entropy

MDPI

Article

Peierls–Bogolyubov's Inequality for Deformed Exponentials

Frank Hansen [1,*,†], Jin Liang [2,†] and Guanghua Shi [2,†]

[1] Institute for Excellence in Higher Education, Tohoku University, Sendai 980-8576, Japan
[2] School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China; jinliang@sjtu.edu.cn (J.L.); sghkanting@163.com (G.S.)
* Correspondence: frank.hansen@m.tohoku.ac.jp; Tel.:+81-22-795-3246
† These authors contributed equally to this work.

Academic Editor: Antonio M. Scarfone
Received: 14 April 2017; Accepted: 5 June 2017; Published: 12 June 2017

Abstract: We study the convexity or concavity of certain trace functions for the deformed logarithmic and exponential functions, and in this way obtain new trace inequalities for deformed exponentials that may be considered as generalizations of Peierls–Bogolyubov's inequality. We use these results to improve previously-known lower bounds for the Tsallis relative entropy.

Keywords: deformed exponential function; Peierls–Bogolyubov's inequality; Tsallis relative entropy

1. Introduction

In statistical mechanics and in quantum information theory, the calculation of the partition function $\operatorname{Tr}\exp H$ of the Hamiltonian H of a physical system is an important issue, but the computation is often difficult. However, it may be simplified by first computing a related quantity $\operatorname{Tr}\exp A$, where A is an easier-to-handle component of the Hamiltonian. Usually, the Hamiltonian is written as a sum $H = A + B$ of two operators, and the Peierls–Bogolyubov inequality states that

$$\log \frac{\operatorname{Tr}\exp(A+B)}{\operatorname{Tr}\exp A} \geq \frac{\operatorname{Tr}\exp(A)B}{\operatorname{Tr}\exp A}, \tag{1}$$

which then provides information about the difficult-to-calculate partition function. In this paper, we give generalizations of Peierls–Bogolyubov's inequality in terms of the so-called deformed exponential and logarithmic functions. We formulate the results for operators on a finite dimensional Hilbert space $\mathcal{H}$, but note that the results with proper modifications also extend to infinite dimensional spaces.

Main Theorem. *Let $A, B \in B(\mathcal{H})$ be self-adjoint operators, and let φ be a positive functional on $B(\mathcal{H})$.*

(i) If $-\infty < q < 1$ and $r \geq q$, and both A and $A+B$ are bounded from above by $-(q-1)^{-1}$, then

$$\log_r \operatorname{Tr}\exp_q(A+B) - \log_r \operatorname{Tr}\exp_q A \geq \bigl(\operatorname{Tr}\exp_q A\bigr)^{r-2}\operatorname{Tr}(\exp_q A)^{2-q}B.$$

(ii) If $-\infty < q \leq 0$ and $r \geq q$, and both A and $A+B$ are bounded from above by $-(q-1)^{-1}$, then

$$\log_r \varphi\bigl(\exp_q(A+B)\bigr) - \log_r \varphi\bigl(\exp_q A\bigr) \geq \varphi\bigl(\exp_q A\bigr)^{r-2}\varphi\bigl(d\exp_q(A)B\bigr).$$

(iii) If $1 < q \leq 2$ and $r \geq q$, and both A and $A+B$ are bounded from below by $-(q-1)^{-1}$, then

$$\log_r \operatorname{Tr}\exp_q(A+B) - \log_r \operatorname{Tr}\exp_q A \geq (\operatorname{Tr}\exp_q A)^{r-2}\operatorname{Tr}(\exp_q A)^{2-q}B.$$

(iv) *If $\frac{3}{2} \leq q \leq 2$ and $r \geq q$, and both A and $A+B$ are bounded from below by $-(q-1)^{-1}$, then*

$$\log_r \varphi(\exp_q(A+B)) - \log_r \varphi(\exp_q A) \geq \varphi(\exp_q A)^{r-2} \varphi(d\exp_q(A)B).$$

(v) *If $q \geq 2$ and $r \leq q$, and both A and $A+B$ are bounded from below by $-(q-1)^{-1}$, then*

$$\log_r \varphi(\exp_q(A+B)) - \log_r \varphi(\exp_q A) \leq \varphi(\exp_q A)^{r-2} \varphi(d\exp_q(A)B).$$

If in particular φ is the trace, this inequality reduces to

$$\log_r \mathrm{Tr}\exp_q(A+B) - \log_r \mathrm{Tr}\exp_q A \leq (\mathrm{Tr}\exp_q A)^{r-2} \mathrm{Tr}(\exp_q A)^{2-q} B.$$

In Section 5.2, we give explicit formulae for the Fréchet differential operators $d\exp_q(A)$ in the parameter ranges $q \leq 0$ and $q \geq 3/2$. Note that the left-hand sides in the above theorem may be written as

$$\frac{\varphi(\exp_q(A+B))^{r-1} - \varphi(\exp_q A)^{r-1}}{r-1},$$

where φ in (i) and (iii) is replaced by the trace. If we in (iii) let q tend to one, we obtain the inequality

$$\log_r \mathrm{Tr}\exp(A+B) - \log_r \mathrm{Tr}\exp A \geq \frac{\mathrm{Tr}\exp(A)B}{(\mathrm{Tr}\exp A)^{2-r}}$$

for $r > 1$ and arbitrary self-adjoint operators A and B. If we furthermore let r tend to one, we recover Peierls–Bogolyubov's inequality (1).

Furuichi ([1], Corollary 3.2) proved (iii) in the case $r = q$ by very different methods. It may be instructive to compare the above results with the first author's study [2] of the deformed Golden–Thompson trace inequality.

In Theorem 6, we obtain another variant Peierls–Bogolyubov type of inequality, and in Theorem 9, we improve previously known lower bounds for the Tsallis relative entropy.

The Peierls–Bogolyubov inequality has been widely used in statistical mechanics and quantum information theory. Recently, Bikchentaev [3] proved that the Peierls–Bogolyubov inequality characterizes the tracial functionals among all positive functionals on a C^*-algebra. Moreover, Carlen and Lieb in [4] combined this inequality with the Golden–Thompson inequality to discover sharp remainder terms in some quantum entropy inequalities.

Deformed Exponentials

The deformed logarithm $\log_q$ is defined by setting

$$\log_q x = \begin{cases} \dfrac{x^{q-1}-1}{q-1} & q \neq 1 \\ \log x & q = 1 \end{cases}$$

for $x > 0$. The deformed logarithm is also denoted as the q-logarithm. The inverse function is called the q-exponential. It is denoted by $\exp_q$, and is given by the formula

$$\exp_q x = \begin{cases} (x(q-1)+1)^{1/(q-1)}, & x > -1/(q-1), \quad q > 1 \\ (x(q-1)+1)^{1/(q-1)}, & x < -1/(q-1), \quad q < 1 \\ \exp x & \quad q = 1. \end{cases}$$

The q-logarithm is for $q > 1$ a bijection of the positive half-line onto the open interval $(-(q-1)^{-1}, \infty)$, and for $q < 1$, a bijection of the positive half-line onto the open interval $(-\infty, -(q-1)^{-1})$. Furthermore,

$$\frac{d}{dx}\log_q(x) = x^{q-2} \qquad \text{and} \qquad \frac{d}{dx}\exp_q(x) = \exp_q(x)^{2-q}\,.$$

Note also that

$$\log_q x - \log_q y = \frac{x^{q-1} - y^{q-1}}{q-1}$$

for $x, y > 0$. If q tends to one, then the q-logarithm and the q-exponential functions converge, respectively, toward the logarithmic and the exponential functions.

2. Preliminaries

Proposition 1. *Let f be a real positive function defined in the cone $B(\mathcal{H})_+$ of positive definite operators acting on a Hilbert space $\mathcal{H}$, and assume f is homogeneous of degree $p \neq 0$.*

(i) If f is convex and $p > 0$, then $f^{1/p}$ is convex.
(ii) If f is convex and $p < 0$, then $f^{1/p}$ is concave.
(iii) If f is convex and $p < 0$ and $r > 0$, then f^r is convex.
(iv) If f is concave and $p > 0$, then $f^{1/p}$ is concave.
(v) If f is concave and $p < 0$, then $f^{1/p}$ is convex.
(vi) If f is concave and $p > 0$ and $r < 0$, then f^r is convex.

Proof. Assume first that f is a convex function. The level set

$$L = \{x \in B(\mathcal{H})_+ \mid f(x) \leq 1\}$$

is then convex. Take $x, y \in B(\mathcal{H})_+$ and assume $p > 0$. Let c and d be any choice of positive numbers such that $f(x)^{1/p} < c$ and $f(y)^{1/p} < d$. We note that $c^{-1}x, d^{-1}y \in L$, and obtain

$$f(x+y)^{1/p} = (c+d) f\Big(\frac{c}{c+d}\cdot\frac{x}{c} + \frac{d}{c+d}\cdot\frac{y}{d}\Big)^{1/p} \leq c+d.$$

Therefore, $f(x+y)^{1/p} \leq f(x)^{1/p} + f(y)^{1/p}$, and by homogeneity, we conclude that $f^{1/p}$ is convex. If $p < 0$, we choose $c, d > 0$ such that $f(x)^{1/p} > c$ and $f(y)^{1/p} > d$. This is possible because f is assumed to be positive. Since the exponent is negative, we obtain $f(x) < c^p$ and $f(y) < d^p$, and therefore by homogeneity

$$f(c^{-1}x) = c^{-p} f(x) < 1 \qquad \text{and} \qquad f(d^{-1}y) = d^{-p} f(y) < 1.$$

It follows that $c^{-1}x, d^{-1}y \in L$, and thus

$$f(x+y)^{1/p} = (c+d) f\Big(\frac{c}{c+d}\cdot\frac{x}{c} + \frac{d}{c+d}\cdot\frac{y}{d}\Big)^{1/p} \geq c+d,$$

where we again used that the exponent is negative. Therefore, $f(x+y)^{1/p} \geq f(x)^{1/p} + f(y)^{1/p}$, and by homogeneity, we conclude that $f^{1/p}$ is concave. This proves (i) and (ii). Under the assumptions in (iii), we proceed as under (ii) to obtain

$$f(x+y)^{1/p} \geq c+d.$$

By homogeneity and since the exponent rp is negative, we obtain the inequality

$$f\Big(\frac{x+y}{2}\Big)^r \leq \Big(\frac{c+d}{2}\Big)^{rp} \leq \frac{c^{rp} + d^{rp}}{2}$$

implying convexity of f^r. We obtain (iv), (v), and (vi) by a variation of the reasoning used to obtain (i), (ii), and (iii). □

Proposition 2. *Consider the function*

$$G(A) = \bigl(\mathrm{Tr} A^p\bigr)^{1/r}$$

defined in positive definite operators. Then,

- *(i)* G *is concave for* $r \le p < 0$,
- *(ii)* G *is convex for* $p < 0$ *and* $r > 0$,
- *(iii)* G *is concave for* $0 < p \le 1$ *and* $r \ge p$,
- *(iv)* G *is convex for* $p \ge 1$ *and* $0 < r \le p$.
- *(v)* G *is convex for* $0 < p \le 1$ *and* $r < 0$.

Proof. Since the real function $t \to t^p$ is convex in positive numbers for $p \le 0$ and $p \ge 1$ and concave for $0 \le p \le 1$, it is well known that the trace function $A \to \mathrm{Tr} A^p$ retains the same properties. A historic account of this result may be found in ([5], Introduction). By (ii) and (i) in Proposition 1, we thus obtain that the function

$$A \to \bigl(\mathrm{Tr} A^p\bigr)^{1/p}$$

is concave for $p < 0$ and convex for $p \ge 1$. Furthermore, since the real function $t \to t^{p/r}$ is concave and increasing for $r \le p < 0$, we derive (i) in the assertion. Part (ii) then follows by Proposition 1 (iii), and Part (iii) follows from Proposition 1 (iv) by noting that $0 < p/r \le 1$. Part (iv) follows from Proposition 1 (i) by noting that $p/r \ge 1$, and part (v) finally follows from Proposition 1 (vi). □

Note that $(\mathrm{Tr}\, A^p)^{1/p}$ for $p \ge 1$ is the Schatten p-norm of the positive definite matrix A. The convexity in this case may also be derived by noting that a norm satisfies the triangle inequality and is positively homogeneous.

Proposition 3. *Let* $B \in B(\mathcal{H})$ *be an arbitrary operator and consider the function*

$$F(A) = \bigl(\mathrm{Tr}\, B^* A^p B\bigr)^{1/r}$$

defined in positive definite operators. Then,

- *(i)* F *is concave for* $-1 \le p < 0$ *and* $r \le p$,
- *(ii)* F *is convex for* $-1 \le p < 0$ *and* $r > 0$,
- *(iii)* F *is concave for* $0 < p \le 1$ *and* $r \ge p$,
- *(iv)* F *is convex for* $1 \le p \le 2$ *and* $0 < r \le p$,
- *(v)* F *is convex for* $0 < p \le 1$ *and* $r < 0$.

Proof. By continuity, we may assume BB^* invertible. Since the function $t \to t^p$ is operator convex for $-1 \le p \le 0$ and for $1 \le p \le 2$, it follows that the trace function $A \to \mathrm{Tr}\, B^* A^p B$ is convex for these parameter values. It then follows by (ii) and (i) in Proposition 1 that the function

$$A \to (\mathrm{Tr}\, B^* A^p B)^{1/p}$$

is concave for $-1 \le p < 0$ and convex for $1 \le p \le 2$. Furthermore, since the real function $t \to t^{p/r}$ is concave and increasing for $r \le p < 0$, we derive part (i) in the assertion. Part (ii) then follows by Proposition 1 (iii). Parts (iii) to (v) now follow by minor variations of the reasoning in the preceding theorem. □

2.1. Some Deformed Trace Functions

Theorem 4. *Consider the function*

$$G(A) = \log_r \operatorname{Tr} \exp_q(A)$$

defined in self-adjoint $A > -(q-1)^{-1}$ for $q > 1$, and in self-adjoint $A < -(q-1)^{-1}$ for $q < 1$.

(i) *If $-\infty < q < 1$ and $r \geq q$, then G is convex,*
(ii) *If $1 < q \leq 2$ and $r \geq q$, then G is convex,*
(iii) *If $q \geq 2$ and $r \leq q$, then G is concave.*

Proof. Note that the conditions on A ensure that $A(q-1)+1 > 0$ for both $q < 1$ and $q > 1$. By calculation, we obtain

$$\begin{aligned} G(A) &= \log_r \operatorname{Tr} \exp_q(A) \\ &= \frac{1}{r-1}\left(\left(\operatorname{Tr}(A(q-1)+1)^{1/(q-1)}\right)^{r-1} - 1\right). \end{aligned}$$

Under the assumptions in (i), we obtain

$$\frac{1}{r-1} \leq \frac{1}{q-1} < 0$$

for $q \leq r < 1$. By Proposition 2(i) and since the factor $(r-1)^{-1}$ is negative, it follows that G is convex. If $r > 1$ then $(r-1)^{-1} > 0$ and the convexity of G follows by Proposition 2 (ii). The case $r = 1$ follows by continuity. This proves the first statement. Under the assumptions in (ii) we obtain

$$\frac{1}{q-1} \geq 1 \qquad \text{and} \qquad 0 < \frac{1}{r-1} \leq \frac{1}{q-1},$$

and thus G is convex by Proposition 2 (iv). Under the assumptions in (iii) we first consider the case $r > 1$ and obtain

$$0 < \frac{1}{q-1} \leq 1 \qquad \text{and} \qquad \frac{1}{r-1} \geq \frac{1}{q-1},$$

and thus G is concave by Proposition 2 (iii). If $r < 1$, then we use Proposition 2(v) to obtain that $(r-1)G$ is convex. Since $r-1 < 0$, we conclude that G is also concave in this case. The case $r = 1$ follows by continuity. □

Theorem 5. *Let B be arbitrary and consider the function*

$$F(A) = \log_r \operatorname{Tr} B^* \exp_q(A) B$$

defined in self-adjoint $A > -(q-1)^{-1}$ for $q > 1$, and in self-adjoint $A \leq -(q-1)^{-1}$ for $q < 1$.

(i) *If $-\infty < q \leq 0$ and $r \geq q$, then F is convex,*
(ii) *If $\frac{3}{2} \leq q \leq 2$ and $r \geq q$, then F is convex,*
(iii) *If $q \geq 2$ and $r \leq q$, then F is concave.*

Proof. By calculation, we obtain

$$\begin{aligned} F(A) &= \log_r \operatorname{Tr} B^* \exp_q(A) B \\ &= \frac{1}{r-1}\left(\left(\operatorname{Tr} B^*(A(q-1)+1)^{1/(q-1)} B\right)^{r-1} - 1\right). \end{aligned}$$

Under the assumptions in (i), we obtain

$$-1 \leq \frac{1}{q-1} < 0 \qquad \text{and} \qquad \frac{1}{r-1} \leq \frac{1}{q-1}$$

for $q \leq r < 1$. By Proposition 3 (i) and since the factor $(r-1)^{-1}$ is negative, it follows that F is convex. If $r > 1$ then $(r-1)^{-1} > 0$ and the convexity of F follows by Proposition 3 (ii). The case $r = 1$ follows by continuity. This proves the first statement. Under the assumptions in (ii), we obtain

$$1 \leq \frac{1}{q-1} \leq 2 \qquad \text{and} \qquad \frac{1}{r-1} \leq \frac{1}{q-1},$$

and thus F is convex by Proposition 3 (iv). The last case is argued as in the preceding theorem by considering the cases $r > 1$ and $r < 1$ separately. □

Note that there is a gap between 0 and $3/2$ for the values of the parameter q in the above theorem. This is unavoidable for a general operator B.

3. Peierls–Bogolyubov-Type Inequalities

We first obtain a variant Peierls–Bogolyubov-type inequality as a consequence of Proposition 2. Take positive definite operators $A, B \in B(\mathcal{H})$ and define the function

$$g(t) = G(A + tB) = \bigl(\mathrm{Tr}\,(A + tB)^p\bigr)^{1/r} \qquad t \in [0, 1].$$

Since $g(t)$ is convex for $p \geq 1$ and $0 < r \leq p$, we obtain the inequality

$$g(1) - g(0) \geq \frac{g(t) - g(0)}{t} \qquad 0 < t \leq 1 \tag{2}$$

for these parameter values. By concavity, we obtain the opposite inequality for the parameter values $0 < p \leq 1$ and $r \geq p$, and for the parameter values $r \leq p < 0$.

Theorem 6. *Let $A, B \in B(\mathcal{H})$ be positive definite operators.*

(i) *For $p \geq 1$ and $0 < r \leq p$, we have the inequality*

$$\bigl(\mathrm{Tr}\,(A+B)^p\bigr)^{1/r} - \bigl(\mathrm{Tr} A^p\bigr)^{1/r} \geq \frac{p}{r}\bigl(\mathrm{Tr}\,A^p\bigr)^{(1-r)/r}\mathrm{Tr} A^{p-1}B.$$

(ii) *For $0 < p \leq 1$ and $r \geq p$, and for $r \leq p < 0$, we have the opposite inequality*

$$\bigl(\mathrm{Tr}\,(A+B)^p\bigr)^{1/r} - \bigl(\mathrm{Tr}\,A^p\bigr)^{1/r} \leq \frac{p}{r}\bigl(\mathrm{Tr}\,A^p\bigr)^{(1-r)/r}\mathrm{Tr}\,A^{p-1}B.$$

Proof. With the parameter values in (i), we may let t tend to zero in (2) and obtain the inequality $g(1) - g(0) \geq g'(0)$. We note that $g(1) - g(0)$ is the left-hand side in the desired inequality. Furthermore,

$$\begin{aligned} g'(0) &= d\bigl(\mathrm{Tr}\,A^p\bigr)^{1/r}B = \frac{1}{r}\bigl(\mathrm{Tr}\,A^p\bigr)^{(1-r)/r}\,d\bigl(\mathrm{Tr}\,A^p\bigr)B \\ &= \frac{1}{r}\bigl(\mathrm{Tr}\,A^p\bigr)^{(1-r)/r}\mathrm{Tr}\,d(A^p)B = \frac{p}{r}\bigl(\mathrm{Tr}\,A^p\bigr)^{(1-r)/r}\mathrm{Tr}\,A^{p-1}B, \end{aligned}$$

where we used the chain rule for Fréchet differentiation, the linearity of the trace, and the formula in ([6], Theorem 2.2). This proves case (i). Case (ii) follows by virtually the same argument using the opposite inequality in (2). □

We then explore consequences of Theorem 4. If $-\infty < q < 1$, we take self-adjoint operators $A, B \in B(\mathcal{H})$ such that both A and $A+B$ are bounded from above by $-(q-1)^{-1}$. For $t \in [0,1]$, we note that $A + tB = (1-t)A + t(A+B) < -(q-1)^{-1}$ such that $(q-1)(A+tB)+1 > 0$. The function

$$h(t) = \log_r \mathrm{Tr}\exp_q(A+tB) \qquad t \in [0,1] \tag{3}$$

is thus well-defined and convex for $-\infty < q < 1$ and $r \geq q$. Therefore,

$$h(1) - h(0) \geq \frac{h(t)-h(0)}{t} \qquad 0 < t \leq 1 \tag{4}$$

for these parameter values.

For $q > 1$, we take self-adjoint operators $A, B \in B(\mathcal{H})$ such that both A and $A+B$ are bounded from below by $-(q-1)^{-1}$. For $t \in [0,1]$, we note that $A + tB = (1-t)A + t(A+B) > -(q-1)^{-1}$ such that $(q-1)(A+tB)+1 > 0$. The function defined in (3) is thus well-defined. It is convex for $1 < q \leq 2$ and $r \geq q$, and it is concave for $q \geq 2$ and $r \leq q$. In the first case, we thus retain the inequality in (4), while the inequality is reversed in the latter case.

Theorem 7. *Let $A, B \in B(\mathcal{H})$ be self-adjoint operators.*

(i) *If $-\infty < q < 1$ and $r \geq q$, and both A and $A+B$ are bounded from above by $-(q-1)^{-1}$, then*

$$\log_r \mathrm{Tr}\exp_q(A+B) - \log_r \mathrm{Tr}\exp_q A \geq \big(\mathrm{Tr}\exp_q A\big)^{r-2}\mathrm{Tr}(\exp_q A)^{2-q}B.$$

(ii) *If $1 < q \leq 2$ and $r \geq q$, and both A and $A+B$ are bounded from below by $-(q-1)^{-1}$, then*

$$\log_r \mathrm{Tr}\exp_q(A+B) - \log_r \mathrm{Tr}\exp_q A \geq \big(\mathrm{Tr}\exp_q A\big)^{r-2}\mathrm{Tr}(\exp_q A)^{2-q}B.$$

(iii) *If $q \geq 2$ and $r \leq q$, and both A and $A+B$ are bounded from below by $-(q-1)^{-1}$, then*

$$\log_r \mathrm{Tr}\exp_q(A+B) - \log_r \mathrm{Tr}\exp_q A \leq \big(\mathrm{Tr}\exp_q A\big)^{r-2}\mathrm{Tr}(\exp_q A)^{2-q}B.$$

Proof. With the parameter values in (i) we may let t tend to zero in (4) and obtain the inequality $h(1) - h(0) \geq h'(0)$. We note that $h(1)-h(0)$ is the left-hand side in the desired inequality. Furthermore,

$$\begin{aligned} h'(0) &= d\big(\log_r \mathrm{Tr}\exp_q A\big)B = (\mathrm{Tr}\exp_q A)^{r-2}\, d\big(\mathrm{Tr}\exp_q A\big)B \\ &= (\mathrm{Tr}\exp_q A)^{r-2}\mathrm{Tr}\, d\exp_q(A)B = (\mathrm{Tr}\exp_q A)^{r-2}\mathrm{Tr}\exp_q'(A)B \\ &= (\mathrm{Tr}\exp_q A)^{r-2}\mathrm{Tr}(\exp_q A)^{2-q}B, \end{aligned}$$

where we used the chain rule for Fréchet differentiation, the derivatives of the deformed logarithmic and exponential functions, the linearity of the trace, and the formula in ([6], Theorem 2.2). This proves case (i). The other cases follow by a variation of this reasoning. □

By a similar line of arguments as in the two previous theorems, we finally obtain the following consequences of Theorem 5.

Theorem 8. *Let $C \in B(\mathcal{H})$ be arbitrary and $A, B \in B(\mathcal{H})$ be self-adjoint.*

(i) *If $-\infty < q \leq 0$ and $r \geq q$, and both A and $A+B$ are bounded from above by $-(q-1)^{-1}$, then*

$$\log_r \mathrm{Tr}\, C^* \exp_q(A+B)C - \log_r \mathrm{Tr}\, C^* \exp_q(A)C \geq \big(\mathrm{Tr}\, C^* \exp_q(A)C\big)^{r-2}\mathrm{Tr}\, C^*\big(d\exp_q(A)B\big)C.$$

(ii) *If* $\frac{3}{2} \leq q \leq 2$ *and* $r \geq q$*, and both* A *and* $A+B$ *are bounded from below by* $-(q-1)^{-1}$*, then*

$$\log_r \operatorname{Tr} C^* \exp_q(A+B)C - \log_r \operatorname{Tr} C^* \exp_q(A)C \geq \big(\operatorname{Tr} C^* \exp_q(A)C\big)^{r-2} \operatorname{Tr} C^* \big(d\exp_q(A)B\big)C.$$

(iii) *If* $q \geq 2$ *and* $r \leq q$*, and both* A *and* $A+B$ *are bounded from below by* $-(q-1)^{-1}$*, then*

$$\log_r \operatorname{Tr} C^* \exp_q(A+B)C - \log_r \operatorname{Tr} C^* \exp_q(A)C \leq \big(\operatorname{Tr} C^* \exp_q(A)C\big)^{r-2} \operatorname{Tr} C^* \big(d\exp_q(A)B\big)C.$$

Proof. We follow a similar path as in the proof of Theorem 7 and consider the function

$$h(t) = \log_r \operatorname{Tr} C^* \exp_q(A+tB)C \qquad t \in [0,1] \tag{5}$$

which by Theorem 5 is convex for the parameter values in (i). We obtain by an argument similar to the one given in the proof of Theorem 7 that $h(1) - h(0) \geq h'(0)$, and we note that $h(1) - h(0)$ is the left-hand side in the desired inequality. Furthermore,

$$\begin{aligned} h'(0) &= d\big(\log_r \operatorname{Tr} C^* \exp_q(A)C\big)B \\ &= (\operatorname{Tr} C^* \exp_q(A)C)^{r-2}\, d\big(\operatorname{Tr} C^* \exp_q(A)C\big)B \\ &= (\operatorname{Tr} C^* \exp_q(A)C)^{r-2} \operatorname{Tr} C^* \big(d\exp_q(A)B\big)C, \end{aligned}$$

where we used the chain rule for Fréchet differentiation, the derivative of the deformed logarithmic function, and the linearity of the trace. This proves case (i). Since the function h in (5) is convex for the parameter values in (ii) and concave for the parameter values in (iii), these cases follow by virtually the same line of arguments as in (i). □

Note that (iii) in Theorem 8 is a generalization of (iii) in Theorem 7. Since C is arbitrary, we may in the above theorem replace the trace by any other positive functional on $B(\mathcal{H})$. The main theorem now follows from Theorem 7 and Theorem 8.

4. The Tsallis Relative Entropy

In this section, we study lower bounds for the (generalized) Tsallis relative entropy. For basic information about the Tsallis entropy and the Tsallis relative entropy, we refer the reader to references [7,8].

The Tsallis relative entropy $D_p(X \mid Y)$ is for positive definite operators $X, Y \in B(\mathcal{H})$ and $p \in [0,1)$ defined by setting

$$D_p(X \mid Y) = \frac{\operatorname{Tr}(X - X^p Y^{1-p})}{1-p} = \operatorname{Tr} X^p (\log_{2-p} X - \log_{2-p} Y).$$

By letting p tend to one, this expression converges to the relative quantum entropy

$$U(X \mid Y) = \operatorname{Tr} X(\log X - \log Y)$$

introduced by Umegaki [9]. It is known ([10], Proposition 2.4) that the Tsallis relative entropy is non-negative for states. This also follows directly from the following:

Lemma 1. *Let* ρ *and* σ *be states. Then,*

$$\operatorname{Tr} \rho^{1-p} \sigma^p \leq 1$$

for $0 \leq p \leq 1$.

Proof. Consider states ρ and σ, and let $E \subseteq [0,1]$ be the set of exponents p such that $\mathrm{Tr}\rho^{1-p}\sigma^p \leq 1$. We take $p, q \in E$ and obtain

$$
\begin{aligned}
&\operatorname{Tr} \rho^{1-(p+q)/2} \sigma^{(p+q)/2} = \operatorname{Tr} \rho^{(1-p)/2} \rho^{(1-q)/2} \sigma^{p/2} \sigma^{q/2} \\
&= \operatorname{Tr} \sigma^{p/2} \rho^{(1-p)/2} \rho^{(1-q)/2} \sigma^{q/2} = \operatorname{Tr}(\rho^{(1-p)/2} \sigma^{p/2})^* \rho^{(1-q)/2} \sigma^{q/2} \\
&\leq \left(\operatorname{Tr}(\rho^{(1-p)/2} \sigma^{p/2})^* \rho^{(1-p)/2} \sigma^{p/2}\right)^{1/2} \left(\operatorname{Tr}(\rho^{(1-q)/2} \sigma^{q/2})^* \rho^{(1-q)/2} \sigma^{q/2}\right)^{1/2} \\
&= \left(\operatorname{Tr} \rho^{1-p} \sigma^p\right)^{1/2} \left(\operatorname{Tr} \rho^{1-q} \sigma^q\right)^{1/2} \leq 1,
\end{aligned}
$$

where we used Cauchy–Schwarz' inequality. This shows that E is midpoint-convex. Since E is also closed and $0, 1 \in E$, we conclude that $E = [0,1]$. □

Theorem 9. *Let $q \in (0,1]$ and take $p \leq q$. Then, for positive definite operators $X, Y \in B(\mathcal{H})$, the inequality*

$$
\frac{\operatorname{Tr} X - (\operatorname{Tr} X)^p (\operatorname{Tr} Y)^{1-p}}{1-p} \leq D_q(X \mid Y)
$$

is valid, where by convention $D_1(X \mid Y) = U(X \mid Y)$.

Proof. Let $X, Y \in B(\mathcal{H})$ be positive definite operators and take $1 < q \leq 2$ and $r \geq q$. By setting

$$
A = \log_q X \qquad \text{and} \qquad B = \log_q Y - \log_q X
$$

we obtain self-adjoint A, B such that both A and $A + B$ are bounded from below by $-(q-1)^{-1}$. We may thus apply (ii) in Theorem 7 and obtain after a little calculation the inequality

$$
\frac{\operatorname{Tr} X - (\operatorname{Tr} X)^{2-r} (\operatorname{Tr} Y)^{r-1}}{r-1} \leq \operatorname{Tr} X^{2-q} (\log_q X - \log_q Y).
$$

By setting $p = 2 - r$ and renaming q by $2 - q$, we obtain the stated inequality for $q \in (0,1]$ and $p \leq q$. □

The lower bound of the Tsallis relative entropy $D_q(X \mid Y)$ in Theorem 9 was obtained in ([10], Theorem 3.3) in the special case $p = q$. The family of lower bounds given above is in general not an increasing function in the parameter p, and may therefore—depending on $\operatorname{Tr} X$ and $\operatorname{Tr} Y$—provide better lower bounds.

5. Various Fréchet Differentials

In order to obtain a more detailed understanding of the bounds obtained in the Main Theorem, we need to provide explicit formulae for the Fréchet differential operator $d\exp_q$ in the parameter ranges $q \leq 0$ and $q \geq 3/2$. The integral representation

$$
t^p = \frac{\sin p\pi}{\pi} \int_0^\infty \frac{t}{t+\lambda} \lambda^{p-1} \, d\lambda \qquad t > 0 \tag{6}
$$

valid for $0 < p < 1$ is well-known. Since $t \to t^p$ is operator monotone, this representation may be quite easily derived by calculating the representing measure; e.g., ([11], Theorem 5.5). Furthermore, since by an elementary calculation

$$
d\left(\frac{x}{x+\lambda}\right) h = \lambda (x+\lambda)^{-1} h (x+\lambda)^{-1},
$$

we obtain the integral representation

$$d(x^p)h = \frac{\sin p\pi}{\pi} \int_0^\infty (x+\lambda)^{-1} h (x+\lambda)^{-1} \lambda^p \, d\lambda, \qquad 0 < p < 1 \tag{7}$$

valid for positive definite x. Since by (6) we have

$$t^{p-1} = \frac{\sin p\pi}{\pi} \int_0^\infty \frac{1}{t+\lambda} \lambda^{p-1} \, d\lambda \qquad t > 0 \tag{8}$$

for $0 < p < 1$ and

$$d\left(\frac{1}{x+\lambda}\right) h = (x+\lambda)^{-1} h (x+\lambda)^{-1},$$

we obtain the integral representation

$$d(x^p)h = \frac{\sin(p+1)\pi}{\pi} \int_0^\infty (x+\lambda)^{-1} h (x+\lambda)^{-1} \lambda^p \, d\lambda, \qquad -1 < p < 0 \tag{9}$$

valid for positive definite x. By using the rule for the Fréchet differential of a product, or by an elementary direct calculation, we obtain the general identity

$$d(x^{p+1})h = hx^p + x\, d(x^p)h, \tag{10}$$

which combined with (7) provides a formula for the Fréchet differential of x^p for $1 < p < 2$. If h is self-adjoint, the formula in (10) may be written in the form

$$d(x^{p+1})h = \frac{hx^p + x^p h}{2} + d(x^p)\frac{xh + hx}{2}$$

which is manifestly self-adjoint.

5.1. The Deformed Logarithm

By setting $t = 1$ in (6), we obtain

$$1 = \frac{\sin p\pi}{\pi} \int_0^\infty \frac{\lambda^{p-1}}{1+\lambda} \, d\lambda$$

and thus

$$t^p - 1 = \frac{\sin p\pi}{\pi} \int_0^\infty \left(\frac{t(1+\lambda)}{t+\lambda} - 1 \right) \frac{\lambda^{p-1}}{1+\lambda} \, d\lambda = \frac{\sin p\pi}{\pi} \int_0^\infty \frac{t-1}{t+\lambda} \frac{\lambda^p}{1+\lambda} \, d\lambda$$

for $0 < p < 1$ and $t > 0$. We therefore obtain the following integral representation of the deformed logarithm

$$\log_q t = \frac{t^{q-1} - 1}{q-1} = \frac{\sin(q-1)\pi}{(q-1)\pi} \int_0^\infty \frac{t-1}{t+\lambda} \frac{\lambda^{q-1}}{1+\lambda} \, d\lambda \qquad t > 0 \tag{11}$$

valid for $1 < q < 2$. Since by an elementary calculation

$$d\left(\frac{x-1}{x+\lambda}\right) h = (1+\lambda)(x+\lambda)^{-1} h (x+\lambda)^{-1}$$

we derive the formula

$$d\log_q(x)h = \frac{\sin(q-1)\pi}{(q-1)\pi} \int_0^\infty (x+\lambda)^{-1} h (x+\lambda)^{-1} \lambda^{q-1} \, d\lambda \tag{12}$$

valid for positive definite x and $1 < q < 2$. Note that

$$d\log_q(x)h = \frac{1}{q-1}\, d(x^{q-1})h \tag{13}$$

for all $q \neq 1$ by the definition of the deformed logarithm. If we in formula (12) let q tend to 1, we obtain

$$d\log(x)h = \int_0^\infty (x+\lambda)^{-1}h(x+\lambda)^{-1}\, d\lambda$$

as expected. If we instead set $h = 1$, we recover the classical integral

$$t^{q-2} = \frac{\sin(q-1)\pi}{(q-1)\pi} \int_0^\infty \frac{\lambda^{q-1}}{(t+\lambda)^2}\, d\lambda$$

valid for $t > 0$ and $1 < q < 2$.

5.2. The Deformed Exponential

We next derive integral representations for the deformed exponential in the parameter ranges $q \leq 0$ and $q \geq 3/2$. We first note that

$$d\exp_q(x)h = (q-1)\, d\big(y^{1/(q-1)}\big)h, \qquad y = x(q-1)+1$$

defined in $x > -(q-1)^{-1}$ for $q > 1$, and defined in $x < -(q-1)^{-1}$ for $q < 1$. We divide the analysis into six cases:

1. If $q < 0$, then $-1 < (q-1)^{-1} < 0$ and we may calculate $d(y^{1/(q-1)})h$ by the formula in (9).
2. If $q = 0$, then $d(y^{1/(q-1)})h = d(y^{-1})h = -y^{-1}hy^{-1}$.
3. If $q = \frac{3}{2}$, then $d(y^{1/(q-1)})h = d(y^2)h = yh + hy$.
4. If $\frac{3}{2} < q < 2$, then $(q-1)^{-1} = p + 1$ for some $p \in (0,1)$, and we may calculate $d(y^{1/(q-1)})h$ by the formulae in (7) and (10).
5. If $q = 2$, then $d(y^{1/(q-1)})h = h$.
6. If $q > 2$, then $0 < (q-1)^{-1} < 1$ and we may calculate $d(y^{1/(q-1)})h$ by the formula in (7).

Let q be arbitrary and take x in the domain of $\exp_q$. Then

$$\mathrm{Tr}\, d\exp_q(x)h = \mathrm{Tr}\exp_q(x)^{2-q}h.$$

Likewise,

$$d\exp_q(x)h = \exp_q(x)^{2-q}h$$

for commuting x and h.

Acknowledgments: The authors would like to thank the anonymous referees for helpful suggestions. The first author acknowledges support by the Japanese Grant-in-Aid for scientific research 17K05267 and by the National Science Foundation of China 11301025. The second and third authors acknowledge support from the National Science Foundation of China 11571229.

Author Contributions: All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Furuichi, S. Trace inequalities in non-extensive statistical mechanics. *Linear Algebra Appl.* **2006**, *418*, 821–827.
2. Hansen, F. Golden-Thompson's inequality for deformed exponentials. *J. Stat. Phys.* **2015**, *159*, 1300–1305.
3. Bikchentaev, A.M. The Peierls-Bogoliubov inequality in C*-algebras and characterization of tracial functionals. *Lobachevskii J. Math.* **2011**, *32*, 175–179.

4. Carlen, E.A.; Lieb, E.H. Remainder terms for some quantum entropy inequalities. *J. Math. Phys.* **2014**, *55*, 042201.
5. Lieb, E.; Pedersen, G.K. Convex multivariable trace functions. *Rev. Math. Phys.* **2002**, *14*, 631–648.
6. Hansen, F.; Pedersen, G.K. Perturbation formulas for traces on C^*-algebras. *Publ. RIMS Kyoto Univ.* **1995**, *31*, 169–178.
7. Tsallis, C. Possible generalization of Bolzmann-Gibbs statistics. *J. Stat. Phys.* **1988**, *52*, 479–487.
8. Tsallis, C. Nonadditive entropy and nonextensive statistical mechanics—An overview after 20 years. *Braz. J. Phys.* **2009**, *39*, 337–356.
9. Umegaki, H. Conditional expectation in an operator algebra, IV (entropy and information). *Kodai Math. Sem. Rep.* **1962**, *14*, 59–85.
10. Furuichi, S.; Yanagi, K.; Kuriyama, K. Fundamental properties of Tsallis relative entropy. *J. Math. Phys.* **2004**, *45*, 4868–4877.
11. Hansen, F. The fast track to Löwner's theorem. *Linear Algebra Appl.* **2013**, *438*, 4557–4571.