathbf{v}w. \tag{3}$$

Another way of obtaining Equation (1) consists of considering the conservation law (for example, for mass concentration) [7,12–14]:

$$\frac{Dc}{Dt} = -\operatorname{div} \mathbf{j}, \tag{4}$$

where:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \tag{5}$$

is the material derivative, $\mathbf{v}$ is a velocity vector and ∇ stands for the gradient operator.

Using the Fick law:

$$\mathbf{j} = -a \operatorname{grad} c, \tag{6}$$

we get:

$$\frac{\partial c}{\partial t} = a\Delta c - \mathbf{v} \cdot \operatorname{grad} c. \tag{7}$$

Investigation of different physical phenomena in media with complex internal structure has led to considering mathematical models involving differential equations with derivatives of fractional order (see [15–27], among others). The space fractional [10,28–39], time fractional [10,28,29,34,35,40–58] and space-time-fractional [59–66] generalizations of the advection-diffusion equation were studied by many authors. Both physical aspects of the fractional advection-diffusion equation and numerical schemes used for their solution were extensively studied in many papers (an extensive literature review can be found in [67]).

Effective implicit numerical methods for the solution of the space-time fractional Fokker–Planck equation and fractional advection diffusion equation were proposed in [68,69]; their stability and convergence were studied in [62] (see also [70,71]).

We can also mention the implicit difference method based on the shifted Grünwald–Letnikov approximation [36], transformation of fractional differential equation into a system of ordinary differential equation [37], the random walk algorithms [38,39], the spectral regularization method [52], the Crank–Nicholson difference scheme [53], Adomian's decomposition [50], a spatial and temporal discretization [64], the fractional variational iteration method [54], the homotopy perturbation method [51,63,72] and the Jacobi collocation method [73].

In this paper, we discuss two possibilities of obtaining the space-time fractional generalization of the advection-diffusion equation. In the case of the time-fractional advection-diffusion equation, for these possibilities, the terms "Galilei variant" and "Galilei invariant" equations are used [34,42,44]. The Caputo time-fractional derivative and Riesz fractional Laplacian are employed. The fundamental solutions to the corresponding Cauchy and source problems in the case of one spatial variable are studied. It should be emphasized that in the case of the classical advection-diffusion equation ($\alpha = 1$, $\beta = 2$), the fundamental solutions to the Cauchy problem and to the source problem coincide for $t > 0$; in the case of the fractional advection-diffusion equation, they are substantially different. The properties of the fundamental solution to the Cauchy problem for the space-time fractional advection-diffusion equation in the case of the first approach were investigated in [74]. In that paper, the explicit representation of the fundamental solution for the space fractional advection-diffusion equation ($\alpha = 1$) was obtained. In the present paper, we supplement the findings of [74] for the Cauchy problem in the case of the first approach by analysis of several particular cases and by the results of numerical calculations. The analytical form of the fundamental solutions for the Cauchy problem in the case of the second approach, as well as of the fundamental solutions to the source problems are obtained in the present paper for the first time. Most attention is concentrated on the solutions of equations with the value of the Caputo derivative $\alpha = 1/2$, which allows us to obtain solutions in the form of integrals amenable for numerical treatment. The numerical results are illustrated graphically.

2. Mathematical Preliminaries

The Riemann–Liouville integral of fractional order α is defined as [75–77]:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \qquad \alpha > 0, \tag{8}$$

where $\Gamma(\alpha)$ is the gamma function.

The Riemann–Liouville derivative of fractional order α is introduced as a left-inverse to the fractional integral I^{α}:

$$D^{\alpha}_{RL} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, \mathrm{d}\tau \right], \qquad n-1 < \alpha < n, \tag{9}$$

whereas the Caputo fractional derivative has the form [75–77]:

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\, \mathrm{d}\tau, \qquad n-1 < \alpha < n. \tag{10}$$

The introduced fractional operators have the following Laplace transform rules:

$$\mathcal{L}\left\{ I^{\alpha} f(t) \right\} = \frac{1}{s^{\alpha}} f^{*}(s), \tag{11}$$

$$\mathcal{L}\left\{ D^{\alpha}_{RL} f(t) \right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} D^k I^{n-\alpha} f(0^{+}) s^{n-1-k}, \qquad n-1 < \alpha < n, \tag{12}$$

$$\mathcal{L}\left\{ \frac{\partial^{\alpha} f}{\partial t^{\alpha}} \right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) s^{\alpha-1-k}, \qquad n-1 < \alpha < n. \tag{13}$$

Here, the asterisk denotes the transform, and s is the Laplace transform variable.

The one-dimensional Riesz derivative can be defined by its Fourier transform rule [77]:

$$\mathcal{F}\left\{ \frac{\mathrm{d}^{\beta} f(x)}{\mathrm{d}|x|^{\beta}} \right\} = -|\xi|^{\beta} \widetilde{f}(\xi), \qquad 0 < \beta \le 2, \tag{14}$$

where the tilde marks the Fourier transform and ξ is the transform variable. For $\beta = 2$, the standard formula is obtained:

$$\mathcal{F}\left\{ \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} \right\} = -\xi^2 \widetilde{f}(\xi). \tag{15}$$

In the case of several spatial variables, the positive powers of the Laplace operator, $-(-\Delta)^{\beta/2}$, $\beta > 0$, are also called the Riesz derivatives and are defined by their Fourier transforms [77–80]:

$$\mathcal{F}\left\{ (-\Delta)^{\beta/2} f(\mathbf{x}) \right\} = |\boldsymbol{\xi}|^{\beta} \mathcal{F}\left\{ f(\mathbf{x}) \right\}, \qquad \beta > 0, \tag{16}$$

where $\boldsymbol{\xi}$ is the transform-variable vector.

Equation (16) is a fractional generalization of the standard formula for the Fourier transform of the Laplace operator corresponding to $\beta = 2$:

$$\mathcal{F}\left\{ (-\Delta) f(\mathbf{x}) \right\} = |\boldsymbol{\xi}|^{2} \mathcal{F}\left\{ f(\mathbf{x}) \right\}. \tag{17}$$

3. Fractional Advection-Diffusion Equations

3.1. The First Approach

Gurtin and Pupkin [81] proposed the general time-nonlocal dependence between the heat flux and the temperature gradient. Nigmatullin [82,83] considered the following general form of such an equation:

$$\mathbf{q}(t) = -k \int_0^t K(t-\tau)\, \mathrm{grad}\, T(\tau)\, \mathrm{d}\tau \tag{18}$$

resulting in the heat conduction equation with memory:

$$\frac{\partial T}{\partial t} = a_{\mathrm{T}} \int_0^t K(t-\tau)\, \Delta T(\tau)\, \mathrm{d}\tau. \tag{19}$$

The constitutive Equation (3) can be generalized in a similar way. We will restrict our consideration to the case $\mathbf{v} = \mathrm{const}$. For example, the constitutive equation:

$$\mathbf{J} = \int_0^t K(t-\tau)\, [-a\, \mathrm{grad}\, w(\tau) + \mathbf{v} w(\tau)]\, \mathrm{d}\tau \tag{20}$$

leads to the advection-diffusion equation with the general memory kernel [27,56]:

$$\frac{\partial w}{\partial t} = \int_0^t K(t-\tau)\, [a\Delta w(\tau) - \mathbf{v} \cdot \mathrm{grad}\, w(\tau)]\, \mathrm{d}\tau \tag{21}$$

(see also [1], where similar equations were obtained for the Fokker–Planck equation in the case of one spatial variable, and the point $-\infty$ was chosen as a starting point (a lower limit) in the integral describing non-Markovian process).

The time-nonlocal constitutive equations for the probability current with the long-tail power kernel [27,56]:

$$\mathbf{J} = D_{RL}^{1-\alpha} \left[-a\, \mathrm{grad}\, w + \mathbf{v} w\right], \quad 0 < \alpha \le 1, \tag{22}$$

give the time-fractional advection-diffusion equation:

$$\frac{\partial^\alpha w}{\partial t^\alpha} = a\Delta w - \mathbf{v} \cdot \mathrm{grad}\, w, \quad 0 < \alpha \le 1. \tag{23}$$

The space-time-fractional advection-diffusion equation takes the form:

$$\frac{\partial^\alpha w}{\partial t^\alpha} = -a\,(-\Delta)^{\beta/2}\, w - \mathbf{v} \cdot \mathrm{grad}\, w, \quad 0 < \alpha \le 1, \quad 1 \le \beta \le 2. \tag{24}$$

The first term in the right-hand side of Equation (24) represents the long-range interaction and provides an attempt to extend the continuum approach to smaller length scales and to link some aspects of lattice mechanics to continuum theory. The left-right side term with the Caputo fractional derivative describes the history effect on the concentration, but there is no memory effect on the drift.

3.2. The Second Approach

In this case, we have the conservation law (4), and instead of the Fick law (6), we assume its time-nonlocal counterpart:

$$\mathbf{j} = -a \int_0^t K(t-\tau)\, \mathrm{grad}\, c(\tau)\, \mathrm{d}\tau, \tag{25}$$

which leads to the general equation:

$$\frac{\partial c}{\partial t} + \mathbf{v} \cdot \mathrm{grad}\, c = a \int_0^t K(t-\tau)\, \Delta c(\tau)\, \mathrm{d}\tau. \tag{26}$$

For the time-fractional generalization of the Fick law:

$$\mathbf{j} = -aD_{RL}^{1-\alpha}\mathrm{grad}\, c, \quad 0 < \alpha \le 1, \tag{27}$$

we arrive at:

$$\frac{\partial c}{\partial t} + \mathbf{v}\cdot \mathrm{grad}\, c = aD_{RL}^{1-\alpha}\Delta c, \quad 0 < \alpha \le 1, \tag{28}$$

or:

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\Delta c - I^{1-\alpha}\left(\mathbf{v}\cdot \mathrm{grad}\, c\right), \quad 0 < \alpha \le 1. \tag{29}$$

The corresponding space-time-fractional equation reads:

$$\frac{\partial^\alpha c}{\partial t^\alpha} = -a\left(-\Delta\right)^{\beta/2} c - I^{1-\alpha}\left(\mathbf{v}\cdot \mathrm{grad}\, c\right), \quad 0 < \alpha \le 1, \quad 1 \le \beta \le 2. \tag{30}$$

In Equation (30), the space-fractional Laplace operator (the Riesz operator) describes the long-range interaction, whereas the time-fractional operators refer to the memory effects (both on the concentration and the drift in contrast to Equation (24).

4. Fundamental Solutions to the Cauchy Problems

4.1. The First Approach

Consider the space-time-fractional advection-diffusion Equation (24) in the domain $-\infty < x < \infty$:

$$\frac{\partial^\alpha w}{\partial t^\alpha} = a\frac{\partial^\beta w}{\partial |x|^\beta} - v\,\frac{\partial w}{\partial x}, \quad 0 < \alpha \le 1, \quad 1 \le \beta \le 2, \tag{31}$$

under initial condition:

$$t = 0: \ \ w = p_0\,\delta(x). \tag{32}$$

The Laplace transform with respect to time t and exponential Fourier transform with respect to the spatial coordinate x give:

$$\widetilde{w}^*(\xi, s) = \frac{p_0}{\sqrt{2\pi}}\,\frac{s^{\alpha-1}}{s^\alpha + a|\xi|^\beta - iv\xi}. \tag{33}$$

The inverse integral transforms lead to:

$$w(x,t) = \frac{p_0}{2\pi}\int_{-\infty}^{\infty} E_\alpha\left[-\left(a|\xi|^\beta - iv\xi\right)t^\alpha\right]\mathrm{e}^{-ix\xi}\,\mathrm{d}\xi, \tag{34}$$

where the following formula has been used [75–77]:

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\gamma}}{s^\alpha + b}\right\} = t^{\gamma-1}E_{\alpha,\gamma}\left(-bt^\alpha\right) \tag{35}$$

with $E_{\alpha,\gamma}(z)$ being the Mittag–Leffler function in two parameters α and γ:

$$E_{\alpha,\gamma}\left(z\right) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\gamma)}, \quad \alpha > 0, \quad \gamma > 0, \quad z \in C, \tag{36}$$

and $E_\alpha(z) \equiv E_{\alpha,1}(z)$.

When $\alpha = 1$, $E_1(z) = \mathrm{e}^z$ and Equation (34) simplifies:

$$w(x,t)=\frac{p_0}{\pi}\int_0^{\infty}\exp\left(-at\xi^{\beta}\right)\cos\left[(x-vt)\xi\right]\mathrm{d}\xi. \tag{37}$$

4.1.1. Standard Diffusion ($\alpha=1$, $\beta=2$)

In this case, the solution is well known (see, for example, [1,9]):

$$w(x,t)=\frac{p_0}{2\sqrt{\pi at}}\exp\left[-\frac{(x-vt)^2}{4at}\right]. \tag{38}$$

4.1.2. Cauchy Diffusion with $\alpha=1$, $\beta=1$

For the so-called Cauchy diffusion [84], which is characterized by the values $\alpha=\beta=1$, we get:

$$w(x,t)=\frac{p_0}{\pi}\frac{at}{a^2t^2+(x-vt)^2}. \tag{39}$$

4.1.3. Subdiffusion with $\alpha=1/2$

The Mittag–Leffler function $E_{1/2}(-z)$ has the following integral representation [85]:

$$E_{1/2}(-z)=\frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp\left(-u^2-2uz\right)\mathrm{d}u, \tag{40}$$

and the solution reads:

$$w(x,t)=\frac{p_0}{\pi^{3/2}}\int_0^{\infty}\int_{-\infty}^{\infty}\exp\left(-u^2-2ua\sqrt{t}|\xi|^{\beta}\right)\cos\left[\left(x-2uv\sqrt{t}\right)\xi\right]\mathrm{d}\xi\,\mathrm{d}u. \tag{41}$$

Taking in (41) $\beta=2$, we obtain [27]:

$$w(x,t)=\frac{p_0}{\pi\sqrt{2a}t^{1/4}}\int_0^{\infty}\frac{1}{\sqrt{u}}\exp\left[-u^2-\frac{\left(x-2uv\sqrt{t}\right)^2}{8au\sqrt{t}}\right]\mathrm{d}u. \tag{42}$$

For $\beta=1$, we have:

$$w(x,t)=\frac{4a\sqrt{t}p_0}{\pi^{3/2}}\int_0^{\infty}\mathrm{e}^{-u^2}\frac{u}{4a^2tu^2+\left(x-2uv\sqrt{t}\right)^2}\mathrm{d}u. \tag{43}$$

The results of numerical calculations are shown in Figures 1–4 for different values of the orders of derivatives α and β and the drift parameter v. In the calculations, we have introduce the following nondimensional quantities:

$$\overline{w}=\frac{a^{1/\beta}t^{\alpha/\beta}}{p_0}w,\qquad \overline{x}=\frac{1}{a^{1/\beta}t^{\alpha/\beta}}x,\qquad \overline{v}=\frac{t^{\alpha-\alpha/\beta}}{a^{1/\beta}}v. \tag{44}$$

4.2. The Second Approach

In this case, we consider the space-time-fractional advection-diffusion equation:

$$\frac{\partial^{\alpha}c}{\partial t^{\alpha}}=a\frac{\partial^{\beta}c}{\partial|x|^{\beta}}-I^{1-\alpha}\left(v\frac{\partial c}{\partial x}\right),\quad 0<\alpha\leq 1,\quad 1\leq\beta\leq 2, \tag{45}$$

under initial condition:

$$t=0:\;\; c=p_0\,\delta(x). \tag{46}$$

The integral transform technique allows us to obtain the solution in the transform domain:

$$\tilde{c}^*(\xi, s) = \frac{p_0}{\sqrt{2\pi}} \frac{1}{s + a|\xi|^\beta s^{1-\alpha} - iv\xi}. \tag{47}$$

In what follows, we confirm ourselves to the case $\alpha = 1/2$:

$$\tilde{c}^*(\xi, s) = \frac{p_0}{\sqrt{2\pi}} \frac{1}{s + a|\xi|^\beta \sqrt{s} - iv\xi}. \tag{48}$$

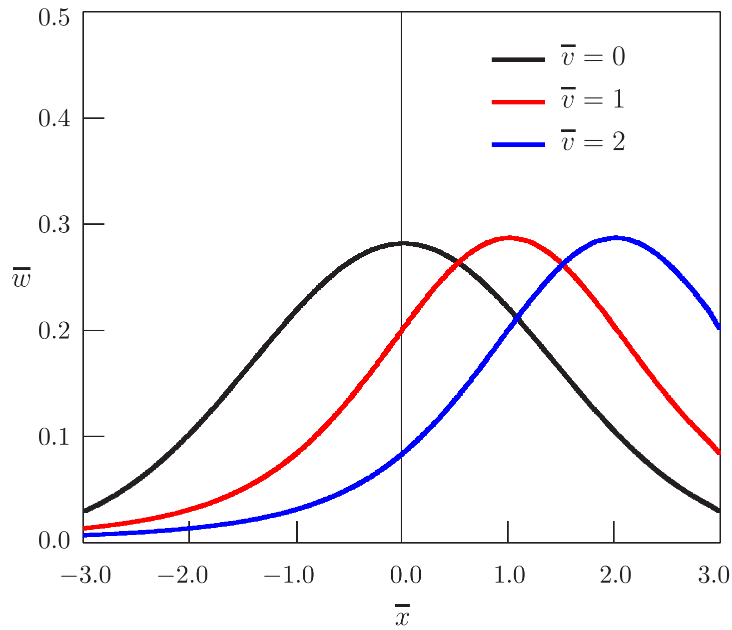

Figure 1. The fundamental solution to the Cauchy problem; Equation (37); $\alpha = 1$, $\beta = 1.5$.

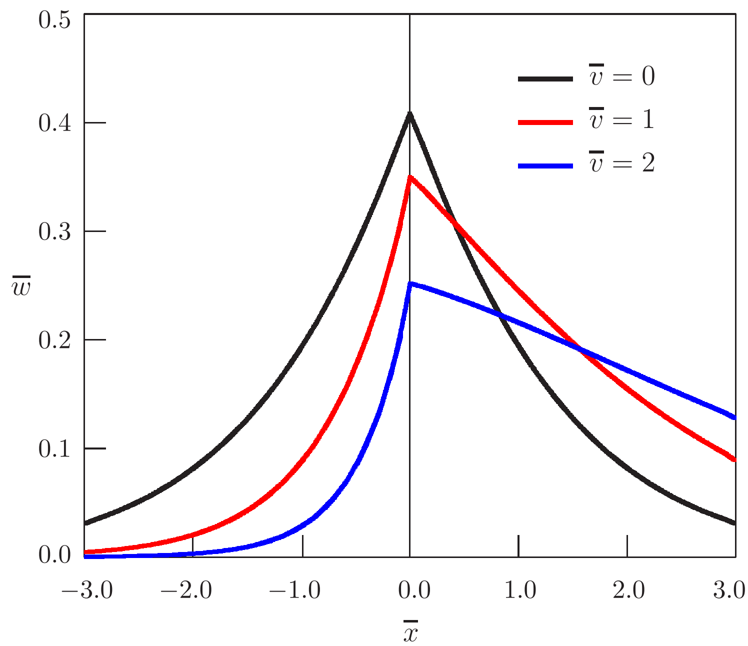

Figure 2. The fundamental solution to the Cauchy problem; the first approach, Equation (42); $\alpha = 0.5$, $\beta = 2$.

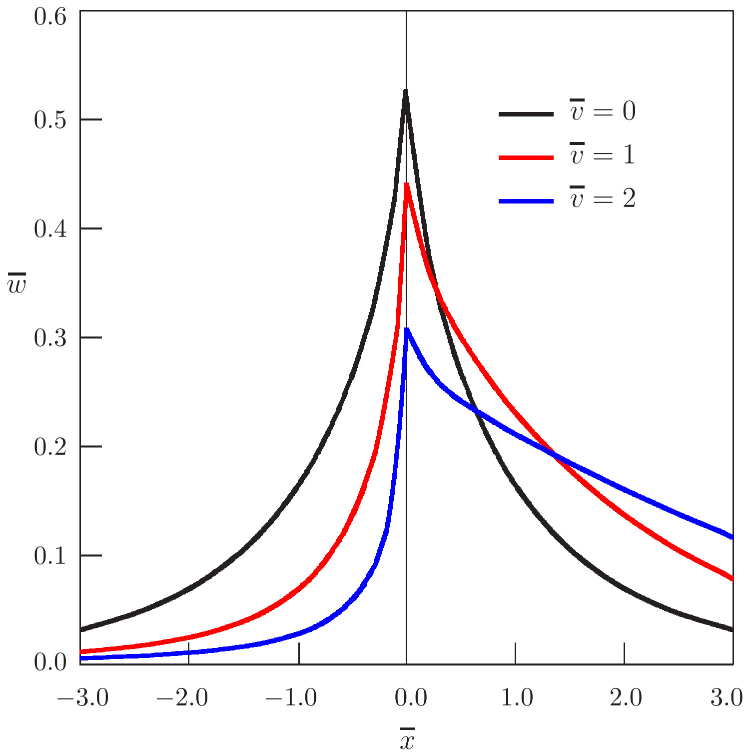

Figure 3. The fundamental solution to the Cauchy problem; the first approach, Equation (41); $\alpha = 0.5$, $\beta = 1.5$.

The partial-fraction decomposition of (48) gives:

$$\begin{aligned}\widetilde{c}^*(\xi, s) &= \frac{p_0}{\sqrt{2\pi}} \frac{1}{\sqrt{a^2|\xi|^{2\beta} + 4iv\xi}} \\ &\times \left[\frac{1}{\sqrt{s} + 0.5\left(a|\xi|^\beta - \sqrt{a^2|\xi|^{2\beta} + 4iv\xi}\right)} - \frac{1}{\sqrt{s} + 0.5\left(a|\xi|^\beta + \sqrt{a^2|\xi|^{2\beta} + 4iv\xi}\right)} \right]\end{aligned} \tag{49}$$

and:

$$\begin{aligned}c(x,t) &= \frac{p_0}{2\pi\sqrt{t}} \int_{-\infty}^{\infty} \mathrm{e}^{-ix\xi} \frac{1}{\sqrt{a^2|\xi|^{2\beta} + 4iv\xi}} \\ &\times \left\{ E_{1/2,1/2}\left[-0.5\left(a|\xi|^\beta - \sqrt{a^2|\xi|^{2\beta} + 4iv\xi}\right)\sqrt{t} \right] \right. \\ &\left. - E_{1/2,1/2}\left[-0.5\left(a|\xi|^\beta + \sqrt{a^2|\xi|^{2\beta} + 4iv\xi}\right)\sqrt{t} \right] \right\} \mathrm{d}\xi.\end{aligned} \tag{50}$$

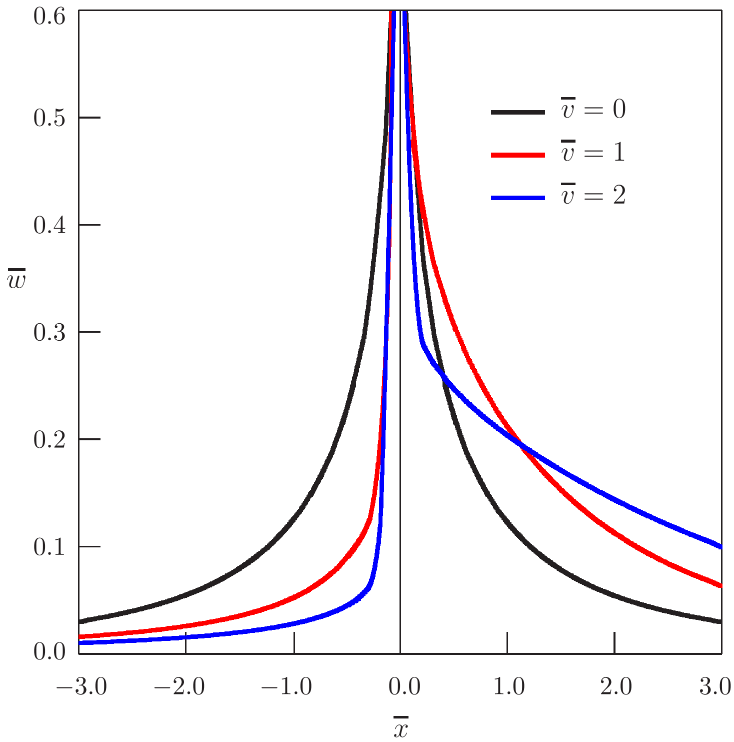

Figure 4. The fundamental solution to the Cauchy problem; the first approach, Equation (43); $\alpha = 0.5$, $\beta = 1$.

Taking into account the integral representation of the Mittag–Leffler function $E_{1/2,1/2}(-z)$ [85]:

$$E_{1/2,1/2}(-z) = \frac{2}{\sqrt{\pi}} \int\limits_0^\infty \exp\left(-u^2 - 2uz\right) u \, \mathrm{d}u, \tag{51}$$

we get:

$$\begin{aligned} c(x,t) \;=\; & \frac{p_0}{\pi^{3/2}\sqrt{t}} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{u}{\sqrt{r}} \exp\left(-u^2 - a\sqrt{t}u|\xi|^\beta\right) \\ & \times \left\{ \exp\left[u\sqrt{rt}\cos(\varphi/2)\right] \cos\left[x\xi + \varphi/2 - u\sqrt{rt}\sin(\varphi/2)\right] \right. \\ & \left. - \exp\left[-u\sqrt{rt}\cos(\varphi/2)\right] \cos\left[x\xi + \varphi/2 + u\sqrt{rt}\sin(\varphi/2)\right] \right\} \mathrm{d}u \, \mathrm{d}\xi, \end{aligned} \tag{52}$$

where:

$$r = \sqrt{a^4|\xi|^{4\beta} + 16v^2\xi^2}, \qquad \varphi = \arctan\left(\frac{4v\xi}{a^2|\xi|^{2\beta}}\right). \tag{53}$$

The solution (52) is shown in Figures 5 and 6 for $\beta = 2$ and $\beta = 1.5$, respectively. The nondimensional quantities are introduced as:

$$\bar{c} = \frac{a^{1/\beta} t^{\alpha/\beta}}{p_0} c, \qquad \bar{x} = \frac{1}{a^{1/\beta} t^{\alpha/\beta}} x, \qquad \bar{v} = \frac{t^{1-\alpha/\beta}}{a^{1/\beta}} v. \tag{54}$$

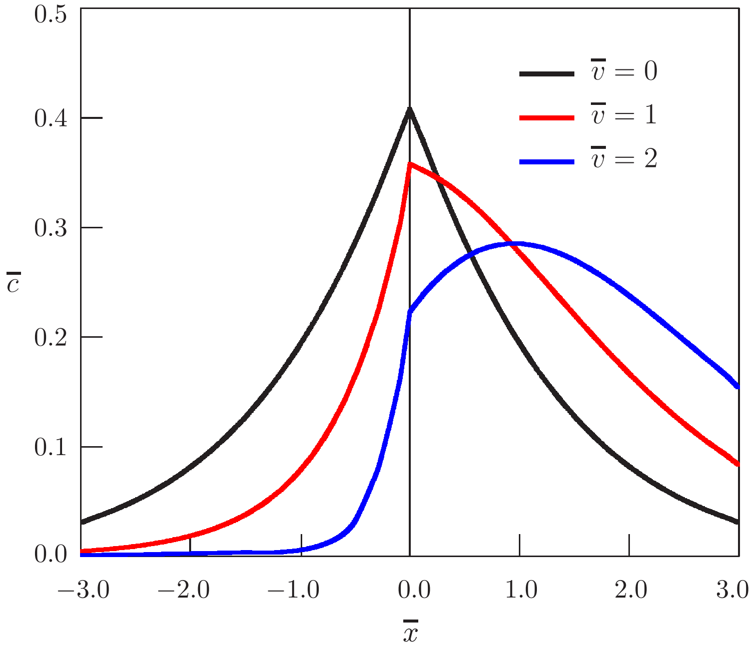

Figure 5. The fundamental solution to the Cauchy problem; the second approach, Equation (52); $\alpha = 0.5$, $\beta = 2$.

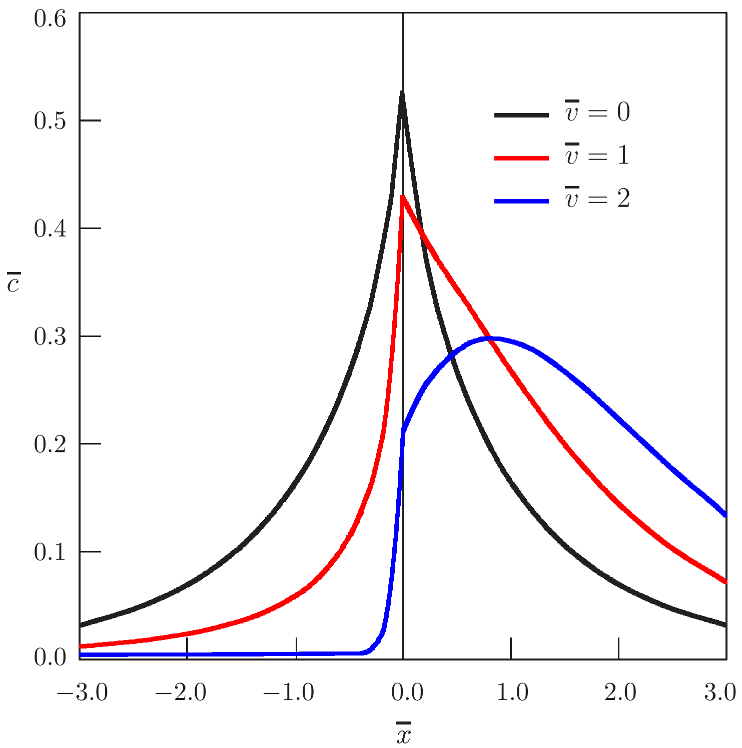

Figure 6. The fundamental solution to the Cauchy problem; the second approach, Equation (52); $\alpha = 0.5$, $\beta = 1.5$.

5. Fundamental Solution to the Source Problem

5.1. The First Approach

Consider the space-time-fractional advection-diffusion Equation (24) with the source term in the domain $-\infty < x < \infty$:

$$\frac{\partial^\alpha w}{\partial t^\alpha} = a\frac{\partial^\beta w}{\partial |x|^\beta} - v\frac{\partial w}{\partial x} + q_0\,\delta(x)\,\delta(t), \quad 0 < \alpha \le 1, \quad 1 \le \beta \le 2, \tag{55}$$

under zero initial condition:

$$t = 0: \quad w = 0. \tag{56}$$

The Laplace transform with respect to time t and exponential Fourier transform with respect to the spatial coordinate x give:

$$\widetilde{w}^*(\xi, s) = \frac{q_0}{\sqrt{2\pi}}\frac{1}{s^\alpha + a|\xi|^\beta - iv\xi}. \tag{57}$$

The inverse integral transforms result in the solution:

$$w(x,t) = \frac{q_0 t^{\alpha-1}}{2\pi}\int_{-\infty}^{\infty} E_{\alpha,\alpha}\left[-\left(a|\xi|^\beta - iv\xi\right)t^\alpha\right]\mathrm{e}^{-ix\xi}\mathrm{d}\xi. \tag{58}$$

Subdiffusion with $\alpha = 1/2$

It follows from Equations (51) and (58) that:

$$w(x,t) = \frac{q_0}{\pi^{3/2}\sqrt{t}}\int_0^\infty\int_{-\infty}^\infty u\,\exp\left(-u^2 - 2ua\sqrt{t}|\xi|^\beta\right)\cos\left[\left(x - 2uv\sqrt{t}\right)\xi\right]\mathrm{d}\xi\,\mathrm{d}u \tag{59}$$

with two particular cases corresponding to $\beta = 2$ [27,56]:

$$w(x,t) = \frac{q_0}{\sqrt{2a}\pi t^{3/4}}\int_0^\infty \exp\left[-u^2 - \frac{\left(x - 2uv\sqrt{t}\right)^2}{8a\sqrt{t}u}\right]\sqrt{u}\,\mathrm{d}u \tag{60}$$

and $\beta = 1$:

$$w(x,t) = \frac{4aq_0}{\pi^{3/2}}\int_0^\infty \mathrm{e}^{-u^2}\frac{u^2}{4a^2tu^2 + \left(x - 2uv\sqrt{t}\right)^2}\mathrm{d}u, \tag{61}$$

respectively.

The results of numerical calculations are shown in Figures 7–9 for different values of the orders of derivatives and the drift parameter v. In the calculations, we have introduce the following nondimensional quantities:

$$\overline{w} = \frac{a^{1/\beta}t^{\alpha/\beta-\alpha+1}}{q_0}w, \qquad \overline{x} = \frac{1}{a^{1/\beta}t^{\alpha/\beta}}x, \qquad \overline{v} = \frac{t^{\alpha-\alpha/\beta}}{a^{1/\beta}}v. \tag{62}$$

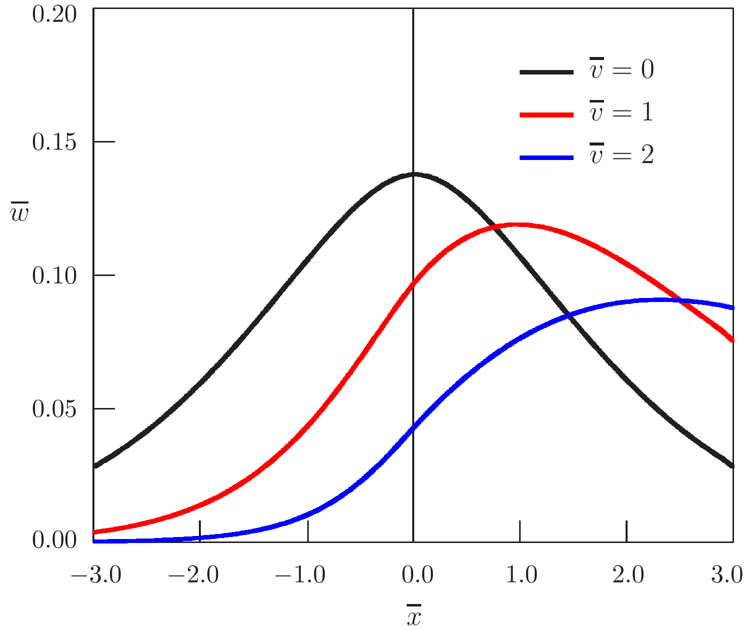

Figure 7. The fundamental solution to the source problem; the first approach, Equation (60); $\alpha = 0.5$, $\beta = 2$.

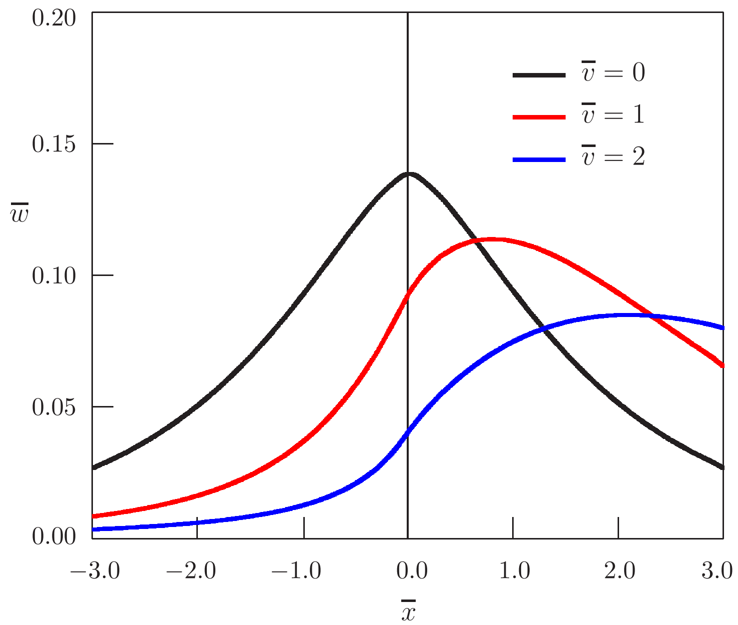

Figure 8. The fundamental solution to the source problem; the first approach, Equation (59); $\alpha = 0.5$, $\beta = 1.5$.

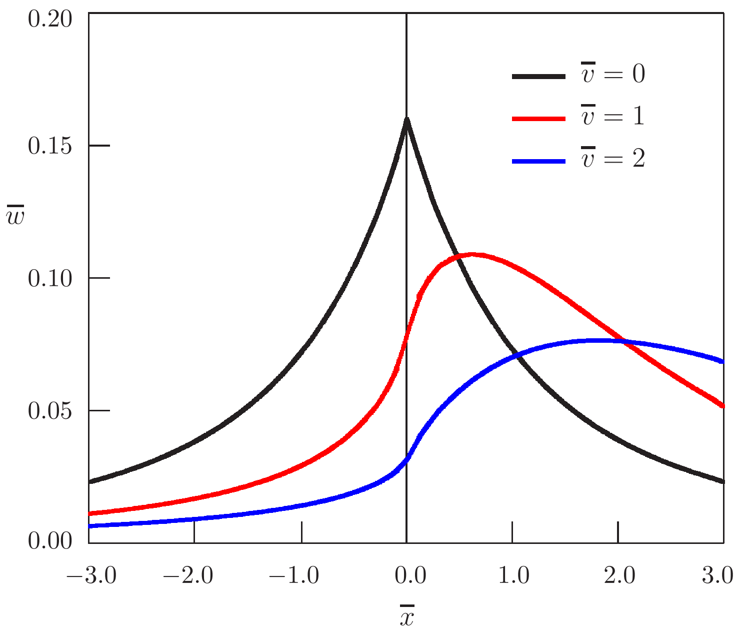

Figure 9. The fundamental solution to the source problem; the first approach, Equation (61); $\alpha = 0.5$, $\beta = 1$.

5.2. The Second Approach

Consider the space-time fractional advection-diffusion Equation (45) with the source term:

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a \frac{\partial^\beta c}{\partial |x|^\beta} - I^{1-\alpha} \left(v \, \frac{\partial c}{\partial x} \right) + q_0 \, \delta(x) \, \delta(t), \quad 0 < \alpha \leq 1, \quad 1 \leq \beta \leq 2, \tag{63}$$

under zero initial condition:

$$t = 0: \quad c = 0. \tag{64}$$

The integral transform technique allows us to obtain the solution in the transform domain:

$$\widetilde{c}^*(\xi, s) = \frac{q_0}{\sqrt{2\pi}} \, \frac{s^{1-\alpha}}{s + a|\xi|^\beta s^{1-\alpha} - iv\xi}. \tag{65}$$

In what follows, we confirm ourselves to the case $\alpha = 1/2$:

$$\widetilde{c}^*(\xi, s) = \frac{q_0}{\sqrt{2\pi}} \, \frac{\sqrt{s}}{s + a|\xi|^\beta \sqrt{s} - iv\xi}. \tag{66}$$

The partial-fraction decomposition of (66) yields:

$$\begin{aligned} \widetilde{c}^*(\xi, s) &= \frac{q_0}{\sqrt{2\pi}} \, \frac{1}{2\sqrt{a^2|\xi|^{2\beta} + 4iv\xi}} \\ &\times \left[\frac{\sqrt{a^2|\xi|^{2\beta} + 4iv\xi} - a|\xi|^\beta}{\sqrt{s} + 0.5\left(a|\xi|^\beta - \sqrt{a^2|\xi|^{2\beta} + 4iv\xi}\right)} + \frac{\sqrt{a^2|\xi|^{2\beta} + 4iv\xi} + a|\xi|^\beta}{\sqrt{s} + 0.5\left(a|\xi|^\beta + \sqrt{a^2|\xi|^{2\beta} + 4iv\xi}\right)} \right] \end{aligned} \tag{67}$$

and:

$$\begin{aligned} c(x,t) &= \frac{q_0}{4\pi\sqrt{t}} \int_{-\infty}^{\infty} \mathrm{e}^{-ix\xi} \frac{1}{\sqrt{a^2|\xi|^{2\beta} + 4iv\xi}} \\ &\times \left\{ \left(\sqrt{a^2|\xi|^{2\beta} + 4iv\xi} - a|\xi|^\beta \right) E_{1/2,1/2} \left[-0.5 \left(a|\xi|^\beta - \sqrt{a^2|\xi|^{2\beta} + 4iv\xi} \right) \sqrt{t} \right] \right. \\ &+ \left. \left(\sqrt{a^2|\xi|^{2\beta} + 4iv\xi} + a|\xi|^\beta \right) E_{1/2,1/2} \left[-0.5 \left(a|\xi|^\beta + \sqrt{a^2|\xi|^{2\beta} + 4iv\xi} \right) \sqrt{t} \right] \right\} \mathrm{d}\xi. \end{aligned} \tag{68}$$

After accounting for the integral representation of the Mittag–Leffler function $E_{1/2,1/2}(-x)$ (51), we arrive at:

$$\begin{aligned} c(x,t) &= \frac{q_0}{2\pi^{3/2}\sqrt{t}} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{u}{\sqrt{r}} \exp\left(-u^2 - a\sqrt{t}u|\xi|^\beta \right) \\ &\times \left\{ \sqrt{r} \exp\left[u\sqrt{rt}\cos(\varphi/2) \right] \cos\left[x\xi - u\sqrt{rt}\sin(\varphi/2) \right] \right. \\ &+ \sqrt{r} \exp\left[-u\sqrt{rt}\cos(\varphi/2) \right] \cos\left[x\xi + u\sqrt{rt}\sin(\varphi/2) \right] \\ &- a|\xi|^\beta \exp\left[u\sqrt{rt}\cos(\varphi/2) \right] \cos\left[x\xi + \varphi/2 - u\sqrt{rt}\sin(\varphi/2) \right] \\ &+ \left. a|\xi|^\beta \exp\left[-u\sqrt{rt}\cos(\varphi/2) \right] \cos\left[x\xi + \varphi/2 + u\sqrt{rt}\sin(\varphi/2) \right] \right\} \mathrm{d}u\,\mathrm{d}\xi. \end{aligned} \tag{69}$$

The solution (69) is presented in Figures 10 and 11 with the nondimensional quantities:

$$\bar{c} = \frac{a^{1/\beta} t^{\alpha/\beta - \alpha + 1}}{q_0} c, \qquad \bar{x} = \frac{1}{a^{1/\beta} t^{\alpha/\beta}} x, \qquad \bar{v} = \frac{t^{1-\alpha/\beta}}{a^{1/\beta}} v. \tag{70}$$

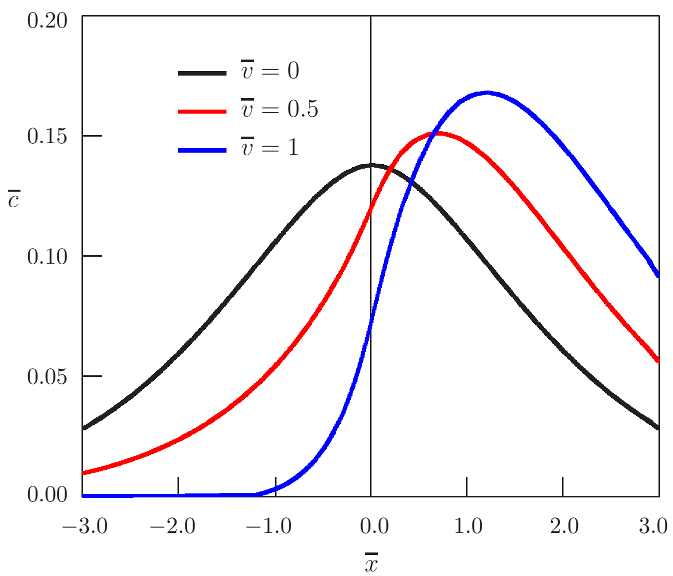

Figure 10. The fundamental solution to the source problem; the second approach, Equation (69); $\alpha = 0.5$, $\beta = 2$.

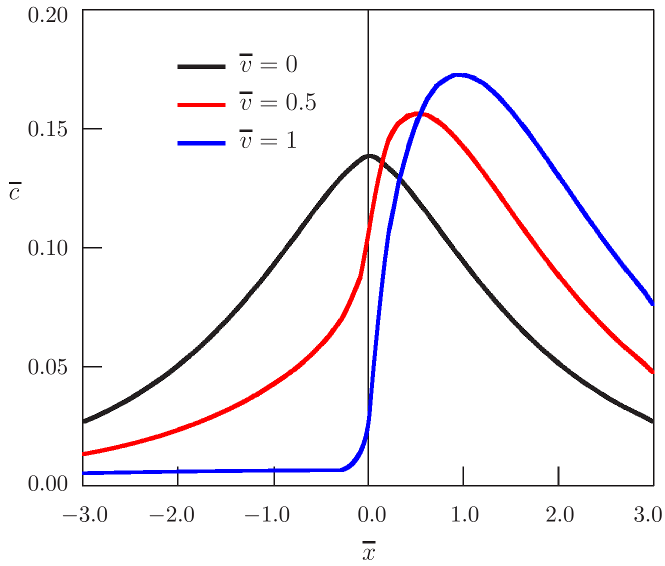

Figure 11. The fundamental solution to the source problem; the second approach, Equation (69); $\alpha = 0.5$, $\beta = 1.5$.

6. Discussion

We have considered two approaches to deriving the space-time fractional advection-diffusion equation. In the case of one spatial dimension, we have studied the fundamental solutions to the Cauchy and source problems for the obtained equations. As is seen from the figures, the solutions corresponding to the first and second approach have different behaviors in the direction of drift ($x > 0$ in the figures). For $\alpha = 1$ and $1 \leq \beta \leq 2$, the solution has no cusp at $x = 0$, the quantity v only causes a drift of the maximum value of the solution in the x-direction ($x - vt$ in the solution (37); the typical curves are shown in Figure 1). For fractional values of α and $1 < \beta \leq 2$, the fundamental solutions to the Cauchy problems have a cusp at $x = 0$. For fractional values of α and $\beta = 1$, the fundamental solution to the Cauchy problem has singularity at $x = 0$, and drift caused by the quantity v is less noticeable as is seen from Figure 4.

For the source problem, the quantity v causes drift of the maximum value of the solution in the x-direction with the significant difference between the solutions in the first and second approaches: in the first approach, the maximum value of the solution decreases with the increasing v, whereas in the second approach, the maximum value of the solution increases with the increasing v. The obtained solutions can also be used for testing numerical algorithms for solving the fractional advection-diffusion equation. The reader interested in evaluation of the Mittag–Leffler functions is referred to the paper [86] and the MATLAB program elaborated by Igor Podlubny [87] that implements the algorithms suggested in [86].

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