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Optimal Noise Enhanced Signal Detection in a Unified Framework

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Abstract: In this paper, a new framework for variable detectors is formulated in order to solve different noise enhanced signal detection optimal problems, where six different disjoint sets of detector and discrete vector pairs are defined according to the two inequality-constraints on detection and false-alarm probabilities. Then theorems and algorithms constructed based on the new framework are presented to search the optimal noise enhanced solutions to maximize the relative improvements of the detection and the false-alarm probabilities, respectively. Further, the optimal noise enhanced solution of the maximum overall improvement is obtained based on the new framework and the relationship among the three maximums is presented. In addition, the sufficient conditions for improvability or non-improvability under the two certain constraints are given. Finally, numerous examples are presented to illustrate the theoretical results and the proofs of the main theorems are given in the Appendix.

Keywords: signal detection; optimal noise; variable detector; different sets

1. Introduction

Noise commonly exists with the useful signal, and more noises in the system often lead to less channel capacity and worse detectability. People usually try to utilize a series of filters and algorithms to remove the unnecessary noise. Hence, understanding and mastering the distribution and characteristics of noise is an essential research topic in traditional signal detection theory. Nevertheless, although it may seem very counterintuitive, noise does play an active role in many signal processing problems, and the performance of some nonlinear systems can be enhanced via adding noise under certain conditions [1–27]. The phenomenon that noise benefits system is the so-called stochastic resonance (SR), which was first proposed by Benzi *et al.* [1] in 1981 when they studied the periodic recurrence of ice gases. The positive effects of noise have drawn the attention of researchers in various fields. For example, the effect of SR on the global stability of complex networks is investigated in [26]. Kohar and Sinha demonstrated how to utilize noise to make a bistable system behave as a memory device in [27].

In the signal detection problem, researchers commonly care about how to increase the output signal-to-noise (SNR) [7–11], the mutual information [12,13], or detection probability with a constant false alarm rate [14–20], or how to decrease the Bayes risk [21,22] or the probability of error [23] by adding additive noise to the input of system or changing the background noise level. As presented in [8], the output SNR obtained by adding suitable noise to the input of system is higher than the input SNR. The research results in [14] indicate that the detection probability of the sign detector can be increased by adding an appropriate amount of white Gaussian noise.

Depending on the detection metrics, the SR phenomenon for the hypothesis testing or detection problems are usually investigated according to the Bayesian, Minimax or Neyman–Pearson criteria.

In [23], the additive noise to optimize the performance of a suboptimal detector is explored according to Bayesian criterion under uniform cost assignment. It is demonstrated that the optimal noise to minimize the probability of error is a constant, and the improvability obtained by adding the constant noise can also be achieved via shifting the decision region without adding any additive noise. The probability distribution of the optimal additive noise to minimize Bayes risk is investigated in [21] according to the restricted Bayesian criterion, which can be extended to Bayesian and Minimax criteria easily. For an M-ary hypothesis testing problem, the optimal noise is determined as a randomization of at most M mass points.

In [15], a mathematical framework is established to search the optimal noise to maximize the detection probability based on Neyman–Pearson criterion. This leads to the very significant conclusion that the optimal noise is determined as the randomization of at most two discrete vectors. In addition, sufficient conditions whether the detection probability can or cannot be increased are given. However, from the perspective of Patel [18], the proof of the optimal noise presented in [15] has a little bit of a drawback. Moreover, the same noise enhanced problem for a fixed detector is researched through establishing a different mathematical framework in [18], where the optimal noise to maximize detection probability is also proved as a random signal consisting of no more than two discrete points and the corresponding probabilities. The researchers in [16] studied the noise enhanced detection performance for variable detectors according to the Neyman–Pearson criterion based on the results in [15]. The optimal noise enhanced solution to maximize the detection probability is determined as a randomization of no more than two detector and constant vector pairs.

Through the comparison and analysis above, it is clear that most researchers have focused on the maximization of detection probability via additive noise and there are few studies which cover the field of the minimization of the false-alarm probability. We cannot exclude the possibility that the false-alarm probability can be decreased by adding noise without deteriorating the original detectability, especially for the case where the detection probability cannot be increased via adding any noise. For example, a noise enhanced model which can increase the detection probability and decrease the false-alarm probability simultaneously by adding noise is first formulated in [25] for a fixed detector. In addition, the model is solved by a convex combination of the optimal noises for two limited cases, *i.e.*, the minimization of false-alarm probability and the maximization of detection probability. Nevertheless, it is obvious that the convex combination is usually not the optimal solution of the maximum overall improvement of the model. In this paper, we explore the optimal solution to maximize the overall improvement of detection and false-alarm probabilities directly instead of the convex combination. In practical applications, though the structure of the detector commonly cannot be replaced in many cases, some parameters of the detector can be varied to obtain a better performance. Moreover, the noise enhanced detection problems for a fixed detector can generally be achieved by simplifying the results for variable detectors directly. Thus, it is necessary to discuss the noise enhanced detection problems on the premise of variable detectors.

In order to obtain the optimal noise enhanced solution to maximize the overall improvement of detection and false-alarm probabilities for variable detectors under two inequality-constraints, we formulate a new framework to define six different disjoint sets of detector and discrete vector pairs based on the signs of the relative improvements of the detection and the false-alarm probabilities. Then we explore the optimal noise enhanced solutions for the maximum detection probability and the minimum false-alarm probability and give the corresponding algorithms in the new framework. Further, through some derivation, the optimal noise enhanced solution for the maximum overall improvement of detection and false-alarm probabilities is proved as a randomization of at most two detector and discrete vector pairs from two different sets, and the relationship among the three maximums is presented. In addition, the theoretical results for the case of allowing the randomization between detectors can be applied straightforwardly to the case where the randomization between detectors cannot be allowed. Namely, the optimization problem for variable detectors is simplified to choose a fixed detector and search the optimal additive noise when the randomization between detectors cannot be allowed. Actually, the

maximization of detection probability in this paper is equivalent to the noise enhanced detection problem for variable detectors studied in [16], which also needs all information of detection and false-alarm probabilities obtained by every detector and discrete vector pair. Indeed, the new framework subdivides the one set in [16] into six subintervals. Based on the definition of the six sets, it is obvious that each detector and discrete vector pair as a component of additive noise is available, partially available, or unavailable to meet the two constraints. Then the available and partial available pairs can be utilized to construct the optimal noise enhanced solutions to satisfy different requirements. Namely, the division of six sets effectively provides the foundation for maximizing the relative improvements of detection and false-alarm probabilities, and the sum of them.

The main contributions of this paper can be summarized as follows:

- Formulation of a new framework, where six different disjoint sets of detector and discrete vector pairs are defined according to two inequality-constraints.
- Algorithms for the noise enhanced solutions to maximize the relative improvements of the detection and the false-alarm probabilities are given based on the new framework.
- Noise enhanced solution of the maximum overall improvability is first provided based on the new framework and the relationship among the three maximums is explored.
- Determination of the sufficient conditions for the improvability and nonimprovability under the two constraints.

The remainder of this paper is organized as follows: in Section 2, three optimization problems for a binary hypothesis testing problem for a variable detector are proposed and the six disjoint sets of detector and discrete vector pairs are defined. In Section 3, the optimal noise enhanced solutions for the three optimization problem are discussed when the randomization between detectors can or cannot be allowed and the corresponding algorithms are given. Numerical results are presented in Section 4 and the conclusions are provided in Section 5.

2. Problem Formulation

Consider a binary hypothesis testing problem as follows:

$$H_i : p_i(\mathbf{x}), i = 0, 1 \quad (1)$$

where $p_i(\mathbf{x})$ is the probability density function (pdf) of the observation \mathbf{x} under H_i , $i = 0, 1$, and $\mathbf{x} \in \mathbb{R}^K$. For any \mathbf{x} , the probability of choosing H_1 can be characterized by $\phi(\mathbf{x})$ and $0 \leq \phi(\mathbf{x}) \leq 1$. Generally, $\phi(\mathbf{x})$ is also treated as a decision function of the detector. In order to investigate the possible enhancement of detectability, a new noise modified observation \mathbf{y} is obtained by adding an independent noise \mathbf{v} to the original observation \mathbf{x} , *i.e.*, $\mathbf{y} = \mathbf{x} + \mathbf{v}$. Correspondingly, the pdf of \mathbf{y} under H_i can be expressed by the convolutions of $p_i(\mathbf{x})$ and $p_{\mathbf{v}}(\mathbf{x})$ as below:

$$p_{\mathbf{y}}(\mathbf{y}; H_i) = p_i(\mathbf{x}) * p_{\mathbf{v}}(\mathbf{x}) = \int_{\mathbb{R}^K} p_i(\mathbf{y} - \mathbf{v}) p_{\mathbf{v}}(\mathbf{v}) d\mathbf{v} \quad (2)$$

where $*$ represents the symbol of convolution.

For the same detector, the decision function for \mathbf{y} is the same as that for \mathbf{x} . Then the detection and false-alarm probabilities based on the new noise modified observation \mathbf{y} are calculated by:

$$P_D^{\mathbf{y}} = \int_{\mathbb{R}^K} \phi(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}; H_1) d\mathbf{y} \quad (3)$$

$$P_{FA}^{\mathbf{y}} = \int_{\mathbb{R}^K} \phi(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}; H_0) d\mathbf{y} \quad (4)$$

According to the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$, where β and α represent the lower limit on the detection probability and the upper limit on the false-alarm probability, respectively, the following three important definitions are given by:

$$z^d = P_D^y - \beta \tag{5}$$

$$z^{fa} = \alpha - P_{FA}^y \tag{6}$$

$$z = z^d + z^{fa} \tag{7}$$

where z^d and z^{fa} can be regarded as the relative improvements of the detection and false-alarm probabilities obtained by adding additive noise, respectively, and z is the sum of z^d and z^{fa} .

In many cases, though the structure of the detector cannot be substituted, some parameters of it can be varied, such as decision threshold. In addition, the whole detector can also be replaced in some special cases. Instead of a fixed decision function $\phi(\cdot)$, we suppose that there exist a set of candidate decision functions written as Φ and any $\phi \in \Phi$ can be utilized. Then for any decision function $\phi_\ell \in \Phi$, $\ell = 1, \dots, L$, the detection and false-alarm probabilities based on y can be obtained by replacing ϕ with ϕ_ℓ , i.e.;

$$P_{D,\phi_\ell}^y = \int_{R^K} \phi_\ell(\mathbf{y}) p_y(\mathbf{y}; H_1) d\mathbf{y} \tag{8}$$

$$P_{FA,\phi_\ell}^y = \int_{R^K} \phi_\ell(\mathbf{y}) p_y(\mathbf{y}; H_0) d\mathbf{y} \tag{9}$$

When the additive noise is a discrete vector with pdf $\delta(\mathbf{v} - v)$, where δ denotes the delta function, i.e., $\delta(\mathbf{v} - v) = 1$ only if $\mathbf{v} = v$ and $\delta(\mathbf{v} - v) = 0$ otherwise, $p_y(\mathbf{y}; H_i) = p_i(\mathbf{y} - v)$. The corresponding noise modified detection and false-alarm probabilities can be rewritten as:

$$P_{D,\phi_\ell}^y(v) = \int_{R^K} \phi_\ell(\mathbf{y}) p_1(\mathbf{y} - v) d\mathbf{y} \tag{10}$$

$$P_{FA,\phi_\ell}^y(v) = \int_{R^K} \phi_\ell(\mathbf{y}) p_0(\mathbf{y} - v) d\mathbf{y} \tag{11}$$

Accordingly, under the constraints of $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$, the relative improvements of the detection and false-alarm probabilities corresponding to the additive noise with pdf $\delta(\mathbf{v} - v)$ can be written as:

$$z_{\phi_\ell}^d(v) = P_{D,\phi_\ell}^y(v) - \beta \tag{12}$$

$$z_{\phi_\ell}^{fa}(v) = \alpha - P_{FA,\phi_\ell}^y(v) \tag{13}$$

Correspondingly:

$$z_{\phi_\ell}(v) = z_{\phi_\ell}^d(v) + z_{\phi_\ell}^{fa}(v) \tag{14}$$

In order to make full use of the information gained by the discrete vector v , we define the following six mutually disjoint sets for each $\phi_\ell \in \Phi$ according to the values of $z_{\phi_\ell}^d(v)$ and $z_{\phi_\ell}^{fa}(v)$ denoted by $M_{1,\phi_\ell} = \{(\phi_\ell, v) | z_{\phi_\ell}^d(v) > 0, z_{\phi_\ell}^{fa}(v) > 0\}$, $M_{2,\phi_\ell} = \{(\phi_\ell, v) | z_{\phi_\ell}^d(v) > 0, z_{\phi_\ell}^{fa}(v) = 0\}$, $M_{3,\phi_\ell} = \{(\phi_\ell, v) | z_{\phi_\ell}^d(v) = 0, z_{\phi_\ell}^{fa}(v) > 0\}$, $M_{4,\phi_\ell} = \{(\phi_\ell, v) | z_{\phi_\ell}^d(v) > 0, z_{\phi_\ell}^{fa}(v) < 0\}$, $M_{5,\phi_\ell} = \{(\phi_\ell, v) | z_{\phi_\ell}^d(v) < 0, z_{\phi_\ell}^{fa}(v) > 0\}$, $M_{6,\phi_\ell} = \{(\phi_\ell, v) | z_{\phi_\ell}^d(v) \leq 0, z_{\phi_\ell}^{fa}(v) \leq 0\}$. Further, define $M_{\phi_\ell} = \bigcup_{j=1}^6 M_{j,\phi_\ell}$ and $M_j = \bigcup_{\ell=1}^L M_{j,\phi_\ell}$, then $M = \bigcup_{\ell=1}^L M_{\phi_\ell} = \bigcup_{j=1}^6 M_j$, where $j = 1, \dots, 6$ and $\ell = 1, \dots, L$.

Accordingly, a framework is formulated by defining the six different sets. As a result, the purpose of this paper is to investigate the optimal noise enhanced solutions for the maximum z^d , z^{fa} and z , respectively, under the two inequality-constraints based on the new framework. Obviously, whether the pair of (ϕ, v) is useful, partially useful or unuseful for the noise enhancement can be determined

according to the definitions of the six sets. For instance, any detector and discrete pair of M_1, M_2 and M_3 can meet the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$, the maximum z^d may be obtained by a suitable detector and discrete pair of M_1 or M_2 , the maximum z^{fa} may be achieved by a detector and discrete pair of M_1 or M_3 , and the maximum z may be reached by a suitable detector and discrete pair of M_1, M_2 or M_3 . In the following sections, the corresponding theorems and algorithms are provided.

3. The Noise Enhanced Solutions

Let z_m^d, z_m^{fa} and z_m be the maximum achievable z^d, z^{fa} and z , respectively, which are obtained by adding a discrete vector as additive noise when the randomization between detectors is allowed. Namely, $z_m^d = \max_{(\phi, v) \in M} z_\phi^d(v), z_m^{fa} = \max_{(\phi, v) \in M} z_\phi^{fa}(v)$ and $z_m = \max_{(\phi, v) \in M} z_\phi(v)$. If anyone of z_m^d, z_m^{fa} and z_m is less than zero, $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ cannot be obtained by adding any noise. So this paper is studied under the conditions that z_m^d, z_m^{fa} and z_m are greater than zero.

In general, when the randomization between different detectors is allowed, the noise enhanced solution can be viewed as a randomization of one or more detector and noise pairs with the corresponding weights. Suppose that the additive noise pdf is for any $\phi_\ell \in \Phi, \ell = 1, \dots, L$, then z^d, z^{fa} and z can be expressed as:

$$z^d = \sum_{\ell=1}^L \lambda_\ell \int_{\mathbb{R}^K} P_{D, \phi_\ell}^y(v) p_{v, \phi_\ell}(v) dv - \beta = \sum_{\ell=1}^L \lambda_\ell \int_{\mathbb{R}^K} z_{\phi_\ell}^d(v) p_{v, \phi_\ell}(v) dv \tag{15}$$

$$z^{fa} = \alpha - \sum_{\ell=1}^L \lambda_\ell \int_{\mathbb{R}^K} P_{FA, \phi_\ell}^y(v) p_{v, \phi_\ell}(v) dv = \sum_{\ell=1}^L \lambda_\ell \int_{\mathbb{R}^K} z_{\phi_\ell}^{fa}(v) p_{v, \phi_\ell}(v) dv \tag{16}$$

$$z = z^d + z^{fa} = \int_{\mathbb{R}^K} [z_{\phi_\ell}^d(v) + z_{\phi_\ell}^{fa}(v)] p_{v, \phi_\ell}(v) dv = \sum_{\ell=1}^L \lambda_\ell \int_{\mathbb{R}^K} z_{\phi_\ell}(v) p_{v, \phi_\ell}(v) dv \tag{17}$$

where $0 \leq \lambda_\ell \leq 1$ and $\sum_{\ell=1}^L \lambda_\ell = 1$. Generally, the additive noise for any $\phi_\ell \in \Phi$ can be viewed as a randomization of finite or infinite discrete vectors, i.e., $p_{v, \phi_\ell}(v) = \sum_{\kappa=1}^N \eta_{\kappa, \phi_\ell} \delta(v - v_{\kappa, \phi_\ell})$, where $0 \leq \eta_{\kappa, \phi_\ell} \leq 1$ and $\sum_{\kappa=1}^N \eta_{\kappa, \phi_\ell} = 1$, and N is a finite or infinite positive integer.

3.1. The Optimal Noise Enhanced Solution to Maximize z^d

From Equation (15), z^d can be rewritten as:

$$\begin{aligned} z^d &= \sum_{\ell=1}^L \lambda_\ell \int_{\mathbb{R}^K} z_{\phi_\ell}^d(v) \sum_{\kappa=1}^N \eta_{\kappa, \phi_\ell} \delta(v - v_{\kappa, \phi_\ell}) dv \\ &= \sum_{i=1}^L \sum_{k=1}^N \lambda_i \eta_{k, \phi_i} \int_{\mathbb{R}^K} z_{\phi_i}^d(v) \delta(v - v_{k, \phi_i}) dv \\ &= \sum_{\ell=1}^L \sum_{\kappa=1}^N \lambda_\ell \eta_{\kappa, \phi_\ell} z_{\phi_\ell}^d(v_{\kappa, \phi_\ell}) \end{aligned} \tag{18}$$

Further, z^d can also be expressed by:

$$z^d = \sum_{\tau=1}^6 \sum_{j=1}^6 \sum_{\kappa=1}^N \tau_{\tau j \kappa} [\xi_{\tau j \kappa} z_{\phi_h}^d(v_{\tau \kappa, \phi_h}) + (1 - \xi_{\tau j \kappa}) z_{\phi_l}^d(v_{j \kappa, \phi_l})] \tag{19}$$

where $(\phi_h, \mathbf{v}_{\tau\kappa, \phi_h}) \in M_\tau$, $(\phi_l, \mathbf{v}_{j\kappa, \phi_l}) \in M_j$, $\tau \neq j$, $\sum_{\tau=1}^6 \sum_{j=1}^6 \sum_{\kappa=1}^N \tau_{\tau j\kappa} = 1$, $0 \leq \tau_{\tau j\kappa} \leq 1$, $0 \leq \xi_{\tau j\kappa} \leq 1$, $h, l = 1, \dots, L$. Let $z_{\tau j\kappa}^d = \xi_{\tau j\kappa} z_{\phi_h}^d(\mathbf{v}_{\tau\kappa, \phi_h}) + (1 - \xi_{\tau j\kappa}) z_{\phi_l}^d(\mathbf{v}_{j\kappa, \phi_l})$. In other words, $z_{\tau j\kappa}^d$ is obtained by the randomization of two detector and discrete vector pairs from two different sets, i.e., $(\phi_h, \mathbf{v}_{\tau\kappa, \phi_h}) \in M_\tau$ and $(\phi_l, \mathbf{v}_{j\kappa, \phi_l}) \in M_j$. Then $z^d = \sum_{\tau=1}^6 \sum_{j=1}^6 \sum_{\kappa=1}^N \tau_{\tau j\kappa} z_{\tau j\kappa}^d$ is the convex combination of multiple $z_{\tau j\kappa}^d$, which means that z^d can be obtained by the randomization of multiple different randomizations consisting of two detector and discrete vector pairs $(\phi_h, \mathbf{v}_{\phi_h})$, $(\phi_l, \mathbf{v}_{\phi_l})$ from two different sets with the corresponding probabilities. From Equation (18), if there exists at least one detector and discrete vector pair which belongs to M_1 , the constraints $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ can be satisfied by choosing the suitable detector and adding the discrete vector. Otherwise, according to Equation (19) and the definitions of $M_1 \sim M_6$, a randomization of two detector and discrete vector pairs from two different sets may satisfy the two constraints $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$.

Let the maximum achievable z^d obtained by any noise solution under the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ be denoted by z_{opt}^d . Define $Q_d = \{(\phi, \mathbf{v}) | \mathbf{v} = \arg \max_{(\phi, \mathbf{v}) \in M} z_\phi^d(\mathbf{v})\}$ be the set of all detector and discrete vector pairs corresponding to z_m^d . Then the following theorem and corollary hold and the corresponding proofs are presented in Appendix A.

Theorem 1. z_{opt}^d can be achieved by the randomization of at most two detector and discrete vector pairs (ϕ_1, \mathbf{v}_1) and (ϕ_2, \mathbf{v}_2) from two different sets, i.e., $(\phi_1, \mathbf{v}_1) \in M_\tau$, $(\phi_2, \mathbf{v}_2) \in M_j$, $\tau, j = 1, \dots, 6$ and $\tau \neq j$.

Corollary 1. (a) If there exists at least one pair (ϕ_o, \mathbf{v}_o) of Q_d which belongs to $M_1 \cup M_2$, z_{opt}^d can be obtained by selecting (ϕ_o, \mathbf{v}_o) and $z_{opt}^d = z_m^d$. (b) When $Q_d \subset M_4$, the z^{fa} corresponding to z_{opt}^d is zero. (c) When $Q_d \subset M_4$, z_{opt}^d is obtained by the randomization of $(\phi_1, \mathbf{v}_1) \in M_4$ and (ϕ_2, \mathbf{v}_2) from M_1 , M_3 or M_5 with the respective probabilities $\xi = \frac{-z_{\phi_2}^{fa}(\mathbf{v}_2)}{z_{\phi_1}^{fa}(\mathbf{v}_1) - z_{\phi_2}^{fa}(\mathbf{v}_2)}$ and $1 - \xi$, or the detector and discrete pair $(\phi_o, \mathbf{v}_o) = \arg \max_{(\phi, \mathbf{v}) \in M_2} z_\phi^d(\mathbf{v})$.

Next, we try to search the maximum achievable z^d obtained by the randomization of $(\phi_1, \mathbf{v}_1) \in M_4$ and (ϕ_2, \mathbf{v}_2) from M_1 , M_3 or M_5 when $Q_d \subset M_4$. Generally, the corresponding z^d and z^{fa} can be expressed by:

$$z^d = \xi z_{\phi_1}^d(\mathbf{v}_1) + (1 - \xi) z_{\phi_2}^d(\mathbf{v}_2) = \sum_{i=1}^2 \xi_i z_{\phi_i}^d(\mathbf{v}_i) \tag{20}$$

$$z^{fa} = \xi z_{\phi_1}^{fa}(\mathbf{v}_1) + (1 - \xi) z_{\phi_2}^{fa}(\mathbf{v}_2) = \sum_{i=1}^2 \xi_i z_{\phi_i}^{fa}(\mathbf{v}_i) \tag{21}$$

where $\xi_1 = \xi$ and $\xi_2 = 1 - \xi$. Under the constraint that $P_{FA}^y \leq \alpha$, the Lagrangian of the optimization problem of maximizing z^d can be formulated as:

$$L(p_{\phi, \mathbf{v}}, k) = z^d + k z^{fa} = \sum_{i=1}^2 \lambda_i [z_{\phi_i}^d(\mathbf{v}_i) + k z_{\phi_i}^{fa}(\mathbf{v}_i)] \tag{22}$$

where $p_{\phi, \mathbf{v}}$ denotes the distribution of (ϕ_1, \mathbf{v}_1) and (ϕ_2, \mathbf{v}_2) . According to the Lagrange duality, we have:

$$\max_{p_{\phi, \mathbf{v}}} z^d = \min_{k \geq 0} \max_{p_{\phi, \mathbf{v}}} L(p_{\phi, \mathbf{v}}, k) \tag{23}$$

So solving the optimal solution to maximize z^d is equivalent to finding $k > 0$ and $p_{\phi,v}$ to make Equation (23) hold. Let us define an auxiliary function $d_{\phi}(v, k)$ such that:

$$d_{\phi}(v, k) = z_{\phi}^d(v) + k \cdot z_{\phi}^{fa}(v) \tag{24}$$

Let $d_1(k)$ and $d_2(k)$ be the respective suprema of $d_{\phi}(v, k)$ over the sets M_4 and $M_1 \cup M_3 \cup M_5$, i.e.,

$$d_1(k) = \sup_{(\phi,v)} \{d_{\phi}(v, k) : (\phi, v) \in M_4\} \tag{25}$$

$$d_2(k) = \sup_{(\phi,v)} \{d_{\phi}(v, k) : (\phi, v) \in M_1 \cup M_3 \cup M_5\} \tag{26}$$

Due to $z_{\phi}^{fa}(v) < 0$ when $(\phi, v) \in M_4$, $d_1(k)$ is a decreasing function of k . When $(\phi, v) \in M_1 \cup M_3 \cup M_5$, $z_{\phi}^{fa}(v) > 0$, which means $d_2(k)$ increases with k . Thus, there exists one $k^* > 0$ such that $d_1(k^*) = d_2(k^*) = d^*$, i.e., there are $(\phi_1, v_1) \in M_4$ and $(\phi_2, v_2) \in M_1 \cup M_3 \cup M_5$ such that $d_{\phi_1}(v_1, k^*) = d_1(k^*) = d_2(k^*) = d_{\phi_2}(v_2, k^*) = d^*$. So the z^{fa} and the z^d obtained by the randomization between (ϕ_1, v_1) and (ϕ_2, v_2) with the respective probabilities $\zeta = \frac{-z_{\phi_2}^{fa}(v_2)}{z_{\phi_1}^{fa}(v_1) - z_{\phi_2}^{fa}(v_2)}$ and $1 - \zeta$ can be calculated by:

$$z^{fa} = \zeta z_{\phi_1}^{fa}(v_1) + (1 - \zeta) z_{\phi_2}^{fa}(v_2) = \frac{-z_{\phi_2}^{fa}(v_2)}{z_{\phi_1}^{fa}(v_1) - z_{\phi_2}^{fa}(v_2)} z_{\phi_1}^{fa}(v_1) + \frac{z_{\phi_1}^{fa}(v_1)}{z_{\phi_1}^{fa}(v_1) - z_{\phi_2}^{fa}(v_2)} z_{\phi_2}^{fa}(v_2) = 0 \tag{27}$$

$$\begin{aligned} z^d &= \zeta z_{\phi_1}^d(v_1) + (1 - \zeta) z_{\phi_2}^d(v_2) \\ &= \zeta (d^* - k^* z_{\phi_1}^{fa}(v_1)) + (1 - \zeta) (d^* - k^* z_{\phi_2}^{fa}(v_2)) \\ &= d^* - k^* (\zeta z_{\phi_1}^{fa}(v_1) + (1 - \zeta) z_{\phi_2}^{fa}(v_2)) \\ &= d^* \end{aligned} \tag{28}$$

Combined with Equations (27) and (28), the k^* and the randomization of (ϕ_1, v_1) and (ϕ_2, v_2) are the solution of Equation (23), i.e., d^* is the maximum achievable z^d obtained by the randomization of $(\phi_1, v_1) \in M_4$ and (ϕ_2, v_2) from M_1, M_3 or M_5 when $Q_d \subset M_4$. Based on the analysis above, Algorithm B1 is provided in Appendix B to search the two detector and discrete vector pairs.

3.2. The Optimal Noise Enhanced Solution to Maximize z^{fa}

In this subsection, the optimal noise enhanced solution to maximize z^{fa} is considered. Let the maximum achievable z^{fa} obtained by any noise solution under the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ be denoted by z_{opt}^{fa} . Define $Q_{fa} = \{(\phi, v) | v = \arg \max_{(\phi,v) \in M} z_{\phi}^{fa}(v)\}$. Then the following theorem and corollary hold and the corresponding proofs are omitted here, which are similar to Theorem 1 and Corollary 1, respectively.

Theorem 2. z_{opt}^{fa} can be obtained by the randomization of at most two detector and discrete vector pairs (ϕ_1, v_1) and (ϕ_2, v_2) from two different sets, i.e., $(\phi_1, v_1) \in M_{\tau}$, $(\phi_2, v_2) \in M_j$, $\tau, j = 1, \dots, 6$ and $\tau \neq j$.

Corollary 2. (a) If there exists at least one pair (ϕ_0, v_0) of Q_{fa} which also belongs to $M_1 \cup M_3$, z_{opt}^{fa} can be achieved by selecting (ϕ_0, v_0) and $z_{opt}^{fa} = z_m^{fa}$. (b) When $Q_{fa} \subset M_5$, the z^d corresponding to z_{opt}^{fa} is zero. (c) When $Q_{fa} \subset M_5$, z_{opt}^{fa} is obtained by the randomization of (ϕ_1, v_1) from M_1, M_2 or M_4 and $(\phi_2, v_2) \in M_5$ with the respective probabilities $\eta = \frac{-z_{\phi_2}^d(v_2)}{z_{\phi_1}^d(v_1) - z_{\phi_2}^d(v_2)}$ and $1 - \eta$, or the detector and discrete pair $(\phi_0, v_0) = \arg \max_{(\phi,v) \in M_3} z_{\phi}^{fa}(v)$.

Then we focus on the maximum z^{fa} obtained by the randomization of (ϕ_1, v_1) from M_1, M_2 or M_4 and $(\phi_2, v_2) \in M_5$ with respective probabilities η and $1 - \eta$ when $Q_{fa} \subset M_5$. The corresponding z^d and z^{fa} can be expressed by:

$$z^d = \eta z_{\phi_1}^d(v_1) + (1 - \eta) z_{\phi_2}^d(v_2) = \sum_{i=1}^2 \eta_i z_{\phi_i}^d(v_i) \tag{29}$$

$$z^{fa} = \eta z_{\phi_1}^{fa}(v_1) + (1 - \eta) z_{\phi_2}^{fa}(v_2) = \sum_{i=1}^2 \eta_i z_{\phi_i}^{fa}(v_i) \tag{30}$$

where $\eta_1 = \eta$ and $\eta_2 = 1 - \eta$. Under the constraint of $P_D^y \geq \beta$, the Lagrangian of the maximization of z^{fa} is:

$$L(p_{\phi,v}, t) = z^{fa} + tz^d = \sum_{i=1}^2 \eta_i [z_{\phi_i}^{fa}(v_i) + tz_{\phi_i}^d(v_i)] \tag{31}$$

The Lagrange duality suggests that:

$$\max_{p_{\phi,v}} z^{fa} = \min_{k \geq 0} \max_{p_{\phi,v}} L(p_{\phi,v}, t) \tag{32}$$

In order to solve Equation (32), let us define an auxiliary function $f_{\phi}(v, t)$ such that:

$$f_{\phi}(v, t) = z_{\phi}^{fa}(v) + t \cdot z_{\phi}^d(v) \tag{33}$$

Suppose that $f_1(t)$ and $f_2(t)$ are the respective suprema of $f_{\phi}(v, t)$ over the sets $M_1 \cup M_2 \cup M_4$ and M_5 , i.e.,

$$f_1(t) = \sup_{(\phi,v)} \{f_{\phi}(v, t) : (\phi, v) \in M_1 \cup M_2 \cup M_4\} \tag{34}$$

$$f_2(t) = \sup_{(\phi,v)} \{f_{\phi}(v, t) : (\phi, v) \in M_5\} \tag{35}$$

When $(\phi, v) \in M_1 \cup M_2 \cup M_4$, $z_{\phi}^d(v) > 0$ and then $f_1(t)$ increases with t . Since $z_{\phi}^d(v) < 0$ when $(\phi, v) \in M_5$, $f_2(t)$ decreases with t . So there exists a $t^* > 0$ such that $f_1(t^*) = f_2(t^*) = f^*$. Namely, there exist $(\phi_1, v_1) \in M_1 \cup M_2 \cup M_4$ and $(\phi_2, v_2) \in M_5$ such that $f_{\phi_1}(v_1, t^*) = f_1(t^*) = f_2(t^*) = f_{\phi_2}(v_2, t^*) = f^*$. The z^d and the z^{fa} obtained by the randomization between (ϕ_1, v_1) and (ϕ_2, v_2) with the respective probabilities $\eta = \frac{-z_{\phi_2}^d(v_2)}{z_{\phi_1}^d(v_1) - z_{\phi_2}^d(v_2)}$ and $1 - \eta$ can be calculated by:

$$z^d = \eta z_{\phi_1}^d(v_1) + (1 - \eta) z_{\phi_2}^d(v_2) = \frac{-z_{\phi_2}^d(v_2)}{z_{\phi_1}^d(v_1) - z_{\phi_2}^d(v_2)} z_{\phi_1}^d(v_1) + \frac{z_{\phi_1}^d(v_1)}{z_{\phi_1}^d(v_1) - z_{\phi_2}^d(v_2)} z_{\phi_2}^d(v_2) = 0 \tag{36}$$

$$\begin{aligned} z^{fa} &= \eta z_{\phi_1}^{fa}(v_1) + (1 - \eta) z_{\phi_2}^{fa}(v_2) \\ &= \eta (f^* - t^* z_{\phi_1}^d(v_1)) + (1 - \eta) (f^* - t^* z_{\phi_2}^d(v_2)) \\ &= f^* - t^* (\eta z_{\phi_1}^d(v_1) + (1 - \eta) z_{\phi_2}^d(v_2)) \\ &= f^* \end{aligned} \tag{37}$$

From Equations (36) and (37), the t^* and the randomization of (ϕ_1, v_1) and (ϕ_2, v_2) are the solution of Equation (32), i.e., f^* is the maximum achievable z^{fa} obtained by the randomization of (ϕ_1, v_1) from M_1, M_2 or M_4 and $(\phi_2, v_2) \in M_5$ when $Q_{fa} \subset M_5$. According to the derivation above, Algorithm B2 presented in Appendix B can be utilized to search the corresponding detector and discrete vector pairs.

3.3. The Optimal Noise Enhanced Solution to Maximize z

Let z_{opt} represent the maximum achievable z under the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$. Define $Q_z = \{(\phi, v) | v = \arg \max_{(\phi, v) \in M} z_\phi(v)\}$. Next, the optimal noise enhanced solution to maximize z is explored in this subsection, the related theorem and corollary are provided as below.

Theorem 3. z_{opt} can be obtained by the randomization of at most two detector and discrete vector pairs (ϕ_1, v_1) and (ϕ_2, v_2) from two different sets such that $(\phi_1, v_1) \in M_\tau, (\phi_2, v_2) \in M_j$ and $\tau \neq j$. The proof of Theorem 3 is also similar to Theorem 1 and omitted here.

Corollary 3. (a) If there exists at least one pair (ϕ_o, v_o) of Q_z belongs to $M_1 \cup M_2 \cup M_3$, the maximum z can be realized by choosing (ϕ_o, v_o) and $z_{opt} = z_m$. (b) If $Q_z \subset M_4 \cup M_5, z_{opt} = \max(z_{opt}^d, z_{opt}^{fa})$. (c) If $Q_z \cap M_4 \neq \emptyset$ and $Q_z \cap M_5 \neq \emptyset$, we have $z_{opt}^d = z_{opt}^{fa} = z_{opt} = z_m$. The proofs are provided in Appendix A.

Especially, when $Q_z \subset M_4 \cup M_5$, we can select the two pairs (ϕ_1, v_1) and (ϕ_2, v_2) directly, according to the analysis above and the properties of $M_1 \sim M_5$, to form an available noise enhanced solution to make the value of z as greater as possible.

If $Q_z \subset M_4$, then we can let $(\phi_1, v_1) \in Q_z$ and $(\phi_2, v_2) \in M_1 \cup M_3 \cup M_5$. Since $z_{\phi_1}(v_1) \geq z_{\phi_2}(v_2)$, the maximum z is achieved when $\lambda = \lambda_2 = \frac{-z_{\phi_2}^{fa}(v_2)}{z_{\phi_1}^{fa}(v_1) - z_{\phi_2}^{fa}(v_2)}$. The greater the value of λ , the greater the value of z . So (ϕ_1, v_1) and (ϕ_2, v_2) can be selected as $(\phi_1, v_1) = \arg \max_{(\phi, v) \in Q_z} z_\phi^{fa}(v)$ and $(\phi_2, v_2) = \arg \max_{(\phi, v) \in T_1} z_\phi^{fa}(v)$, where $T_1 = \{(\phi, v) | (\phi, v) = \arg \max_{(\phi, v) \in M_1 \cup M_3 \cup M_5} z_\phi(v)\}$.

Similarly, when $Q_z \subset M_5$, let $(\phi_2, v_2) = \arg \min_{(\phi, v) \in Q_z} z_\phi^d(v)$ and $(\phi_1, v_1) = \arg \min_{(\phi, v) \in T_2} z_\phi^d(v)$, where $T_2 = \{(\phi, v) | (\phi, v) = \arg \max_{(\phi, v) \in M_1 \cup M_2 \cup M_4} z_\phi(v)\}$.

3.4. Sufficient Conditions for $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$

In this section, according to the analysis from Section 3.1 to Section 3.3 and the properties of the six sets, the sufficient conditions which can or cannot satisfy the two constraints $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ are determined as below.

Theorem 4. (a) If $M_1 \cup M_2 \cup M_3 \neq \emptyset$, any pair $(\phi, v) \in M_1 \cup M_2 \cup M_3$ can meet $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$; (b) When $M_1 \cup M_2 \cup M_3 = \emptyset$, if there exist $(\phi_1, v_1) \in M_4$ and $(\phi_2, v_2) \in M_5$ such that:

$$|z_{\phi_1}^{fa}(v_1)| \cdot |z_{\phi_2}^d(v_2)| < |z_{\phi_1}^d(v_1)| \cdot |z_{\phi_2}^{fa}(v_2)| \tag{38}$$

then $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ can be realized by the randomization of (ϕ_1, v_1) and (ϕ_2, v_2) , otherwise there exists no noise enhanced solution to make $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ hold. The proofs are presented in Appendix A.

When no randomization between detectors is allowed, only one detector can be selected to utilize. Suppose that the selected detector is ϕ_o , the conclusions for the case of allowing randomization between detectors can be applied to the case of nonrandomization between detectors straightforwardly by replacing M_j with M_{j, ϕ_o} and $\phi_1 = \phi_2 = \phi_o$, where $j = 1, \dots, 6$.

4. Numerical Results

In this section, a binary hypothesis-testing problem is considered to verify the theoretical results explored in the previous sections. The binary hypotheses are given by:

$$\begin{cases} H_0 : x[i] = \omega[i] \\ H_1 : x[i] = A + \omega[i] \end{cases} \tag{39}$$

where $i = 0, \dots, K - 1$, $A > 0$ is a known constant, and $\omega[i]$ are independent identically distributed (i.i.d.) symmetric Gaussian mixture noise samples such that:

$$p_{\omega}(\omega) = \frac{1}{2}\gamma(\omega; -\mu, \sigma^2) + \frac{1}{2}\gamma(\omega; \mu, \sigma^2) \tag{40}$$

where $\gamma(\omega; \mu, \sigma^2) = (1/\sqrt{2\pi\sigma^2})\exp(-(\omega - \mu)^2/2\sigma^2)$. The test statistic of a suboptimal detector is shown as:

$$T(x) = \frac{1}{K} \sum_{i=0}^{K-1} \left(\frac{1}{2} + \frac{1}{2} \text{sgn}(x[i]) \right) = \frac{1}{K} \sum_{i=0}^{K-1} (S(x[i])) \underset{H_0}{\overset{H_1}{>}} \gamma \tag{41}$$

where $S(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases}$. From Equation (41), the test result in this case is obtained through twice decision. Firstly, use the sign detector $S(\cdot)$ to determine the sign of each observation component $x[i]$. Secondly, compute the proportion of the positive observation components in the observation vector and then compare it with the decision threshold γ to obtain the final result.

Let $K = 2$, then the detector shown in Equation (41) can be substituted by two decision thresholds $\gamma = 1$ and $\gamma = 0.5$, the corresponding decision function are ϕ_1 and ϕ_2 , respectively. When $\gamma = 1$, the detector chooses H_1 only if $x[0] > 0$ and $x[1] > 0$ at the same time. When $\gamma = 0.5$, the detector chooses H_1 if $x[0] > 0$ or $x[1] > 0$. Assume $v = (v_1, v_2)^T$ be a discrete vector without any constraints. Then the detection and false-alarm probabilities of the sign detector $S(\cdot)$ choosing the noise modified observation component $y[i] = x[i] + v[i] > 0, i = 0, 1$, can be calculated by $P_{1,s}(v[i]) = \int_{-\infty}^{\infty} S(y[i])p_1(y[i] - v[i])dy[i] = \frac{1}{2}Q(\frac{-v[i]-\mu-A}{\sigma}) + \frac{1}{2}Q(\frac{-v[i]+\mu-A}{\sigma})$ and $P_{0,s}(v[i]) = \int_{-\infty}^{\infty} S(y[i])p_0(y[i] - v[i])dy[i] = \frac{1}{2}Q(\frac{-v[i]-\mu}{\sigma}) + \frac{1}{2}Q(\frac{-v[i]+\mu}{\sigma})$, where $v[0] = v_1, v[1] = v_2$ and $Q(x) = \int_x^{+\infty} (1/\sqrt{2\pi})\exp(-t^2/2)dt$. Further, the detection and false-alarm probabilities of y for ϕ_1 are computed as $P_{D,\phi_1}^y(v) = P_{1,s}(v_1)P_{1,s}(v_2)$ and $P_{FA,\phi_1}^y(v) = P_{0,s}(v_1)P_{0,s}(v_2)$, respectively. The detection and false-alarm probabilities of y for ϕ_2 can be expressed by $P_{D,\phi_2}^y(v) = 1 - (1 - P_{1,s}(v_1))(1 - P_{1,s}(v_2))$ and $P_{FA,\phi_2}^y(v) = 1 - (1 - P_{0,s}(v_1))(1 - P_{0,s}(v_2))$. Correspondingly, $z_{\phi_i}^d(v) = P_{D,\phi_i}^y(v) - \beta$ and $z_{\phi_i}^{fa}(v) = \alpha - P_{FA,\phi_i}^y(v), i = 1, 2$.

Let $\mu = 3$ and $A = 1$. Under the two constraints that $P_{FA}^y \leq \alpha$ and $P_D^y \geq \beta$, for any σ , we can determine the six sets $M_{j,\phi_i}, j = 1, \dots, 6$, for ϕ_1 and ϕ_2 according to the definitions of the six sets and the values of $z_{\phi_i}^d(v)$ and $z_{\phi_i}^{fa}(v)$, respectively. Naturally, the six sets obtained by allowing the randomization between ϕ_1 and ϕ_2 can be determined by $M_j = M_{j,\phi_1} \cup M_{j,\phi_2}, j = 1, \dots, 6$. Afterwards, we can search the maximum z^d, z^{fa}, z and the corresponding noise enhanced solutions according to the algorithm provided in Section 3.

Figure 1 illustrates that the maximum achievable z^d, z^{fa} and z for ϕ_1, ϕ_2 and the case of allowing the randomization between ϕ_1 and ϕ_2 for different values of σ when $\alpha = 0.3$ and $\beta = 0.7$. The z^d plotted in Figure 1a is actually the relative improvement of the maximum achievable detection probability $P_{D,opt}^y$ compared to $\beta = 0.7$ under the constraint $P_{FA}^y \leq \alpha = 0.3$, i.e., $z^d = P_{D,opt}^y - \beta$. Hence, $P_{FA}^y \leq \alpha$ and $P_D^y \geq \beta$ can be realized only when $z^d \geq 0$. As shown in Figure 1a, $z^d < 0$ when σ increases to a certain extent, which means the feasible range of σ for the noise enhanced phenomenon is limited. When σ is close to 0, the maximum achievable z^d for ϕ_1 is 0.3, which equals to that for the case of allowing the randomization between detectors, and the corresponding $P_{D,opt}^y$ is close to 1 while the maximum P_D^y for ϕ_2 can only reach 0.9. With the increase of σ, z^d for ϕ_1, ϕ_2 and the case of allowing the randomization between them gradually decrease. When, $\sigma > 0.34$ the maximum achievable z^d for ϕ_1 is lower than that for the case of allowing the randomization between detectors. The maximum achievable $z^d = 0$ for ϕ_1 and ϕ_2 when $\sigma = 0.63$, and the maximum P_D^y for ϕ_2 is gradually greater than that for ϕ_1 when $\sigma > 0.63$. In particular, for the case where the randomization between detectors is allowed, the maximum achievable z^d decreases to 0 when $\sigma = 0.71$. Consequently, for $\sigma \in [0.63, 0.71]$,

the noise enhanced phenomenon, under the constraints $P_{FA}^y \leq \alpha$ and $P_D^y \geq \beta$, can still happen through allowing the randomization between ϕ_1 and ϕ_2 , which is on account of the fusion of M_{j,ϕ_1} and M_{j,ϕ_2} , $j = 1, \dots, 6$, providing more available noise enhanced solutions.

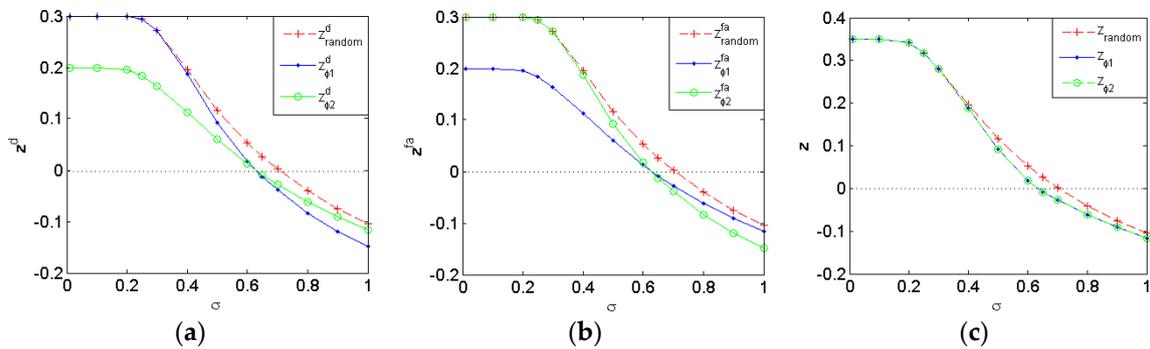


Figure 1. The maximum achievable z^d , z^{fa} and z for ϕ_1 , ϕ_2 and the case of allowing the randomization between ϕ_1 and ϕ_2 for different values of σ when $\alpha = 0.3$ and $\beta = 0.7$ Plotted in (a), (b) and (c), respectively.

The z^{fa} depicted in Figure 1b is actually the relative improvement of the minimum achievable false-alarm probability $P_{FA,opt}^y$ compared to $\alpha = 0.3$ under the constraint that $P_{FA}^y \geq \beta = 0.7$, i.e., $z^{fa} = \alpha - P_{FA,opt}^y$. Similarly, there exists noise enhanced solution to meet the two constraints $P_{FA}^y \leq \alpha$ and $P_D^y \geq \beta$ only if $z^{fa} \geq 0$. When σ approaches to 0, the maximum z^{fa} for ϕ_2 is equal to that for the case of allowing the randomization between ϕ_1 and ϕ_2 , the corresponding minimum P_{FA}^y is close to 0 while the minimum P_{FA}^y for ϕ_1 can only reach 0.1. From Figure 1b, as σ increases, the maximum achievable z^{fa} for ϕ_1 , ϕ_2 and the case of allowing the randomization between them gradually decrease. The maximum achievable z^{fa} for ϕ_1 and ϕ_2 are lower than zero when $\sigma > 0.63$, while z^{fa} obtained in the case of allowing the randomization between ϕ_1 and ϕ_2 is still greater than zero for $0.63 \leq \sigma \leq 0.71$. In other words, for $0.63 \leq \sigma \leq 0.71$, compared to the nonrandomization case where the noise enhanced phenomenon cannot happen, $P_{FA}^y \leq \alpha$ and $P_D^y \geq \beta$ can still be realized by allowing the randomization between ϕ_1 and ϕ_2 .

From Figure 1a,b, it is clear that under the constraints $P_{FA}^y \leq 0.3$ and $P_D^y \geq 0.7$, the maximum achievable P_D^y for ϕ_1 is greater than that for ϕ_2 and the minimum achievable P_{FA}^y for ϕ_2 is smaller than that for ϕ_1 when $\sigma \leq 0.63$. In such case, we can choose ϕ_1 for a greater P_D^y or select ϕ_2 for a smaller P_{FA}^y when the randomization between detectors cannot be allowed. As illustrated in Figure 1c, the maximum z for ϕ_1 is equal to that for ϕ_2 . When σ is close to 0, the maximum z for the case of allowing the randomization can reach 0.35, which is greater than the corresponding maximum z^d and z^{fa} . Obviously, there exists $(\phi, v) \in M_1$ to obtain the maximum $z_\phi(v)$ in the whole M . As σ increases, the number of the elements in the set M_1 decreases. When $\sigma > 0.33$, $Q_z \subset M_4 \cup M_5$, then the maximum z obtained in the case of allowing the randomization is equal to the maximum z^d or z^{fa} according to Corollary 3, i.e., $z_{opt} = \max(z_{opt}^d, z_{opt}^{fa})$.

As a comparison, Figures 2 and 3 show the maximum achievable z^d , z^{fa} and z for ϕ_1 , ϕ_2 and the case of allowing the randomization between them for different values of σ when $\alpha = 0.3$, $\beta = 0.6$ and $\alpha = 0.2$, $\beta = 0.7$, respectively. Compared Figures 1a and 2a, both of them plot the z^d corresponding to the maximum P_D^y under the constraint $P_{FA}^y \leq 0.3$. So the z^d in the two figures indicate the same change trend. In Figure 2b, the maximum z^{fa} obtained for the case of allowing randomization between detectors equals to that for ϕ_2 when $\sigma < 0.59$. Compared to Figure 1b, when σ is close to 0, the minimum P_{FA}^y in Figure 2b for ϕ_2 still maintains zero, while the minimum P_{FA}^y for ϕ_1 decreases from 0.1 to 0.05 as the corresponding z^{fa} increases from 0.2 to 0.25. Further, compared the minimum P_{FA}^y for ϕ_2 when $\beta = 0.7$ and $\beta = 0.6$, they are equal when $\sigma < 0.20$ and then the latter one is gradually greater

than the former one as σ increases, which is consistent with the description as shown in Figures 1b and 2b. From the definition of $z_\phi^d(v)$, i.e., $z_\phi^d(v) = P_{D,\phi}^y(v) - \beta$, with the decrease of β , the value of $z_\phi^d(v)$ increases and some $z_\phi^d(v)$ may change from negative to positive. In other words, the decrease of β changes the distribution of the detector and discrete pair (ϕ, v) in $M_1 \sim M_6$. For any σ , some (ϕ, v) belonged to M_6 for $\beta = 0.7$ are reallocated to the set M_2 or M_4 when β decreases to 0.6. In addition, some $(\phi, v) \in M_5$ when $\beta = 0.7$ may belong to M_1 or M_3 when $\beta = 0.6$. Further, these new elements in $M_1 \sim M_5$ can be utilized to construct more available noise enhanced solutions to obtain a superior false-alarm probability. However, we need to note that behind the improvement of P_{FA}^y is the decrease of the corresponding P_D^y .

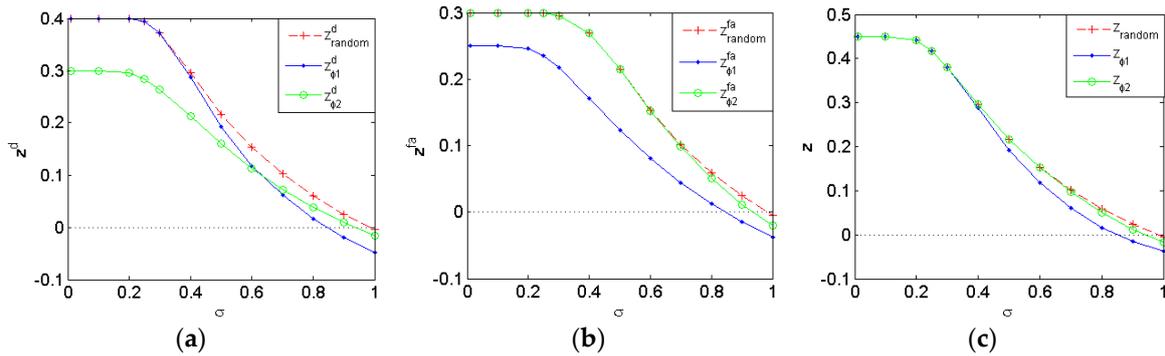


Figure 2. The maximum achievable z^d , z^{fa} and z for ϕ_1 , ϕ_2 and the case of allowing the randomization between ϕ_1 and ϕ_2 for different values of σ when $\alpha = 0.3$ and $\beta = 0.6$. Plotted in (a), (b) and (c), respectively.

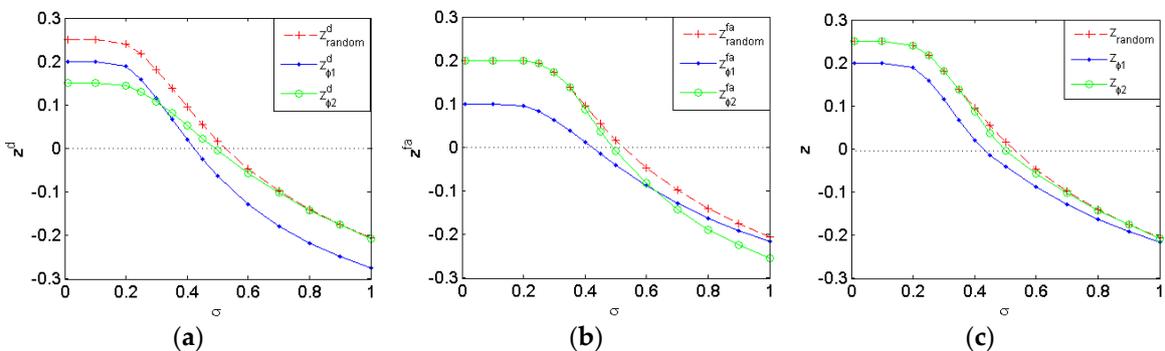


Figure 3. The maximum achievable z^d , z^{fa} and z for ϕ_1 , ϕ_2 and the case of allowing the randomization between ϕ_1 and ϕ_2 for different values of σ when $\alpha = 0.2$ and $\beta = 0.7$ Plotted in (a), (b) and (c), respectively.

Compared Figures 1 and 3, as α decreases from 0.3 to 0.2, some $z_\phi^{fa}(v)$ will change from positive to negative, i.e., the distribution of (ϕ, v) changes. Consequently, for any σ , there may be some $(\phi, v) \in M_1$ change to M_2 and M_4 , $(\phi, v) \in M_2$ change to M_4 , or $(\phi, v) \in M_3 \cup M_5$ change to M_6 . As shown in Figure 3a, when σ closes to 0, the maximum available z^d for ϕ_1 , ϕ_2 and the case of allowing the randomization between them are 0.2, 0.15 and 0.25, and the corresponding maximum P_D^y can reach 0.9, 0.85 and 0.95, respectively. As σ increases, the maximum z^d for ϕ_1 , ϕ_2 and the case of allowing the randomization decrease, where the maximum z^d decreases fastest for ϕ_1 and slowest for ϕ_2 . Further, the maximum achievable P_D^y for ϕ_2 is greater than that for ϕ_1 when $\sigma > 0.32$ and the difference between the maximum z^d for ϕ_2 and the case of allowing the randomization are smaller and smaller with the increase of σ . Compared Figures 3b and 1b, both of them plot the z^{fa} corresponding to the minimum P_{FA}^y under the constraint $P_D^y \geq 0.7$. Especially, $Q_z \subset M_4 \cup M_5$ for any σ , i.e., $z_{opt} = \max(z_{opt}^d, z_{opt}^{fa})$

according to Corollary 3. In addition, compared with Figure 1, under the two constraints that $P_{FA}^y \leq \alpha$ and $P_D^y \geq \beta$, the feasible ranges of σ for ϕ_1, ϕ_2 and the case of allowing the randomization between them become smaller.

In conclusion, as σ decreases, the values of z^d, z^{fa} and z increase. This is mainly on account of the noise enhanced phenomenon generally occurs when the observation has multimodal pdf and the multimodal structure is more obvious for a smaller σ [21]. In order to investigate the simulation results of Figures 1–3 further, Tables 1–3 present the optimal noise enhanced solutions to maximize z^d, z^{fa} and z for ϕ_1, ϕ_2 and the case of allowing the randomization, respectively, for different σ when $\alpha = 0.3$ and $\beta = 0.7$.

Table 1. The optimal additive noises to maximize z^d, z^{fa} and z for various values of σ for ϕ_1 when $\alpha = 0.3$ and $\beta = 0.7$.

σ	$\max z^d$	$\max z^{fa}$	$\max z$
	$v_1/v_2/\lambda$	$v_1/v_2/\lambda$	$v_1/v_2/\lambda$
0.01	[2.5, 2.5]/-/1	[2.5, 2.5]/[-3.75, 2.5]/0.4	[2.5, 2.5]/-/1
0.2	[2.7, 2.75]/-/1	[2.5, 2.5]/[2.7, -3.5]/0.4088	[2.55, 2.5]/-/1
0.4	[2.5, 2.5]/[-3.5, 2.75]/0.9794	[2.5, 2.5]/[-3.5, 2.75]/0.5684	[2.5, 2.5]/[-3.5, 2.75]/0.9794
0.65	-	-	-

Table 2. The optimal additive noises to maximize z^d, z^{fa} and z for various values of σ for ϕ_2 when $\alpha = 0.3$ and $\beta = 0.7$.

σ	$\max z^d$	$\max z^{fa}$	$\max z$
	$v_1/v_2/\lambda$	$v_1/v_2/\lambda$	$v_1/v_2/\lambda$
0.01	[-5.75, 2.75]/[-3.5, -3.25]/0.6	[-3.75, -3.75]/-/1	[-3.7, -3.75]/-/1
0.2	[2.55, 2.5]/[2.7, -3.5]/0.4088	[-3.7, -3.75]/-/1	[-3.55, -3.5]/-/1
0.4	[-3.75, 2.5]/[-3.5, -3.5]/0.4316	[-3.75, 2.5]/[-3.5, -3.5]/0.0206	[-3.75, 2.5]/[-3.5, -3.5]/0.0206
0.65	-	-	-

Table 3. The optimal additive noises to maximize z^d, z^{fa} and z for various values of σ for the case of allowing the randomization between ϕ_1 and ϕ_2 when $\alpha = 0.3$ and $\beta = 0.7$.

σ	$\max z^d$	$\max z^{fa}$	$\max z$
	$(\phi, v_1)/-/ \lambda$	$(\phi, v_1)/(\phi, v_2)/\lambda$	$(\phi, v_1)/(\phi, v_2)/\lambda$
0.01	$(\phi_1, [2.3, 2.75])/-/1$	$-(\phi_2, [-3.35, -3.25])/1$	$(\phi_1, [2.3, 2.75])/-/1$
0.2	$(\phi_1, [2.75, 2.75])/(\phi_1, [2.7, 2.75])/0.4770$	$(\phi_2, [-3.7, -3.75])/(\phi_2, [-3.75, -3.75])/0.5230$	$-(\phi_2, [-3.55, -3.5])/1$
0.4	$(\phi_1, [2.6, 2.5])/(\phi_2, [-3.6, -3.5])/0.9141$	$(\phi_1, [2.6, 2.5])/(\phi_2, [-3.6, -3.5])/0.0859$	$(\phi_1, [2.6, 2.5])/(\phi_2, [-3.6, -3.5])/0.0859, 0.9141$
0.65	$(\phi_1, [2.65, 2.75])/(\phi_2, [-3.65, -3.75])/0.5437$	$(\phi_1, [2.65, 2.75])/(\phi_2, [-3.65, -3.75])/0.4563$	$(\phi_1, [2.65, 2.75])/(\phi_2, [-3.65, -3.75])/0.4563, 0.5437$

It is worthy to note that the optimal noise enhanced solutions to maximize z^d, z^{fa} and z , respectively, are not unique in many cases. Moreover, due to the property of the detector, the noise modified detectability for $\phi_i, i = 1, 2$, obtained by adding $v = (v_1, v_2)^T$ is the same with that achieved via adding $v = (v_2, v_1)^T$. As a demonstration, for each σ , there only lists one noise enhanced solution for the maximum z^d , as well as the maximum z^{fa} and z . As shown in Tables 1–3, the optimal noise enhanced solutions to maximize z^d, z^{fa} and z , respectively, are the randomization of at most two detector and discrete vector pairs (ϕ_1, v_1) and (ϕ_2, v_2) from two different sets, which are consistent with Theorems 1–3.

Next, the noise enhanced solution for ϕ_1 is taken as an example to illustrate firstly. When $\sigma = 0.01$, the maximum $z_{\phi_1}^d(v)$ obtained by $(\phi_1, v) \in M_{1,\phi_1}$ is equal to the maximum $z_{\phi_1}^d(v)$ obtained by $(\phi_1, v) \in M_{\phi_1}$. Through some calculations, $v = [2.5, 2.5] \in M_{1,\phi_1}$ is one of the discrete vectors to maximize $z_{\phi_1}^d(v)$, so $v = [2.5, 2.5]$ is the optimal noise to maximize z^d for ϕ_1 when $\alpha = 0.3$ and $\beta = 0.7$. At the same time, the $z_{\phi_1}(v)$ obtained by $v = [2.5, 2.5]$ is also the maximum $z_{\phi_1}(v)$ for ϕ_1 , thus $v = [2.5, 2.5]$ is the optimal noise to maximize z for ϕ_1 . Besides, the maximum $z_{\phi_1}^{fa}(v)$ obtained from M_{1,ϕ_1} and M_{3,ϕ_1} are smaller than the maximum $z_{\phi_1}^{fa}(v)$ for ϕ_1 , then the maximum z^{fa} under the two constraints $P_{FA}^y \leq 0.3$ and $P_D^y \geq 0.7$ is obtained by the randomization of $v_1 = [2.5, 2.5]$ from M_{1,ϕ_1} and $v_2 = [-3.75, 2.5]$ from M_{5,ϕ_1} with probabilities 0.4 and 0.6, respectively. The case of $\sigma = 0.2$ is similar with the case of $\sigma = 0.01$.

When $\sigma = 0.4$, both M_{2,ϕ_1} and M_{3,ϕ_1} are null, and the maximum $z_{\phi_1}^d(v)$, $z_{\phi_1}^{fa}(v)$ and $z_{\phi_1}(v)$ for ϕ_1 cannot be obtained by the discrete vector from M_{1,ϕ_1} . Based on Theorems 1–3, the maximum z^d , z^{fa} and z can be achieved by the randomization of two detector and discrete vector pairs from M_{4,ϕ_1} and M_{5,ϕ_1} . Further, the noise enhanced solutions for the maximum z^d and z^{fa} have the same additive noise components, *i.e.*, v_1 and v_2 , but with different probabilities. Moreover, according to Corollary 3(b), $z_{opt} = \max(z_{opt}^d, z_{opt}^{fa})$. When the randomization between ϕ_1 and ϕ_1 is allowed, the $z_{\phi_1}(v_1)$ obtained by $(\phi_1, v_1 = [2.6, 2.5]) \in M_4$ is equal to the $z_{\phi_2}(v_2)$ obtained by $(\phi_2, v_2 = [-3.6, -3.5]) \in M_5$, and it is the maximum $z_{\phi}(v)$ obtained in M . According to Corollary 3(c), the maximum z^d , z^{fa} and z can be obtained by the randomization of (ϕ_1, v_1) and (ϕ_2, v_2) with different probabilities. Especially, the probability λ of (ϕ_1, v_1) for the maximum z^d or z^{fa} is unique, while the probability λ of (ϕ_1, v_1) for the maximum z can be chosen in a certain interval. When $\sigma = 0.65$, no noise enhanced solution exists to meet the two constraints for the nonrandomization case in Tables 1 and 2, while there still exist noise enhanced solutions to improve the detectability under the same constraints by allowing randomization between ϕ_1 and ϕ_2 shown in Table 3 and the corresponding solutions are also obtained according to Corollary 3(c).

In order to discuss the effect of the decision threshold γ on the detection and false-alarm probabilities, the proposed noise enhanced method is operated on different values of γ . Further, the relationships between the maximum achievable detection probability and β , the minimum achievable false-alarm probability and α are explored for different γ . The different results of the original detector and the noise enhanced detector for different γ are given in Figures 4–6.

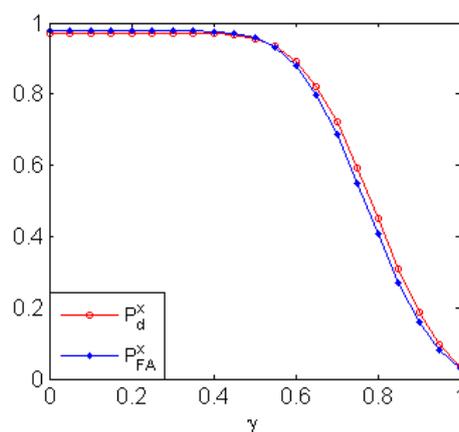


Figure 4. P_D^x and P_{FA}^x for different γ when $\mu = 3$, $A = 1$, $\sigma = 1$ and $K = 20$.

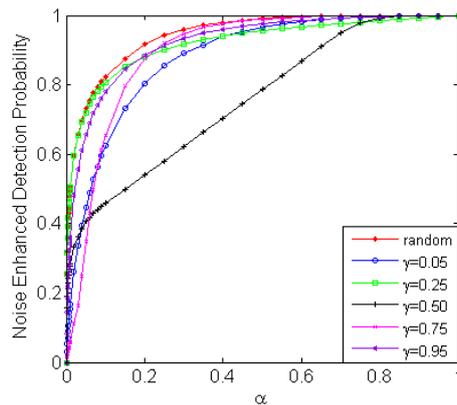


Figure 5. The maximum achievable P_D^y as a function of α for $\gamma = 0.05, 0.25, 0.5, 0.75$ and 0.95 , and the case of allowing the randomization between thresholds, when $\mu = 3, A = 1, \sigma = 1$ and $K = 20$.

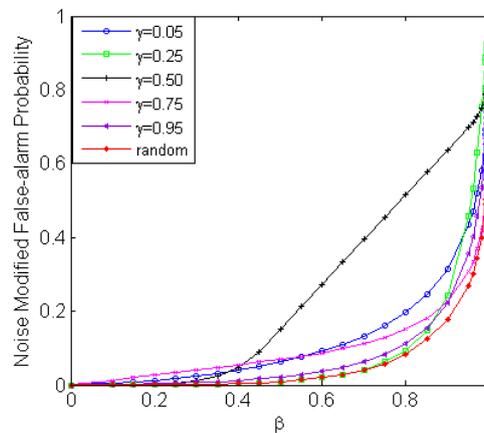


Figure 6. The minimum achievable P_{FA}^y as a function of β for $\gamma = 0.05, 0.25, 0.5, 0.75$ and 0.95 , and the case of allowing the randomization between thresholds, when $\mu = 3, A = 1, \sigma = 1$ and $K = 20$.

Figure 4 gives the original detection and false-alarm probabilities for different γ when $K = 20$. From Figure 4, we can see that both the original detection and false-alarm probabilities decrease with the increase of γ and the value of the original detection probability is close to that of the original false-alarm probability for any γ . In other words, a better detection probability is obtained for a smaller γ and a lower false-alarm probability is achieved for a greater γ . Figure 5 plots the maximum achievable P_D^y obtained by adding noise as a function of α for $\gamma = 0.05, 0.25, 0.5, 0.75$ and 0.95 , and the case of allowing the randomization between thresholds, when $\mu = 3, A = 1, \sigma = 1$ and $K = 20$. Compared Figure 5 with Figure 4, the detection probabilities for $\gamma = 0.95$ and 0.75 can be increased significantly by adding suitable additive noises with a lower false-alarm probability. Figure 6 presents the minimum achievable P_{FA}^y obtained by adding noise as a function of β for $\gamma = 0.05, 0.25, 0.5, 0.75$ and 0.95 , and the case of allowing the randomization between thresholds, when $\mu = 3, A = 1, \sigma = 1$ and $K = 20$. Comparing Figure 6 with Figure 4, the false-alarm probabilities for $\gamma = 0.05, 0.25$ and 0.5 can be decreased significantly by adding suitable additive noises with a higher detection probability. From Figures 5 and 6, different detection performance can be realized by adding noise. As shown in Figures 5 and 6, for the cases of $\alpha \in (0.055, 0.725)$ and $\beta \in (0.405, 0.975)$, the detector of $\gamma = 0.5$ shows the worst performance compared to others. Thus, in such cases, $\gamma = 0.5$ is not a suitable threshold. From Figures 5 and 6, different detection performance can be realized by adding noises. Namely, various noise enhanced solutions can be provided with our method to satisfy different performance requirements. As a result, for any decision threshold γ , we can determine whether the performance of

the detector can be improved or not, and search a noise enhanced solution to realize the improvement according to the method proposed in this paper.

It is worthy to note that there is no limit on the detector in the method proposed in this manuscript. Furthermore, it only depends on detector itself whether the detection performance of the detector can or cannot be improved by adding noise. The algorithms proposed in this paper not only provide ways to prove the improvability or nonimprovability, but also analyze how to search the optimal noise enhanced solutions. For any detector, no matter an optimal Neyman–Pearson (Bayesian, Minimax) detector or other suboptimal detector, we first calculate all information of $(P_{D,\phi_\ell}(\mathbf{v}), P_{FA,\phi_\ell}(\mathbf{v}))$ obtained by every detector and discrete vector pair (ϕ_ℓ, \mathbf{v}) . Then, we divide each pair (ϕ_ℓ, \mathbf{v}) into six sets according to the values of $z_{\phi_\ell}^d(\mathbf{v})$ and $z_{\phi_\ell}^{fa}(\mathbf{v})$, where $z_{\phi_\ell}^d(\mathbf{v}) = P_{D,\phi_\ell}^y(\mathbf{v}) - \beta$ and $z_{\phi_\ell}^{fa}(\mathbf{v}) = \alpha - P_{FA,\phi_\ell}^y(\mathbf{v})$. If there exist detector and discrete vector pairs to satisfy the sufficient conditions as given in Section 3.4, noise enhanced solutions to maximize z^d, z^{fa} and z can be obtained according to Section 3.1, Section 3.2, and Section 3.3, respectively, on the premise that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$. Otherwise, no noise enhanced solution exists to satisfy $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ simultaneously.

5. Conclusions

In this paper, a framework consisting of six mutually disjoint sets is established according to two inequality-constraints on detection and false-alarm probabilities. The maximization of z^d, z^{fa} and z are searched based on the framework. Theorems 1–3 give the forms of the optimal noise enhanced solutions to maximize z^d, z^{fa} and z . The calculations of maximum z^d, z^{fa} and z are presented in Corollaries 1, 2 and 3, respectively. Especially, the maximum z is equal to the maximum z^d or z^{fa} under certain conditions according to Corollary 3. Furthermore, sufficient conditions for $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ are given in Theorem 4.

Compared with the method proposed in [16], which only focuses on the maximization of z^d with a constant false-alarm rate (CFAR), both methods require all the information of $(P_{D,\phi_\ell}(\mathbf{v}), P_{FA,\phi_\ell}(\mathbf{v}))$ obtained by every detector and discrete vector pair (ϕ_ℓ, \mathbf{v}) , and our method may use the information more effectively to realize the overall improved performance or decrease the false alarm probability due to the division of six sets.

Furthermore, the theoretical results in this paper can be extended to a broad class of noise enhanced optimal problems subject to two inequality constraints, such as the minimization of Bayes risk under the constraint on condition risk about the binary hypothesis testing problem, and the minimization of linear combinations of error probabilities under constraints on type I and II error probabilities as discussed in [28].

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

A.1. Proofs of Theorems and Corollaries

Theorem A1. z_{opt}^d can be achieved by the randomization of at most two detector and discrete vector pairs (ϕ_1, \mathbf{v}_1) and (ϕ_2, \mathbf{v}_2) from two different sets, i.e., $(\phi_1, \mathbf{v}_1) \in M_\tau, (\phi_2, \mathbf{v}_2) \in M_j, \tau, j = 1, \dots, 6$ and $\tau \neq j$.

Proof. Combining Equation (19) and the definitions of $M_1 \sim M_6$, if there exists a randomization of two detector and discrete vector pairs from two different sets to make $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ hold, the two constraints of $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ can be satisfied by the randomization consisting of one or more randomization consisting of two detector and discrete vector pairs from two different sets with corresponding probabilities. Otherwise, no noise enhanced solution exists to meet $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$.

Obviously, under the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$, the z^d obtained by any randomization of detector and discrete vector pairs cannot be greater than the maximum z^d obtained by a randomization of two detector and discrete vector pairs (ϕ_1, v_1) and (ϕ_2, v_2) with probabilities ξ and $1 - \xi$ from two different sets, respectively, where $0 \leq \xi \leq 1$. Especially, $\xi = 0$ and $\xi = 1$ represent the case where the maximum z^d under the two constraints is obtained by one pair of detector and discrete vector. Namely, under the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$, the maximum z^d can be obtained by the randomization of at most two detector and discrete vector pairs from two different sets. □

Corollary A1. (a) If there exists at least one pair (ϕ_0, v_0) of Q_d which belongs to $M_1 \cup M_2$, z_{opt}^d can be obtained by selecting (ϕ_0, v_0) and $z_{opt}^d = z_m^d$. (b) When $Q_d \subset M_4$, the z^{fa} corresponding to z_{opt}^d is zero. (c) When $Q_d \subset M_4$, z_{opt}^d is obtained by the randomization of $(\phi_1, v_1) \in M_4$ and (ϕ_2, v_2) from M_1, M_3 or M_5 with the respective probabilities $\xi = \frac{-z_{\phi_2}^{fa}(v_2)}{z_{\phi_1}^{fa}(v_1) - z_{\phi_2}^{fa}(v_2)}$ and $1 - \xi$, or the detector and discrete pair $(\phi_0, v_0) = \arg \max_{(\phi, v) \in M_2} z_{\phi}^d(v)$.

Proof.

Part (a): According to the definition of Q_d , the z^d obtained by each pair $(\phi, v) \in Q_d$ is equal to z_m^d . Since $z_m^d > 0$, the (ϕ, v) of Q_d can only come from M_1, M_2 or M_4 . If there exists at least one pair (ϕ_0, v_0) of Q_d which belongs to $M_1 \cup M_2$, it is obvious that $z_{opt}^d = z_m^d$ can be obtained by choosing (ϕ_0, v_0) .

Part (b): The contradiction method is used here. When $Q_d \subset M_4$, suppose that the z^{fa} corresponding to z_{opt}^d is greater than zero and denoted by $z_1^{fa} > 0$. From the definition of Q_d , for any pair $(\phi_d, v_d) \in Q_d$, we have $z_{\phi_d}^{fa}(v_d) < 0$ and $z_{\phi_d}^d(v_d) = z_m^d > z_{opt}^d$ as z_m^d is defined without the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$. Then the z^{fa} and the z^d , which are obtained by the randomization of (ϕ_d, v_d) and the optimal noise enhanced solution for z_{opt}^d with probabilities $\xi = \frac{-z_1^{fa}}{z_{\phi_d}^{fa}(v_d) - z_1^{fa}}$ and $1 - \xi$, respectively, can be calculated as follows,

$$z^{fa} = \xi z_{\phi_d}^{fa}(v_d) + (1 - \xi) z_1^{fa} = \frac{-z_1^{fa}}{z_{\phi_d}^{fa}(v_d) - z_1^{fa}} z_{\phi_d}^{fa}(v_d) + \frac{z_{\phi_d}^{fa}(v_d)}{z_{\phi_d}^{fa}(v_d) - z_1^{fa}} z_1^{fa} = 0 \tag{A1}$$

$$z^d = \xi z_{\phi_d}^d(v_d) + (1 - \xi) z_{opt}^d = \frac{-z_1^{fa}}{z_{\phi_d}^{fa}(v_d) - z_1^{fa}} z_m^d + \frac{z_{\phi_d}^{fa}(v_d)}{z_{\phi_d}^{fa}(v_d) - z_1^{fa}} z_{opt}^d > z_{opt}^d \tag{A2}$$

because $z_m^d > z_{opt}^d$. Obviously, $z^d > z_{opt}^d$ is in conflict with the definition of z_{opt}^d . So the z^{fa} corresponding to z_{opt}^d is zero when $Q_d \subset M_4$.

Part (c): Firstly, according to Theorem 1, z_{opt}^d is obtained by the randomization of at most two detector and discrete pairs from two different sets. Secondly, according to the conclusion (a) of Corollary 1, the z^{fa} corresponding to z_{opt}^d is zero. Then combined the definitions of $M_1 \sim M_6$, $z^{fa} = 0$ can only be realized by a detector and discrete pair from M_2 or a randomization of (ϕ_1, v_1) from M_4 and (ϕ_2, v_2) from M_1, M_3 or M_5 with the respective probabilities ξ and $1 - \xi$, and vice versa. Further, $z^{fa} = 0$ means that $z^{fa} = \xi z_{\phi_1}^{fa}(v_1) + (1 - \xi) z_{\phi_2}^{fa}(v_2) = 0$, i.e., $\xi = \frac{-z_{\phi_2}^{fa}(v_2)}{z_{\phi_1}^{fa}(v_1) - z_{\phi_2}^{fa}(v_2)}$. If the maximum z^d obtained by

the above randomization is greater than the maximum z^d obtained by (ϕ, v) from M_2 , the optimal noise enhanced solution to maximize z^d is the corresponding randomization. Otherwise, the optimal noise enhanced solution to maximize z^d is the detector and discrete vector pair $(\phi_0, v_0) = \arg \max_{(\phi, v) \in M_2} z_{\phi}^d(v)$. \square

Corollary A3. (a) If there exists at least one pair (ϕ_0, v_0) of Q_z belongs to $M_1 \cup M_2 \cup M_3$, the maximum z can be realized by choosing (ϕ_0, v_0) and $z_{opt} = z_m$. (b) If $Q_z \subset M_4 \cup M_5$, $z_{opt} = \max(z_{opt}^d, z_{opt}^{fa})$. (c) If $Q_z \cap M_4 \neq \emptyset$ and $Q_z \cap M_5 \neq \emptyset$, we have $z_{opt}^d = z_{opt}^{fa} = z_{opt} = z_m$.

Proof.

Part (a): According to the definition of Q_z , the z obtained by each pair $(\phi, v) \in Q_z$ is equal to z_m . Since $z_m > 0$, the (ϕ, v) of Q_z can come from $M_1 \sim M_5$. If there exists at least one pair (ϕ_0, v_0) of Q_z which belongs to $M_1 \cup M_2 \cup M_3$, it is obvious that $z_{opt} = z_m$ can be achieved by choosing (ϕ_0, v_0) .

Part (b): When $Q_z \subset M_4 \cup M_5$, the maximum z is obtained by the randomization of (ϕ_1, v_1) and (ϕ_2, v_2) with probabilities λ and $1 - \lambda$, so:

$$z = \lambda z_{\phi_1}(v_1) + (1 - \lambda) z_{\phi_2}(v_2) \tag{A3}$$

In order to satisfy the two constraints that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$, (ϕ_1, v_1) and (ϕ_2, v_2) need to meet the following inequalities:

$$z^d = \lambda z_{\phi_1}^d(v_1) + (1 - \lambda) z_{\phi_2}^d(v_2) \geq 0 \tag{A4}$$

$$z^{fa} = \lambda z_{\phi_1}^{fa}(v_1) + (1 - \lambda) z_{\phi_2}^{fa}(v_2) \geq 0 \tag{A5}$$

Accordingly, $\lambda = \lambda_1 = \frac{-z_{\phi_2}^d(v_2)}{z_{\phi_1}^d(v_1) - z_{\phi_2}^d(v_2)}$ when $z^d = 0$ and $\lambda = \lambda_2 = \frac{-z_{\phi_2}^{fa}(v_2)}{z_{\phi_1}^{fa}(v_1) - z_{\phi_2}^{fa}(v_2)}$ when $z^{fa} = 0$.

Since $Q_z \subset M_4 \cup M_5$, there is at least one pair of (ϕ_1, v_1) and (ϕ_2, v_2) which belongs to $M_4 \cup M_5$. If (ϕ_2, v_2) can be selected only from M_1, M_3 or M_5 .

When $(\phi_1, v_1) \in M_4$ and $(\phi_2, v_2) \in M_1 \cup M_3$, the randomization is available if and only if $z_{\phi_1}(v_1) \geq z_{\phi_2}(v_2)$. From the definitions of M_1 and M_3 , Equation (A4) holds for any $0 \leq \lambda \leq 1$, and Equation (A5) holds only for $0 \leq \lambda \leq \lambda_2$. So the maximum z is obtained when $\lambda = \lambda_2$ as $z_{\phi_1}(v_1) \geq z_{\phi_2}(v_2)$.

When $(\phi_1, v_1) \in M_4$ and $(\phi_2, v_2) \in M_5$, Equations (A4) and (A5) can hold at the same time if and only if $\lambda_1 < \lambda_2$. If $z_{\phi_1}(v_1) \geq z_{\phi_2}(v_2)$, z gets the maximum when $\lambda = \lambda_2$, otherwise z reaches the maximum when $\lambda = \lambda_1$.

When $(\phi_1, v_1) \in M_1 \cup M_2$ and $(\phi_2, v_2) \in M_5$, suppose $z_{\phi_1}(v_1) \leq z_{\phi_2}(v_2)$, based on the definitions about M_1 and M_2 , Equation (A4) holds for $\lambda_1 \leq \lambda \leq 1$ and Equation (A5) holds for any $0 \leq \lambda \leq 1$. So the maximum z is reached when $\lambda = \lambda_1$.

If $Q_z \subset M_4$ or $Q_z \cap M_4 \neq \emptyset$, $z_m < z_m^d$ and $Q_d \subset M_4$. According to Corollary 1, z_{opt}^d is obtained when $\lambda = \lambda_2$ and the corresponding $z^{fa} = 0$. If $Q_z \subset M_5$ or $Q_z \cap M_5 \neq \emptyset$, $z_m < z_m^{fa}$ and $Q_{fa} \subset M_5$. According to Corollary 2, z_{opt}^{fa} is obtained when $\lambda = \lambda_1$ and the corresponding z^d equals to zero, i.e., $z^d = 0$. As a result, the maximum z is equal to the maximum z^d or z^{fa} , i.e., $z_{opt} = \max(z_{opt}^d, z_{opt}^{fa})$.

Part (c): When $Q_z \cap M_4 \neq \emptyset$ and $Q_z \cap M_5 \neq \emptyset$, let $(\phi_1, v_1) \in Q_z \cap M_4$ and $(\phi_2, v_2) \in Q_z \cap M_5$. Then we have $z_{\phi_1}(v_1) = z_{\phi_2}(v_2) = z_m$. As $z_m > 0$, $|z_{\phi_1}^d(v_1)| > |z_{\phi_1}^{fa}(v_1)|$ and $|z_{\phi_2}^d(v_2)| < |z_{\phi_2}^{fa}(v_2)|$. It is obvious that $\lambda_1 < \lambda_2$ through some simple calculations, which means that $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ can be satisfied by the randomization of (ϕ_1, v_1) and (ϕ_2, v_2) with the respective probabilities λ and $1 - \lambda$, where $\lambda \in [\lambda_1, \lambda_2]$. Accordingly, we always have $z_{opt} = \lambda z_{\phi_1}(v_1) + (1 - \lambda) z_{\phi_2}(v_2) = z_m$ for any $\lambda \in [\lambda_1, \lambda_2]$. Furthermore, z^d and z^{fa} reach the maximum when $\lambda = \lambda_2$ and $\lambda = \lambda_1$, respectively, and $z_{opt}^d = z_{opt}^{fa} = z_{opt} = z_m$. \square

Theorem 4. (a) If $M_1 \cup M_2 \cup M_3 \neq \emptyset$, any pair $(\phi, v) \in M_1 \cup M_2 \cup M_3$ can meet $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$; (b) When $M_1 \cup M_2 \cup M_3 = \emptyset$, if there exist $(\phi_1, v_1) \in M_4$ and $(\phi_2, v_2) \in M_5$ such that

$$|z_{\phi_1}^{fa}(v_1)| \cdot |z_{\phi_2}^d(v_2)| < |z_{\phi_1}^d(v_1)| \cdot |z_{\phi_2}^{fa}(v_2)| \tag{A6}$$

Then $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ can be realized by the randomization of (ϕ_1, v_1) and (ϕ_2, v_2) , otherwise there exists no noise enhanced solution to make $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$ hold.

Proof.

Part (a): If $M_1 \cup M_2 \cup M_3 \neq \emptyset$, from the definitions of $M_1 \sim M_3$, $z_{\phi}^d(v) \geq 0$ and $z_{\phi}^{fa}(v) \geq 0$ for any $(\phi, v) \in M_1 \cup M_2 \cup M_3$. Accordingly, the corresponding $P_D^y = \beta + z_{\phi}^d(v) \geq \beta$ and $P_{FA}^y = \alpha - z_{\phi}^{fa}(v) \leq \alpha$.

Part (b): When $M_1 \cup M_2 \cup M_3 = \emptyset$, if we want to make the P_D^y and P_{FA}^y obtained by the randomization of $(\phi_1, v_1) \in M_4$ and $(\phi_2, v_2) \in M_5$ meet the two constraints $P_D^y \geq \beta$ and $P_{FA}^y \leq \alpha$, the two inequalities (A3) and (A4) should be satisfied simultaneously. Further, Equations (A3) and (A4) hold if and only if $\lambda_1 < \lambda_2$. Due to $z_{\phi_1}^d(v_1) > 0$ and $z_{\phi_1}^{fa}(v_1) < 0$ for any $(\phi_1, v_1) \in M_4$, $z_{\phi_2}^d(v_2) < 0$ and $z_{\phi_2}^{fa}(v_2) > 0$ for any $(\phi_2, v_2) \in M_5$, then:

$$\lambda_1 = \frac{|z_{\phi_2}^d(v_2)|}{|z_{\phi_1}^d(v_1)| + |z_{\phi_2}^d(v_2)|} < \lambda_2 = \frac{|z_{\phi_2}^{fa}(v_2)|}{|z_{\phi_1}^{fa}(v_1)| + |z_{\phi_2}^{fa}(v_2)|} \tag{A7}$$

$$|z_{\phi_2}^d(v_2)| \cdot (|z_{\phi_1}^{fa}(v_1)| + |z_{\phi_2}^{fa}(v_2)|) < |z_{\phi_2}^{fa}(v_2)| \cdot (|z_{\phi_1}^d(v_1)| + |z_{\phi_2}^d(v_2)|) \tag{A8}$$

$$|z_{\phi_1}^{fa}(v_1)| \cdot |z_{\phi_2}^d(v_2)| < |z_{\phi_1}^d(v_1)| \cdot |z_{\phi_2}^{fa}(v_2)| \tag{A9}$$

□

Appendix B

Algorithm B1. Optimal noise solution to maximize z^d .

```

d1 = max{z_{\phi}^d(v) + z_{\phi}^{fa}(v) : (\phi, v) \in M_4}; dt = d1;
d2 = max{z_{\phi}^d(v) + z_{\phi}^{fa}(v) : (\phi, v) \in M_1 \cup M_3 \cup M_5}; dp = d2;
while |d2 - d1| > \epsilon
    dc = (dt + dp)/2; d1 = dc;
    k = max{(dc - z_{\phi}^d(v))/z_{\phi}^{fa}(v) : (\phi, v) \in M_4};
    d2 = max{z_{\phi}^d(v) + k \cdot z_{\phi}^{fa}(v) : (\phi, v) \in M_1 \cup M_3 \cup M_5};
    if d2 > d1
        dp = min(d2, max(dt, dp));
    else
        dp = max(d2, min(dt, dp));
    end
    dt = dc;
end
(\phi_1, v_1) = arg max_{(\phi, v) \in M_4} (z_{\phi}^d(v) + k \cdot z_{\phi}^{fa}(v)); (\phi_2, v_2) = arg max_{(\phi, v) \in M_1 \cup M_3 \cup M_5} (z_{\phi}^d(v) + k \cdot z_{\phi}^{fa}(v)).

```

It is noted that the parameter ϵ in Algorithm B1 is a near-zero positive, which is used to ensure accuracy of the algorithm.

Algorithm B2. Optimal noise solution to maximize z^{fa} .

```

f1 = max{ $z_{\phi}^{fa}(v) + z_{\phi}^d(v) : (\phi, v) \in M_1 \cup M_2 \cup M_4$ };  $ft = f1$ ,
f2 == max{ $z_{\phi}^{fa}(v) + z_{\phi}^d(v) : (\phi, v) \in M_5$ };  $fp = f2$ ;
while  $|f2 - f1| > \varepsilon$ 
   $fc = (ft + fp)/2$ ;  $f1 = fc$ ;
   $t = \min\{(fc - z_{\phi}^{fa}(v))/z_{\phi}^d(v) : (\phi, v) \in M_1 \cup M_2 \cup M_4\}$ ;
   $f2 = \max\{z_{\phi}^{fa}(v) + t \cdot z_{\phi}^d(v) : (\phi, v) \in M_5\}$ ;
  if  $f2 > f1$ 
     $fp = \min(f2, \max(ft, fp))$ ;
  else
     $fp = \max(f2, \min(ft, fp))$ ;
  end
   $ft = fc$ ;
end
 $(\phi_1, v_1) = \arg \max_{(\phi, v) \in M_1 \cup M_2 \cup M_4} (z_{\phi}^{fa}(v) + t \cdot z_{\phi}^d(v))$ ;  $(\phi_2, v_2) = \arg \max_{(\phi, v) \in M_5} (z_{\phi}^{fa}(v) + t \cdot z_{\phi}^d(v))$ .

```

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