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Bowen Lemma in the Countable Symbolic Space

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Abstract: We consider the sets of quasi-regular points in the countable symbolic space. We measure the sizes of the sets by Billingsley-Hausdorff dimension defined by Gibbs measures. It is shown that the dimensions of those sets, always bounded from below by the convergence exponent of the Gibbs measure, are given by a variational principle, which generalizes Li and Ma's result and Bowen's result.

Keywords: measure theoretic entropy; Gibbs measure; Hausdorff dimension; symbolic dynamical system; Bowen Lemma; quasi-regular point

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1. Introduction

Let (X, T) be a topological system, where $T : X \to X$ is a continuous transformation on a metric space X. Denote the set of Borel probability measures on X by $\mathcal{M}(X)$ and endow it with weak*-topology induced by $C_b(X)$, which stands for the set of all bounded continuous functions on X. For any $x \in X$, and $n \geq 1$, define the orbit measure

$$\Delta_{x,n} := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x}.$$

Write $V(x)$ for the collection of all accumulation points of the sequence orbit measures $\{\Delta_{x,n}\}_{n\geq 1}$ in the sense of weak*-topology of $\mathcal{M}(X)$. Denote by $\mathcal{M}_T(X)$ the set of T-invariant Borel probability measures on X. For $\mu \in \mathcal{M}_T(X)$, the set G_μ of μ-generic points is defined by

$$G_\mu := \left\{ x \in X : \Delta_{x,n} \xrightarrow{w^*} \mu,\ n \to \infty \right\}$$

When X is a compact metric space, Bowen [1] proved that for any $\mu \in \mathcal{M}_T(X)$, the topological entropy $h_{top}(G_\mu)$ does not exceed the measure-theoretic entropy h_μ. In his proof, Bowen consider the following quasi-regular point set: for $t \geq 0$, define

$$QR(t) := \left\{ x \in X : \exists \mu \in V(x) \text{ with } h_\mu \leq t \right\}.$$

Then he obtained $h_{top}(QR(t)) \leq t$ which implied that $h_{top}(G_\mu) \leq h_\mu$.

As a basic dynamic model, the symbolic system always attract much attention for its intrinsic property. In the setting of finite symbolic space, it is well known that there is a close relationship between the indexes topological entropy and dimension, i.e., topological entropy is different from Hausdorff dimension by a multiplier under the usual metric. However, as a generalization of Hausdorff dimension, the index Billingsley dimension (see the definition in Section 2) provides more ways to measure the fractal sets. In this note, we only consider Billingsley dimensions defined by Gibbs measures. Denote by $X_N = \{1, \cdots, N\}^{\mathbb{N}}$ $(N \geq 2)$ the finite symbolic space endowed with the product topology and define the shift map $T : X_N \to X_N$ by

$$T(x_1 x_2 \cdots) = (x_2 x_3 \cdots).$$

Let $\varphi : X_N \to \mathbb{R}$ be a potential with summable variations. It is well known that φ admits a unique Gibbs measure ν. Li and Ma in [2] consider the quasi-regular point set $Q_\nu(t)$ with respect to a Gibbs measure ν :

$$Q_\nu(t) := \left\{ x \in X_N : \exists \mu \in V(x) \text{ with } \frac{h_\mu}{P_\varphi - \int \varphi \mathrm{d}\mu} \leq t \right\},$$

where P_φ denotes the Gurevich pressure of φ and $0 \leq t \leq 1$.

Theorem 1 ([2]). *Assume that φ has summable variations and denote the unique Gibbs measure by ν. Then for any $0 \leqslant t \leqslant 1$ we have*

$$\dim_\nu Q_\nu(t) = t.$$

When ν is the uniform distribution on X_N Billingsley dimension is just Hausdorff dimension under the usual metric on X_N. Hence, the above dimension formula generalizes Bowen's result [3].

When X is the countable symbolic space, the situation changes. One key point is that X is non-compact space under the product topology, which leads to a phenomenon that the potential φ which admits a Gibbs measure has upper bound but no lower bound. Therefore, there exists T-invariant measure μ such that the integral $\int \varphi \mathrm{d}\mu$ is infinite. In the sequel, unless otherwise stated, we always denote by X the countable symbolic space.

We still denote by φ a potential on X and by ν the corresponding Gibbs measure. For any $\mu \in \mathcal{M}_T(X)$, define the entropy dimension of ν with respect to μ by

$$\beta(\nu|\mu) := \limsup_{k\to\infty} \limsup_{N\to\infty} \frac{-\sum_{\omega\in\Sigma_N^k} \mu([\omega]) \ln \mu([\omega])}{-\sum_{\omega\in\Sigma_N^k} \mu([\omega]) \ln \nu([\omega])},$$

where $\Sigma_N^k = \{1, 2, \ldots, N\}^k$ and $[\omega]$ denotes a cylinder (see the definition in Section 2). We define the convergence exponent of ν by

$$\alpha_\nu := \inf \left\{ t > 0 : \sum_{n=1}^{\infty} \nu^t([n]) < +\infty \right\}.$$

For $0 \leq t \leq 1$, define the quasi-regular point set with respect to ν by

$$QR_\nu(t) = \left\{ x \in X : \exists \mu \in V(x) \text{ with } \beta(\nu|\mu) \leq t \right\}.$$

Our main result in this note is the following.

Theorem 2. *Let φ be a potential function of summable variations admitting a unique Gibbs measure ν with convergence exponent α_ν. For $0 \leq t \leq 1$, we have*

$$\dim_\nu QR_\nu(t) = \max\{\alpha_\nu, t\}. \tag{1}$$

We give four remarks:

(1) In the setting of finite symbolic space, according to Propositions 3 and 4 (see Section 2) we have $\beta(\nu|\mu) = \frac{h_\mu}{P_\varphi - \int \varphi \mathrm{d}\mu}$, thus the set $QR_\nu(t)$ on the above is the same as the set $Q_\nu(t)$ in Theorem 1. By a fact that the convergence exponent of Gibbs measure is zero in the finite symbolic space, we have generalized Theorem 1 and Bowen 's result in [1].

(2) The dimension formula (1) implies the dimension upper bound of generic point set. In fact, for any $\mu \in \mathcal{M}_T(X)$, we have $G_\mu \subset QR_\nu(t)$ for $t = \beta(\nu|\mu)$. Thus, we obtain $\dim_\nu G_\mu \leq \max\{\alpha_\nu, \beta(\nu|\mu)\}$, which was proved by Fan, Li and Ma in [3]. More precisely, they showed

$$\dim_\nu G_\mu = \max\Big\{\alpha_\nu, \beta(\nu|\mu)\Big\}. \tag{2}$$

(3) There are two main reasons to choose Gibbs measure to define Billingsley dimension. One reason is that Gibbs measure has quasi-Bernoulli property (see Lemma 1). On the other hand, note a fact that Lebsgue measure is boundedly equivalent with Gauss measure in the continued fraction system on $[0, 1)$. Hence, by taking a suitable Gibbs measure on X, one can apply Theorem 2 to the Gauss map related to the continued fractions. Moreover, by transferring dimension result from the symbolic space to the interval $[0, 1)$ [3], we can obtain the corresponding dimension formula for quasi-regular point set in expanding Markov Rényi dynamical system ([4], p. 148).

(4) When (X, T) is a topologically mixing countable Markov shift with BIP property ([5], p. 97), the corresponding dimension formula can be proved by a parallel argument.

Let us present the idea of the proof. We first use set operation to describe the limit property possessed by the point in $QR_\nu(t)$. Next, we adopt the finite-symbol-approximation technique developed in [3] to obtain the dimension upper-bound. There are two main problems to deal with. One problem is that the relative entropy $h(\nu|\mu)$ may be infinite. The other one is that we need approximate a class of sets depending on infinite T-invariant measures. However, the authors in [3] approximated a class of sets depending on only one T-invariant measure. In fact, we prove Theorem 3 by combining the ideas in [2,3]. For the lower bound, we draw support from the dimension formula (2).

The article is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the main result of this article.

2. Preliminaries

In this section, we introduce notations and some useful known facts. We denote by X the countable symbolic space $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology and define the *shift map* $T : X \to X$ by

$$(Tx)_n = x_{n+1}.$$

An element $(x_1 \cdots x_n) \in \mathbb{N}^n$ is called an n-length *word*. Let $\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathbb{N}^n$ stand for the set of all finite words, where $\mathbb{N}^0$ denotes the set of empty word. Given $x = (x_1 x_2 \cdots) \in X$ and $m \geqslant n \geqslant 1$,

$$x|_n^m = (x_n \cdots x_m)$$

denotes a subword of x. For $\omega = (\omega_1 \cdots \omega_n) \in \mathbb{N}^n$, the *n-cylinder* $[\omega]$ is defined by

$$[\omega] = \{x \in X : x|_1^n = \omega\}.$$

We will denote by $\mathcal{C}^n$ the set of all n-cylinders for $n \geqslant 0$. There is a one-to-one correspondence between $\mathbb{N}^n$ and $\mathcal{C}^n$. Let $\mathcal{C}^* = \bigcup_{n=0}^{\infty} \mathcal{C}^n$ denote the set of all cylinders. For $j, N \geqslant 1$ we will write

$$\Sigma_N^j = \{1, \cdots, N\}^j, \qquad \mathcal{C}_N^j = \{[\omega] : \omega \in \Sigma_N^j\}.$$

2.1. Billingsley Dimension

Let $A \subset X$ and $v \in \mathcal{M}(X)$ be a non-atomic Borel probability measure. Define

$$\mathcal{H}_\delta{}^s(A) = \inf\left\{\sum_{i=1}^{\infty} v^s(U_i) : A \subset \bigcup_{i=1}^{\infty} U_i, v(U_i) < \delta \text{ for } i \geq 1\right\} \text{ and } \mathcal{H}^s(A) = \liminf_{\delta \to 0} \mathcal{H}_\delta^s(A).$$

Thus, there exists s_0 such that $\mathcal{H}^s(A) = \infty$ for $s < s_0$ and $\mathcal{H}^s(A) = 0$ for $s > s_0$. This s_0 we define to be Billingsley dimension of A with respect to v and denote it by $\dim_v A$. In fact, Billingsley dimension $\dim_v$ is just Hausdorff dimension $\dim_H$ under the metric ρ_v: for any $x, y \in X$, if $x = y$, we define $\rho_v(x, y) = 0$; otherwise

$$\rho_v(x, y) = v([x|_1^n]),$$

where $n = \min\{k \geqslant 0 : x_{k+1} \neq y_{k+1}\}$. One can show that ρ_v is a ultrametric. Let $n \geqslant 1$ be an integer. Define

$$\delta_n = \sup\{v([\omega]) : \omega \in \mathbb{N}^n\}.$$

The following proposition means that the ρ_v-distance of two points uniformly tends to zero when they approach each other in the sense of Bowen.

Proposition 1 ([3]). *For $\{\delta_n\}_{n \geqslant 1}$ defined on the above, one has*

$$\lim_{n \to \infty} \delta_n = 0.$$

The reader may consult [6] for more information on Billingsley dimension.

2.2. Metrization of the w^*-Topology

Recall that we endow $\mathcal{M}(X)$ with the w^*-topology induced by $C_b(X)$. In this subsection, we introduce a metric to describe the w^*-topology of $\mathcal{M}(X)$.

For every cylinder $[\omega] \in \mathcal{C}^*$, we choose a positive number $a_{[\omega]}$ so that

$$\sum_{[\omega] \in \mathcal{C}^*} a_{[\omega]} = 1,$$

where the sum is taken over all cylinders.

For $\mu, v \in \mathcal{M}(X)$, define

$$d^*(\mu, v) = \sum_{[\omega] \in \mathcal{C}^*} a_{[\omega]} |\mu([\omega]) - v([\omega])|.$$

The following proposition shows that the metric d^* is compatible with the w^*-topology of $\mathcal{M}(X)$.

Proposition 2 ([3]). *Let $\{\mu_n\}_{n \geqslant 1} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Then μ_n converges in w^*-topology to μ if and only if* $\lim_{n \to \infty} d^*(\mu_n, \mu) = 0$.

According to [7], we call a collection $\mathcal{F}$ of Borel subset of X a convergence-determining class if, for every $\mu \in \mathcal{M}(X)$ and $\{\mu_n\} \subset \mathcal{M}(X)$, convergence $\mu_n(A) \to \mu(A)$ for all $\mu-$continuity set in $\mathcal{F}$ implies $\mu_n \xrightarrow{w^*} \mu$. As a corollary of Proposition 2, the cylinder set $\mathcal{C}^*$ is a convergence-determining class.

2.3. Gibbs Measure

We use Gibbs measures to induce metrics on X. The following facts about Gibbs measures can be found in [5].

Recall that for a function $\varphi : X \to \mathbb{R}$, called *potential function*, the *n-order variation* of φ is defined by

$$\mathrm{var}_n \varphi := \sup\{|\varphi(x) - \varphi(y)| : x, y \in X,\ x|_1^n = y|_1^n\}.$$

We say that a potential φ has summable variations if

$$\sum_{n=1}^{\infty} \mathrm{var}_n \varphi < +\infty.$$

It is easy to see that a potential φ with summable variations is uniformly continuous on X. The Gurevich pressure of φ with summable variations is defined to be the limit

$$P_\varphi := \lim_{n\to\infty} \frac{1}{n} \ln \sum_{T^n x = x} e^{S_n\varphi(x)} 1_{[a]}(x),$$

where $a \in \mathbb{N}$ and it can be shown that the limit exists and is independent of a (see [8]).

An invariant probability measure ν is called a Gibbs measure associated to a potential function φ if it satisfies the Gibbsian property: there exist constants $C > 1$ and $P \in \mathbb{R}$ such that

$$\frac{1}{C} \leqslant \frac{\nu([x_1 x_2 \cdots x_n])}{\exp(S_n\varphi(x) - nP)} \leqslant C \tag{3}$$

holds for any $n \geqslant 1$ and any $x \in X$. It is known ([9]) that a potential function φ with summable variations admits a unique Gibbs measure ν iff the Gurevich pressure $P_\varphi < +\infty$. Assume that φ admits a unique Gibbs measure ν_φ. Then the constant P in (3) is equal to the Gurevich pressure P_φ. Let $\varphi^* = \varphi - P_\varphi$, we have

$$P_{\varphi^*} = 0 \text{ and } \nu_{\varphi^*} = \nu_\varphi.$$

Hence, without loss of generality, we always suppose $P_\varphi = 0$ in the rest of this paper. A trivial fact is that the Gibbsian property (3) implies that φ has no lower bound but upper bound, i.e.,

$$\forall x \in X, \quad \varphi(x) \leqslant \ln C. \tag{4}$$

The Gibbsian property implies the quasi-Bernoulli property which is a key point in the proof of dimension upper bound in Section 3.

Lemma 1 ([3])**.** *Let ν be a Gibbs measure associated to potential φ. For any k words $\omega_1, \cdots, \omega_k$, we have*

$$C^{-(k+1)}\nu([\omega_1 \cdots \omega_k]) \leqslant \nu([\omega_1]) \cdots \nu([\omega_k]) \leqslant C^{k+1}\nu([\omega_1 \cdots \omega_k]).$$

For any T-invariant Borel probability measure μ, define the *relative entropy* of ν with respect to μ by

$$h(\nu|\mu) = \limsup_{k\to\infty} -\frac{1}{k} \sum_{\omega\in\mathbb{N}^k} \mu([\omega]) \ln \nu([\omega]).$$

It is trivially true that $h(\mu|\mu) = h_\mu$.

Proposition 3 ([3])**.** *Assume that φ has summable variations and admits a unique Gibbs measure ν. Then for any invariant measure $\mu \in \mathcal{M}_T(X)$, we have*

$$h(\nu|\mu) = \lim_{k\to\infty} -\frac{1}{k} \sum_{\omega\in\mathbb{N}^k} \mu([\omega]) \ln \nu([\omega]) = -\int_X \varphi \, d\mu.$$

By Proposition 3, we can rewrite the variational principle ([5], p. 86) in the following form

$$P_\varphi = \sup_{\mu\in\mathcal{M}_T(X)} \left\{ h_\mu - h(\nu|\mu) : h(\nu|\mu) < +\infty \right\}. \tag{5}$$

Recall that we assume that $P_\varphi = 0$. It is known that the supremum in the variational principle (5) is attained only by a Gibbs measure ν with $h_\nu < \infty$ if such Gibbs measure exits ([5], p. 89). It follows that when $\nu \neq \mu$, we have $h(\nu|\mu) > h_\mu$, which implies $h(\nu|\mu) > 0$.

Now we introduce two exponents. Define the *convergence exponent* of ν by

$$\alpha_\nu := \inf\Big\{t > 0 : \sum_{n=1}^{\infty} \nu([n])^t < +\infty\Big\}.$$

For any $\mu \in \mathcal{M}_T(X)$, define the *entropy dimension* of ν with respect to μ by

$$\beta(\nu|\mu) := \limsup_{k\to\infty} \limsup_{N\to\infty} \frac{H_{k,N}(\mu,\mu)}{H_{k,N}(\nu,\mu)},$$

where

$$H_{k,N}(\nu,\mu) := -\sum_{\omega\in\Sigma_N^k} \mu([\omega]) \ln \nu([\omega]).$$

The convergence exponent α_ν reflects the mass distribution of ν. Moreover, α_ν has the following property.

Lemma 2 ([3]). *Let α_ν be the convergence exponent of Gibbs measure ν associated to a potential function φ. Then for any $\epsilon > 0$ there exist constants C_0 and M such that*

$$\sum_{\omega\in\mathbb{N}^k} \nu([\omega])^{\alpha_\nu+\epsilon} \leqslant C_0 M^k, \ (\forall k \geqslant 1). \tag{6}$$

If $\mu = \nu$, it is clear that $\beta(\nu|\mu) = 1$. However, we have the following claim.

Proposition 4 ([3]). *Let $\mu \in \mathcal{M}_T(X)$ and φ be a potential function of summable variations. Assume that φ admits a unique Gibbs measure ν with convergence exponent α_ν. If $\nu \neq \mu$ and $h(\nu|\mu) < +\infty$, then*

$$\beta(\nu|\mu) = \frac{h_\mu}{h(\nu|\mu)}; \tag{7}$$

if $h(\nu|\mu) = +\infty$, we have

$$\beta(\nu|\mu) \leqslant \alpha_\nu. \tag{8}$$

3. Proof of the Main Result

The proof is divided into two parts. First, we shall prove

$$\dim_\nu QR_\nu(t) \leq \max\{\alpha_\nu, t\}.$$

We introduce a technical lemma. For every word $\omega \in \Sigma_N^n$ of length n and every word $u \in \Sigma_N^k$ of length k with $k \leqslant n$, denote by $p(u|\omega)$ the frequency of appearances of u in ω, i.e.,

$$p(u|\omega) = \frac{\tau_u(\omega)}{n-k+1},$$

where $\tau_u(\omega)$ denotes the number of j with $1 \leqslant j \leqslant n-k+1$, so that $\omega_j \cdots \omega_{j+k-1} = u$.

Lemma 3 ([10]). *For any $h > 0$, $\delta > 0$, $k \in \mathbb{N}$ and $n \in \mathbb{N}$ large enough, we have*

$$\sharp\Big\{\omega \in \Sigma_N^n : \sum_{u\in\Sigma_N^k} -p(u|\omega)\ln p(u|\omega) \leqslant kh\Big\} \leqslant \exp(n(h+\delta)).$$

Theorem 3. *Let φ be a potential function of summable variations admitting a unique Gibbs measure ν with convergence exponent α_ν. For $0 \le t \le 1$, we have*

$$\dim_\nu QR_\nu(t) \le \max\{\alpha_\nu, t\}.$$

Proof. Fix $\epsilon_1 > 0$, we define, for $N_1 > 1$,

$$\mathcal{M}_{\epsilon_1}(N_1) = \left\{\mu \in \mathcal{M}_T(X) : \sum_{u \in \Sigma_{N_1}^{N_1}} \mu([u]) > 1 - \epsilon_1\right\}.$$

It is clear that $\{\mathcal{M}_{\epsilon_1}(N_1)\}_{N_1 \ge 1}$ monotonically converges to $\mathcal{M}_T(X)$. Hence we have

$$QR_\nu(t) = \bigcup_{N_1 \ge 1} H_{\epsilon_1}(N_1),$$

where

$$H_{\epsilon_1}(N_1) = \left\{x \in X : \exists \mu \in V(x) \cap \mathcal{M}_{\epsilon_1}(N_1) \text{ with } \beta(\nu|\mu) \le t\right\}.$$

Fix $\epsilon_2 > 0$. According to the definition of $\beta(\nu|\mu)$, we have

$$H_{\epsilon_1}(N_1) \subset \bigcup_{l \ge 1} \bigcap_{k \ge l} \bigcup_{r \ge 1} \bigcap_{N \ge r} A_{\epsilon_1,\epsilon_2}(N_1, k, N),$$

where

$$A_{\epsilon_1,\epsilon_2}(N_1, k, N) = \left\{x \in X : \exists \mu \in V(x) \cap \mathcal{M}_{\epsilon_1}(N_1) \text{ with } \frac{H_{k,N}(\mu,\mu)}{H_{k,N}(\nu,\mu)} \le t + \epsilon_2\right\}.$$

By the σ-stability and monotonicity of Hausdorff dimension, we have

$$\dim_\nu QR_\nu(t) \le \sup_{N_1 \ge 1} \liminf_{k \to \infty} \liminf_{N \to \infty} \dim_\nu A_{\epsilon_1,\epsilon_2}(N_1, k, N).$$

According to the definition of $V(x)$ and Proposition 2, we have

$$A_{\epsilon_1,\epsilon_2}(N_1, k, N) = \bigcap_{m \ge 1} \bigcap_{j \ge 1} \bigcup_{n \ge j} B_{\epsilon_1,\epsilon_2}(N_1, k, N, m, n),$$

where

$$B_{\epsilon_1,\epsilon_2}(N_1, k, N, m, n) = \left\{x \in X : \exists \mu \in \mathcal{M}_{\epsilon_1}(N_1) \text{ s.t. } d^*(\Delta_{x,n}, \mu) < \frac{1}{m}, \frac{H_{k,N}(\mu,\mu)}{H_{k,N}(\nu,\mu)} \le t + \epsilon_2\right\}.$$

For m large enough, by the uniformness of $H_{k,N}(\cdot,\cdot)$, we have

$$B_{\epsilon_1,\epsilon_2}(N_1, k, N, m, n) \subset C_{\epsilon_1,\epsilon_2}(N_1, k, N, m, n),$$

where

$$C_{\epsilon_1,\epsilon_2}(N_1, k, N, m, n) = \left\{x \in X : \exists \mu \in \mathcal{M}_{\epsilon_1}(N_1) \text{ s.t. } d^*(\Delta_{x,n}, \mu) < \frac{1}{m}, \frac{H_{k,N}(\Delta_{x,n},\Delta_{x,n})}{H_{k,N}(\nu,\Delta_{x,n})} \le t + 2\epsilon_2\right\}.$$

Clearly, the parameters n, m, N, k and N_1 successively go to the infinity. In the sequel, we will write $C(k, N, n)$ for $C_{\epsilon_1,\epsilon_2}(N_1, k, N, m, n)$ for simplicity.

Now we shall give a suitable covering of $A_{\epsilon_1,\epsilon_2}(N_1, k, N)$. Consider the $(n + k - 1)$-prefixes of the points in $C(k, N, n)$:

$$\Lambda_n(k,N) := \left\{x_1 \cdots x_{n+k-1} \in \mathbb{N}^{n+k-1} : x \in C(k,N,n)\right\}.$$

Let $\delta_n = \sup\{\nu([\omega]) : \omega \in \mathbb{N}^n\}$. According to Proposition 1 we have $\lim_{n\to\infty} \delta_n = 0$. Then the cylinder set $\{[\omega] : \omega \in \bigcup_{n\geq j} \Lambda_n(k,N)\}$ forms a δ_{j+k-1}-covering of $A_{\epsilon_1,\epsilon_2}(N_1,k,N)$. Assume that $\gamma > \max\{\alpha_\nu, t\}$ and $\gamma < 3/2$ without loss of generality. By the definition of γ- Hausdorff measure, we have

$$\mathcal{H}^\gamma_{\delta_{j+k-1}}(A_{\epsilon_1,\epsilon_2}(n_1,k,N)) \leqslant \sum_{n\geq j} \sum_{\omega\in\Lambda_n(k,N)} \nu^\gamma([\omega]).$$

It suffices to dominate the outer sum in the above inequality by a constant. Recall that $\tau_u(\omega)$ denotes the number of times that the subword u appears in ω. Given a word $\omega \in \Lambda_n(k,N)$, we say that $(\tau_u(\omega))_{u\in\Sigma_N^k}$ is the appearance distribution with respect to Σ_N^k of ω. Write $\mathfrak{D}_n(k,N)$ for the set of such appearance distributions of all elements of $\Lambda_n(k,N)$. Clearly we have $\sharp\mathfrak{D}_n(k,N) \leqslant n^{N^k}$ since there are at most N^k possible words u in Σ_N^k. Given a distribution $(\tau_u) \in \mathfrak{D}_n(k,N)$, write

$$D((\tau_u)) := \left\{\omega \in \Lambda_n(k,N) : \tau_u(\omega) = \tau_u,\ \forall u \in \Sigma_N^k\right\}.$$

Thus, we have

$$\sum_{\omega\in\Lambda_n(k,N)} \nu^\gamma([\omega]) \leqslant n^{N^k} \max_{(\tau_u)\in\mathfrak{D}_n(k,N)} \sum_{\omega\in D((\tau_u))} \nu^\gamma([\omega]). \tag{9}$$

In order to dominate the sum on the right-hand side in the above inequality, we shall decompose $D((\tau_u))$ into disjoint union of some sets. Given $\omega = \omega_1 \cdots \omega_{n+k-1} \in D((\tau_u))$, a subword $\omega_j\omega_{j+1}\cdots\omega_{j+m-1}$ is called a maximal (N,k)-run subword of ω if $m \geqslant k$, $\omega_{j-1} > N$, $\omega_{j+m} > N$, and $\omega_{j+i} \leqslant N$ for $0 \leqslant i \leqslant m-1$. On the other hand, we say that a subword between maximal (N,k)-run subwords is a "bad subword". Thus, the element of $D((\tau_u))$ is just like

$$\omega = B_{r_1} W_{n_1} B_{r_2} \cdots W_{n_t} B_{r_{t+1}}, \tag{10}$$

where B_{r_i} stands for "bad subword" with length r_i and W_{n_i} stands for maximal (N,k)-run subword with length n_i. Write

$$K := \sum_{u\in\Sigma_N^k} \tau_u \text{ and } s := \left\lfloor \frac{n-K}{k} \right\rfloor + 1.$$

It is clear that every element in $D((\tau_u))$ has at most s maximal (N,k)-run subwords, which implies that $t \leqslant s$. Moreover, by setting $K_t := \sum_{i=1}^t n_i$, we have

$$K_t = K + t(k-1)$$

and

$$r_1 \geqslant 0, r_{t+1} \geqslant 0, r_i \geqslant 1\ (2 \leqslant i \leqslant t) \text{ and } K_t + \sum_{i=1}^{t+1} r_i = n+k-1. \tag{11}$$

For $1 \leqslant t \leqslant s$, let D_t be the set of words in $D((\tau_u))$ with t maximal (N,k)-run subwords. Clearly, $D((\tau_u))$ is a disjoint union of D_t's, i.e.,

$$D((\tau_u)) = \bigsqcup_{t=1}^{s} D_t. \tag{12}$$

Next, we continue to partition D_t by the length pattern of "bad subword" and maximal (N,k)-run subword. Recall that every word $\omega \in D_t$ has the form (10). We say that $(\mathbf{r},\mathbf{n}) := (r_1, n_1, r_2, \cdots, n_t, r_{t+1})$ is the length pattern of "bad subword" and maximal (N,k)-run subword. Let $\mathfrak{L}_t$ stand for the set of all such length pattern of ω in D_t. Given a length pattern $(\mathbf{r},\mathbf{n}) \in \mathfrak{L}_t$, we denote by $B(\mathbf{r},\mathbf{n})$ the set of elements of D_t with the length pattern $(\mathbf{r},\mathbf{n})$. Thus, D_t is partitioned into $B(\mathbf{r},\mathbf{n})$'s. It follows that

$$\sum_{\omega\in D_t} \nu^{\gamma}([\omega]) \leqslant \sharp\mathfrak{L}_t \max_{(\mathbf{r},\mathbf{n})\in\mathfrak{L}_t} \sum_{\omega\in B(\mathbf{r},\mathbf{n})} \nu^{\gamma}([\omega]). \tag{13}$$

We shall estimate the cardinal of $\mathfrak{L}_t$ and the sum on the right-hand side in the above inequality. First, note that every length pattern $(\mathbf{r},\mathbf{n}) \in \mathfrak{L}_t$ is just corresponding to the integer solution of the following equation set

$$\begin{cases} \sum_{i=1}^{t} n_i = K_t,\ n_i \geqslant k(1\leqslant i \leqslant t), \\ \sum_{i=1}^{t+1} r_i = n+k-1-K_t,\ r_1 \geqslant 0,\ r_{t+1}\geqslant 0,\ r_i \geqslant 1\ (2\leqslant i\leqslant t). \end{cases}$$

By the elementary combinatorial theory, we have

$$\sharp\mathfrak{L}_t \leqslant \frac{(K-1)!}{(K-t)!(t-1)!}\frac{(n-K-(t-1)k+t)!}{t!(n-K-(t-1)k)!}.$$

According to the definition of $\mathcal{M}_{\epsilon_1}(N_1)$, for any $\mu \in \mathcal{M}_{\epsilon_1}(N_1)$ and $k, N \geq N_1$, we have

$$\sum_{u\in\Sigma_N^k} \mu([u]) > 1-\epsilon_1.$$

By the definition of $C(k,N,n)$, for $x \in C(k,N,n)$ there exists $\mu \in \mathcal{M}_{\epsilon_1}(N_1)$ such that $d^*(\Delta_{x,n},\mu) < \frac{1}{m}$. Thus, when m is taken large enough, we have

$$\frac{K}{n} = \sum_{u\in\Sigma_N^k} \frac{\tau_u}{n} = \sum_{u\in\Sigma_N^k} \Delta_{x,n}([u]) > 1-2\epsilon_1.$$

In other word, we have

$$1-\frac{K}{n} < 2\epsilon_1.$$

Note that the parameter n goes to the infinity before k. For any $\delta > 0$, when n is taken large enough, by the Stirling formula we have

$$\frac{(K-1)!}{(K-t)!(t-1)!}\frac{(n-K-(t-1)k+t)!}{t!(n-K-(t-1)k)!} \leqslant e^{n\delta/2},$$

which leads to

$$\sharp\mathfrak{L}_t \leqslant e^{n\delta/2}. \tag{14}$$

On the other hand, denote by D_t' the set of finite words by deleting all "bad subwords" of ω in D_t, i.e.,

$$D_t' = \left\{W_{n_1}\cdots W_{n_t} : \omega = B_{r_1}W_{n_1}\cdots W_{n_r}B_{r_{t+1}} \in D_t\right\}.$$

By the quasi-Bernoulli property of Gibbs measure ν, we have

$$\begin{aligned} \sum_{\omega\in B(\mathbf{r},\mathbf{n})} \nu^{\gamma}([\omega]) &\leqslant C^{2\gamma(t+1)} \sum_{\omega\in B(\mathbf{r},\mathbf{n})} \prod_{i=1}^{t+1} \nu^{\gamma}([B_{r_i}(\omega)]) \prod_{i=1}^{t} \nu^{\gamma}([W_{n_i}(\omega)]) \\ &\leqslant C^{2\gamma(t+1)} \sum_{\omega\in B(\mathbf{r},\mathbf{n})} \prod_{i=1}^{t+1} \nu^{\gamma}([B_{r_i}(\omega)]) \sum_{\omega\in B(\mathbf{r},\mathbf{n})} \prod_{i=1}^{t} \nu^{\gamma}([W_{n_i}(\omega)]) \\ &\leqslant C^{\gamma(4t+5)} V \sum_{\omega\in D_t'} \nu^{\gamma}([\omega]), \end{aligned} \tag{15}$$

where

$$V := \sum_{\omega\in B(\mathbf{r},\mathbf{n})} \nu^{\gamma}([B_{r_1}(\omega)\cdots B_{r_{t+1}}(\omega)]).$$

By (11), we have $\sum_{i=1}^{t+1} r_i \leqslant n-K$. Thus, by Lemma 2, there exist constants $C_0>0$ and $M>0$ such that

$$V \leqslant \sum_{\omega\in\mathbb{N}^{n-K}} \nu^{\gamma}([\omega]) \leqslant C_0 M^{n-K}.$$

This, together with (13), (14) and (15), yields

$$\sum_{\omega\in D_t} \nu^{\gamma}([\omega]) \leqslant C_0 C^{\gamma(4t+5)} M^{n-K} e^{n\delta/2} \sum_{\omega\in D_t'} \nu^{\gamma}([\omega]). \tag{16}$$

By the definition of s and the fact that $t\leqslant s$, for the same $\delta>0$ as above, when ϵ_1 is taken small enough we have

$$C_0 C^{\gamma(4t+5)} M^{n-K} \leqslant e^{n\delta/2},$$

which, in combination with (16), gives

$$\sum_{\omega\in D_t} \nu^{\gamma}([\omega]) \leqslant e^{n\delta} \sum_{\omega\in D_t'} \nu^{\gamma}([\omega]). \tag{17}$$

In order to bound the sum on the right-hand side in (17), we estimate the cardinal of D_t' and the ν-size of the cylinder, respectively. First, let $\widetilde{D}_t$ be the set of finite words obtained by replacing each "bad subword" B_{r_i} of ω in D_t by a finite word composed of digit $N+1$ with length r_i. Clearly both sets $\widetilde{D}_t$ and D_t' have the same cardinal and each subword $u\in\Sigma_N^k$ appears τ_u times in ω of $\widetilde{D}_t$. It is time to use Lemma 3 by taking

$$h = \frac{1}{k}\Big(\sum_{u\in\Sigma_N^k} -\frac{\tau_u}{n}\ln\frac{\tau_u}{n} - \frac{n-K}{n}\ln\frac{n-K}{n}\Big).$$

Then, for the same $\delta>0$ as above and for n large enough we have

$$\sharp D_t' = \sharp\widetilde{D}_t \leqslant \exp(n(h+\delta)). \tag{18}$$

Given $\omega\in D_t'$, let (τ_u') stand for the appearance distribution with respect to Σ_N^k of ω. Then, we have

$$|\omega| = K_t \text{ and } \tau_u \leqslant \tau_u' \leqslant \tau_u + (t-1)(k-1) \leqslant \tau_u + n - K.$$

That leads to

$$0 \leq \frac{\tau_u'}{n} - \frac{\tau_u}{n} \leq 1 - \frac{K}{n} < 2\epsilon_1. \tag{19}$$

For any $\omega=\omega_1\cdots\omega_{K_t}\in D_t'$, by the Gibbsian property and (4), we have

$$\begin{aligned}
k\ln\nu([\omega_1\cdots\omega_{K_t}]) &\leqslant k\ln C + k\sum_{i=0}^{K_t-1}\varphi(T^i x)\\
&= k\ln C + \sum_{i=0}^{k-2}(k-1-i)\varphi(T^i x) + \sum_{i=K_t-k+1}^{K_t-1}(K_t-i)\varphi(T^i x) + \sum_{i=0}^{K_t-k}\sum_{l=0}^{k-1}\varphi(T^{i+l}x)\\
&\leqslant (K_t+k^2-k+1)\ln C + \sum_{u\in\Sigma_N^k}\tau_u'\ln\nu([u]),
\end{aligned}$$

where $x \in [\omega]$ and the last inequality follows from the Gibbsian property and (4). In combination with (18), this gives

$$\begin{aligned} \sum_{\omega \in D'_t} \nu^{\gamma}([\omega]) &\leq \sharp D'_t \max_{\omega \in D'_t} \nu^{\gamma}([\omega]) \\ &\leq \exp\left\{ n(h+\delta) + \frac{\gamma}{k}\left\{ \sum_{u \in \Sigma_N^k} \tau'_u \ln \nu([u]) + (K_t + k^2)\ln C \right\}\right\} \\ &= \exp\left\{ nL(\gamma, n, k, (\tau'_u))\right\}, \end{aligned}$$

where

$$L(\gamma, n, k, (\tau'_u)) = h + \frac{\gamma}{k} \sum_{u \in \Sigma_N^k} \frac{\tau'_u}{n} \ln \nu([u]) + \frac{\gamma}{kn}(K_t + k^2)\ln C + \delta.$$

Now we shall give a negative upper-bound of $L(\gamma, n, k, (\tau'_u))$. According to (19), we can take $\epsilon_1 > 0$ small enough and n large enough such that

$$\frac{1}{k} \sum_{u \in \Sigma_N^k} \frac{\tau'_u}{n} \ln \nu([u]) \leq \frac{1}{k} \sum_{u \in \Sigma_N^k} \frac{\tau_u}{n} \ln \nu([u]) + \delta = -\frac{1}{k} H_{k,N}(\nu, \Delta_{x,n}) + \delta.$$

At the same time, by noting that $K_t \leq n + k - 1$, we can take k and n large enough such that

$$\frac{\gamma}{kn}(K_t + k^2)\ln C \leqslant \delta/2 \ .$$

In combination with the last two inequalities, we have

$$L(\gamma, n, k, (\tau'_u)) \leq h - \gamma\frac{1}{k} H_{k,N}(\nu, \Delta_{x,n}) + (3/2+\gamma)\delta \leq h - \gamma\frac{1}{k} H_{k,N}(\nu, \Delta_{x,n}) + 3\delta.$$

Recall that $\gamma > t$. We can take $\epsilon_2 > 0$ small enough such that $\gamma > t + 2\epsilon_2$. Then, we can take $\delta > 0$ small enough and k, N large enough such that

$$\gamma \geqslant \frac{\frac{1}{k} H_{k,N}(\Delta_{x,n}, \Delta_{x,n}) + 7\delta}{\frac{1}{k} H_{k,N}(\nu, \Delta_{x,n})}.$$

It follows that

$$L(\gamma, n, k, (\tau'_u)) \leq h - \frac{1}{k} H_{k,N}(\Delta_{x,n}, \Delta_{x,n}) - 4\delta = -\frac{1}{k}\frac{n-K}{n} \ln \frac{n-K}{n} - 4\delta.$$

Observe that the function $x \ln x$ is bounded in $(0, 1]$. Then we can take k large enough such that

$$-\frac{1}{k}\frac{n-K}{n} \ln \frac{n-K}{n} \leq \delta.$$

Thus, we have

$$L(\gamma, n, k, (\tau'_u)) \leq -3\delta,$$

and

$$\sum_{\omega \in D'_t} \nu^{\gamma}([\omega]) \leq \exp(-3n\delta).$$

In combination with (9), (12) and (17), this yields

$$\sum_{\omega \in \Lambda_n(k,N)} \nu^{\gamma}([\omega]) \leqslant sn^{N^k} e^{-2n\delta} \leq e^{-n\delta}.$$

Thus, we have for any $\gamma > \max\{\alpha_\nu, t\}$,

$$\mathcal{H}^{\gamma}_{\delta_{j+k-1}}(A_{\epsilon_1,\epsilon_2}(N_1,k,N)) \leqslant \sum_{n\geq j}\sum_{\omega\in\Lambda_n(k,N)} \nu^{\gamma}([\omega]) \leq \sum_{n\geq j} e^{-n\delta}.$$

Then it follows that

$$\dim_\nu QR_\nu(t) \leqslant \gamma.$$

Thus, we obtain

$$\dim_\nu QR_\nu(t) \leqslant \max\{\alpha_\nu, t\}.$$

□

Now we shall show the lower bound.

Theorem 4. *Let φ be a potential function of summable variations admitting a unique Gibbs measure ν with convergence exponent α_ν. For $0 \leq t \leq 1$, we have*

$$\dim_\nu QR_\nu(t) \geq \max\{\alpha_\nu, t\}.$$

Proof. According to the dimension formula (2) (see also [3]), it suffices to find a T-invariant measure $\mu \in \mathcal{M}_T(X)$ satisfying $\beta(\nu|\mu) = t$. In fact, the generic point set $G_\mu \subset QR_\nu(t)$, thus we have

$$\dim_\nu QR_\nu(t) \geq \dim_\nu G_\mu = \max\{\alpha_\nu, t\}.$$

Consider the subspace $X_N := \{1, 2, \cdots, N\}^\infty$, where $N > 1$ is a positive integer. Let $\mathfrak{B}$ denote the set of Bernoulli measures on X_N. Then $\mathfrak{B}$ is homeomorphic to a simplex $\{(p_1, p_2, \cdots, p_N) \in [0,1]^N : \Sigma_{i=1}^N p_i = 1\}$. By the entropy formula of Bernoulli measure it is clear that h_μ is continuous on $\mathfrak{B}$. Bearing in mind that the restriction $\varphi|_{X_N}$ of φ on X_N is a continuous bounded function, by Proposition 4, the entropy dimension $\beta(\nu|\mu) = \frac{h_\mu}{-\int \varphi d\mu}$ is continuous on $\mathfrak{B}$. Thus, there exists a T-invariant measure $\mu^* \in \mathcal{M}_T(X_N) \subset \mathcal{M}_T(X)$ such that $\beta(\nu|\mu^*) = t$. We finish the proof. □

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