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Multiplicity of Homoclinic Solutions for Fractional Hamiltonian Systems with Subquadratic Potential

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Abstract: In this paper, we study the existence of homoclinic solutions for the fractional Hamiltonian systems with left and right Liouville–Weyl derivatives. We establish some new results concerning the existence and multiplicity of homoclinic solutions for the given system by using Clark's theorem from critical point theory and fountain theorem.

Keywords: homoclinic solutions; fractional Hamiltonian systems; critical point theory

1. Introduction

In this paper, we consider the following fractional Hamiltonian system

$$\begin{cases} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u(t)) + L(t)u(t) = \nabla W(t,u(t)), & t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}), \end{cases} \tag{1}$$

where ${}_{-\infty}D^\alpha_t$ and ${}_tD^\alpha_\infty$ are left and right Liouville–Weyl fractional derivatives of order $\alpha \in (\frac{1}{2}, 1)$ on the whole axis $\mathbb{R}$ respectively, $u \in \mathbb{R}^n$, $W(t,u)$ is of indefinite sign and subquadratic as $|u| \to +\infty$ and $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$.

As usual, we say that a solution $u(t)$ of (1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u(t) \neq 0$ then $u(t)$ is called a nontrivial homoclinic solution.

In particular, if $\alpha = 1$, (1) reduces to the standard second order Hamiltonian system of the following form

$$u''(t) - L(t)u(t) + \nabla W(t,u(t)) = 0, \quad t \in \mathbb{R}. \tag{2}$$

The existence of homoclinic solutions for Hamiltonian systems and their importance in the study of behavior of dynamical systems can be recognized from Poincaré [1]. Since then the investigation of existence and multiplicity of homoclinic solutions has become one of most important areas of research in dynamical systems. The existence of homoclinic orbits of (2) has been studied by several researchers by using critical point theory. Examples and details can be found in a series of papers [2–14] and the references cited therein.

It is important to study the multiplicity of homoclinic solutions for Hamiltonian systems. Moreover, ones can show that Hamiltonian system has rich dynamics, in particular a positive entropy. In [5,15] a more complete description of the dynamics is given. Precisely Séré proved the existence of a class of solutions, called multibump solutions. The existence of such a class of solutions implies that the dynamics of the system is chaotic (in particular that its topological entropy is positive). Such a result has been obtained under a nondegeneracy condition which is verified when the set of homoclinic solutions is countable. Bolle and Buffoni [16] show that the existence of a homoclinic orbit that is the transverse intersection of the stable and unstable manifolds, implies the existence of an infinite number of 'multibump' homoclinic solutions. In particular the topological entropy of the system is positive.

On the other hand, fractional calculus is playing a very important role in various scientific fields in the last years. In fact, fractional calculus has been recognized as an excellent instrument for description of memory and hereditary properties of various physical and engineering processes. Fractional-order models are interesting not only for engineers and physicists, but also for mathematicians. There is an increasing interest in the generalization of the classical concepts of entropy. Tenreiro Machado [17] studied several entropy definitions and types of particle dynamics with fractional behavior where traditional Shannon entropy has presented limitations. These concepts allow a fruitful interplay in the analysis of system dynamics. Indeed, applying fractional calculus theory to entropy theory has become a significant research work [17–26], since the fractional entropy could be used in the formulation of algorithms for image segmentation and in the analysis of anomalous diffusion processes and fractional diffusion equations [22–26].

It should be noted that critical point theory has become an effective tool in studying the existence of solutions to fractional differential equations by constructing fractional variational structures. Hamiltonian systems driven by fractional Laplacian operators have been considered by Dipierro, Patrizi and Valdinoci in [27]. In such paper, the fractional setting was motivated by problems atom dislocation in crystals, according to the so-called Peierls-Nabarro model. A throughout discussion on this motivation can be found in Section 2 of [28]. In this paper, we instead consider a fractional framework due to memory effect in the time evolution of the system. For the first time, Jiao and Zhou [29,30] showed that the critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla F(t,u(t)),\ t \in [0,T], \\ u(0) = u(T) = 0. \end{cases} \tag{3}$$

Inspired by this work, Torres [31], Zhang and Yuan [32], Zhou [33], Nyamoradi and Zhou [34], Zhou and Zhang [35] considered the fractional Hamiltonian system (1). The authors [31,32] recently established the following results on the existence of solutions of system (1).

Theorem 1 ([31]). *Suppose that L and W satisfy the following assumptions:*

(L) *$L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, (0, +\infty))$ such that $l(t) \to +\infty$ as $|t| \to +\infty$ and*

$$(L(t)u, u) \geq l(t)|u|^2, \quad \text{for all } t \in \mathbb{R},\ u \in \mathbb{R}^n.$$

(H1) *$W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, and there is a constant $\mu > 2$ such that*

$$0 < \mu W(t,u) \leq (\nabla W(t,u), u), \quad \forall\, t \in \mathbb{R},\ u \in \mathbb{R}^n \setminus \{0\}.$$

(H2) *$|\nabla W(t,u)| = o(|u|)$ as $|u| \to 0$ uniformly with respect to $t \in \mathbb{R}$.*

(H3) *There exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that*

$$|W(t,u)| + |\nabla W(t,u)| \leq |\overline{W}(u)|, \quad \forall\, t \in \mathbb{R},\ u \in \mathbb{R}^n.$$

Then system (1) possesses at least one nontrivial solution.

Theorem 2 ([32]). *Suppose that (L) is satisfied. Moreover, assume that*

(H4) $W(t,0) = 0$ *for* $t \in \mathbb{R}$ *and* $W(t,u) \geq a(t)|u|^{\nu}$, *and* $|\nabla W(t,u)| \leq b(t)|u|^{\nu-1}$ *for all* $(t,u) \in \mathbb{R} \times \mathbb{R}^N$, *where* $1 < \nu < 2$ *is constant,* $a : \mathbb{R} \to \mathbb{R}^+$ *is a bounded continuous function, and* $b : \mathbb{R} \to \mathbb{R}^+$ *is a continuous function such that* $b \in L^{\frac{2}{2-\nu}}(\mathbb{R},\mathbb{R})$.

(H5) *There exist constants* $1 < \sigma \leq \nu < 2$ *such that*

$$(\nabla W(t,u),u) \leq \sigma W(t,u), \quad \forall\, t \in \mathbb{R}, \;\; u \in \mathbb{R}^N \setminus \{0\}.$$

(H6) $W(t,-u) = W(t,u), \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^N$.

Then system (1) has infinitely many nontrivial solutions $\{u_j\}$ *such that*

$$\frac{1}{2}\int_{\mathbb{R}} \left(|_{-\infty}D_t^{\alpha} u_j(t)|^2 + (L(t)u_j(t),u_j(t))\right)dt - \int_{\mathbb{R}} W(t,u_j(t))dt \to 0^-$$

as $j \to +\infty$.

In [31,32], the authors worked on X^{α} which is compactly embedded in $L^q(\mathbb{R},\mathbb{R}^n)$ for $q \in [2,+\infty)$. However, in this paper, X^{α} is compactly embedded in $L^q(\mathbb{R},\mathbb{R}^n)$ for $1 \leq q \in (2/(3-\nu),2)$ or for $\varpi \in [0,\nu)$ and $1 \leq q \in (2(1+\varpi)/(3-\nu),2)$, which is the novelty of the present work.

For the statement of our main results, also we suppose the following conditions for $L(t)$ and $W(t,u)$:

(L_ν) $L(t)$ is $n \times n$ real symmetric positive definite matrix for all $t \in \mathbb{R}$ and there exists a constant $\nu < 2$ such that

$$\liminf_{|t|\to+\infty}\left[|t|^{\nu-2}\inf_{|\xi|=1}(L(t)\xi,\xi)\right] > 0;$$

(W1) $W(t,0) = 0$ for all $t \in \mathbb{R}$ and there exist constants $\max\{1,2/(3-\nu)\} < \gamma_i < 2$ and $a_i \geq 0$ $(i = 1,2,\ldots,m)$ such that

$$|W(t,u)| \leq \sum_{i=1}^{m} a_i |u|^{\gamma_i}, \quad \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^n;$$

(W2) There exists a function $\varphi \in C([0,+\infty),[0,+\infty))$ such that

$$|\nabla W(t,u)| \leq \varphi(|u|), \quad \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^n,$$

where $\varphi(x) = O(x^{\gamma_{m+1}-1})$ as $x \to 0^+$, $\max\{1,2/(3-\nu)\} < \gamma_{m+1} < 2$;

(W3) There exists a constant $\delta_0 > 0$ such that

$$W(t,u) \geq \sum_{k=1}^{l} b_k(t)|u|^{\nu_k}, \quad \forall\, t \in \Omega, \; u \in \mathbb{R}^n, \; |u| \leq \delta_0,$$

for some positive measure subset Ω of $\mathbb{R}$, where $\max\{1,2/(3-\nu)\} < \nu_k < 2$ are constants, $b_k : \mathbb{R} \to \mathbb{R}^+$ are bounded continuous functions for $k = 1,2,\ldots,l$;

(W4) There exist $t_0 \in \mathbb{R}$ and $\max\{1,2/(3-\nu)\} < \vartheta < 2$ such that

$$\lim_{(t,u)\to(t_0,0)} \frac{W(t,u)}{|u|^{\vartheta}} > 0;$$

(W5) $W(t,-u) = W(t,u)$ for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^n$.

Now, we can state our main results.

Theorem 3. *Suppose that L and W satisfy (L_ν) and (W1)–(W3). Then, (1) has at least one nontrivial homoclinic solution.*

Theorem 4. *Suppose that L and W satisfy (L_ν), (W1), (W2), (W4) and (W5). Then, (1) has at least d ($\in \mathbb{N}$) distinct pairs of nontrivial homoclinic solutions.*

Next, we replace the conditions (W1)–(W4) with the following conditions:

(W6) $W(t,0) = 0$ for all $t \in \mathbb{R}$, there exist constants $\varpi_i \in [0, 2-\nu)$, $g_i \geq 0$ and $\max\{1, 2(1+\varpi_i)/(3-\nu)\} < \tau_i < 2$ $(i = 1, 2, \ldots, r)$ such that

$$|W(t,u)| \leq \sum_{i=1}^{r} g_i(1+|t|^{\varpi_i})|u|^{\tau_i}, \quad \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^n;$$

(W7) There exist r functions $\chi_i \in C([0,+\infty),[0,+\infty))$ such that

$$|\nabla W(t,u)| \leq \sum_{i=1}^{r}(1+|t|^{\varpi_i})\chi_i(|u|), \quad \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^n,$$

where $\chi(x) = O(x^{\tau_{r+i}-1})$ as $x \to 0^+$, $\max\{1, 2(1+\varpi_i)/(3-\nu)\} < \tau_{r+i} < 2$ $(i = 1, 2, \ldots, r)$;

(W8) There exists a constant $\delta_0^1 > 0$ such that

$$W(t,u) \geq \sum_{k=1}^{l} b_k^1(t)|u|^{\nu_k^1}, \quad \forall\, t \in \Omega,\ u \in \mathbb{R}^n,\ |u| \leq \delta_0^1,$$

for some positive measure subset Ω of $\mathbb{R}$, and where $\max\{1, 2(1+\varpi_i)/(3-\nu)\} < \nu_k^1 < 2$ are constants, $b_k^1 : \mathbb{R} \to \mathbb{R}^+$ are bounded continuous functions for $k = 1, 2, \ldots, l$;

(W9) There exist $t_0 \in \mathbb{R}$ and $\max\{1, 2(1+\varpi_i)/(3-\nu)\} < \vartheta < 2$ such that

$$\lim_{(t,u)\to(t_0,0)} \frac{W(t,u)}{|u|^{\vartheta}} > 0.$$

Then, we have the following results.

Theorem 5. *Suppose that L and W satisfy (L_ν) and (W6)–(W8). Then, (1) has at least one nontrivial homoclinic solution.*

Theorem 6. *Suppose that L and W satisfy (L_ν), (W5), (W6), (W7) and (W9). Then, (1) has at least d ($\in \mathbb{N}$) distinct pairs of nontrivial homoclinic solutions.*

We will use the following conditions on $W(t,u)$ to fined infinitely many homoclinic solutions:

(W10) $\lim_{|u|\to\infty} \frac{W(t,u)}{|u|^2} = +\infty$ uniformly for all $t \in \mathbb{R}$.

(W11) There exists $\varrho > 0$ such that $W(t,u) \geq -\varrho|u|^2$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$.

(W12) $W(t,0) = 0$ and there exist $D > 0$ and $\gamma_j > 2$ $(j = 1, \ldots, l)$ such that

$$|\nabla W(t,u)| \leq D\Big(|u| + \sum_{j=1}^{l}|u|^{\gamma_j-1}\Big), \quad \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^n;$$

(W13) There exist $\rho > 0, p_j, q_j > 0, 0 \leq \sum_{j=1}^{l} q_j < \frac{\rho-2}{2}$ and $0 < \theta_j < 2$ $(j = 1, \ldots, l)$ such that

$$(\nabla W(t,u), u) - \rho W(t,u) \geq -\sum_{j=1}^{l}\Big[p_j|u|^2 + q_j(L(t)u,u) + M_j(t)|u|^{\theta_j}\Big], \quad \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^n,$$

where $M_j \in L^{\frac{2}{2-\theta_j}}(\mathbb{R}, \mathbb{R}^+)$ $(j = 1, \ldots, l)$.

(W14) There exist $\vartheta \geq \gamma_j - 1$ $(j = 1, \ldots, l), c > 0$ and $R_1 > 0$ such that

$$(\nabla W(t,u), u) - 2W(t,u) \geq c|u|^{\vartheta}, \quad \forall\, t \in \mathbb{R}, \ \forall\, |u| \geq R_1,$$
$$(\nabla W(t,u), u) \geq 2W(t,u), \quad \forall\, t \in \mathbb{R}, \ \forall\, |u| \leq R_1.$$

Remark 1. *In view of (W12), we have*

$$W(t,u) = \int_0^1 (\nabla W(t,su), u) ds \leq D\Big(\frac{1}{2}|u|^2 + \sum_{j=1}^{l} \frac{1}{\gamma_j}|u|^{\gamma_j}\Big), \quad \forall\, (t,u) \in \mathbb{R} \times \mathbb{R}^n.$$

Now, we can state our main results.

Theorem 7. *Suppose that L and W satisfy (L), (W5) and (W10)–(W13). Then, system (1) possesses an unbounded sequence of homoclinic solutions.*

Theorem 8. *Suppose that L and W satisfy (L), (W5), (W10)–(W12) and (W14). Then, system (1) possesses an unbounded sequence of homoclinic solutions.*

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Section 3 contains our results on existence and multiplicity of homoclinic solutions.

2. Preliminaries

Here we present some basic concepts and lemmas that we need in the sequel.

Definition 1 ([36]). *The left and right Liouville–Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by*

$$_{-\infty}I_x^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1}\phi(\xi) d\xi, \tag{4}$$
$$_{x}I_{\infty}^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi-x)^{\alpha-1}\phi(\xi) d\xi, \tag{5}$$

respectively, where $x \in \mathbb{R}$.

The left and right Liouville–Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by

$$_{-\infty}D_x^{\alpha}\phi(x) = \frac{d}{dx}\, {}_{-\infty}I_x^{1-\alpha}\phi(x), \tag{6}$$
$$_{x}D_{\infty}^{\alpha}\phi(x) = -\frac{d}{dx}\, {}_{x}I_{\infty}^{1-\alpha}\phi(x), \tag{7}$$

respectively, where $x \in \mathbb{R}$.

The Definitions (6) and (7) may be written in an alternative form as follows:

$$_{-\infty}D_x^{\alpha}\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\phi(x) - \phi(x-\xi)}{\xi^{\alpha+1}} d\xi, \tag{8}$$
$$_{x}D_{\infty}^{\alpha}\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\phi(x) - \phi(x+\xi)}{\xi^{\alpha+1}} d\xi. \tag{9}$$

According to the results in [37,38], we note that solutions of equations with fractional derivatives (differently from the classical setting) can approximate any smooth function.

Also, we define the Fourier transform $\mathcal{F}(u)(\xi)$ of $u(x)$ as

$$\mathcal{F}(u)(\xi) = \int_{-\infty}^{\infty} e^{-ix\cdot\xi} u(x)dx.$$

For any $\alpha > 0$, we define the semi-norm and norm respectively as [31]

$$\begin{aligned} |u|_{I^{\alpha}_{-\infty}} &= ||{}_{-\infty}D^{\alpha}_{x}u||_{L^2}, \\ ||u||_{I^{\alpha}_{-\infty}} &= \left(||u||^2_{L^2} + |u|^2_{I^{\alpha}_{-\infty}}\right)^{\frac{1}{2}}, \end{aligned} \tag{10}$$

and let the space $I^{\alpha}_{-\infty}(\mathbb{R})$ denote the completion of $C^{\infty}_{0}(\mathbb{R})$ with respect to the norm $||\cdot||_{I^{\alpha}_{-\infty}}$.

Next, for $0 < \alpha < 1$, we give the relationship between classical fractional Sobolev space $H^{\alpha}(\mathbb{R})$ and $I^{\alpha}_{-\infty}(\mathbb{R})$, where $H^{\alpha}(\mathbb{R})$ is defined by

$$H^{\alpha}(\mathbb{R}) = \overline{C^{\infty}_{0}(\mathbb{R})}^{||\cdot||_{\alpha}},$$

with the norm

$$||u||_{\alpha} = \left(||u||^2_{L^2} + |u|^2_{\alpha}\right)^{\frac{1}{2}}, \tag{11}$$

and semi-norm

$$|u|_{\alpha} = |||\xi|^{\alpha}\mathcal{F}(u)||_{L^2}.$$

Observe that the spaces $H^{\alpha}(\mathbb{R})$ and $I^{\alpha}_{-\infty}(\mathbb{R})$ are isomorphic and have equivalent norms (see [31]). Therefore, we define

$$H^{\alpha}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) | \ |\xi|^{\alpha}\mathcal{F}(u) \in L^2(\mathbb{R})\right\}.$$

Now we recall the following results of critical point theory.

Lemma 1 ([39]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy Palais–Smale condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

Lemma 2 (Clark Theorem [40]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ with I even, bounded from below, and satisfy Palais–Smale condition. Suppose that $I(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to S^{d-1} (unit sphere) by an odd map, and $\sup_K I < 0$. Then I possesses at least d distinct pairs of critical points.*

3. Proofs of Theorems

In order to establish our results via variational methods and the critical point theory, we firstly describe some properties of the space on which the variational associated with (1) is defined. Let

$$X^{\alpha} = \left\{u \in H^{\alpha}(\mathbb{R}) | \int_{\mathbb{R}} \left(|{}_{-\infty}D^{\alpha}_{t}u(t)|^2 + (L(t)u(t), u(t))\right)dt < \infty\right\}.$$

The space X^{α} is a Hilbert space with the inner product

$$\langle u, v\rangle_{X^{\alpha}} = \int_{\mathbb{R}} \left(({}_{-\infty}D^{\alpha}_{t}u(t), {}_{-\infty}D^{\alpha}_{t}v(t)) + (L(t)u(t), v(t))\right)dt,$$

and the corresponding norm

$$\|u\|_{X^\alpha}^2 = \langle u, u\rangle_{X^\alpha}.$$

Lemma 3 (See Theorem 2.1 in [31]). *Let $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R},\mathbb{R}^n) \subset C(\mathbb{R},\mathbb{R}^n)$ and there is a constant $C = C_\alpha$ such that*

$$\|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)| \le C||u||_\alpha. \tag{12}$$

So by Lemma 3, there exists a constant $C_\alpha > 0$ such that

$$\|u\|_\infty \le C_\alpha ||u||_{X^\alpha}. \tag{13}$$

By (L_ν), there exist integers $T_0 > 0$ and $M_0 > 0$ such that

$$|t|^{\nu-2} \inf_{|\xi|=1} (L(t)\xi,\xi) \ge M_0, \quad |t| > T_0,$$

which implies that

$$|t|^{\nu-2}(L(t)\xi,\xi) \ge M_0|\xi|^2, \quad |t| > T_0, \ \ \xi \in \mathbb{R}^n. \tag{14}$$

Lemma 4. *Suppose that L satisfies (L_ν). Then, for $1 \le q \in (2/(3-\nu), 2)$, X^α is compactly embedded in $L^q(\mathbb{R},\mathbb{R}^n)$. Moreover*

$$\int_{|t|>T} |u(t)|^q dt \le \frac{\rho(q)}{T^{\frac{(3-\nu)q-2}{2}}}\|u\|_{X^\alpha}^q, \quad \forall\, u \in X^\alpha, \ \ T \ge T_0. \tag{15}$$

and

$$\|u\|_q^q \le \left[\left(\int_{|t|\le T}[l(t)]^{\frac{-q}{2-q}}dt\right)^{1-\frac{q}{2}} + \frac{\rho(q)}{T^{\frac{(3-\nu)q-2}{2}}}\right]\|u\|_{X^\alpha}^q, \quad \forall\, u \in X^\alpha, \ \ T \ge T_0, \tag{16}$$

where

$$\rho(q) = \left[\frac{2(2-q)}{(3-\nu)q-2}\right]^{1-\frac{q}{2}} M_0^{-\frac{q}{2}}, \tag{17}$$

and

$$l(t) = \inf_{x\in\mathbb{R}^n,\ |x|=1} (L(t)x, x). \tag{18}$$

Proof. Let $\varsigma = \frac{(3-\nu)q-2}{2-q}$. Then $\varsigma > 0$. For $u \in X^\alpha$ and $T \ge T_0$, it follows from (14) and (17) together with the Hölder inequality that

$$\begin{aligned}
\int_{|t|>T} |u(t)|^q dt &\le \left(\int_{|t|>T} |t|^{-\frac{(2-\nu)q}{2-q}} dt\right)^{1-\frac{q}{2}} \left(\int_{|t|>T} |t|^{2-\nu}|u(t)|^2 dt\right)^{\frac{q}{2}} \\
&\le \left(\frac{2}{\varsigma T^\varsigma}\right)^{1-\frac{q}{2}} \left(\frac{1}{M_0}\int_{|t|>T}(L(t)u(t), u(t))dt\right)^{\frac{q}{2}} \\
&\le \frac{2^{\frac{2-q}{2}}}{M_0^{\frac{q}{2}}\varsigma^{\frac{2-q}{2}} T^{\frac{(3-\nu)q-2}{2}}}\|u\|_{X^\alpha}^q \\
&\le \frac{\rho(q)}{T^{\frac{(3-\nu)q-2}{2}}}\|u\|_{X^\alpha}^q.
\end{aligned}$$

This shows that (15) holds. Hence, from (15) and (18) and the Hölder inequality, one can get

$$\begin{aligned}
\|u\|_q^q &= \int_{|t|\le T} |u(t)|^q dt + \int_{|t|>T} |u(t)|^q dt \\
&\le \left(\int_{|t|\le T} [l(t)]^{-\frac{q}{2-q}} dt\right)^{1-\frac{q}{2}} \left(\int_{|t|\le T} l(t)|u(t)|^2 dt\right)^{\frac{q}{2}} + \frac{\rho(q)}{T^{\frac{(3-\nu)q-2}{2}}} \|u\|_{X^\alpha}^q \\
&\le \left(\int_{|t|\le T} [l(t)]^{-\frac{q}{2-q}} dt\right)^{1-\frac{q}{2}} \|u\|_{X^\alpha}^q + \frac{\rho(q)}{T^{\frac{(3-\nu)q-2}{2}}} \|u\|_{X^\alpha}^q.
\end{aligned}$$

This shows that (16) holds.

Finally, we prove that X^α is compactly embedded in $L^q(\mathbb{R}, \mathbb{R}^n)$. Let $\{u_k\} \subset X^\alpha$ be a bounded sequence. Then by (13), there exists a constant $\Lambda > 0$ such that

$$\|u_k\|_\infty \le C_\alpha \|u_k\|_{X^\alpha} \le \Lambda, \quad k \in \mathbb{N}. \tag{19}$$

Since X^α is reflexive, $\{u_k\}$ possesses a weakly convergent subsequence in X^α. Passing to a subsequence if necessary, we may assume that $u_k \rightharpoonup u_0$ weakly in X^α. It is easy to verify that

$$\lim_{k\to\infty} u_k(t) = u_0(t), \quad \forall\, t \in \mathbb{R}. \tag{20}$$

For any given number $\varepsilon > 0$, we can choose $T_\varepsilon > 0$ such that

$$\frac{2^{q-1}\rho(q)}{T_\varepsilon^{\frac{(3-\nu)q-2}{2}}} \left[\left(\frac{\Lambda}{C_\alpha}\right)^q + \|u_0\|_{X^\alpha}^q\right] < \varepsilon. \tag{21}$$

It follows from (20) that there exists $k_0 \in \mathbb{N}$ such that

$$\int_{|t|\le T_\varepsilon} |u_k(t) - u_0(t)|^q dt < \varepsilon, \quad \forall\, k \ge k_0. \tag{22}$$

On the other hand, it follows from (15), (19) and (21) that

$$\begin{aligned}
\int_{|t|>T_\varepsilon} |u_k(t) - u_0(t)|^q dt &\le 2^{q-1} \int_{|t|>T_\varepsilon} \left(|u_k(t)|^q + |u_0(t)|^q\right) dt \\
&\le \frac{2^{q-1}\rho(q)}{T_\varepsilon^{\frac{(3-\nu)q-2}{2}}} \left(\|u_k\|_{X^\alpha}^q + \|u_0\|_{X^\alpha}^q\right) \\
&\le \frac{2^{q-1}\rho(q)}{T_\varepsilon^{\frac{(3-\nu)q-2}{2}}} \left[\left(\frac{\Lambda}{C_\alpha}\right)^q + \|u_0\|_{X^\alpha}^q\right] < \varepsilon, \quad k \in \mathbb{N}.
\end{aligned} \tag{23}$$

Since $\varepsilon > 0$ is arbitrary, we obtain by (22) and (23) that

$$\|u_k - u_0\|_q^q = \int_{\mathbb{R}} |u_k(t) - u_0(t)|^q dt \to 0, \quad \text{as } k \to +\infty.$$

This shows that $\{u_k\}$ possesses a convergent subsequence in $L^q(\mathbb{R}, \mathbb{R}^n)$. Therefore, X^α is compactly embedded in $L^q(\mathbb{R}, \mathbb{R}^n)$ for $1 \le q \in (2/(3-\nu), 2)$. Therefore, the proof is complete. □

Also, by (L), since $l \in C(\mathbb{R}, (0,\infty))$ and l is coercive, then $l_{\min} = \min_{t\in\mathbb{R}} l(t)$ exists, then we have

$$(L(t)u(t), u(t)) \ge l(t)|u(t)|^2 \ge l_{\min}|u(t)|^2, \quad \forall\, t \in \mathbb{R}. \tag{24}$$

Lemma 5. *Suppose that L satisfies (L). Then for $2 \le q < \infty$, X^α is compactly embedded in $L^q(\mathbb{R}, \mathbb{R}^n)$; moreover*

$$\int_{|t|>T} |u(t)|^q dt \le \frac{C_\alpha^{q-2}}{\min_{|s|\ge T} l(s)} \|u\|_{X^\alpha}^q, \quad \forall\, u \in X^\alpha, \quad T \ge 1, \tag{25}$$

and

$$\|u\|_{L^q}^q \le l_{\min}^{-1} C_\alpha^{q-2} \|u\|_{X^\alpha}^q, \quad \forall\, u \in X^\alpha. \tag{26}$$

Proof. From (13) and (24), one can get

$$\begin{aligned} \int_{|t|>T} |u(t)|^q dt &\le \|u\|_\infty^{q-2} \int_{|t|>T} |u(t)|^2 dt \\ &\le \|u\|_\infty^{q-2} \int_{|t|>T} [l(t)]^{-1} (L(t)u(t), u(t)) dt \\ &\le \frac{\|u\|_\infty^{q-2}}{\min_{|s|\ge T} l(s)} \|u\|_{X^\alpha}^2 \\ &\le \frac{C_\alpha^{q-2}}{\min_{|s|\ge T} l(s)} \|u\|_{X^\alpha}^q, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \|u\|_{L^q}^q &\le \|u\|_\infty^{q-2} \int_{t\in\mathbb{R}} |u(t)|^2 dt \\ &\le l_{\min}^{-1} \|u\|_\infty^{q-2} \int_{t\in\mathbb{R}} (L(t)u(t), u(t)) dt \\ &\le l_{\min}^{-1} C_\alpha^{q-2} \|u\|_{X^\alpha}^q, \end{aligned}$$

which, together with (27), shows that (25) and (26) holds.

We now can prove that X^α is compactly embedded in $L^q(\mathbb{R}, \mathbb{R}^n)$ for $2 \le q < \infty$ by (L). By Lemma 2.2 in [31], we know that the embedding of X^α in $L^2(\mathbb{R}, \mathbb{R}^n)$ is continuous and compact. On the other hand, from Lemma 3, we know that if $u \in H^\alpha$ with $\frac{1}{2} < \alpha < 1$, then $u \in L^q(\mathbb{R}, \mathbb{R}^n)$ for all $q \in [2, +\infty)$, because

$$\int_{\mathbb{R}} |u(x)|^q dx \le \|u\|_\infty^{q-2} \|u\|_{L^2}^2.$$

So, it is easy to verify that the embedding of X^α in $L^q(\mathbb{R}, \mathbb{R}^n)$ is also continuous and compact for $2 < q < \infty$. Therefore, combining this with Lemma 2.2 in [31], we have the desired conclusion for $2 \le q < \infty$. Therefore, the proof is complete. □

Now, we establish the corresponding variational framework to obtain solutions of (1). To this end, define the functional $I : X^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}} \Big(|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t)) \Big) dt - \int_{\mathbb{R}} W(t, u(t)) dt \\ &= \frac{1}{2} \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, u(t)) dt. \end{aligned} \tag{28}$$

Lemma 6. *Assume that the conditions (L_ν), (W1) and (W2) hold. Then the functional I is well defined and of class $C^1(X^\alpha, \mathbb{R})$ with*

$$I'(u)v = \int_{\mathbb{R}} \Big(({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t)) \Big) dt. \tag{29}$$

Furthermore, the critical points of I in X^α are solutions of (1) with $u(t) \to 0$ as $t \to \pm\infty$.

Proof. We firstly show that $I : X^\alpha \to \mathbb{R}$. For $T \geq T_0$, in view of (W1) and (26), we have

$$\begin{aligned}\left|\int_{\mathbb{R}} W(t,u(t))dt\right| &\leq \sum_{i=1}^{m} a_i \int_{\mathbb{R}} |u(t)|^{\gamma_i} dt \\ &\leq \sum_{i=1}^{m} a_i \left[\left(\int_{|t|\leq T} [l(t)]^{\frac{-\gamma_i}{2-\gamma_i}} dt\right)^{1-\frac{\gamma_i}{2}} + \frac{\rho(\gamma_i)}{T^{\frac{(3-\nu)\gamma_i-2}{2}}}\right] \|u\|_{X^\alpha}^{\gamma_i} \\ &\leq \sum_{i=1}^{m} \phi_i(T)\|u\|_{X^\alpha}^{\gamma_i}, \end{aligned} \tag{30}$$

where $\phi_i(T) := a_i \left[\left(\int_{|t|\leq T}[l(t)]^{\frac{-\gamma_i}{2-\gamma_i}} dt\right)^{1-\frac{\gamma_i}{2}} + \frac{\rho(\gamma_i)}{T^{\frac{(3-\nu)\gamma_i-2}{2}}}\right]$. Combining this with (28), it follows that $I : X^\alpha \to \mathbb{R}$.

Next, we prove that $I \in C^1(X^\alpha, \mathbb{R})$. Rewrite I as $I = I_1 - I_2$, where

$$\begin{aligned} I_1(u) &:= \frac{1}{2}\int_{\mathbb{R}} \Big(|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))\Big)dt, \\ I_2(u) &:= \int_{\mathbb{R}} W(t,u(t))dt. \end{aligned} \tag{31}$$

It is easy to check that $I_1 \in C^1(X^\alpha, \mathbb{R})$, and that

$$I_1'(u)v = \int_{\mathbb{R}} \Big(({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t))\Big)dt.$$

Then, it is sufficient to show that $I_2 \in C^1(X^\alpha, \mathbb{R})$. So, we have

$$I_2'(u)v = \int_{\mathbb{R}} (\nabla W(t,u(t)), v(t))dt, \quad \forall\, u,v \in X^\alpha. \tag{32}$$

By (W2), one can choose a constant $\varphi_0 > 0$ such that

$$\varphi(|u|) \leq \varphi_0 |u|^{\gamma_{m+1}-1}, \quad \forall\, u \in \mathbb{R}^n,\ |u| \leq 1. \tag{33}$$

For any $u, v \in X^\alpha$, there exists $T_1 > 0$ such that $|u(t)| + |v(t)| < 1$ as $|t| > T_1$. Then for any function $\theta : \mathbb{R} \to (0,1)$ and any number $h \in (0,1)$, by (W2), (33) and Lemma 4, we have

$$\begin{aligned} &\int_{\mathbb{R}} |(\nabla W(t,u(t)+\theta(t)hv(t)), v(t))|dt \\ &\leq \int_{|t|\leq T_1} |(\nabla W(t,u(t)+\theta(t)hv(t))||v(t))|dt \\ &\quad + \int_{|t|>T_1} |(\nabla W(t,u(t)+\theta(t)hv(t))||v(t))|dt \\ &\leq \int_{|t|\leq T_1} \max_{|x|\leq \|u\|_\infty + \|v\|_\infty} |(\nabla W(t,x)||v(t))|dt + \varphi_0 \int_{|t|>T_1} (|u(t)|+|v(t)|)^{\gamma_{m+1}-1}|v(t))|dt \\ &\leq \int_{|t|\leq T_1} \max_{|x|\leq \|u\|_\infty + \|v\|_\infty} |(\nabla W(t,x)||v(t))|dt + \varphi_0 \int_{|t|>T_1} |v(t)|^{\gamma_{m+1}} dt \\ &\quad + \varphi_0 \left(\int_{|t|>T_1} |u(t)|^{\gamma_{m+1}} dt\right)^{1-\frac{1}{\gamma_{m+1}}} \left(\int_{|t|>T_1} |v(t)|^{\gamma_{m+1}} dt\right)^{\frac{1}{\gamma_{m+1}}} \\ &\leq \int_{|t|\leq T_1} \max_{|x|\leq \|u\|_\infty + \|v\|_\infty} |(\nabla W(t,x)||v(t))|dt \\ &\quad + \varphi_0 \frac{\rho(\gamma_{m+1})}{T^{\frac{(3-\nu)\gamma_{m+1}-2}{2}}} (\|u\|_{X^\alpha}^{\gamma_{m+1}-1} + \|v\|_{X^\alpha}^{\gamma_{m+1}-1})\|v\|_{X^\alpha} < +\infty. \end{aligned} \tag{34}$$

Then by (28) and (34), the mean value theorem and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
I_2'(u)v &= \lim_{h\to 0^+}\frac{I_2(u+hv)-I_2(u)}{h}\\
&= \lim_{h\to 0^+}\left[\int_{\mathbb{R}}\frac{W(t,u(t)+hv(t))-W(t,u(t))}{h}dt\right]\\
&= \lim_{h\to 0^+}\left[\int_{\mathbb{R}}(\nabla W(t,u(t)+\theta(t)hv(t)),v(t))dt\right]\\
&= \int_{\mathbb{R}}(\nabla W(t,u(t)),v(t))dt.
\end{aligned}$$

This shows that (32) holds.

It remains to prove that I_2' is continuous. Suppose that $u_k \to u_0$ in X^α, then, by the Banach-Steinhaus theorem, there exists a constant $\varrho > 0$ such that

$$\|u_0\|_{X^\alpha} \le \frac{1}{C_\alpha}\varrho, \qquad \sup_{k\in\mathbb{N}}\|u_k\|_{X^\alpha} \le \frac{1}{C_\alpha}\varrho. \tag{35}$$

In view of (13), we have

$$\|u_0\|_\infty \le \varrho, \qquad \sup_{k\in\mathbb{N}}\|u_k\|_\infty \le \varrho. \tag{36}$$

Now, by (W2), we can choose a constant $\varphi_1 > 0$ such that

$$\varphi(|u|) \le \varphi_1|u|^{\gamma_{m+1}-1}, \quad \forall\, u\in\mathbb{R}^n,\ |u|\le\varrho. \tag{37}$$

Thus by (15), (29), (35)–(37), (W2) and the Hölder inequality, we obtain

$$\begin{aligned}
|I_2'(u_k)v - I_2'(u_0)v| &= \int_{\mathbb{R}}|(\nabla W(t,u_k(t))-\nabla W(t,u_0(t)),v(t))|dt\\
&\le \int_{|t|\le T}|(\nabla W(t,u_k(t))-\nabla W(t,u_0(t))||v(t))|dt\\
&\quad+\int_{|t|>T}|(\nabla W(t,u_k(t))-\nabla W(t,u_0(t))||v(t))|dt\\
&\le o(1)+\varphi_1\int_{|t|>T_1}\left(|u_k(t)|^{\gamma_{m+1}-1}+|u_0(t)|^{\gamma_{m+1}-1}\right)|v(t))|dt\\
&\le o(1)+\varphi_1\left(\int_{|t|>T}|u_k(t)|^{\gamma_{m+1}}dt\right)^{1-\frac{1}{\gamma_{m+1}}}\left(\int_{|t|>T_1}|v(t)|^{\gamma_{m+1}}dt\right)^{\frac{1}{\gamma_{m+1}}}\\
&\quad+\varphi_1\left(\int_{|t|>T}|u_0(t)|^{\gamma_{m+1}}dt\right)^{1-\frac{1}{\gamma_{m+1}}}\left(\int_{|t|>T_1}|v(t)|^{\gamma_{m+1}}dt\right)^{\frac{1}{\gamma_{m+1}}}\\
&\le o(1)+\varphi_1\frac{\rho(\gamma_{m+1})}{T^{\frac{(3-\nu)\gamma_{m+1}-2}{2}}}(\|u_k\|_{X^\alpha}^{\gamma_{m+1}-1}+\|u_0\|_{X^\alpha}^{\gamma_{m+1}-1})\|v\|_{X^\alpha}\\
&= o(1),\ \text{ as } k\to+\infty,\ T\to+\infty,\ \forall\, v\in X^\alpha,
\end{aligned}$$

which shows the continuity of I_2'.

Finally, by a standard argument, it is easy to show that the critical points of I in X^α are solutions of (1) with $u(\pm\infty)=0$. Therefore, the proof is complete. □

Proof of Theorem 3. In view of Lemma 6, $I\in C^1(X^\alpha,\mathbb{R})$. We show that I satisfies the hypotheses of Lemma 1.

Claim 1. We first show that I is bounded from below. Selecting $T_2 > T_0$, it follows from (30) that

$$\int_{\mathbb{R}} W(t,u(t))dt \leq \sum_{i=1}^{m} \phi_i(T_2)\|u\|_{X^\alpha}^{\gamma_i}, \quad \forall\, u \in X^\alpha. \tag{38}$$

From (28) and (38), we get

$$I(u) = \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,u(t))dt \geq \frac{1}{2}\|u\|_{X^\alpha}^2 - \sum_{i=1}^{m} \phi_i(T_2)\|u\|_{X^\alpha}^{\gamma_i}. \tag{39}$$

Since $\max\{1, 2/(3-\nu)\} < \gamma_i < 2$, (39) implies that $I(u) \to +\infty$ as $\|u\|_{X^\alpha} \to +\infty$. Therefore, I is bounded from below.

Claim 2. We show that I satisfies the Palais–Smale condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset X^\alpha$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. So by (13) and (39), there exists a constant $\Lambda_1 > 0$ such that

$$\|u_k\|_\infty \leq C_\alpha \|u_k\|_{X^\alpha} \leq \Lambda_1, \quad k \in \mathbb{N}. \tag{40}$$

Hence, passing to a subsequence if necessary, one may assume that $u_k \rightharpoonup u$ weakly in X^α. It is easy to verify that

$$\lim_{k\to\infty} u_k(t) = u(t), \quad \forall\, t \in \mathbb{R}. \tag{41}$$

So,

$$(I'(u_k) - I'(u))(u_k - u) \to 0 \;\text{ as }\; k \to \infty, \tag{42}$$

it follows from (40) and (41) that

$$\|u\|_{X^\alpha} \leq \Lambda_1. \tag{43}$$

By (W2), we can choose $\varphi_2 > 0$ such that

$$\varphi(|u|) \leq \varphi_2 |u|^{\gamma_{m+1}-1}, \quad \forall\, u \in \mathbb{R}^n,\ |u| \leq \Lambda_1. \tag{44}$$

For any given number $\varepsilon > 0$, we can choose $T_3 > 0$ such that

$$\frac{\rho(\gamma_{m+1})}{T_3^{\frac{(3-\nu)\gamma_{m+1}-2}{2}}} \left[\left(\frac{\Lambda_1}{C_\alpha}\right)^{\gamma_{m+1}} + \|u\|_{X^\alpha}^{\gamma_{m+1}} \right] < \varepsilon. \tag{45}$$

It follows from (41) and the continuity of $\nabla W(t,x)$ on x that there exists $k_1 \in \mathbb{N}$ such that

$$\int_{|t|\leq T_3} |\nabla W(t,u_k(t)) - \nabla W(t,u(t))||u_k(t) - u(t)|dt < \varepsilon, \quad \forall\, k \geq k_1. \tag{46}$$

Therefore, in view of (15), (40), (43)–(45) and (W2), we obtain

$$\begin{aligned} &\int_{|t|>T_3} |\nabla W(t,u_k(t)) - \nabla W(t,u(t))||u_k(t) - u(t)|dt \\ &\leq \varphi_2 \int_{|t|>T_3} \left(|u_k(t)|^{\gamma_{m+1}-1} + |u(t)|^{\gamma_{m+1}-1}\right)\left(|u_k(t)| + |u(t)|\right)dt \\ &\leq 2\varphi_2 \int_{|t|>T_3} \left(|u_k(t)|^{\gamma_{m+1}} + |u(t)|^{\gamma_{m+1}}\right) dt \end{aligned} \tag{47}$$

$$
\begin{aligned}
&\leq 2\varphi_2 \frac{\rho(\gamma_{m+1})}{T_3^{\frac{(3-\nu)\gamma_{m+1}-2}{2}}} \left[\|u_k\|_{X^\alpha}^{\gamma_{m+1}} + \|u\|_{X^\alpha}^{\gamma_{m+1}} \right] \\
&\leq 2\varphi_2 \frac{\rho(\gamma_{m+1})}{T_3^{\frac{(3-\nu)\gamma_{m+1}-2}{2}}} \left[\left(\frac{\Lambda_1}{C_\alpha}\right)^{\gamma_{m+1}} + \|u\|_{X^\alpha}^{\gamma_{m+1}} \right] \\
&< 2\varphi_2\varepsilon, \quad k \in \mathbb{N}.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, so by (46) and (47), we get

$$
\int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t))dt \to 0, \quad \text{as } k \to +\infty. \tag{48}
$$

On the other hand, we have

$$
\begin{aligned}
&(I'(u_k) - I'(u))(u_k - u) \\
&\quad = \|u_k - u\|_{X^\alpha}^2 - \int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t))dt.
\end{aligned} \tag{49}
$$

Hence, it follows from (42), (48) and (49) that $\|u_k - u\|_{X^\alpha} \to 0$ as $k \to +\infty$. Therefore, I satisfies Palais–Smale condition.

Then, by Lemma 1, $c = \inf_{X^\alpha} I(u)$ is a critical value of I, that is, there exists a critical point e such that $I(e) = c$.

Finally, we show that $e \neq 0$. Take some $u \in X^\alpha$ such that $\|u\|_{X^\alpha} = 1$. Then there exists a subset Ω of positive measure $|\Omega| < \infty$ of $\mathbb{R}$ such that $u(t) \neq 0$ for $t \in \Omega$. Take $\sigma > 0$ small enough so that $\sigma|u(t)| \leq \delta_0$ for $t \in \Omega$. By (W3), there exists a constant $\eta > 0$ such that

$$
W(t, u) \geq \eta \sum_{k=1}^{l} |u|^{\nu_k}, \quad \forall\, t \in \Omega,\ u \in \mathbb{R}^n,\ |u| \leq \delta_0. \tag{50}
$$

Then by (50), one can get

$$
\begin{aligned}
I(\sigma u) &= \frac{\sigma^2}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, \sigma u(t))dt \\
&\leq \frac{\sigma^2}{2} - \eta \sum_{k=1}^{l} \sigma^{\nu_k} \int_{\Omega} |u(t)|^{\nu_k} dt.
\end{aligned} \tag{51}
$$

Since $\max\{1, 2/(3-\nu)\} < \nu_k < 2$ $(k = 1, 2, \ldots, l)$ and $\int_\Omega |u(t)|^\mu dt > 0$, (51) implies that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\sigma|u(t)| \leq \delta_0$ for $t \in \Omega$. Thus, $I(e) = c = \inf_{X^\alpha} I(u) < 0$, therefore e is a nontrivial critical point of I, and hence $e = e(t)$ is a nontrivial homoclinic solution of system (1). The proof is complete. □

Proof of Theorem 4. In view of Lemma 6 and the Proof of Theorem 3, $I \in C^1(X^\alpha, \mathbb{R})$ is bounded from below and satisfies the Palais–Smale condition. It is obvious that I is even and $I(0) = 0$. In order to apply Lemma 2, we show that there is a set $K \subset X^\alpha$ such that K is homeomorphic to S^{d-1} by an odd map, and $\sup_K I < 0$.

By (W4), there exist an open set $D \subset \mathbb{R}$ with $t_0 \in D$, $\sigma_1 > 0$ and $\eta > 0$ such that

$$
W(t, u) \geq \eta |u|^\vartheta, \quad \forall\, (t, u) \in D \times \mathbb{R}^n,\ |u| < \sigma_1. \tag{52}
$$

For any $d \in \mathbb{N}$, we take d disjoint open sets D_i such that $\bigcup_{i=1}^d D_i \subset D$. For $i = 1, 2, \ldots, d$, let $u_i \in \left(H_0^\alpha(D_i) \bigcap X^\alpha\right) \setminus \{0\}$ (for detail of $H_0^\alpha(D_i)$, see [41]) and $\|u_i\|_{X^\alpha} = 1$, and

$$
X_d = \mathrm{span}\{u_1, \ldots, u_d\}, \qquad S_d = \{u \in X_d : \ \|u\|_{X^\alpha} = 1\}. \tag{53}
$$

For a $u \in X_d$, there exist $\lambda_i \in \mathbb{R},\ i = 1, 2, \ldots, d$ such that

$$u(t) = \sum_{i=1}^{d} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{R}. \tag{54}$$

So

$$\|u\|_\vartheta = \left(\int_{\mathbb{R}} |u(t)|^\vartheta dt \right)^{\frac{1}{\vartheta}} = \left(\sum_{i=1}^{d} |\lambda_i|^\vartheta \int_{D_i} |u_i(t)|^\vartheta dt \right)^{\frac{1}{\vartheta}}, \tag{55}$$

and

$$\begin{aligned} \|u\|_{X^\alpha}^2 &= \int_{\mathbb{R}} \left(|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t)) \right) dt \\ &= \sum_{i=1}^{d} \lambda_i^2 \int_{D_i} \left(|{}_{-\infty}D_t^\alpha u_i(t)|^2 + (L(t)u_i(t), u_i(t)) \right) dt \\ &= \sum_{i=1}^{d} \lambda_i^2 \int_{\mathbb{R}} \left(|{}_{-\infty}D_t^\alpha u_i(t)|^2 + (L(t)u_i(t), u_i(t)) \right) dt \\ &= \sum_{i=1}^{d} \lambda_i^2 \|u_i\|_{X^\alpha}^2 = \sum_{i=1}^{d} \lambda_i^2. \end{aligned} \tag{56}$$

As all norms of a finite dimensional normed space are equivalent, there is a constant $C' > 0$ such that

$$C' \|u\|_{X^\alpha} \le \|u\|_{L^\vartheta} \quad \text{for } u \in X_d. \tag{57}$$

Note that $W(t, 0) = 0$, and so according to (52), (54), (55)–(57), one can get

$$\begin{aligned} I(su) &= \frac{s^2}{2} \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, su(t)) dt \\ &= \frac{s^2}{2} \|u\|^2 - \sum_{i=1}^{d} \int_{D_i} W(t, s\lambda_i u_i(t)) \\ &\le \frac{s^2}{2} \|u\|_{X^\alpha}^2 - \eta s^\vartheta \sum_{i=1}^{d} |\lambda_i|^\vartheta \int_{D_i} |u_i(t)|^\vartheta dt \\ &\le \frac{s^2}{2} \|u\|_{X^\alpha}^2 - \eta s^\vartheta \|u\|_\vartheta^\vartheta \\ &\le \frac{s^2}{2} \|u\|_{X^\alpha}^2 - \eta (C's)^\vartheta \|u\|_{X^\alpha}^\vartheta, \quad \forall\, u \in S_d, \end{aligned} \tag{58}$$

and sufficiently small $s > 0$. In this case (52) is applicable, since u is continuous on $\overline{D}$ and so $|s\lambda_i u_i(t)| \le \sigma_1$ for any $t \in D$, $i = 1, 2, \ldots, d$ can be true for sufficiently small s. Hence, it follows from (58) that there exist $\varepsilon > 0$ and $\sigma_2 > 0$ such that

$$I(\sigma_2 u) < -\varepsilon \quad \forall\, u \in S_d. \tag{59}$$

Let

$$S_d^{\sigma_2} = \{\sigma_2 u :\ u \in S_d\}, \quad S^{d-1} = \left\{ \left(\frac{\lambda_1}{\sigma_2}, \frac{\lambda_2}{\sigma_2}, \ldots, \frac{\lambda_d}{\sigma_2} \right)^T \in \mathbb{R}^d : \sum_{i=1}^{d} \frac{\lambda_i^2}{\sigma_2^2} = 1 \right\}.$$

Then it follows from (56) that

$$S_d^{\sigma_2} = \left\{ \sum_{i=1}^{d} \lambda_i u_i : \sum_{i=1}^{d} \lambda_i^2 = \sigma_2^2 \right\}.$$

By (52), we define a map $\Psi : S_d^{\sigma_2} \to S^{d-1}$ as follows

$$\Psi(u) = \sigma_2^{-1} \left(\frac{\lambda_1}{\sigma_2}, \frac{\lambda_2}{\sigma_2}, \dots, \frac{\lambda_d}{\sigma_2} \right)^T, \quad \forall\, u \in S_d^{\sigma}.$$

It is easy to verify that $\Psi : S_d^{\sigma_2} \to S^{d-1}$ is an odd homeomorphic map. On the other hand, by (59), we have

$$I(u) < -\varepsilon \quad \forall\, u \in S_d^{\sigma_2},$$

and thus $\sup_{S_d^{\sigma_2}} I < -\varepsilon < 0$. By Lemma 2, I has at least d distinct pairs of critical points, and so system (1) possesses at least d distinct pairs of nontrivial homoclinic solutions. The proof is complete. □

Lemma 7. *Suppose that L satisfies (L_ν). Then for $\varpi \in [0, \nu)$ and $1 \le q \in (2(1+\varpi)/(3-\nu), 2)$, X^α is compactly embedded in $L^q(\mathbb{R}, \mathbb{R}^n)$; moreover*

$$\int_{|t|>T} (1+|t|^{\varpi})|u(t)|^q dt \le \frac{\rho(\varpi, q)}{T^{\frac{(3-\nu)q-2(1+\varpi)}{2}}} \|u\|_{X^\alpha}^q, \quad \forall\, u \in X^\alpha, \;\; T \ge T_0, \tag{60}$$

and

$$\int_{\mathbb{R}} (1+|t|^{\varpi})|u(t)|^q dt \le \left[\left(\int_{|t|\le T} (1+|t|^{\varpi})^{\frac{2}{2-q}} [l(t)]^{\frac{-q}{2-q}} dt \right)^{1-\frac{q}{2}} + \frac{\rho(\varpi, q)}{T^{\frac{(3-\nu)q-2(1+\varpi)}{2}}} \right] \|u\|_{X^\alpha}^q,$$
$$\forall\, u \in X^\alpha, \;\; T \ge T_0, \tag{61}$$

where

$$\rho(\varpi, q) = 2 \left[\frac{2(2-q)}{(3-\nu)q - 2(1+\varpi)} \right]^{1-\frac{q}{2}} M_0^{-\frac{q}{2}}, \tag{62}$$

and $l(t)$ is defined in (18).

Proof. Let $\zeta = \frac{(3-\nu)q-2(1+\varpi)}{2-q}$. Then $\zeta > 0$. For $u \in X^\alpha$ and $T \ge T_0$, it follows from (14) and (62) and the Hölder inequality that

$$\begin{aligned}
\int_{|t|>T} (1+|t|^{\varpi})|u(t)|^q dt &\le 2 \left(\int_{|t|>T} |t|^{-\frac{(2-\nu)q-2\varpi}{2-q}} dt \right)^{1-\frac{q}{2}} \left(\int_{|t|>T} |t|^{2-\nu} |u(t)|^2 dt \right)^{\frac{q}{2}} \\
&= 2 \left(\int_{|t|>T} |t|^{-(\zeta+1)} dt \right)^{1-\frac{q}{2}} \left(\int_{|t|>T} |t|^{2-\nu} |u(t)|^2 dt \right)^{\frac{q}{2}} \\
&\le 2 \left(\frac{2}{\zeta T^{\zeta}} \right)^{1-\frac{q}{2}} \left(\frac{1}{M_0} \int_{|t|>T} (L(t)u(t), u(t)) dt \right)^{\frac{q}{2}} \\
&\le \frac{2^{1+\frac{2-q}{2}}}{M_0^{\frac{q}{2}} \zeta^{\frac{2-q}{2}} T^{\frac{(3-\nu)q-2(1+\varpi)}{2}}} \|u\|_{X^\alpha}^q \\
&= \frac{\rho(\varpi, q)}{T^{\frac{(3-\nu)q-2(1+\varpi)}{2}}} \|u\|_{X^\alpha}^q.
\end{aligned}$$

This shows that (60) holds. Hence, from (60) and (18) and the Hölder inequality, one can get

$$\begin{aligned}\int_{\mathbb{R}}(1+|t|^{\varpi})|u(t)|^q dt &= \int_{|t|\le T}(1+|t|^{\varpi})|u(t)|^q dt + \int_{|t|>T}(1+|t|^{\varpi})|u(t)|^q dt \\ &\le \left(\int_{|t|\le T}(1+|t|^{\varpi})^{\frac{2}{2-q}}[l(t)]^{-\frac{q}{2-q}}dt\right)^{1-\frac{q}{2}}\left(\int_{|t|\le T} l(t)|u(t)|^2 dt\right)^{\frac{q}{2}} \\ &\quad + \frac{\rho(\varpi,q)}{T^{\frac{(3-\nu)q-2(1+\varpi)}{2}}}\|u\|^q_{X^\alpha} \\ &\le \left(\int_{|t|\le T}(1+|t|^{\varpi})^{\frac{2}{2-q}}[l(t)]^{-\frac{q}{2-q}}dt\right)^{1-\frac{q}{2}}\|u\|^q_{X^\alpha} \\ &\quad + \frac{\rho(\varpi,q)}{T^{\frac{(3-\nu)q-2(1+\varpi)}{2}}}\|u\|^q_{X^\alpha}.\end{aligned}$$

This shows that (61) holds.

Finally, by similar argument in the proof of Lemma 4, it is easy to show that X^α is compactly embedded in $L^q(\mathbb{R},\mathbb{R}^n)$. Therefore, the proof is complete. □

In this case Lemma 7 holds again with replacing (W1) and (W2) by (W6) and (W7), and in view of (W6) and (61), we have

$$\begin{aligned}\left|\int_{\mathbb{R}} W(t,u(t))dt\right| &\le \sum_{i=1}^{r} g_i\int_{\mathbb{R}}(1+|t|^{\varpi_i})|u(t)|^{\tau_i}dt \\ &\le \sum_{i=1}^{r} g_i\Bigg[\left(\int_{|t|\le T}(1+|t|^{\varpi_i})^{\frac{2}{2-\tau_i}}[l(t)]^{\frac{-\tau_i}{2-\tau_i}}dt\right)^{1-\frac{\tau_i}{2}} \\ &\quad + \frac{\rho(\varpi_i,\tau_i)}{T^{\frac{(3-\nu)\gamma_i-2(1+\varpi_i)}{2}}}\Bigg]\|u\|^{\tau_i}_{X^\alpha} \\ &\le \sum_{i=1}^{r}\Pi_i(T)\|u\|^{\tau_i}_{X^\alpha}, \qquad (63)\end{aligned}$$

where $\Pi_i(T) := g_i\left[\left(\int_{|t|\le T}(1+|t|^{\varpi_i})^{\frac{2}{2-\tau_i}}[l(t)]^{\frac{-\tau_i}{2-\tau_i}}dt\right)^{1-\frac{\tau_i}{2}} + \frac{\rho(\varpi_i,\tau_i)}{T^{\frac{(3-\nu)\gamma_i-2(1+\varpi_i)}{2}}}\right]$.

Therefore, the proof of Theorems 5 and 6 are similar to Theorems 3 and 4, respectively, and are omitted.

Let X be a Banach space with the norm $\|\cdot\|$ and $X=\overline{\oplus_{j\in\mathbb{N}}X_j}$, where X_j are finite-dimensional subspace of X, for each $k\in\mathbb{N}$, assume that $Y_k=\oplus_{j=0}^{k}X_j$ and $Z_k=\overline{\oplus_{j=k}^{\infty}X_j}$. The functional Φ is said to satisfy the Palais–Smale condition if any sequence $\{u_j\}_{j\in\mathbb{N}}\subset X$ such that $\{\Phi(u_j)\}_{n\in\mathbb{N}}$ is bounded and $\Phi'(u_j)\to 0$ as $j\to+\infty$ has a convergent subsequence.

Now, let us recall, for the reader's convenience, a critical point result as follow:

Theorem 9 ([42,43]). *Suppose that the functional $\Phi\in C^1(X,\mathbb{R})$ is even. If, for every $k\in\mathbb{N}$, there exist $\varrho_k>r_k>0$ such that*

(F1) $a_k := \max_{u\in Y_k,\|u\|=\varrho_k}\Phi(u)\le 0.$
(F2) $b_k := \inf_{u\in Z_k,\|u\|=r_k}\Phi(u)\to+\infty$ *as* $k\to\infty$.
(F3) *Φ satisfies the Palais–Smale condition.*

Then Φ possesses an unbounded sequence of critical values.

Proof of Theorem 7. Let $\{e_j\}_{j=1}^{\infty}$ be the standard orthogonal basis of X^α and define $X_j := \mathbb{R}e_j$, then Z_k and Y_k can be defined as that in Theorem 9. From (29) and (W5), we can obtain that $\Phi \in C^1(X^\alpha, \mathbb{R})$ is even. Let us prove that the functionals Φ satisfy the required conditions in Theorem 9.

We firstly verify condition (F2) in Theorem 9. Let

$$\lambda_k = \sup_{u \in Z_k, \|u\|_{X^\alpha}=1} \|u\|_{L^2},$$
$$\beta_k^j = \sup_{u \in Z_k, \|u\|_{X^\alpha}=1} \|u\|_{L^{\gamma_j}}, \quad \text{for any } j = 1, \ldots, l,$$

then $\lambda_k \to 0$ and $\beta_k^j \to 0$ as $k \to +\infty$ for any $j = 1, \ldots, l$. Clearly the sequence $\{\lambda_k\}$ is nonnegative and nonincreasing, so we assume that $\lambda_k \to \overline{\lambda} \geq 0$, $k \to +\infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\|_{X^\alpha} = 1$ and $\|u_k\|_{L^2} > \frac{\lambda_k}{2}$. Then, up to a subsequence, we may assume that $u_k \rightharpoonup u$ weakly in X^α. Noticing that Z_k is a closed subspace of X^α, by Mazur's theorem, we have $u \in Z_k$, for all $k > \widetilde{n}$. Consequently, we get $u \in \bigcap_{k=\widetilde{n}}^{\infty} Z_k = \{0\}$, which implies $u_k \rightharpoonup 0$ weakly in X^α. By Lemma 5, we have $u_k \to 0$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. Thus we have proved that $\overline{\lambda} = 0$. Similarly, we can prove that $\beta_k^j \to 0$ as $k \to +\infty$ for any $j = 1, \ldots, l$. In view of (28) and (W3), one can get

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, u(t))dt \\ &\geq \frac{1}{2}\|u\|_{X^\alpha}^2 - D\left(\frac{1}{2}\|u\|_{L^2}^2 + \sum_{j=1}^{l}\frac{1}{\gamma_j}\|u\|_{L^{\gamma_j}}^{\gamma_j}\right) \\ &\geq \frac{1}{2}\|u\|_{X^\alpha}^2 - \frac{1}{2}D\lambda_k^2\|u\|_{X^\alpha}^2 - D\sum_{j=1}^{l}\frac{1}{\gamma_j}\beta_k^{\gamma_j}\|u\|_{X^\alpha}^{\gamma_j}. \end{aligned} \tag{64}$$

Since $\lambda_k \to 0$ as $k \to +\infty$, there exists a positive constant N_0 such that

$$D\lambda_k^2 \leq \frac{1}{2}, \quad \forall\, k \geq N_0. \tag{65}$$

By (64) and (65), we have

$$\Phi(u) \geq \frac{1}{4}\|u\|_{X^\alpha}^2 - D\sum_{j=1}^{l}\frac{1}{\gamma_j}\beta_k^{\gamma_j}\|u\|_{X^\alpha}^{\gamma_j}, \quad \forall\, k \geq N_0. \tag{66}$$

If we choose $r_k = \frac{1}{l}\max\left\{\left(8\frac{D}{\gamma_1}\beta_k^{\gamma_1}\right)^{\frac{1}{2-\gamma_1}}, \ldots, \left(8\frac{D}{\gamma_l}\beta_k^{\gamma_l}\right)^{\frac{1}{2-\gamma_l}}\right\}$, then

$$b_k = \inf_{u \in Z_k, \|u\|_{X^\alpha}=r_k} \Phi(u) \geq \frac{1}{8}r_k^2, \quad \forall\, k \geq N_0. \tag{67}$$

Since $\beta_k \to 0$ as $k \to \infty$ and $\gamma_j > 2$ for any $j = 1, \ldots, l$, we can obtain

$$b_k \to +\infty, \quad \text{as } k \to \infty.$$

We now verify condition (F1) in Theorem 9. Since $\dim Y_k < \infty$ and all norms of a finite-dimensional normed space are equivalent, there exists a constant $M_0 > 0$ such that

$$\|u\|_{X^\alpha} \leq M_0 \|u\|_{L^2}, \quad \forall\, u \in Y_k. \tag{68}$$

By (W1), for $\varepsilon_1 = \left(1 + \varrho l_{\min}^{-1}\right) M_0^2$ where ϱ is given in (W2), there exists $\delta = \delta(\varepsilon_1) > 0$ such that

$$W(t,u) \geq \varepsilon_1 |u|^2, \quad \forall\, |u| \geq \delta, \quad \forall\, t \in \mathbb{R}. \tag{69}$$

Then, for any $u \in Y_k$, in view of (26), (28) and (69), one has

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,u(t))dt \\
&= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\{t\in\mathbb{R};\ |u(t)|\geq\delta\}} W(t,u(t))dt - \int_{\{t\in\mathbb{R};\ |u(t)|<\delta\}} W(t,u(t))dt \\
&\leq \frac{1}{2}\|u\|_{X^\alpha}^2 - \varepsilon_1 \|u\|_{L^2}^2 + \varrho \|u\|_{L^2}^2 \\
&\leq \frac{1}{2}\|u\|_{X^\alpha}^2 - \frac{\varepsilon_1}{M_0^2}\|u\|_{X^\alpha}^2 + \varrho l_{\min}^{-1}\|u\|_{X^\alpha}^2 \\
&= \left(\frac{1}{2} - \frac{\varepsilon_1}{M_0^2} + \varrho l_{\min}^{-1}\right)\|u\|_{X^\alpha}^2 = -\frac{1}{2}\|u\|_{X^\alpha}^2.
\end{aligned}$$

Hence, we can choose $\|u\|_{X^\alpha} = \varrho_k$ large enough $(\varrho_k > r_k > 0)$ such that

$$a_k = \max_{u\in Y_k, \|u\|=\varrho_k} \Phi(u) \leq 0.$$

Finally, We prove that Φ satisfies the Palais–Smale condition. Let $\{u_i\}_{i\in\mathbb{N}} \subset X^\alpha$ be a Palais–Smale sequence, that is, $\{\Phi(u_i)\}_{n\in\mathbb{N}}$ is bounded and $\Phi'(u_i) \to 0$ as $i \to +\infty$. Then there exists a constant $M_1 > 0$ such that

$$|\Phi(u_i)| \leq M_1, \quad \|\Phi'(u_i)\|_{(X^\alpha)^*} \leq M_1 \tag{70}$$

for every $i \in \mathbb{N}$, where $(X^\alpha)^*$ is the dual space of X^α.

We now prove that $\{u_i\}$ is bounded in X^α. In fact, if not, we may assume that by contradiction that $\|u_i\|_{X^\alpha} \to \infty$ as $i \to +\infty$. Set $v_i = \frac{u_i}{\|u_i\|_{X^\alpha}}$. Clearly, $\|v_i\|_{X^\alpha} = 1$ and there is $v_0 \in X^\alpha$ such that, up to a subsequence

$$\begin{cases} v_i \rightharpoonup v_0, & \text{weakly in } X^\alpha, \\ v_i \to v_0, & \text{strongly in } L^q(\mathbb{R},\mathbb{R}^n),\ \ 2 \leq q < +\infty, \end{cases} \tag{71}$$

as $i \to +\infty$. Since $v_i \rightharpoonup v_0$ in X^α, it is easy to verify that

$$\lim_{i\to+\infty} v_i(t) = v_0(t) \quad \forall\, t \in \mathbb{R}. \tag{72}$$

Now, we consider the following two cases:

Case 1. $v_0 = 0$. From (26), (70), (W13) and the Hölder's inequality, we can obtain

$$\begin{aligned}
\rho M_1 + M_1\|u_i\|_{X^\alpha} &\geq \rho\Phi(u_i) - \Phi'(u_i)u_i \\
&= \left(\frac{\rho}{2} - 1\right)\|u_i\|^2_{X^\alpha} + \int_{\mathbb{R}}[(\nabla W(t, u_i(t)), u_i(t)) - \rho W(t, u_i(t))]dt \\
&\geq \left(\frac{\rho}{2} - 1\right)\|u_i\|^2_{X^\alpha} - \sum_{j=1}^{l}\int_{\mathbb{R}}\left[p_j|u_i(t)|^2 + q_j(L(t)u_i(t), u_i(t)) + M_j(t)|u_i(t)|^{\theta_j}\right]dt \\
&\geq \left(\frac{\rho-2}{2} - \sum_{j=1}^{l} q_j\right)\|u_i\|^2_{X^\alpha} - \sum_{j=1}^{l} p_j\|u_i\|^2_{L^2} - \sum_{j=1}^{l}\|M_j\|_{L^{\frac{2}{2-\theta_j}}}\|u_i\|^{\theta_j}_{L^2} \\
&\geq \left(\frac{\rho-2}{2} - \sum_{j=1}^{l} q_j\right)\|u_i\|^2_{X^\alpha} - \sum_{j=1}^{l} p_j\|u_i\|^2_{L^2} - \sum_{j=1}^{l}\|M_j\|_{L^{\frac{2}{2-\theta_j}}}(l^{-1}_{\min})^{\frac{\theta_j}{2}}\|u_i\|^{\theta_j}_{X^\alpha}.
\end{aligned} \tag{73}$$

Divided by $\|u_i\|^2_{X^\alpha}$ on both sides of (73), noting that $0 \leq \sum_{j=1}^{l} q_j < \frac{\rho-2}{2}$ and $0 < \theta_j < 2$ $(j = 1, \ldots, l)$, one has

$$\|v_i\|^2_{L^2} \geq \frac{\frac{\rho-2}{2} - \sum_{j=1}^{l} q_j}{\sum_{j=1}^{l} p_j} > 0, \quad \text{as} \quad i \to \infty. \tag{74}$$

It follows from (71) and (74) that $v_0 \neq 0$. This is a contradiction.

Case 2. $v_0 \neq 0$. Since $\{\Phi(u_i)\}_{n\in\mathbb{N}}$ is bounded, then by (70), we have

$$\Phi(u_i) = \frac{1}{2}\|u_i\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t, u_i(t))dt \geq -M_1. \tag{75}$$

Divided by $\|u_i\|^2_{X^\alpha}$ on both sides of (75), we have

$$\int_{\mathbb{R}}\frac{W(t, u_i(t))}{\|u_i\|^2_{X^\alpha}}dt \leq \frac{1}{2} + \frac{M_1}{\|u_i\|^2_{X^\alpha}} < +\infty. \tag{76}$$

Let $\Lambda := \{t \in \mathbb{R} : \ v_0(t) \neq 0\}$, then $\Lambda \neq \emptyset$. Hence, by (72), we can obtain

$$\lim_{i\to+\infty} u_i(t) = \lim_{i\to+\infty} v_i(t)\|u_i\|_{X^\alpha} = +\infty \quad \forall\, t \in \Lambda.$$

Combining (W10) and (W11), one has

$$\lim_{i\to+\infty}\left(\frac{W(t, u_i(t))}{|u_i(t)|^2} + \varrho\right)|v_i(t)|^2 = +\infty \quad \forall\, t \in \Lambda. \tag{77}$$

So, by (W11), (71), (77) and Fatou's lemma, one can get

$$\begin{aligned}
\int_{\mathbb{R}}\frac{W(t, u_i(t))}{\|u_i\|^2_{X^\alpha}}dt &= \int_{t\in\Lambda}\frac{W(t, u_i(t))}{\|u_i\|^2_{X^\alpha}}dt + \int_{t\in\mathbb{R}\setminus\Lambda}\frac{W(t, u_i(t))}{\|u_i\|^2_{X^\alpha}}dt \\
&\geq \int_{t\in\Lambda}\frac{W(t, u_i(t))}{\|u_i\|^2_{X^\alpha}}dt - \varrho\int_{t\in\mathbb{R}\setminus\Lambda}|v_i(t)|^2dt \\
&= \int_{t\in\Lambda}\frac{W(t, u_i(t)) + \varrho|u_i(t)|^2}{|u_i(t)|^2}|v_i(t)|^2dt - \varrho\int_{\mathbb{R}}|v_i(t)|^2dt \to +\infty
\end{aligned}$$

as $i \to +\infty$. This contradicts (76). Therefore, $\{u_i\}$ is bounded in X^α, that is, there exists $\xi_1 > 0$ such that

$$\|u_i\|_{X^\alpha} \leq \xi_1. \tag{78}$$

Then the sequence $\{u_i\}$ has a subsequence, again denoted by $\{u_i\}$, and there exists $u \in X^\alpha$ such that $u_i \rightharpoonup u$ in X^α. Hence we will prove that $u_i \to u$ in X^α. By (W13), (26) and (78), we have

$$\begin{aligned}
&\int_{\mathbb{R}} (\nabla W(t,u_i(t)) - \nabla W(t,u(t)), u_i(t) - u(t))dt \\
&\leq \int_{\mathbb{R}} (|\nabla W(t,u_i(t))| + |\nabla W(t,u(t))|)|u_i(t) - u(t)|dt \\
&\leq D \int_{\mathbb{R}} \Big(|u_i(t)| + \sum_{j=1}^{l} |u_i(t)|^{\gamma_j - 1}\Big)|u_i(t) - u(t)|dt \\
&\quad + D \int_{\mathbb{R}} \Big((|u(t)| + \sum_{j=1}^{l} |u(t)|^{\gamma_j - 1}\Big)|u_i(t) - u(t)|dt \\
&\leq D\Big(\|u_i\|_{L^2} + \sum_{j=1}^{l} \|u_i\|^{\gamma_j - 1}_{L^{2(\gamma_j - 1)}}\Big)\|u_i - u\|_{L^2} \\
&\quad + D\Big(\|u\|_{L^2} + \sum_{j=1}^{l} \|u\|^{\gamma_j - 1}_{L^{2(\gamma_j - 1)}}\Big)\|u_i - u\|_{L^2} \\
&\leq D\Big(\sqrt{l_{\min}^{-1}}\|u_i\|_{X^\alpha} + \sum_{j=1}^{l} \sqrt{l_{\min}^{-1}} C_\alpha^{\gamma_j - 2}\|u_i\|^{\gamma_j - 1}_{X^\alpha}\Big)\|u_i - u\|_{L^2} \\
&\quad + D\Big(\|u\|_{L^2} + \sum_{j=1}^{l} \|u\|^{\gamma_j - 1}_{L^{2(\gamma_j - 1)}}\Big)\|u_i - u\|_{L^2} \\
&\leq D\Big(\sqrt{l_{\min}^{-1}}\xi_1 + \sum_{j=1}^{l} \sqrt{l_{\min}^{-1}} C_\alpha^{\gamma_j - 2}\xi_1^{\gamma_j - 1}\Big)\|u_i - u\|_{L^2} \to 0, \quad \text{as } i \to +\infty.
\end{aligned} \tag{79}$$

It follows from $u_i \rightharpoonup u$ weakly in X^α and (79) that

$$(\Phi'(u_i) - \Phi'(u), u_i - u) = \|u_i - u\|^2_{X^\alpha} - \int_{\mathbb{R}} (\nabla W(t,u_i(t)) - \nabla W(t,u(t)), u_i(t) - u(t))dt,$$

it is easy to deduce that $\|u_i - u\|_{X^\alpha} \to 0$ as $i = +\infty$. Therefore, Φ satisfies the Palais–Smale condition.

Therefore, it follows from Theorem 9 that Φ possesses an unbounded sequence $\{d_i\}$ of critical values with $d_i = \Phi(u_i)$, where u_i is such that $\Phi'(u_i) = 0$ for $i = 1, 2, \ldots$. If $\|u_i\|_{X^\alpha}$ is bounded, then there exists $R > 0$ such that

$$\|u_i\|_{X^\alpha} \leq R, \quad \text{for } i \in \mathbb{N}. \tag{80}$$

Hence, by virtue of (26) and (W12), we have

$$\begin{aligned}
\frac{1}{2}\|u_i\|^2_{X^\alpha} &= d_i + \int_{\mathbb{R}} W(t,u_i(t))dt \\
&\geq d_i - D\int_{\mathbb{R}} \Big(\frac{1}{2}|u_i(t)|^2 + \sum_{j=1}^{l} \frac{1}{\gamma_j}|u_i(t)|^{\gamma_j}\Big)dt \\
&\geq d_i - D\Big(\frac{1}{2} l_{\min}^{-1}\|u_i\|^2_{X^\alpha} + \sum_{j=1}^{l} \frac{1}{\gamma_j} l_{\min}^{-1} C_\alpha^{\gamma_j - 2}\|u_i\|^{\gamma_j}_{X^\alpha}\Big)
\end{aligned}$$

Thus, this follows that

$$d_i \leq \frac{1}{2}\|u_i\|^2_{X^\alpha} + D\Big(\frac{1}{2} l_{\min}^{-1}\|u_i\|^2_{X^\alpha} + \sum_{j=1}^{l} \frac{1}{\gamma_j} l_{\min}^{-1} C_\alpha^{\gamma_j - 2}\|u_i\|^{\gamma_j}_{X^\alpha}\Big) < +\infty.$$

This contradicts the fact that $\{d_i\}$ is unbounded, and so $\|u_i\|_{X^\alpha}$ is unbounded. The proof is completed. □

Proof of Theorem 8. By a similar argument as that in Theorem 4, we can prove Theorem 8. In fact, we only need to prove that Φ satisfies the Palais–Smale condition. Let $\{u_i\}_{i\in\mathbb{N}} \subset X^\alpha$ be a Palais–Smale sequence, that is, $\{\Phi(u_i)\}_{n\in\mathbb{N}}$ is bounded and $\Phi'(u_i) \to 0$ as $i \to +\infty$. We now prove that $\{u_i\}$ is bounded in X^α. In fact, if not, we may assume that by contradiction that $\|u_i\|_{X^\alpha} \to \infty$ as $i \to +\infty$. We take v_i as in the proof of Theorem 4.

Case 1. $v_0 = 0$. From (W14), one has

$$\begin{aligned}
2\Phi(u_i) - \Phi'(u_i)u_i &= \int_{\mathbb{R}} [(\nabla W(t, u_i(t)), u_i(t)) - 2W(t, u_i(t))]dt \\
&\geq \int_{\{t\in\mathbb{R}, |u_i(t)|\geq R_1\}} [(\nabla W(t, u_i(t)), u_i(t)) - 2W(t, u_i(t))]dt \\
&\geq c \int_{\{t\in\mathbb{R}, |u_i(t)|\geq R_1\}} |u_i(t)|^\vartheta dt,
\end{aligned} \tag{81}$$

which implies that

$$\frac{\int_{t\in\mathbb{R}, |u_i(t)|\geq R_1} |u_i(t)|^\vartheta dt}{\|u_i\|_{X^\alpha}} \to 0, \quad \text{as} \quad i \to \infty. \tag{82}$$

It follows from (13), (W12), (W14) and Remark 1 that

$$\begin{aligned}
M_2 \geq \Phi(u_i) &= \frac{1}{2}\|u_i\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t, u_i(t))dt \\
&\geq \frac{1}{2}\|u_i\|^2_{X^\alpha} - D\int_{\mathbb{R}} \left(\frac{1}{2}|u_i(t)|^2 + \sum_{j=1}^{l} \frac{1}{\gamma_j}|u_i(t)|^{\gamma_j}\right)dt \\
&\geq \frac{1}{2}\|u_i\|^2_{X^\alpha} - \frac{1}{2}D\|u_i\|^2_{L^2} - D\sum_{j=1}^{l} \frac{1}{\gamma_j} \int_{\{t\in\mathbb{R}, |u_i(t)|\geq R_1\}} |u_i(t)|^{\gamma_j} dt \\
&\quad - D\sum_{j=1}^{l} \frac{1}{\gamma_j} \int_{\{t\in\mathbb{R}, |u_i(t)|< R_1\}} |u_i(t)|^{\gamma_j} dt \\
&\geq \frac{1}{2}\|u_i\|^2_{X^\alpha} - \frac{1}{2}D\|u_i\|^2_{L^2} - D\|u_i\|_\infty \sum_{j=1}^{l} \frac{1}{\gamma_j} \int_{\{t\in\mathbb{R}, |u_i(t)|\geq R_1\}} |u_i(t)|^{\gamma_j-1} dt \\
&\quad - D\sum_{j=1}^{l} \frac{1}{\gamma_j} R_1^{\gamma_j-2} \int_{\{t\in\mathbb{R}, |u_i(t)|< R_1\}} |u_i(t)|^2 dt \\
&\geq \frac{1}{2}\|u_i\|^2_{X^\alpha} - \frac{1}{2}D\|u_i\|^2_{L^2} - D\|u_i\|_\infty \sum_{j=1}^{l} \frac{1}{\gamma_j} R_1^{\gamma_j-\vartheta-1} \int_{\{t\in\mathbb{R}, |u_i(t)|\geq R_1\}} |u_i(t)|^\vartheta dt \\
&\quad - D\sum_{j=1}^{l} \frac{1}{\gamma_j} R_1^{\gamma_j-2} \int_{\{t\in\mathbb{R}, |u_i(t)|< R_1\}} |u_i(t)|^2 dt \\
&\geq \frac{1}{2}\|u_i\|^2_{X^\alpha} - D\left(\frac{1}{2} + \sum_{j=1}^{l} \frac{1}{\gamma_j} R_1^{\gamma_j-2}\right)\|u_i\|^2_{L^2} \\
&\quad - DC_\alpha\|u_i\|_{X^\alpha} \sum_{j=1}^{l} \frac{1}{\gamma_j} R_1^{\gamma_j-\vartheta-1} \int_{\{t\in\mathbb{R}, |u_i(t)|\geq R_1\}} |u_i(t)|^\vartheta dt,
\end{aligned} \tag{83}$$

for some $M_2 > 0$. Divided by $\|u_i\|^2_{X^\alpha}$ on both sides of (83), noting that (82), we have

$$\|v_i\|^2_{L^2} \geq \frac{1}{2D\left(\frac{1}{2} + \sum_{j=1}^{l} \frac{1}{\gamma_j} R_1^{\gamma_j - 2}\right)} > 0, \quad \text{as} \quad i \to \infty. \tag{84}$$

It follows from (71) and (84) that $v_0 \neq 0$. This is a contradiction.

Case 2. $v_0 \neq 0$. The proof is the same as that in Theorem 4, and we omit it here. Hence, $\{u_i\}$ is bounded in X^α. Similar to the proof of Theorem 4, we can prove that $\{u_i\}$ has a convergent subsequence in X^α. Hence, Φ satisfies the Palais–Smale condition. The proof is completed. □

4. Conclusions

Using variational methods, we have obtained homoclinic solutions for fractional Hamiltonian systems. The fractional component of the equation is due to a memory effect modeled by means of Liouville–Weyl type derivative in time. The introduction provides an overview about the state of the fractional Hamiltonian systems and authors' motivation. In Section 2, we have recalled some related preliminary concepts for the convenience of the reader. Section 3 contains main theorems, which are proved by applying Clark's theorem from critical point theory and fountain theorem.

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