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On Two-Distillable Werner States

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Abstract: We consider bipartite mixed states ρ in a $d \otimes d$ quantum system. We say that ρ is PPT if its partial transpose $1 \otimes T(\rho)$ is positive semidefinite, and otherwise ρ is NPT. The well-known Werner states are divided into three types: (a) the separable states (the same as the PPT states); (b) the one-distillable states (necessarily NPT); and (c) the NPT states which are not one-distillable. We give several different formulations and provide further evidence for the validity of the conjecture that Werner states of type (c) are not two-distillable.

Keywords: Werner states; bipartite entangled states; distillable states; hermitian biquadratic forms

1. Introduction

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the Hilbert space for the quantum system consisting of two parties, A and B (Alice and Bob). We assume that the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ have the same finite dimension, which we denote by d. A *product state* is a tensor product $\rho_A \otimes \rho_B$ of the states ρ_A and ρ_B of the first and second party, respectively. A bipartite state ρ is *separable* if it can be written as a convex linear combination of product states. We say that a bipartite state is *entangled* if it is not separable. We say that ρ is PPT if its partial transpose $\sigma = 1 \otimes T(\rho)$, computed in some fixed orthonormal (o.n.) basis of $\mathcal{H}_B$, is a positive semidefinite operator. Otherwise, σ has a negative eigenvalue, and we say that ρ is NPT.

It is more complicated to give the definition of distillability for bipartite states ρ. For that purpose, we have to consider multiple copies of ρ. For k copies, the density matrix is the k-th tensor power $\rho^{\otimes k}$ which acts on the Hilbert space $\mathcal{H}^{\otimes k}$. We can identify $\mathcal{H}^{\otimes k}$ with the tensor product of the Hilbert spaces $\mathcal{H}_A^{\otimes k}$ and $\mathcal{H}_B^{\otimes k}$. In this way, we can view $\rho^{\otimes k}$ as a bipartite state. Thus, any vector $|\psi\rangle \in \mathcal{H}^{\otimes k}$ has its Schmidt decomposition and a well-defined Schmidt rank.

The definition of distillability given below is not the original one, but it is the only one that we are going to use. Replacing the original definition with this one was nontrivial (see [1]).

Definition 1. For a bipartite state ρ acting on $\mathcal{H}$ and an integer $k \geq 1$, we say that ρ is *k-distillable* if there exists a (non-normalized) pure state $|\psi\rangle \in \mathcal{H}^{\otimes k}$ of Schmidt rank that is at most two, such that

$$\langle\psi|\sigma^{\otimes k}|\psi\rangle < 0, \quad \sigma = 1 \otimes T(\rho). \tag{1}$$

We say that ρ is *distillable* if it is k-distillable for some integer $k \geq 1$.

The entanglement of a state ρ which is not distillable is known as *bound entanglement*.

If a bipartite state ρ is separable, then it is PPT, *i.e.*, σ is positive semidefinite, and consequently ρ is not distillable. For the same reason, the entangled bipartite PPT states are not distillable, *i.e.*, their entanglement is bound. Equivalently, every distillable bipartite state is necessarily NPT. It is not known whether the converse holds, *i.e.*, whether every bipartite NPT state is distillable. However, it is widely believed that the converse is false. Actually, the following conjecture has been raised in [2,3] (see also [4], (p. 62)).

Conjecture 1. There exist bipartite NPT states which are not distillable, *i.e.*, bound NPT entanglement exists.

It is known [5] that for each integer $k \geq 1$, there exist examples of bipartite states which are distillable but not k-distillable.

We fix an o.n. basis $|i\rangle$, $i = 1, 2, \ldots, d$ of $\mathcal{H}_A$, and an o.n. basis of $\mathcal{H}_B$ for which we use the same notation. The context will make clear which basis is used. After fixing these bases, we can define the *flip operator* $F : \mathcal{H} \to \mathcal{H}$ by

$$F = \sum_{i,j} |i, j\rangle\langle j, i|.$$

The (non-normalized) Werner states on $\mathcal{H}$ (see [6], Example 1) can be parametrized as follows:

$$\rho_W(t) = 1 - tF, \quad -1 \leq t \leq 1. \tag{2}$$

Several different parametrizations of Werner states appear in the literature (see e.g., [6–8]). We have chosen the one above because of its simplicity. It is easy to express the parameter used in these and other references in terms of our parameter t.

Let $|\psi_{\max}\rangle \in \mathcal{H}$ be the maximally entangled (pure) state given by

$$|\psi_{\max}\rangle = \frac{1}{\sqrt{d}} \sum_i |i, i\rangle.$$

Its density matrix is the projector

$$P = \frac{1}{d} \sum_{i,j} |i, i\rangle\langle j, j|.$$

Since dP is the partial transpose of F, the partial transpose of $\rho_W(t)$ is

$$\sigma_W(t) = 1 - tdP.$$

The following facts about the Werner states are well-known.

Proposition 1. *The Werner states $\rho_W(t)$ are:*

(a) *separable for $-1 \leq t \leq 1/d$;*
(b) *1-distillable for $1/2 < t \leq 1$;*
(c) *NPT but not one-distillable for $1/d < t \leq 1/2$.*

For (a) and (c), see [7] (p. 59) and [9], and, for (b), see [2] (Theorem 2) and [3,8].

From now on, unless stated otherwise, we assume that $d \geq 3$. (In Section 4, we will consider briefly the case $d = 2$.) The importance of Werner states for the distillability problem for bipartite states was first established in [7].

Proposition 2. *Conjecture 1 is equivalent to the assertion that some NPT Werner states $\rho_W(t)$ are not distillable.*

In fact, the following stronger conjecture is believed to be true [2,3,10].

Conjecture 2. None of the Werner states $\rho_W(t)$, $1/d < t \leq 1/2$, are distillable.

The above two conjectures have been open for more than 15 years. In order to stimulate further research related to these conjectures, we propose yet another one. Namely, we shall consider a very weak version of Conjecture 2.

Conjecture 3. None of the Werner states $\rho_W(t)$, $1/d < t \leq 1/2$, are 2-distillable.

For the k-distillability problem, the following fact [2] (Lemma 4) is useful.

Proposition 3. *If $\rho_W(1/2)$ is not k-distillable, then none of the states $\rho_W(t)$, $1/d < t < 1/2$, is k-distillable.*

In view of this proposition, it suffices to prove Conjecture 3 for $t = 1/2$ only. Extensive numerical evidence for the validity of this conjecture in the case $d = 3$ is presented in [2,3,8] and [11]. In [11], it is also claimed that their numerical proof is rigorous. The case $d = 4$ was analyzed in [12], but it remains open.

For an alternative approach to Conjecture 1, see the very recent paper [13]. Actually, the authors of that paper study the positive linear maps between matrix algebras which remain positive under tensoring of n copies of themselves for each $n = 2, 3, \ldots$. Completely positive and completely co-positive linear maps are trivial examples. They show that the existence of non-trivial examples implies the existence of bound NPT entanglement. Moreover, they construct a one-parameter family of candidates for non-trivial maps of that kind, which is reminiscent of the family of Werner states.

Our paper is organized as follows. In Section 2, we construct a hermitian biquadratic form Φ and show that Conjecture 3 is equivalent to Φ being positive semidefinite, $\Phi \geq 0$. The form Φ depends on $4d$ arbitrary vectors $x_i, y_i \in \mathcal{H}_A$ and $u_i, v_i \in \mathcal{H}_B$, $i = 1, 2, \ldots, d$.

In Section 3, we obtain a formula which expresses Φ as a function of four matrices X, Y, U, V of order d, where $X = [\, x_1 \; x_2 \; \cdots \; x_d \,]$, *etc.* From that formula, we deduce that Φ is invariant under an action of the product of two copies of the unitary group $\mathrm{U}(d)$.

In Section 4, we compute the matrix $H = H(X, Y)$ of Φ when the latter is viewed as a hermitian quadratic form in the $2d^2$ complex entries of U and V. The entries of X and Y play the role of parameters. Conjecture 3 is equivalent to the claim that $H \geq 0$. After partitioning H into four square blocks of order d^2, we show that the two diagonal blocks are positive definite matrices. We reduce the task of proving that $H \geq 0$ to the case where X is a diagonal matrix with positive diagonal entries. In the case $d = 2$, we prove that $H \geq 0$.

In Section 5, we prove that, for any d, $H(X, Y) \geq 0$ when X and Y are diagonal matrices. We point out that $H(X, Y)$ is not diagonal even when both X and Y are. Since this is done for arbitrary d, and the proof is nontrivial, we view this fact as an important piece of evidence for the validity of Conjecture 3.

In Section 6, we prove that the inequality $H(X, Y) \geq 0$ is equivalent to $H(\alpha X + \beta Y, \gamma X + \delta Y) \geq 0$, where $\alpha\delta - \beta\gamma \neq 0$. Hence, it suffices to prove the inequality $H(X, Y) \geq 0$ when X is singular.

In Section 7, we consider the case $d = 3$. To prove that $H(X, Y) \geq 0$, we may assume that X is singular. Hence, X has rank 1 or 2. We prove that $H(X, Y) \geq 0$ when X has rank 1. We also show that the leading principal minor of H of order 10 is a positive semidefinite polynomial.

The superscripts $*$, T and $\dagger$ denote the complex conjugation, the transposition and the adjoint, respectively. We denote by M_m the algebra of complex matrices of order m, and by I_m the identity matrix of M_m.

2. The Hermitian Biquadratic Form Φ

Since we are going to use only one Werner state, the one for $t = 1/2$, we set

$$\rho_W = \rho_W(1/2) = 1 - F/2, \quad \sigma_W = \sigma_W(1/2) = 1 - dP/2.$$

Conjecture 3 is equivalent to the claim that the inequality

$$\langle\psi|\sigma_W^{\otimes 2}|\psi\rangle \geq 0 \tag{3}$$

is valid for all $|\psi\rangle \in \mathcal{H}^{\otimes 2}$ of Schmidt rank ≤ 2. Such $|\psi\rangle$ can be written as $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$, where

$$|\psi_1\rangle = |x\rangle \otimes |u\rangle, \quad |\psi_2\rangle = |y\rangle \otimes |v\rangle.$$

Note that $|x\rangle, |y\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A$ while $|u\rangle, |v\rangle \in \mathcal{H}_B \otimes \mathcal{H}_B$. We point out that we do not require $|\psi_1\rangle + |\psi_2\rangle$ to be the Schmidt decomposition of $|\psi\rangle$, *i.e.*, we do not require that $\langle x|y\rangle = \langle u|v\rangle = 0$. The reason for this is to allow the vectors $|x\rangle, |y\rangle, |u\rangle, |v\rangle$ to be completely arbitrary.

We can rewrite $|\psi_1\rangle$ and $|\psi_2\rangle$ as

$$|\psi_1\rangle = \sum_{i,j} |i, j, x_i, u_j\rangle, \quad |\psi_2\rangle = \sum_{i,j} |i, j, y_i, v_j\rangle.$$

The vectors $|x_i\rangle$ and $|y_i\rangle$ live in Alice's second copy of $\mathcal{H}_A$, while $|u_i\rangle$ and $|v_i\rangle$ live in Bob's second copy of $\mathcal{H}_B$. The summation is taken over all i and j in $\{1, 2, \ldots, d\}$. Consequently, we can view the left-hand side (LHS) of Equation (3) as a function of $4d$ vectors x_i, y_j, u_r, v_s:

$$\Phi(x_1, \ldots, x_d, y_1, \ldots, y_d, u_1, \ldots, u_d, v_1, \ldots, v_d) = \langle\psi|\sigma_W^{\otimes 2}|\psi\rangle.$$

As

$$\sigma_W^{\otimes 2} = 1 - \frac{1}{2}(1 \otimes dP + dP \otimes 1) + \frac{1}{4} dP \otimes dP,$$

we have

$$\Phi = \Phi_1 - \frac{1}{2}(\Phi_2 + \Phi_3) + \frac{1}{4}\Phi_4,$$

where

$$\begin{aligned}
\Phi_1 &= \langle\psi|\psi\rangle, \\
\Phi_2 &= \langle\psi|1 \otimes dP|\psi\rangle, \\
\Phi_3 &= \langle\psi|dP \otimes 1|\psi\rangle, \\
\Phi_4 &= \langle\psi|dP \otimes dP|\psi\rangle.
\end{aligned}$$

After the substitution $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$, each of the Φ_k breaks up into four pieces. For instance, we have

$$\begin{aligned}
\Phi_2 = & \sum_{i,j,r,s} \langle i, j, x_i, u_j|1 \otimes dP|r, s, x_r, u_s\rangle \\
& + \sum_{i,j,r,s} \langle i, j, x_i, u_j|1 \otimes dP|r, s, y_r, v_s\rangle \\
& + \sum_{i,j,r,s} \langle i, j, y_i, v_j|1 \otimes dP|r, s, x_r, u_s\rangle \\
& + \sum_{i,j,r,s} \langle i, j, y_i, v_j|1 \otimes dP|r, s, y_r, v_s\rangle.
\end{aligned}$$

We have computed each of the resulting 16 pieces. For instance, the second piece, say E, in the above formula for Φ_2, is computed as follows. We first observe that $\langle i, j, x_i, u_j|1 \otimes dP|r, s, y_r, v_s\rangle = 0$ if $r \neq i$ or $s \neq j$. Thus, we have

$$
\begin{aligned}
E &= \sum_{i,j} \langle x_i, u_j | dP | y_i, v_j \rangle \\
&= \sum_{i,j,r,s} \langle x_i, u_j | r, r \rangle \langle s, s | y_i, v_j \rangle \\
&= \sum_{i,j} \left(\sum_r \langle x_i, u_j | r, r \rangle \cdot \sum_s \langle s, s | y_i, v_j \rangle \right) \\
&= \sum_{i,j} \langle x_i | u_j^* \rangle \langle y_i | v_j^* \rangle^* .
\end{aligned}
$$

The final formulas are:

$$
\begin{aligned}
\Phi_1 &= \sum_i \|x_i\|^2 \cdot \sum_j \|u_j\|^2 + \sum_i \langle x_i | y_i \rangle \cdot \sum_j \langle u_j | v_j \rangle \\
&\quad + \sum_i \langle y_i | x_i \rangle \cdot \sum_j \langle v_j | u_j \rangle + \sum_i \|y_i\|^2 \cdot \sum_j \|v_j\|^2 , \\
\Phi_2 &= \sum_{i,j} |\langle x_i | u_j^* \rangle|^2 + \sum_{i,j} \langle x_i | u_j^* \rangle \langle y_i | v_j^* \rangle^* \\
&\quad + \sum_{i,j} \langle y_i | v_j^* \rangle \langle x_i | u_j^* \rangle^* + \sum_{i,j} |\langle y_i | v_j^* \rangle|^2 , \\
\Phi_3 &= \sum_{i,j} \langle x_i | x_j \rangle \langle u_i | u_j \rangle + \sum_{i,j} \langle x_i | y_j \rangle \langle u_i | v_j \rangle \\
&\quad + \sum_{i,j} \langle y_j | x_i \rangle \langle v_j | u_i \rangle + \sum_{i,j} \langle y_i | y_j \rangle \langle v_i | v_j \rangle , \\
\Phi_4 &= \left| \sum_i \langle x_i | u_i^* \rangle \right|^2 + \sum_{i,j} \langle x_i | u_i^* \rangle \langle y_j | v_j^* \rangle^* \\
&\quad + \sum_{i,j} \langle x_i | u_i^* \rangle^* \langle y_j | v_j^* \rangle + \left| \sum_j \langle y_j | v_j^* \rangle \right|^2 .
\end{aligned}
$$

These formulas show that each Φ_k, viewed as a function of the components of the x_i and y_j, is a hermitian quadratic form. The same is true when we view them as functions of the components of the u_i and v_j. Hence, we shall refer to the Φ_k (and Φ) as hermitian biquadratic forms. The next proposition follows immediately from Equation (3) and the definition of the form Φ.

Proposition 4. *Conjecture 3 is equivalent to the assertion that $\Phi \geq 0$.*

3. Φ as a Function of Four Matrices

Let X denote the $d \times d$ matrix whose successive columns are the vectors $x_1, \ldots, x_d$. Define similarly the matrices $Y, U,$ and V. Let M_d denote the space of complex matrices of order d. Define the inner product on M_d by $\langle A|B \rangle = \mathrm{tr}\,(A^\dagger B)$. For the corresponding norm, we have $\|A\|^2 = \mathrm{tr}\,(A^\dagger A)$. The tensor product of matrices $A = [a_{ij}]$ and B is defined as the block-matrix $A \otimes B = [a_{i,j} B]$.

Now the formulas for Φ can be rewritten in terms of the matrices $X, Y, U,$ and V. We obtain that

$$
\begin{aligned}
\Phi_1(X,Y,U,V) &= \|X\|^2 \|U\|^2 + \|Y\|^2 \|V\|^2 + 2\Re(\mathrm{tr}\,(X^\dagger Y) \cdot \mathrm{tr}\,(U^\dagger V)), \\
\Phi_2(X,Y,U,V) &= \|X^T U + Y^T V\|^2 , \\
\Phi_3(X,Y,U,V) &= \mathrm{tr}\, \left(X^T X^* U^\dagger U + X^T Y^* V^\dagger U + Y^T X^* U^\dagger V + Y^T Y^* V^\dagger V \right) , \\
\Phi_4(X,Y,U,V) &= |\mathrm{tr}\,(X^T U + Y^T V)|^2 ,
\end{aligned}
$$

where $\Re$ stands for "the real part of".

The first expression can be further simplified by using the standard Frobenius norm on the tensor product of matrices

$$\Phi_1(X, Y, U, V) = \|X \otimes U + Y \otimes V\|^2.$$

The third expression also simplifies to

$$\Phi_3(X, Y, U, V) = \|UX^T + VY^T\|^2.$$

Consequently, we have

$$\begin{aligned} \Phi(X, Y, U, V) \quad &= \quad \|X \otimes U + Y \otimes V\|^2 \\ &\quad -\frac{1}{2}\left(\|X^T U + Y^T V\|^2 + \|UX^T + VY^T\|^2\right) \\ &\quad +\frac{1}{4}\left|\mathrm{tr}\,(X^T U + Y^T V)\right|^2. \end{aligned} \tag{4}$$

The next proposition follows immediately from the above formulas.

Proposition 5. *The identity*

$$\Phi(AXB, AYB, A^*UB^*, A^*VB^*) = \Phi(X, Y, U, V), \tag{5}$$

holds true for arbitrary $X, Y, U, V \in M_d$ *and* $A, B \in \mathrm{U}(d)$.

4. The Matrix H of the Form Φ

We shall consider the entries of X and Y as parameters and those of U and V as complex variables. Then, Φ (and each Φ_k) becomes a family of hermitian quadratic forms depending on the mentioned parameters. Let $H = H(X, Y)$ and $H_k = H_k(X, Y)$, $k = 1, 2, 3, 4$, be the matrices of the corresponding forms Φ and Φ_k. These are hermitian matrices of order $2d^2$.

For any complex matrix Z, let $\tilde{Z}$ denote the column vector obtained by writing the columns of Z one below the other starting with the first column, then the second, *etc.* Now, we can express the relationship between the form Φ and its matrix H by the formula

$$\Phi(X, Y, U, V) = \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix}^\dagger H(X, Y) \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix}. \tag{6}$$

By using the formulas given in Section 2, we obtain the following simple formulas:

$$H_1 = \begin{bmatrix} \|X\|^2 & \mathrm{tr}\,(X^\dagger Y) \\ \mathrm{tr}\,(Y^\dagger X) & \|Y\|^2 \end{bmatrix} \otimes I_{d^2}, \tag{7}$$

$$H_2 = \begin{bmatrix} X^\dagger X & X^\dagger Y \\ Y^\dagger X & Y^\dagger Y \end{bmatrix} \otimes I_d, \tag{8}$$

$$H_3 = \begin{bmatrix} I_d \otimes X^* X^T & I_d \otimes X^* Y^T \\ I_d \otimes Y^* X^T & I_d \otimes Y^* Y^T \end{bmatrix}, \tag{9}$$

$$H_4 = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}^* \cdot \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}^T, \tag{10}$$

for the matrices H_k. Those for H_1 and H_4 are obvious. We omit the tedious but straightforward verification of the formulas for H_2 and H_3.

For H, we obtain the formula

$$H(X,Y) = H_1 - \frac{1}{2}(H_2 + H_3) + \frac{1}{4}H_4, \tag{11}$$

and for its trace

$$\text{tr } H(X,Y) = \left(d - \frac{1}{2}\right)^2 \left(\|X\|^2 + \|Y\|^2\right). \tag{12}$$

In view of Proposition 4, we can restate Conjecture 3 in the following equivalent form.

Conjecture 4. $H(X,Y) \geq 0, \quad \forall X, Y \in M_d$.

If $A \in \mathrm{U}(d)$, and we replace X and Y with AX and AY, respectively, then the H_k undergo the transformation $Z \to (I_{2d} \otimes A^*)Z(I_{2d} \otimes A^T)$. In fact, H_1 and H_2 remain fixed under this transformation.

Similarly, if $B \in \mathrm{U}(d)$, and we replace X and Y with XB and YB, respectively, then the H_k undergo the transformation $Z \to (I_2 \otimes B^\dagger \otimes I_d)Z(I_2 \otimes B \otimes I_d)$. This time, H_1 and H_3 remain fixed. In the case of H_4, one should use the formulas

$$\widetilde{AX} = (I_d \otimes A) \cdot \tilde{X}, \quad (\widetilde{YB})^T = (\tilde{Y})^T \cdot (B \otimes I_d),$$

which are not hard to verify.

Hence, the following proposition holds.

Proposition 6. *For $A, B \in \mathrm{U}(d)$, we have*

$$H(AXB, AYB) = (I_2 \otimes B^\dagger \otimes A^*)H(X,Y)(I_2 \otimes B \otimes A^T). \tag{13}$$

Thanks to this proposition (or Proposition 5) we can simplify the task of proving Conjecture 4. Indeed, it suffices to prove this conjecture when the matrix X is diagonal and its diagonal entries are positive.

Let us partition $H(X,Y)$ into four square blocks of size d^2. The first diagonal block depends only on X and the second one only on Y. By using Equation (11) and the formulas (7)–(9), we obtain that

$$H(X,Y) = \begin{bmatrix} L(X) & L(X,Y) \\ L(X,Y)^\dagger & L(Y) \end{bmatrix}, \tag{14}$$

where

$$L(X,Y) = \text{tr } (X^\dagger Y)I_{d^2} - \frac{1}{2}\left(X^\dagger Y \otimes I_d + I_d \otimes X^* Y^T\right) + \frac{1}{4}\tilde{X}^* \tilde{Y}^T, \tag{15}$$

and $L(X) := L(X,X)$.

If X and Y are nonzero matrices, then the two diagonal blocks in Equation (14) are positive definite matrices. This is shown in the next proposition.

Proposition 7. *If $X \neq 0$ then $L(X) > 0$.*

Proof. By Proposition 5, we may assume that $X = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$. Let $s = \|X\|^2 = \sum \lambda_i^2$. It follows from Equation (15) that $L(X) = M + (1/4)\tilde{X}\tilde{X}^T$, where

$$M = \bigoplus_{i=1}^{d} \left((s - \frac{\lambda_i^2}{2})I_d - \frac{1}{2}X^2 \right)$$

is a diagonal matrix with the diagonal entries

$$\mu_{ij} = s - (\lambda_i^2 + \lambda_j^2)/2, \quad i,j = 1,2,\ldots,d.$$

Since

$$\mu_{ij} \geq \mu_{1,1} = \lambda_2^2 + \cdots + \lambda_d^2 \geq 0$$

for all i,j, we have $L(X) \geq 0$. As $X \neq 0$, we have $\lambda_1 > 0$. If $\lambda_2 > 0$, then all $\mu_{ij} > 0$ and so $L(X) > 0$. Otherwise, $\lambda_i = 0$ for $i > 1$ and $L(X)$ is a diagonal matrix with positive diagonal entries. Hence, again $L(X) > 0$. $\square$

The matrix H has order $2d^2$, but one can reduce the proof of Conjecture 4 to matrices of order d^2. This does not come for free since the smaller matrix will have a more complicated structure. Recall that we may assume that X is a diagonal matrix with positive diagonal entries. For simplicity, we set $A = L(X)$, $B = L(X,Y)$ and $C = L(Y)$t in Equation (14). Since $A > 0$, it suffices to show that $S := C - B^\dagger A^{-1} B \geq 0$, see e.g., [14] (Proposition 8.2.3). (As X is diagonal, one can easily compute A^{-1}.) Proving that $S \geq 0$ may be somewhat easier than proving that $H \geq 0$. We shall use this simplification to handle the case $d = 2$ below.

Recall that $d \geq 3$ by the assumption made earlier, but Conjecture 4 also makes sense for $d = 1$ and $d = 2$. However, in these two cases, the determinant of $H(X,Y)$ is identically 0. For $d = 1$, we have $H_1 = H_2 = H_3 = H_4$ and the conjecture is obviously valid. It is also valid for $d = 2$.

Proposition 8. *Conjecture 4 is true for $d = 2$.*

Proof. We may assume that $X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a,b > 0$. Let $Y = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$, and let us partition H as in Equation (14) and set again $A = L(X)$, $B = L(X,Y)$ and $C = L(Y)$. Let $t^4 - c_1 t^3 + c_2 t^2 - c_3 t + c_4$ be the characteristic polynomial of $S := C - B^\dagger A^{-1} B$. A computation shows that $c_4 = 0$. Set

$$\begin{aligned} p &= a^2 + b^2, \\ q &= a^4 + 4a^2b^2 + b^4, \\ r &= p(|u_2|^2 + |v_1|^2) + |av_2 - bu_1|^2. \end{aligned}$$

After some tedious computations, we found the following formulas for the c_i:

$$\begin{aligned} 2pqc_1 &= 4(p^2 + a^2b^2)|av_2 - bu_1|^2 + p(2a^2b^2 + 3q)(|u_2|^2 + |v_1|^2), \\ 4p^2qc_2 &= q|av_2 - bu_1|^4 + p(7a^4 + 22a^2b^2 + 7b^4)(|u_2|^2 + |v_1|^2)|av_2 - bu_1|^2 \\ &\quad + 2p^2 \left((q + 3a^2b^2) \left(|u_2|^2 + |v_1|^2 \right)^2 + 2(a^4 + a^2b^2 + b^4)\, |u_2 v_1|^2 \right. \\ &\qquad \left. + 2 \left| abu_2v_1 + (av_2 - bu_1)^2 \right|^2 \right), \\ 4pqc_3 &= r \left(\left| 2abu_2v_1 + (av_2 - bu_1)^2 \right|^2 + 2a^2b^2(|u_2|^4 + |v_1|^4) \right. \\ &\quad \left. + p(|u_2|^2 + |v_1|^2)|av_2 - bu_1|^2 + 2p^2\, |u_2v_1|^2 \right). \end{aligned}$$

Since $p,q,r > 0$, we conclude that all coefficients $c_i \geq 0$. Hence, $S \geq 0$ (see e.g., [14] (Proposition 8.2.6)). $\square$

We shall consider the case $d = 3$ in Section 7.

5. The Diagonal Case

We say that a matrix pair (X, Y) is *generic* if the matrices X and Y are linearly independent and some linear combination of them is nonsingular.

In this section, we prove that $H(X, Y) \geq 0$ when both X and Y are diagonal matrices, while d is arbitrary. This appears to be a trivial case, but it is not so as $H(X, Y)$ is not diagonal even if X and Y are. We prove a slightly stronger result.

Theorem 1. *If (X, Y) is a generic pair of diagonal matrices, then $H(X, Y) > 0$.*

Proof. We denote the diagonal entries of X and Y by $\lambda_1, \ldots, \lambda_d$ and $\mu_1, \ldots, \mu_d$ respectively. The hypothesis implies that $\lambda_k \neq 0$ or $\mu_k \neq 0$ for each k. After replacing H with $\Pi H \Pi^T$ where Π is a suitable permutation matrix, H becomes direct sum of $d^2 - d$ blocks of order 2 and an additional block of order $2d$. It suffices to show that each of these blocks is positive definite.

The blocks of order 2 are indexed by the integers $p = (i-1)d + j$, where $i, j \in \{1, 2, \ldots, d\}$ and $j \neq i$. For such index p, the corresponding block of order 2 is the principal submatrix $H(p)$ of the original matrix H corresponding to indices p and $p + d^2$. Explicitly, we have

$$H(p) = \sum_{k=1}^{d} c_k \begin{bmatrix} |\lambda_k|^2 & \lambda_k^* \mu_k \\ \lambda_k \mu_k^* & |\mu_k|^2 \end{bmatrix},$$

where $c_k = 1$ for $k \neq i, j$ and $c_i = c_j = 1/2$. Each matrix on the right-hand side is positive semidefinite of rank 1. If $H(p)$ is singular, then all of these matrices must be singular and must have the same kernel. This contradicts the linear independence of X and Y. Hence, $H(p)$ must be positive definite.

It remains to consider the block B of size $2d$, *i.e.*, the principal submatrix of H corresponding to the indices $(i-1)d + i$ and $(i-1)d + i + d^2$ for $1 \leq i \leq d$. We have $B = B_1 - (B_2 + B_3)/2 + B_4/4$, where B_k denotes the corresponding principal submatrix of H_k. Let us first consider the matrix $B' = B_1 - (B_2 + B_3)/2$. After a suitable simultaneous permutation of rows and columns, B' breaks up into the direct sum of d blocks $G(i)$ of order 2, where $i \in \{1, 2, \ldots, d\}$. Explicitly, we have

$$G(i) = \sum_{k \neq i} \begin{bmatrix} |\lambda_k|^2 & \lambda_k^* \mu_k \\ \lambda_k \mu_k^* & |\mu_k|^2 \end{bmatrix}.$$

Each $G(i)$ is positive semidefinite of rank 1 or 2. Thus, in the decomposition $B = B' + B_4/4$, we have $B' \geq 0$ and $B_4 \geq 0$. If all $G(i) > 0$, then $B' > 0$, and so $B > 0$.

It remains to consider the case where some $G(i)$, say $G(1)$, is singular. By Cauchy–Schwarz inequality, the vectors $(\lambda_2, \lambda_3, \ldots, \lambda_d)$ and $(\mu_2, \mu_3, \ldots, \mu_d)$ are linearly dependent. It follows that all other $G(i)$ must be positive definite. Consequently, the nullspace of B' is one-dimensional and is spanned by the column vector having all components 0 except the first which is $-\mu_2$ and $(d+1)$-th which is λ_2. This vector is not killed by B_4, because $\lambda_1 \mu_2 - \lambda_2 \mu_1 \neq 0$. Hence, we conclude that $B > 0$. □

Corollary 1. *Conjecture 4 is valid when X and Y are diagonal matrices.*

Proof. This follows from the theorem because any pair of diagonal matrices can be approximated by a generic pair of diagonal matrices. □

6. Reduction to the Singular Case

Let us show that $H(X, Y)$ satisfies yet another identity. Let

$$\Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}_2(\mathbf{C}), \tag{16}$$

and

$$(\Lambda^T)^{-1} = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix}. \tag{17}$$

By using

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we deduce that

$$\begin{aligned} &(\alpha X + \beta Y) \otimes (\alpha' U + \beta' V) + (\gamma X + \delta Y) \otimes (\gamma' U + \delta' V) = X \otimes U + Y \otimes V, \\ &(\alpha X + \beta Y)^T (\alpha' U + \beta' V) + (\gamma X + \delta Y)^T (\gamma' U + \delta' V) = X^T U + Y^T V, \\ &(\alpha' U + \beta' V)(\alpha X + \beta Y)^T + (\gamma' U + \delta' V)(\gamma X + \delta Y)^T = U X^T + V Y^T. \end{aligned}$$

Consequently, Equation (4) implies that

$$\Phi(\alpha X + \beta Y, \gamma X + \delta Y, \alpha' U + \beta' V, \gamma' U + \delta' V) = \Phi(X, Y, U, V).$$

By using Equation (6) and the formula

$$\begin{bmatrix} \alpha' \tilde{U} + \beta' \tilde{V} \\ \gamma' \tilde{U} + \delta' \tilde{V} \end{bmatrix} = \left((\Lambda^T)^{-1} \otimes I_{d^2} \right) \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix},$$

we obtain the new identity

$$H(\alpha X + \beta Y, \gamma X + \delta Y) = (\Lambda^* \otimes I_{d^2}) H(X, Y) (\Lambda^T \otimes I_{d^2}). \tag{18}$$

It suffices to prove the inequality $H(X,Y) \geq 0$ for generic pairs (X,Y) only. If (X,Y) is generic, we can choose $\Lambda \in \mathrm{GL}_2(\mathbf{C})$ such that $\alpha X + \beta Y$ is a singular matrix. Thus, the identity Equation (18) shows that it suffices to prove $H(X,Y) \geq 0$ when X is singular and Y is invertible.

Yet another conjecture, which is simpler and stronger than Conjecture 4, may be of interest. Let us introduce the real valued polynomial $D(X,Y) = \det H(X,Y)$. By taking the determinants in Equation (13), we obtain that

$$D(AXB, AYB) = D(X,Y), \quad \forall A, B \in \mathrm{U}(d). \tag{19}$$

From Equation (18), we deduce that

$$D(\alpha X + \beta Y, \gamma X + \delta Y) = |\alpha\delta - \beta\gamma|^{2d^2} D(X,Y) \tag{20}$$

is valid when Λ is invertible. Since both sides are polynomials, this identity must be valid for arbitrary Λ.

Note that $D(X,0) = 0$ for all matrices X. More generally, we claim that $D(X,Y) = 0$ if X and Y are linearly dependent. Indeed, it suffices to choose a matrix Λ as in Equation (16) such that $\gamma X + \delta Y = 0$ and apply Equation (20). The converse of this claim is false, but we conjecture that it is true in a weaker form.

Conjecture 5. If $d \geq 3$, then $D(X,Y) \neq 0$ for generic (X,Y).

Theorem 1 shows that this conjecture is true when the matrices X and Y are diagonal. As this conjecture deals with only one polynomial and has no positivity conditions whatsoever, it should be much easier to prove (or disprove).

Proposition 9. *Conjecture 4 is a consequence of Conjecture 5.*

Proof. Let X_1 and Y_1 be any matrices in M_d. We have to show that $H(X_1, Y_1)$ is positive semidefinite. Clearly, it suffices to prove this when the pair (X_1, Y_1) is generic. Let (X_0, Y_0) be a generic pair of diagonal matrices. Then, $H(X_0, Y_0)$ is positive definite by Theorem 1. Consequently, $D(X_0, Y_0) > 0$, and all eigenvalues of $H(X_0, Y_0)$ are positive. We can join the pairs (X_0, Y_0) and (X_1, Y_1) by a continuous path (X_t, Y_t), $0 \le t \le 1$, such that (X_t, Y_t) is generic for each t. By Conjecture 5, $D(X_t, Y_t) \neq 0$ for all t. Hence, $H(X_t, Y_t)$ has no zero eigenvalues. Since the eigenvalues of $H(X_t, Y_t)$ are continuous functions of t, and they are all positive for $t = 0$, they must all remain positive for all values of t. In particular, this is true for $t = 1$. We thus conclude that $H(X_1, Y_1)$ is positive definite. □

7. The Case $d = 3$

In this section, we consider only the case $d = 3$. As mentioned earlier, in order to prove that $H(X, Y) \ge 0$, it suffices to do that in the case when X is singular. Thus, the rank of X is 1 or 2. We shall prove the inequality in the case when this rank is 1.

Proposition 10. *If $X, Y \in M_3$, and some linear combination of X and Y has rank one, then $H(X, Y) \ge 0$.*

Proof. We may assume that X and Y are linearly independent and that X has rank one. Since we can multiply X by a nonzero scalar, by applying Proposition 6, we may assume that

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By applying the same proposition, we may also assume that

$$Y = \begin{bmatrix} a & u & v \\ x & b & 0 \\ y & 0 & c \end{bmatrix},$$

where $b, c, u, v \ge 0$.

We partition the matrix $H = H(X, Y)$ as in Equation (14) and set $A = L(X)$, $B = L(X, Y)$, $C = L(Y)$. As explained in Section 4, it suffices to show that the matrix $S := C - B^\dagger A^{-1} B$ is positive semidefinite. Let

$$p(t) = \sum_{k=0}^{9} (-1)^k c_k t^{9-k}, \quad c_0 = 1,$$

be the characteristic polynomial of S. The c_k are polynomials in the real variables b, c, u, v and the complex variables x, y and their conjugates x^*, y^*. (The variable a does not occur.)

Set $c_k = p_k / d_k$, where $d_k = 2^k$ for $k < 9$ and $d_9 = d_8 = 256$. Then, the p_k are polynomials with integer coefficients. All these computations were performed by using Maple since the p_k may have several thousand terms. We claim that the polynomials p_k are positive semidefinite, *i.e.*, they have nonnegative values for all real b, c, u, v and all complex x, y. The inequality $H(X, Y) \ge 0$ is a consequence of this claim.

To prove our claim, we construct positive semidefinite polynomials q_k, $k \in \{1, 2, \ldots, 9\}$, such that the difference $p_k - q_k |bux - cvy|^2$ is also a positive semidefinite polynomial. We have $q_1 = q_2 = 0$. The other q_k are given in the Appendix. The q_k are obviously positive semidefinite. The proof that the differences $p_k - q_k |bux - cvy|^2$ are positive semidefinite requires the use of Maple (or some other software for symbolic algebraic computations). We just expand $p_k - q_k |bux - cvy|^2$ and check that all coefficients are nonnegative integers and all monomials that occur in the expansion are hermitian squares. For instance, we have

$$
\begin{aligned}
p_1 &= 5(u^2+v^2+|x|^2+|y|^2)+6(b^2+c^2), \\
p_2 &= 41\left((u^2+v^2)^2+(|x|^2+|y|^2)^2\right) \\
&\quad +62(b^2+c^2)^2+6b^2c^2 \\
&\quad +91(u^2+v^2)(|x|^2+|y|^2) \\
&\quad +102\left(b^2(u^2+|x|^2)+c^2(v^2+|y|^2)\right) \\
&\quad +108\left(b^2(v^2+|y|^2)+c^2(u^2+|x|^2)\right).
\end{aligned}
$$

□

As an aside, we mention that in the case when

$$
X=\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad Y=\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix},
$$

where $a,b,c>0$ and $u_i,v_i,w_i\in\mathbf{C}$, the leading principal minor μ_{10} of H of order 10 is a positive semidefinite polynomial. This follows from the following explicit expression for μ_{10} as a sum of squares of real polynomials:

$$
\mu_{10} = \frac{1}{512}(2a^2+b^2+c^2)^2(a^2+2b^2+c^2)(a^2+b^2+2c^2)\cdot p,
$$

where

$$
\begin{aligned}
p &= 2(a^2+2b^2+c^2)(a^2+b^2+2c^2)\cdot\Big((4a^4+b^4+c^4+5a^2(b^2+c^2)+4b^2c^2)|bw_3-cv_2|^2 \\
&\quad +(a^2+b^2)(a^2+c^2)(|cu_1-aw_3|^2+|av_2-bu_1|^2)\Big) \\
&\quad +(a^6+b^6+c^6+11a^2b^2c^2+5(a^4(b^2+c^2)+b^4(a^2+c^2)+c^4(a^2+b^2))\Big)\cdot q,
\end{aligned}
$$

and

$$
\begin{aligned}
q &= 2(a^2+2b^2+c^2)(a^2+b^2+2c^2)(|v_3|^2+|w_2|^2) \\
&\quad +(a^2+2b^2)(a^2+b^2+2c^2)(|u_3|^2+|w_1|^2) \\
&\quad +(a^2+2c^2)(a^2+2b^2+c^2)(|u_2|^2+|v_1|^2).
\end{aligned}
$$

Note that the equality $\mu_{10}=0$ implies that Y is a scalar multiple of X.

8. Results and Discussion

We consider the question whether the Werner $d\otimes d$ states $\rho_W(t)=1-tF$, $1/d<t\leq 1/2$, where F is the flip operator, are two-distillable. The question whether these states are distillable has been considered previously in references [2,8,11,12], and it has been conjectured that they are not distillable, which implies that they are not two-distillable. All evidence so far supports Conjecture 3 saying that these states are not two-distillable. We present in this paper a novel method to attack this conjecture, and we obtain further evidence for its validity. In view of the well-known fact stated as Proposition 3, it suffices to prove Conjecture 3 for $t=1/2$ only.

We first construct a hermitian biquadratic form depending on $2d$ vectors $x_1, \ldots, x_d, y_1, \ldots, y_d \in \mathcal{H}_A$ and $2d$ vectors $u_1, \ldots, u_d, v_1, \ldots, v_d \in \mathcal{H}_B$ and show that Conjecture 3 is equivalent to Φ being positive semidefinite.

Next, we organize the vectors $x_1, \ldots, x_d$ into the matrix $X = [\, x_1 \ \cdots \ x_d \,]$, and, similarly, we construct the matrices Y, U, V from the remaining $3d$ vectors. It turns out that the form Φ has relatively simple expression Equation (4) in terms of the matrices X, Y, U, V. By using this expression, we deduce that Φ is invariant under the action of the product of two copies of the unitary group $\mathrm{U}(d)$. More precisely, $\Phi(X, Y, U, V)$ is invariant under the transformation, which sends

$$X \to AXB,\ Y \to AYB,\ U \to A^*UB^*,\ V \to A^*VB^*,$$

where $A, B \in \mathrm{U}(d)$.

If we fix the matrices X and Y, then $\Phi(X, Y, U, V)$ becomes an ordinary hermitian quadratic form in the $2d^2$ complex entries of the matrices U and V. We compute the matrix $H = H(X, Y)$ of this hermitian quadratic form (see the formula (11)). Then, Conjecture 3 reduces to the claim that $H(X, Y) \geq 0$ for all matrices $X, Y \in M_d$.

Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\alpha\delta - \beta\gamma \neq 0$. We prove in Proposition 6 that $H(X, Y) \geq 0$ if and only if $H(\alpha X + \beta Y, \gamma X + \delta Y) \geq 0$. We can choose such $\alpha, \beta, \gamma, \delta$ so that the matrix $\alpha X + \beta Y$ becomes singular. Hence, it suffices to prove the inequality $H(X, Y) \geq 0$ when X is singular. By using the action of $\mathrm{U}(d) \times \mathrm{U}(d)$, we can additionally assume that X is a diagonal matrix with nonnegative diagonal entries.

Even when both X and Y are diagonal matrices, the matrix $H(X, Y)$ is not diagonal in general. However, we did prove that $H(X, Y) \geq 0$ in that case (see Theorem 1). Since this is true for any d and the proof is nontrivial, we view this fact as an important piece of evidence for the validity of Conjecture 3 in the general case.

Recall that $H = H(X, Y)$ is a hermitian matrix of order $2d^2$. After partitioning H into four square blocks of order d^2, we show that the two diagonal blocks are positive definite matrices (assuming that X and Y are nonzero matrices). By using the four blocks of H, one can easily construct a hermitian matrix S of order d^2 such that $H \geq 0$ if and only if $S \geq 0$. By using this trick, we proved by brute force that $H \geq 0$ is true in the case $d = 2$. This also follows from the fact that $\rho_W(1/2)$ is separable when $d = 2$.

Assume now that $d = 3$. Since we may assume that X is singular, its rank is 1 or 2. We prove that $H(X, Y) \geq 0$ when X has rank 1. This is done by using the above mentioned trick which replaces H by S, which is of order 9. We compute the characteristic polynomial of S and prove that $S \geq 0$ by showing that this polynomial has no negative roots. We also show that the leading principal minor of H of order 10 is a positive semidefinite polynomial.

To finish off the case $d = 3$, it remains to consider the case when the matrix X has rank 2. We may assume that X is a diagonal matrix with the diagonal entries $1, a, 0$, and $a > 0$. We were not able to compute the characteristic polynomial of S. Then, we made the additional assumption that Y is real. By subtracting a multiple of X from Y, we can also assume that the first entry of Y vanishes. After these simplifications, we succeeded with computing the determinant of S. Its denominator is

$$256(a^2+1)^3(a^2+2)^2(2a^2+1)^2(a^4+4a^2+1).$$

The numerator is a (non-homogeneous) polynomial of degree 36 in nine real variables, having 487,056 terms. We stopped at this point, short of reaching our goal to write this numerator as a sum of squares.

9. Conclusions

The old conjecture that the bipartite bound NPT entanglement exists is still open. We have proposed a much simpler conjecture that, in $d \otimes d$, the NPT Werner states which are not one-distillable are also not two-distillable. We have reformulated this conjecture in several different ways and provided new evidence for its validity, especially for $d = 3$.

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Appendix

We list here the polynomials q_k, $k > 2$, used in Section 7.

$$
\begin{aligned}
q_3 &= 2,\\
q_4 &= 11(u^2+v^2+|x|^2+|y|^2)+14(b^2+c^2),\\
q_5 &= 22(u^4+v^4+|x|^4+|y|^4)+38(b^4+c^4)\\
&\quad +44(u^2+v^2+|x|^2+|y|^2)+52(u^2+v^2)(|x|^2+|y|^2)\\
&\quad +59\left(b^2(u^2+|x|^2)+c^2(v^2+|y|^2)\right)\\
&\quad +65\left(b^2(v^2+|y|^2)+c^2(u^2+|x|^2)\right)+86b^2c^2,\\
q_6 &= 296b^2c^2(u^2+v^2+|x|^2+|y|^2)\\
&\quad +254(b^2v^2|y|^2+c^2u^2|x|^2)\\
&\quad +225(b^2+c^2)(u^2|y|^2+v^2|x|^2)\\
&\quad +202(b^2u^2|x|^2+c^2v^2|y|^2)+198b^2c^2(b^2+c^2)\\
&\quad +192(b^2+c^2)(u^2v^2+|x|^2|y|^2)\\
&\quad +168(u^2|x|^2(v^2+|y|^2)+v^2|y|^2(u^2+|x|^2))\\
&\quad +141(b^4(v^2+|y|^2)+c^4(u^2+|x|^2))\\
&\quad +116(b^4(u^2+|x|^2)+c^4(v^2+|y|^2))\\
&\quad +106(b^2(v^4+|y|^4)+c^2(u^4+|x|^4))\\
&\quad +86(b^2(u^4+|x|^4)+c^2(v^4+|y|^4))\\
&\quad +84((u^2+v^2)(|x|^4+|y|^4)+(|x|^2+|y|^2)(u^4+v^4))\\
&\quad +60(u^2v^2(u^2+v^2)+|x|^2|y|^2(|x|^2+|y|^2))\\
&\quad +50(b^6+c^6)+20(u^6+v^6+|x|^6+|y|^6)+3|bux+cvy|^2,\\
q_7 &= 802b^2c^2(u^2|x|^2+v^2|y|^2)+778b^2c^2(u^2|y|^2+v^2|x|^2)\\
&\quad +688b^2c^2(u^2v^2+|x|^2|y|^2)\\
&\quad +574(b^2v^2|y|^2(u^2+|x|^2)+c^2u^2|x|^2(v^2+|y|^2))\\
&\quad +515b^2c^2(b^2(v^2+|y|^2)+c^2(u^2+|x|^2))\\
&\quad +488(b^2u^2|x|^2(v^2+|y|^2)+c^2v^2|y|^2(u^2+|x|^2))\\
&\quad +470b^2c^2(b^2(u^2+|x|^2)+c^2(v^2+|y|^2))+418(b^4v^2|y|^2+c^4u^2|x|^2)\\
&\quad +384u^2v^2|x|^2|y|^2+364b^4c^4+336b^2c^2(u^4+v^4+|x|^4+|y|^4)\\
&\quad +331(b^4+c^4)(u^2|y|^2+v^2|x|^2)\\
&\quad +324(b^2v^2|y|^2(v^2+|y|^2)+c^2u^2|x|^2(u^2+|x|^2))\\
&\quad +278(b^4+c^4)(u^2v^2+|x|^2|y|^2)\\
&\quad +274(u^2|y|^2(b^2|y|^2+c^2u^2)+v^2|x|^2(b^2v^2+c^2|x|^2))\\
&\quad +260(b^4u^2|x|^2+c^4v^2|y|^2)
\end{aligned}
$$

$$
\begin{aligned}
&+250(u^2|y|^2(b^2u^2+c^2|y|^2)+v^2|x|^2(b^2|x|^2+c^2v^2)) \\
&+214(b^2u^2|x|^2(u^2+|x|^2)+c^2v^2|y|^2(v^2+|y|^2))+210b^2c^2(b^4+c^4) \\
&+204(u^2v^2(b^2v^2+c^2u^2)+|x|^2|y|^2(b^2|y|^2+c^2|x|^2)) \\
&+192(u^2v^2(|x|^4+|y|^4)+|x|^2|y|^2(u^4+v^4)) \\
&+180(u^2v^2(b^2u^2+c^2v^2)+|x|^2|y|^2(b^2|x|^2+c^2|y|^2)) \\
&+174(b^4(v^4+|y|^4)+c^4(u^4+|x|^4)) \\
&+168(u^2+v^2)(|x|^2+|y|^2)(u^2v^2+|x|^2|y|^2) \\
&+135(b^6(v^2+|y|^2)+c^6(u^2+|x|^2))+112(b^4(u^4+|x|^4)+c^4(v^4+|y|^4)) \\
&+100(b^6(u^2+|x|^2)+c^6(v^2+|y|^2))+96(u^4+v^4)(|x|^4+|y|^4) \\
&+76(b^2(v^6+|y|^6)+c^2(u^6+|x|^6)) \\
&+56((u^2+v^2)(|x|^6+|y|^6)+(|x|^2+|y|^2)(u^6+v^6)) \\
&+52(b^2(u^6+|x|^6)+c^2(v^6+|y|^6))+48(u^4v^4+|x|^4|y|^4) \\
&+32(b^8+c^8+u^2v^2(u^4+v^4)+|x|^2|y|^2(|x|^4+|y|^4)) \\
&+(10(b^2+c^2)+7(u^2+v^2+|x|^2+|y|^2))|bux+cvy|^2 \\
&+8(u^8+v^8+|x|^8+|y|^8), \\
q_8 \;=\;& 1248b^2c^2(u^2v^2(|x|^2+|y|^2)+|x|^2|y|^2(u^2+v^2)) \\
&+960b^2c^2(b^2v^2|y|^2+c^2u^2|x|^2)+820b^2c^2(b^2+c^2)(u^2|y|^2+v^2|x|^2) \\
&+780b^2c^2(b^2u^2|x|^2+c^2v^2|y|^2)+776u^2v^2|x|^2|y|^2(b^2+c^2) \\
&+748b^2c^2(b^2+c^2)(u^2v^2+|x|^2|y|^2) \\
&+628b^2c^2(u^2|x|^2(u^2+|x|^2)+v^2|y|^2(v^2+|y|^2)) \\
&+586b^4c^4(u^2+v^2+|x|^2+|y|^2) \\
&+576b^2c^2(u^2|y|^2(u^2+|y|^2)+v^2|x|^2(v^2+|x|^2)) \\
&+570(b^4v^2|y|^2(u^2+|x|^2)+c^4u^2|x|^2(v^2+|y|^2)) \\
&+488b^2c^2(u^2v^2(u^2+v^2)+|x|^2|y|^2(|x|^2+|y|^2)) \\
&+470(b^2v^2|y|^2+c^2u^2|x|^2)(u^2|y|^2+v^2|x|^2) \\
&+428(b^2v^2|y|^2+c^2u^2|x|^2)(u^2v^2+|x|^2|y|^2) \\
&+398b^2c^2(b^2(v^4+|y|^4)+c^2(u^4+|x|^4)) \\
&+394(b^4u^2|x|^2(v^2+|y|^2)+c^4v^2|y|^2(u^2+|x|^2)) \\
&+382(b^2v^2|y|^2(u^4+|x|^4)+c^2u^2|x|^2(v^4+|y|^4)) \\
&+366(b^4v^2|y|^2(v^2+|y|^2)+c^4u^2|x|^2(u^2+|x|^2)) \\
&+358b^2c^2(b^4(v^2+|y|^2)+c^4(u^2+|x|^2)) \\
&+332b^2c^2(b^2(u^4+|x|^4)+c^2(v^4+|y|^4)) \\
&+320(b^2u^2|x|^2(v^4+|y|^4)+c^2v^2|y|^2(u^4+|x|^4)) \\
&+306(b^2u^2|x|^2+c^2v^2|y|^2)(v^2|x|^2+u^2|y|^2) \\
&+304b^2c^2(b^4(u^2+|x|^2)+c^4(v^2+|y|^2)) \\
&+284(b^2v^4|y|^4+c^2u^4|x|^4)+276b^4c^4(b^2+c^2) \\
&+274(b^2u^2|x|^2+c^2v^2|y|^2)(u^2v^2+|x|^2|y|^2) \\
&+268(b^4(u^2|y|^4+v^4|x|^2)+c^4(u^4|y|^2+v^2|x|^4))+256(b^6v^2|y|^2+c^6u^2|x|^2) \\
&+234(b^4(u^4|y|^2+v^2|x|^4)+c^4(u^2|y|^4+v^4|x|^2))
\end{aligned}
$$

$$
\begin{aligned}
&+192(b^4(u^2v^4+|x|^2|y|^4)+c^4(u^4v^2+|x|^4|y|^2)\\
&\quad +u^2v^2|x|^2|y|^2(u^2+v^2+|x|^2+|y|^2))\\
&+190(b^6+c^6)(u^2|y|^2+v^2|x|^2)+186(b^2+c^2)(u^4|y|^4+v^4|x|^4)\\
&+158(b^2v^2|y|^2(v^4+|y|^4)+c^2u^2|x|^2(u^4+|x|^4))\\
&+152b^2c^2(u^6+v^6+|x|^6+|y|^6)\\
&+140(b^4u^2|x|^2(u^2+|x|^2)+c^4v^2|y|^2(v^2+|y|^2)+(b^6+c^6)(u^2v^2+|x|^2|y|^2))\\
&+136(b^4(u^4v^2+|x|^4|y|^2)+c^4(u^2v^4+|x|^2|y|^4))\\
&+122(b^2(u^2|y|^6+v^6|x|^2)+c^2(u^6|y|^2+v^2|x|^6))+120(b^2u^4|x|^4+c^2v^4|y|^4)\\
&+112(b^2|x|^2(b^4u^2+v^2|x|^4)+c^2v^2(c^4|y|^2+v^4|x|^2)+u^2|y|^2(b^2u^4+c^2|y|^4))\\
&+110(b^6(v^4+|y|^4)+c^6(u^4+|x|^4))+100b^2c^2(b^6+c^6)\\
&+96((b^2+c^2)(u^4v^4+|x|^4|y|^4)+(u^2+v^2)(|x|^4|y|^4+u^2v^2(|x|^4+|y|^4))\\
&\quad +(|x|^2+|y|^2)(u^4v^4+|x|^2|y|^2(u^4+v^4)))\\
&+88(b^4(v^6+|y|^6)+c^4(u^6+|x|^6))\\
&+80(b^2(u^2v^6+|x|^2|y|^6)+c^2(u^6v^2+|x|^6|y|^2))\\
&+76(b^2u^2|x|^2(u^4+|x|^4)+c^2v^2|y|^2(v^4+|y|^4))\\
&+64(u^2v^2(|x|^6+|y|^6+(|x|^2+|y|^2)(u^4+v^4))\\
&\quad +|x|^2|y|^2(u^6+v^6+(u^2+v^2)(|x|^4+|y|^4)))\\
&+52(b^8(v^2+|y|^2)+c^8(u^2+|x|^2))\\
&+48(b^6(u^4+|x|^4)+c^6(v^4+|y|^4)+u^2v^2(b^2u^4+c^2v^4)\\
&\quad +|x|^2|y|^2(b^2|x|^4+c^2|y|^4))\\
&+32((u^6+v^6)(|x|^4+|y|^4)+(|x|^6+|y|^6)(u^4+v^4)\\
&\quad +b^8(u^2+|x|^2)+c^8(v^2+|y|^2)+b^4(u^6+|x|^6)+c^4(v^6+|y|^6))\\
&+28b^2c^2|bux+cvy|^2+24(b^2(v^8+|y|^8)+c^2(u^8+|x|^8))\\
&+18(b^2(v^2+|y|^2)+c^2(u^2+|x|^2))|bux+cvy|^2\\
&+16((u^8+v^8)(|x|^2+|y|^2)+(u^2+v^2)(|x|^8+|y|^8)\\
&\quad +(u^2+v^2)(|x|^2+|y|^2)|bux+cvy|^2)\\
&+10(b^2(u^2+|x|^2)+c^2(v^2+|y|^2))|bux+cvy|^2\\
&+8(b^{10}+c^{10}+b^2(u^8+|x|^8)+c^2(v^8+|y|^8)+(b^4+c^4)|bux+cvy|^2),
\end{aligned}
$$

$$
\begin{aligned}
q_9 \;=\;& 492b^2c^2u^2v^2|x|^2|y|^2\\
&+349b^2c^2(b^2v^2|y|^2(u^2+|x|^2)+c^2u^2|x|^2(v^2+|y|^2))\\
&+305b^2c^2(b^2u^2|x|^2(v^2+|y|^2)+c^2v^2|y|^2(u^2+|x|^2))\\
&+260b^4c^4(u^2|x|^2+v^2|y|^2)\\
&+231b^2c^2(u^2|x|^2+v^2|y|^2)(u^2v^2+|x|^2|y|^2)\\
&+230b^4c^4(u^2+|x|^2)(v^2+|y|^2)\\
&+228b^2c^2(u^2v^2(|x|^4+|y|^4)+|x|^2|y|^2(u^4+v^4))\\
&+216b^2c^2(u^2|x|^2(v^4+|y|^4)+v^2|y|^2(u^4+|x|^4))\\
&+200b^2c^2(b^2v^2|y|^2(v^2+|y|^2)+c^2u^2|x|^2(u^2+|x|^2))\\
&+164b^2c^2(b^4v^2|y|^2+c^4u^2|x|^2)\\
&+149b^2c^2(b^2(u^2|y|^4+v^4|x|^2)+c^2(u^4|y|^2+v^2|x|^4))\\
&+146u^2v^2|x|^2|y|^2(b^4+c^4)
\end{aligned}
$$

$$
\begin{aligned}
&+144b^2c^2(b^2(u^4|y|^2+u^2v^4+v^2|x|^4+|x|^2|y|^4)\\
&\qquad +c^2(u^2|y|^4+u^4v^2+v^4|x|^2+|x|^4|y|^2))\\
&+140b^2c^2(b^2u^2|x|^2(u^2+|x|^2)+c^2v^2|y|^2(v^2+|y|^2))\\
&+131b^2c^2(b^4+c^4)(u^2|y|^2+v^2|x|^2)\\
&+125b^2c^2(b^2(u^4v^2+|x|^4|y|^2)+c^2(u^2v^4+|x|^2|y|^4))\\
&+120(b^2c^2(u^4|x|^4+v^4|y|^4)+(b^4v^2|y|^2+c^4u^2|x|^2)(u^2|y|^2+v^2|x|^2))\\
&+117b^2c^2(b^4+c^4)(u^2v^2+|x|^2|y|^2)\\
&+112(b^2c^2(b^4u^2|x|^2+c^4v^2|y|^2)\\
&\quad +u^2v^2|x|^2|y|^2(b^2(v^2+|y|^2)+c^2(u^2+|x|^2)))\\
&+111(b^4v^2|y|^2+c^4u^2|x|^2)(u^2v^2+|x|^2|y|^2)\\
&+110b^4c^4(b^2(v^2+|y|^2)+c^2(u^2+|x|^2))\\
&+108b^4c^4(u^4+v^4+|x|^4+|y|^4)\\
&+104b^4c^4(b^2(u^2+|x|^2)+c^2(v^2+|y|^2))\\
&+93(b^4v^2|y|^2(u^4+|x|^4)+c^4u^2|x|^2(v^4+|y|^4))\\
&+92(b^4v^4|y|^4+c^4u^4|x|^4)+90b^2c^2(u^4|y|^4+v^4|x|^4)\\
&+80(b^2c^2(u^4v^4+|x|^4|y|^4)+b^2v^4|y|^4(u^2+|x|^2)+c^2u^4|x|^4(v^2+|y|^2)\\
&\quad +u^2v^2|x|^2|y|^2(b^2(u^2+|x|^2)+c^2(v^2+|y|^2)))\\
&+76b^2c^2(u^2|x|^2(u^4+|x|^4)+v^2|y|^2(v^4+|y|^4))\\
&+74(b^6v^2|y|^2(u^2+|x|^2)+c^6u^2|x|^2(v^2+|y|^2))\\
&+72(u^4v^4(b^2|y|^2+c^2|x|^2)+|x|^4|y|^4(b^2v^2+c^2u^2))\\
&+70b^2c^2(b^4(v^4+|y|^4)+c^4(u^4+|x|^4))\\
&+64(u^4|y|^4(b^2v^2+c^2|x|^2)+v^4|x|^4(b^2|y|^2+c^2u^2))\\
&+61b^2c^2(u^2|y|^2(u^4+|y|^4)+v^2|x|^2(v^4+|x|^4))\\
&+57(b^4u^2|x|^2(v^4+|y|^4)+c^4v^2|y|^2(u^4+|x|^4))\\
&+56(b^6c^6+b^2v^2|y|^2(u^2v^4+|x|^2|y|^4)+c^2u^2|x|^2(u^4v^2+|x|^4|y|^2))\\
&+54(b^6v^2|y|^2(v^2+|y|^2)+c^6u^2|x|^2(u^2+|x|^2))\\
&+48(b^2c^2(u^2v^2(u^4+v^4)+|x|^2|y|^2(|x|^4+|y|^4)\\
&\qquad +b^4(u^4+|x|^4)+c^4(v^4+|y|^4))+b^2(v^6+|y|^6)+c^2(u^6+|x|^6)\\
&\quad +b^2v^2|y|^2(u^2|y|^4+v^4|x|^2)+c^2u^2|x|^2(u^4|y|^2+v^2|x|^4))\\
&+46(b^4v^2|y|^2(v^4+|y|^4)+c^4u^2|x|^2(u^4+|x|^4))\\
&+45(b^4+c^4)(u^4|y|^4+v^4|x|^4)\\
&+44(b^4u^2|x|^2+c^4v^2|y|^2)(u^2|y|^2+v^2|x|^2)\\
&+40(b^2v^2|y|^2(u^6+|x|^6)+c^2u^2|x|^2(v^6+|y|^6)\\
&\quad +b^2c^2(b^6(v^2+|y|^2)+c^6(u^2+|x|^2)))\\
&+39(b^6(u^2|y|^4+v^4|x|^2)+c^6(v^2|x|^4+u^4|y|^2))+36b^4c^4(b^4+c^4)\\
&+34(b^6(u^4|y|^2+v^2|x|^4)+c^6(v^4|x|^2+u^2|y|^4))\\
&+33(b^4(v^6|x|^2+u^2|y|^6)+c^4(u^6|y|^2+v^2|x|^6))
\end{aligned}
$$

$$
\begin{aligned}
&+32(b^2v^4|y|^4(v^2+|y|^2)+c^2u^4|x|^4(u^2+|x|^2)\\
&\quad+b^4c^2(u^6+|x|^6)+b^2c^4(v^6+|y|^6)+b^8c^2(u^2+|x|^2)\\
&\quad+b^2c^8(v^2+|y|^2)+b^6u^2|x|^2(v^2+|y|^2)+c^6v^2|y|^2(u^2+|x|^2)\\
&\quad+b^2u^2|x|^2(u^2|y|^4+v^4|x|^2)+c^2v^2|y|^2(v^2|x|^4+u^4|y|^2))\\
&+28(b^4(u^6|y|^2+v^2|x|^6)+c^4(v^6|x|^2+u^2|y|^6))\\
&+24(b^2u^2|x|^2(v^6+|y|^6)+c^2v^2|y|^2(u^6+|x|^6)\\
&\quad+(b^4u^2|x|^2+c^4v^2|y|^2)(u^2v^2+|x|^2|y|^2)\\
&\quad+u^4v^4(b^2|x|^2+c^2|y|^2)+|x|^4|y|^4(b^2u^2+c^2v^2))\\
&+20(b^8v^2|y|^2+c^8u^2|x|^2)+17(b^6(u^2v^4+|x|^2|y|^4)+c^6(u^4v^2+|x|^4|y|^2))\\
&+16((b^2+c^2)(u^4|y|^4(u^2+|y|^2)+v^4|x|^4(v^2+|x|^2))\\
&\quad+(b^8+c^8)(u^2|y|^2+v^2|x|^2)+u^2v^2(b^2|x|^6+c^2|y|^6)\\
&\quad+|x|^2|y|^2(b^2u^6+c^2v^6)+b^2v^2|y|^2(v^6+|y|^6)+c^2u^2|x|^2(u^6+|x|^6)\\
&\quad+b^2u^4|x|^4(v^2+|y|^2)+c^2v^4|y|^4(u^2+|x|^2)\\
&\quad+b^4(u^2v^6+|x|^2|y|^6)+c^4(u^6v^2+|x|^6|y|^2)\\
&\quad+b^6(v^6+|y|^6)+c^6(u^6+|x|^6)+(b^2v^2|y|^2+c^2u^2|x|^2)|bux+cvy|^2)\\
&+10(b^8(v^4+|y|^4)+c^8(u^4+|x|^4)\\
&\quad+b^2c^2(u^2+v^2+|x|^2+|y|^2)|bux+cvy|^2)\\
&+8(b^{10}c^2+b^2c^{10}+b^4(v^8+|y|^8)+c^4(u^8+|x|^8)\\
&\quad+(b^4+c^4)(u^4v^4+|x|^4|y|^4)+v^2|x|^2(b^2+c^2)(v^6+|x|^6)\\
&\quad+u^2|y|^2(b^2+c^2)(u^6+|y|^6)+b^2c^2(|x|^8+|y|^8+u^8+v^8)\\
&\quad+b^2u^2|x|^2(u^4v^2+|x|^4|y|^2)+c^2v^2|y|^2(|x|^2|y|^4+u^2v^4)\\
&\quad+(b^2+c^2)(b^2c^2+u^2|y|^2+v^2|x|^2)|bux+cvy|^2)\\
&+4((b^8+c^8)(u^2v^2+|x|^2|y|^2)+(b^4(v^2+|y|^2)+c^4(u^2+|x|^2))|bux+cvy|^2)\\
&+2(b^{10}(v^2+|y|^2)+c^{10}(u^2+|x|^2)\\
&\quad+b^6(u^4v^2+|x|^4|y|^2)+c^6(u^2v^4+|x|^2|y|^4)).
\end{aligned}
$$

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