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Optimum Accelerated Degradation Tests for the Gamma Degradation Process Case under the Constraint of Total Cost

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Abstract: An accelerated degradation test (ADT) is regarded as an effective alternative to an accelerated life test in the sense that an ADT can provide more accurate information on product reliability, even when few or no failures may be expected before the end of a practical test period. In this paper, statistical methods for optimal designing ADT plans are developed assuming that the degradation characteristic follows a gamma process (GP). The GP-based approach has an advantage that it can deal with more frequently encountered situations in which the degradation should always be nonnegative and strictly increasing over time. The optimal ADT plan is developed under the total experimental cost constraint by determining the optimal settings of variables such as the number of measurements, the measurement times, the test stress levels and the number of units allocated to each stress level such that the asymptotic variance of the maximum likelihood estimator of the q-th quantile of the lifetime distribution at the use condition is minimized. In addition, compromise plans are developed to provide means to check the adequacy of the assumed acceleration model. Finally, sensitivity analysis procedures for assessing the effects of the uncertainties in the pre-estimates of unknown parameters are illustrated with an example.

Keywords: accelerated degradation test; gamma process; optimal plan; compromise plan; maximum likelihood estimation

1. Introduction

Strong pressure from customers and intense global competition among manufacturers have resulted in the production of highly reliable products. Reliability inferences based on the results of life tests or

accelerated life tests can be inaccurate when used to evaluate highly reliable products since few or no failures may be expected before the end of a practical test period. In contrast, if a degradation characteristic related to the failure mechanism exists, a test which monitors the behavior of the characteristic may provide more accurate information on product reliability. In order to accelerate the degradation process for highly reliable products, such a test is usually conducted under stress levels higher than the normal use condition. This type of test is called an accelerated degradation test (ADT) in the literature [1]. For statistical modeling and analysis of various degradation phenomena, the reader is referred to the papers in Escobar *et al.* [2].

An ADT, as all other reliability tests, must be carefully designed beforehand to obtain the estimates of the quantities of interest as precisely as possible. A recent review of the literature by Yum *et al.* [3] on designing ADT plans shows that many researchers have designed ADT plans based on the general degradation path (GDP) model. For instance, see Boulanger and Escobar [4], Park and Yum [5], Yu [6,7], Park and Yum [8], and Shi *et al.* [9]. The GDP model consists of an actual degradation path and an error term. The actual degradation path is represented as a deterministic function of time, and the error term usually represents the measurement error assumed to be independent over time. Given this formulation, most of the GDP models developed to date do not consider the time-dependent error structure (Tseng and Peng [10]). For this reason, a stochastic process (SP) model that naturally incorporates the correlation among degradation measurements over time can represent a useful alternative.

Currently used SP models for degradation include the Wiener process (WP), geometric Brownian motion (GBM), and gamma process (GP) models, among others. Tang *et al.* [11], Liao and Tseng [12], and Lim and Yum [13] developed optimal ADT plans under the assumption of a WP model for degradation. Liao and Elsayed [14] designed optimal ADT plans based on an accelerated geometric Brownian motion degradation rate (AGBMDR) model. In a WP model, the degradation may take negative values and does not always increase with time, while in the GBM model the degradation is always positive, but not strictly increasing over time. However, in certain physical situations where the measurement error is relatively small, the degradation should always be nonnegative and strictly increasing over time. In such situations, a GP model which always gives nonnegative, strictly increasing degradation over time is considered to be a more adequate model. Bagdonavicius and Nikulin [15] and Lawless and Crowder [16] modeled degradation as a GP which allows covariates. Park and Padgett [17,18] organized a basic framework for the degradation model using the GP as well as the WP and GBM.

Despite the appropriateness of GP models for degradation, little research has been conducted on designing optimal ADT plans based on a GP model (Yum *et al.* [3]). The notable exceptions is Tseng *et al.* [19], and Pan and Sun [20] in which optimal step-stress ADT plans are developed for a gamma degradation process by determining the sample size, measurement frequency and termination time such that the asymptotic variance of the estimated mean time to failure or *q*-th quantile of the lifetime distribution is minimized under the total cost constraint, and Tsai *et al.* [21] in which optimal ADT plans with two stress variable are developed. In this paper, optimal ADT plans are developed based on the assumptions that a single stress variable is considered for ease of conducting the test and interpreting the result, the constant-stress loading method is employed and the degradation characteristic follows a GP. The number of measurements, the measurement times, the test stress levels and the number of units allocated to each stress level are determined under the total experimental cost

constraint such that the asymptotic variance of the maximum likelihood estimator (MLE) of the *q*-th quantile of the lifetime distribution at the use condition is minimized.

This paper is organized as follows: Section 2 introduces the assumed accelerated degradation model, and derives the lifetime distribution. In Section 3, the asymptotic variance of the MLE of the *q*-th quantile of the lifetime distribution at the use condition and the total experimental cost are described, and the corresponding optimization problem is formulated. The procedure for obtaining the optimal ADT plan is presented in Section 4. In Section 5, a compromise plan with three stress levels is developed for checking the adequacy of the assumed acceleration function. Sensitivity analyses of the test plans with respect to the uncertainties involved in the pre-estimates of unknown parameters are given in Section 6 with an example. Finally, conclusions and future research directions are presented in Section 7.

2. Accelerated Degradation Model

2.1. Gamma Process Degradation Model

In the present investigation, $y(t)$, the degradation characteristic at time t, is assumed to follow a GP with shape coefficient $\alpha'(>0)$ and scale coefficient $\beta(>0)$. Then, $y(t)$ has the following properties:

1. $y(0)=0$,

2. $\{y(t)\,|\,t\geq 0\}$ has stationary independent increments,

3. $y(t)$ follows a gamma distribution with a probability density function (pdf):

$$f_y(y)=\frac{1}{\Gamma(\alpha't)}\frac{1}{\beta^{\alpha't}}y^{\alpha't-1}e^{-y/\beta}, \qquad t>0$$

where $\alpha't$ and β are the shape and scale parameters, respectively, and each increment, Δy_{ab} $(=y(t_b)-y(t_a))$, follows a gamma distribution with shape parameter $\alpha'\Delta t$ and scale parameter β for $0\leq t_a<t_b$ where $\Delta t=t_b-t_a$ (see Figure 1).
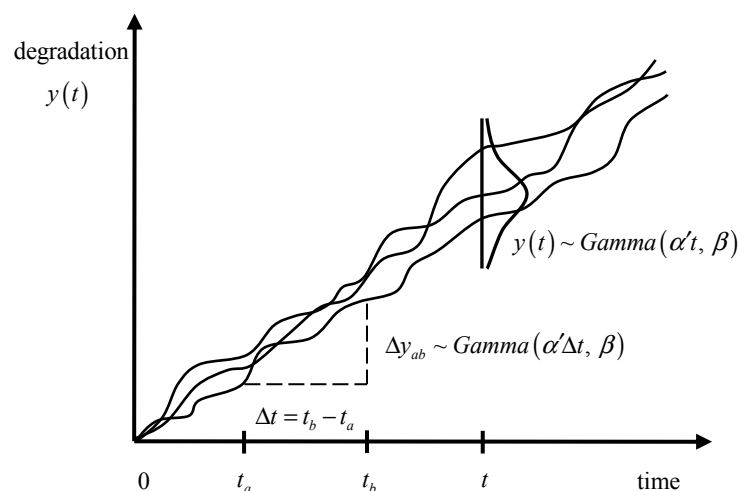

Figure 1. Representative sample paths of a gamma process with shape coefficient α' and scale coefficient β.

2.2. Acceleration Function and Standardization

In this paper, it is assumed that the relationship between the shape coefficient α' of a GP and the stress variable s' can be described by any one of the following (Nelson [1]):

- Arrhenius model: $\alpha'(s') = \delta'_1 \exp(-\delta'_2/s')$, e.g., s' is absolute temperature
- Power model: $\alpha'(s') = \delta'_1(s')^{\delta'_2}$, e.g., s' is voltage
- Exponential model: $\alpha'(s') = \delta'_1 \exp(\delta'_2 s')$, e.g., s' is weathering variable

where $\delta'_1(>0)$ and $\delta'_2(>0)$ are unknown constants and β does not depend on s'.

It is assumed that the use stress level s'_0 and maximum stress level s'_M are pre-specified. For simplicity and without loss of generality, the stress level is standardized as follows:

$$s = \begin{cases} \dfrac{1/s'_0 - 1/s'}{1/s'_0 - 1/s'_M}, & \text{for the Arrhenius model} \\[3mm] \dfrac{\ln s' - \ln s'_0}{\ln s'_M - \ln s'_0}, & \text{for the power model} \\[3mm] \dfrac{s' - s'_0}{s'_M - s'_0}, & \text{for the exponential model} \end{cases}$$

Under the above standardization, $s_0 = 0$, $s_M = 1$ and $0 \le s \le 1$. In addition, the shape coefficient $\alpha'(s')$ can be re-expressed in terms of s as follows:

$$\alpha(s) = \exp(\delta_1 + \delta_2 s) \tag{1}$$

where $\delta_1 = \ln(\delta'_1) - \delta'_2/s'_0$, $\delta_2 = \delta'_2(1/s'_0 - 1/s'_M)$ for the Arrhenius model; $\delta_1 = \ln(\delta'_1) + \delta'_2 \ln s'_0$, $\delta_2 = \delta'_2(\ln s'_M - \ln s'_0)$ for the power model; $\delta_1 = \ln(\delta'_1) + \delta'_2 s'_0$, $\delta_2 = \delta'_2(s'_M - s'_0)$ for the exponential model. Note that δ_2 is always positive since $\delta'_2 > 0$ and $s'_M > s'_0$.

2.3. Lifetime Distribution

When the degradation characteristic $y(t)$ follows a GP with shape coefficient $\alpha(s)$ and scale coefficient β, $y(t)$ strictly increases over time. Let T be the lifetime defined as the first passage time to the failure level y_c. Then, the cumulative distribution function (cdf) of T can be obtained as follows:

$$\begin{aligned} G_s(t) = P\{T < t\} &= P\{y(t) > y_c\} \\ &= \int_{y_c}^{\infty} \frac{1}{\Gamma[\alpha(s)t]} \frac{1}{\beta^{\alpha(s)t}} y^{\alpha(s)t-1} e^{-y/\beta} dy \\ &= \frac{1}{\Gamma[\alpha(s)t]} \int_{y_c/\beta}^{\infty} x^{\alpha(s)t-1} e^{-x} dx \\ &= \frac{\Gamma[\alpha(s)t, y_c/\beta]}{\Gamma[\alpha(s)t]} \end{aligned} \tag{2}$$

where $\Gamma(a, b)$ is the incomplete gamma function defined as:

$$\Gamma(a, b) = \int_b^\infty z^{a-1} e^{-z} dz .$$

In addition, the pdf of T is obtained as follows (Park and Padgett [17]):

$$g_s(t) = \frac{\partial}{\partial t} \frac{\Gamma[\alpha(s)t, y_c/\beta]}{\Gamma[\alpha(s)t]}$$

$$= \frac{\alpha(s)}{\Gamma[\alpha(s)t]} \left\{ \begin{array}{l} \Gamma[\alpha(s)t, y_c/\beta] \ln(y_c/\beta) + \dfrac{(y_c/\beta)^{\alpha(s)t}}{[\alpha(s)t]^2} \cdot {}_2F_2\left(\begin{array}{cc} \alpha(s)t & \alpha(s)t \\ \alpha(s)t+1 & \alpha(s)t+1 \end{array} \middle| -y_c/\beta \right) \\ +\Gamma[\alpha(s)t]\Psi_0[\alpha(s)t] - \Gamma[\alpha(s)t]\ln(y_c/\beta) - \Gamma[\alpha(s)t, y_c/\beta]\Psi_0[\alpha(s)t] \end{array} \right\}$$

where $\Psi_0(x)$ is the digamma function defined as $\Psi_0(x) = \partial \ln \Gamma(x)/\partial x = \Gamma'(x)/\Gamma(x)$ and ${}_pF_q(\cdot)$ is the generalized hypergeometric function defined by:

$${}_pF_q\left(\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} \middle| z \right) = \sum_{k=0}^\infty \frac{(a_1)_k, \cdots, (a_p)_k}{(b_1)_k, \cdots, (b_q)_k} \cdot \frac{z^k}{k!}$$

with $(a_i)_k = a_i \cdot (a_i+1) \cdots (a_i+k-1)$ and $(a_i)_0 = 1$.

3. Formulation of ADT Design Problem

3.1. Optimization Criterion and Constraint

Various optimization criteria have been proposed for designing ADT plans. The most frequently used criteria include the (asymptotic) variance and mean squared error (MSE) of the estimator of the q-th quantile (or the mean) of the lifetime distribution at the use condition (Park and Yum [5]; Yu [6,7]; Li and Kececioglu [22,23]; Park and Yum [8]; Liao and Tseng [12]; Shi *et al.* [9]; and Tseng *et al.* [19]). In addition to these criteria, Boulanger and Escobar [4] and Liao and Elsayed [14] considered the generalized variance, namely, the determinant of the Fisher information matrix of the MLEs of unknown parameters, and Tang *et al.* [11] considered the total experimental cost. These optimization criteria have been also used as constraint. Among constraints, the total experimental cost is frequently used (Li and Kececioglu [22]; Liao and Tseng [12]; and Tseng *et al.* [19]). In addition to the constraint, Tang *et al.* [11] considered the precision constraint using the asymptotic variance of the estimator of the mean lifetime at the use condition. In this paper, the asymptotic variance of the MLE of the q-th quantile of the lifetime distribution at the use condition is adopted as an optimization criterion and the total experimental cost as a constraint.

3.2. The Asymptotic Variance of the MLE of the q-th Quantile of the Lifetime Distribution

In the following, we consider a constant-stress loading ADT where n_i test units among $n\left(=\sum_{i=1}^r n_i\right)$ total test units are allocated to each stress level s_i ($i = 1, 2, \cdots, r$) and the stress is loaded constantly until the test is over. Let m_{ij} be the number of measurements for the j-th unit ($j = 1, 2, \cdots, n_i$) at the stress level s_i. It is assumed that $m_{ij} = m$ for all i and j. In addition, the measurement times $(t_{ijk}, k = 1, 2, \cdots, m)$ for the j-th test unit at the stress level s_i are determined such

that the measurement time interval, $\Delta t_{ijk}\left(=t_{ijk}-t_{ij(k-1)}\right)$, is the same as Δt for all i, j and k where $t_{ij(k-1)}<t_{ijk}$, $t_{ij0}=0$, and $t_{ijm}=\Delta t\cdot m$. For the j-th unit at the stress level s_i, let y_{ijk} be the degradation characteristic measured at t_{ijk} for $i=1,2,\cdots,r$, $j=1,2,\cdots,n_i$ and $k=1,2,\cdots,m$. Then, each degradation increment $\Delta y_{ijk}=y_{ijk}-y_{ij(k-1)}$ $\left(y_{ijk}>y_{ij(k-1)}\right)$ follows a gamma distribution with the following pdf:

$$f_{\Delta y}\left(\Delta y_{ijk}\right)=\frac{1}{\Gamma\left[\alpha(s_i)\Delta t\right]}\frac{1}{\beta^{\alpha(s_i)\Delta t}}\left(\Delta y_{ijk}\right)^{\alpha(s_i)\Delta t-1}\exp\left(-\frac{\Delta y_{ijk}}{\beta}\right),\ \Delta y_{ijk}>0,\ \alpha(s_i),\beta>0 \qquad (3)$$

From Equations (1) and (3), the log likelihood function for n test units is given by:

$$\ln L=\sum_{i=1}^{r}\sum_{j=1}^{n_i}\sum_{k=1}^{m}\left[(A_i-1)\ln\Delta y_{ijk}-\frac{\Delta y_{ijk}}{\beta}-\ln\Gamma(A_i)-A_i\ln\beta\right]$$

where $A_i=\alpha(s_i)\Delta t=\left[\exp(\delta_1+\delta_2 s_i)\right]\Delta t$. The MLEs of δ_1, δ_2 and β, which is $\hat{\delta}_1$, $\hat{\delta}_2$ and $\hat{\beta}$, can be obtained by solving the simultaneous equations: $\partial\ln L/\partial\delta_1=0$, $\partial\ln L/\partial\delta_2=0$ and $\partial\ln L/\partial\beta=0$. Hence, the corresponding MLE of the q-th quantile of the lifetime distribution at the use condition ($t_{q,0}$) is $\hat{t}_{q,0}=\hat{G}_0^{-1}(q)$. Let **F** be the Fisher information matrix obtained by taking expectations of the negative second partial derivatives of $\ln L$ with respect to unknown parameters δ_1, δ_2 and β (Lawless [24]). Then, it is shown in Appendix A that:

$$\mathbf{F}=m\begin{bmatrix}\sum_{i=1}^{r}n_i A_i^2 B_i & \sum_{i=1}^{r}n_i A_i^2 B_i s_i & \sum_{i=1}^{r}n_i\dfrac{A_i}{\beta}\\[2ex] & \sum_{i=1}^{r}n_i A_i^2 B_i s_i^2 & \sum_{i=1}^{r}n_i\dfrac{A_i s_i}{\beta}\\[2ex] symmetric & & \sum_{i=1}^{r}n_i\dfrac{A_i}{\beta^2}\end{bmatrix} \qquad (4)$$

where $B_i=\Psi_1(A_i)$ is the trigamma function defined as $\partial^2\ln\Gamma(A_i)/\partial A_i^2$ for $i=1,2,\cdots,r$. Define:

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}\dfrac{\partial G_0(t_{q,0})}{\partial\delta_1}\\[2ex]\dfrac{\partial G_0(t_{q,0})}{\partial\delta_2}\\[2ex]\dfrac{\partial G_0(t_{q,0})}{\partial\beta}\end{bmatrix}=\begin{bmatrix}\dfrac{A_0^{t_q}}{\Gamma(A_0')}\left\{\begin{array}{l}\Gamma\left(A_0^{t_q},\beta_c\right)\ln\beta_c+\dfrac{\beta_c^{A_0^{t_q}}}{\left(A_0'\right)^2}\cdot{}_2F_2\left(\begin{array}{cc}A_0^{t_q} & A_{00}^{t_q}\\A_{00}^{t_q}+1 & A_{00}^{t_q}+1\end{array}\middle|-\beta_c\right)\\[2ex]+\Gamma\left(A_0^{t_q}\right)\Psi_0\left(A_0^{t_q}\right)-\Gamma\left(A_{00}^{t_q}\right)\ln\beta_c-\Gamma\left(A_{00}^{t_q},\beta_c\right)\Psi_0\left(A_0^{t_q}\right)\end{array}\right\}\\[6ex]0\\[4ex]\dfrac{\beta_c^{A_0^{t_q}+1}}{y_c\exp(\beta_c)\Gamma\left(A_0^{t_q}\right)}\end{bmatrix}$$

where $A_0^{t_q}=\exp(\delta_1)t_{q,0}$ and $\beta_c=y_c/\beta$. Then, by using the delta method, the asymptotic variance of the MLE of the q-th quantile of the lifetime distribution at the use condition is obtained as follows (Liao and Tseng [12]):

$$\mathrm{Avar}\left(\hat{t}_{q,0}\right)=\frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2}\,\mathrm{Avar}\left[\hat{G}_0\left(t_{q,0}\right)\right]=\frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2}\,\mathbf{u}^t\mathbf{F}^{-1}\mathbf{u}$$

where the superscript *t* represents a transposition.

3.3. Total Experimental Cost

The total experimental cost comprises three parts:

1. Operation cost which includes the labor cost can be formulated as $C_{op}\cdot\Delta t\cdot m$, where C_{op} is the unit cost of operation.
2. Measurement cost which involves the cost of measurement equipments and test materials can be generated as $C_m\cdot m\cdot n$, where C_m is the unit cost of measurement.
3. Sample cost which is related to number of test samples can be expressed as $C_s\cdot n$, where C_s is the unit cost of the device.

Therefore the total cost of conducting the ADT $\left(TC\right)$ is given by $TC=C_{op}\cdot\Delta t\cdot m+C_m\cdot m\cdot n+C_s\cdot n$.

3.4. Formulation of the Problem

The ADT design problem with two stress levels is formulated in this section. For the case of two stress levels, the objective function is given by (see Appendix B):

$$\mathrm{Avar}\left(\hat{t}_{q,0}\right)=\frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2}\frac{1}{m}\frac{u_1^2P+2u_1u_3Q+u_3^2R}{K} \tag{5}$$

where

$$K=n_1n_2A_1^2A_2^2\left(s_2-s_1\right)^2\left[n_1B_2\left(A_1B_1-1\right)+n_2B_1\left(A_2B_2-1\right)\right],$$

$$P=\left(n_1A_1^2B_1s_1^2+n_2A_2^2B_2s_2^2\right)\left(n_1A_1+n_2A_2\right)-\left(n_1A_1s_1+n_2A_2s_2\right)^2,$$

$$Q=\frac{y_c}{\beta_c}\left[n_1n_2A_1A_2\left(s_2-s_1\right)\left(A_1B_1s_1-A_2B_2s_2\right)\right],$$

$$R=\frac{y_c^2}{\beta_c^2}\left[n_1n_2A_1^2A_2^2B_1B_2\left(s_2-s_1\right)^2\right]$$

Since $1/\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2$ in Equation (5) is a scaling factor, the objective function can be reduced to:

$$v=\frac{1}{m}\frac{u_1^2P+2u_1u_3Q+u_3^2R}{K} \tag{6}$$

Finally, the ADT design problem with two stress levels is formulated as follows:

$$\begin{aligned}
\text{Minimize}\quad & v=\frac{1}{m}\frac{u_1^2P+2u_1u_3Q+u_3^2R}{K}\\
\text{Subject to}\quad & TC=C_{op}\cdot\Delta t\cdot m+C_m\cdot m\cdot n+C_s\cdot n\le C_b\\
& n_1,n_2,\Delta t,m\in N=\{1,2,3,\cdots\}\\
& 0\le s_1<s_2\le 1
\end{aligned}$$

Note that v is a function of δ_1, δ_2, β_c and q as well as of the decision variables s_1, s_2, n_1, n_2, Δt and m.

4. Optimum ADT Plans

4.1. Pre-Estimation

The objective function v in Equation (6) depends on the unknown parameters δ_1, δ_2 and β_c, which need to be estimated in advance. Pre-estimates of these parameters could be obtained based on engineering judgement and/or preliminary experiments. In this paper, it is suggested to conduct a preliminary experiment at the maximum stress level ($s = 1$), and in addition guess the value of the failure probability (p_0) until a specific time point (τ) at the use condition. Using the MLE method for the data from the preliminary experiment at the maximum stress level, the three unknown parameters, δ_1, δ_2 and β_c cannot be estimated separately, but $\delta_1 + \delta_2$ and β_c can be estimated. For separation, the information on p_0 can be utilized. First, p_0 can be expressed as follows using Equation (2):

$$p_0 = \frac{\Gamma\left[\exp(\delta_1)\tau, \beta_c\right]}{\Gamma\left[\exp(\delta_1)\tau\right]}. \tag{7}$$

Then, δ_1 can be numerically estimated by inserting the pre-estimate of β_c obtained from the preliminary experiment into (7), and δ_2 can be estimated by subtracting the pre-estimate of δ_1 from that of $\delta_1 + \delta_2$.

4.2. Optimization

It is difficult to obtain the analytic expression of the optimal solution since v is highly complicated. There is simulation method such as simulated annealing to search the optimal solution, but it takes relatively long time to obtain the global optimum. In this paper, considering the simplicity in the constraint structure and the integer restriction on the decision variables except the stress levels, the optimal solution can be fast determined through the following algorithm (as part of which a simple grid search method is employed for determining the optimal stress levels where the distance between adjacent grid points is 0.01) and Figure 2 shows the flow chart of the algorithm (see Tseng *et al.* [19])

Algorithm:

Step1. Obtain the upper bound for the possible number of test units

$$n_b = \left\lfloor \frac{C_b - C_{op}}{C_m + C_s} \right\rfloor$$

where $\lfloor x \rfloor$ is the largest integer less than x.

Step 2. Set $n = 1$.

Step 3. Obtain the upper bound of the measurement time interval for fixed n.

$$\Delta t_b = \left\lfloor \frac{C_b - (C_m \cdot n + C_s \cdot n)}{C_{op}} \right\rfloor$$

Step 4. Set $\Delta t = 1$.

Step 5. Set the number of measurements (m) as large as possible since v is the decreasing function of m.

$$m = \left\lfloor \left| \frac{C_b - C_s \cdot n}{C_{op} \cdot \Delta t + C_m \cdot n} \right| \right\rfloor$$

Step 6. Find the combination of $n_1, n_2 \in N$ such that $n_1 + n_2 = n$.

Step 7. Calculate v for all possible combinations of grid values of each stress level which satisfy $0 \le s_1 < s_2 \le 1$.

Step 8. Set $\Delta t = \Delta t + 1$, and repeat step 5 through 7 until $\Delta t = \Delta t_b$.

Step 9. Set $n = n + 1$, and repeat step 3 through 8 until $n = n_b$.

Step 10. The optimal solution is determined as the combination of the decision variables $(\Delta t,\ m,\ n_1, n_2,\ s_1,\ s_2)$ for which v is minimized
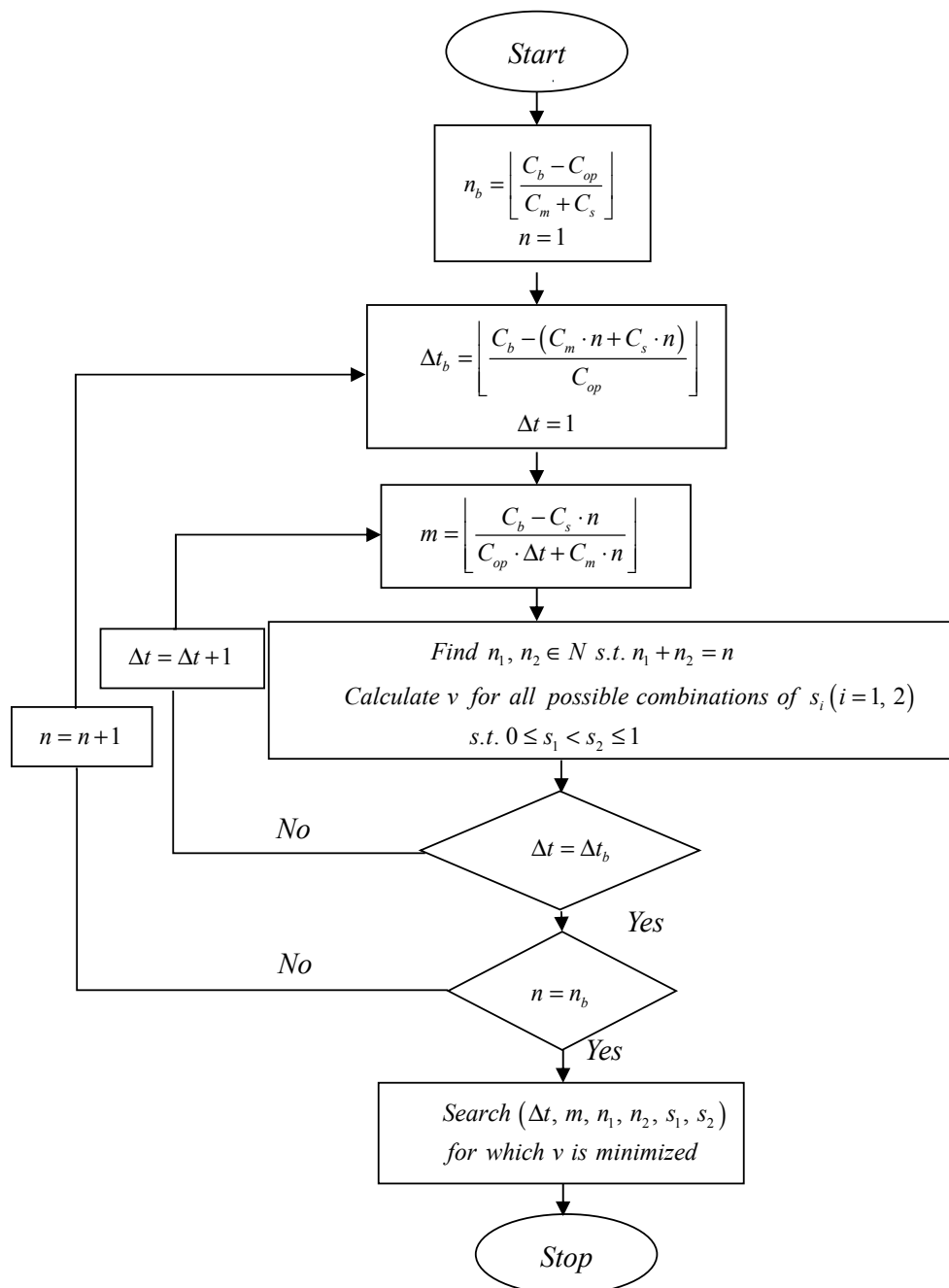

Figure 2. Flow chart of obtaining the optimal solution.

5. Compromise Plans

Since the optimal ADT plan developed in Section 4 involves two stress levels, it does not provide means to check the validity of the assumed acceleration model. For the case of three stress levels, the asymptotic variance of $\hat{t}_{q,0}$ can be obtained in a similar manner as for the case of two stress levels as follows:

$$\text{Avar}\left(\hat{t}_{q,0}\right) = \frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2} \frac{1}{m} \frac{u_1^2 P^c + 2u_1 u_3 Q^c + u_3^2 R^c}{K^c} = \frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2} v^c$$

where:

$$K^c = n_1 n_2 A_1^2 A_2^2 \left(s_2 - s_1\right)^2 \left[B_1 B_2 \left(n_1 A_1 + n_2 A_2 + n_3 A_3\right) - \left(n_1 B_2 + n_2 B_1\right)\right]$$
$$+ n_1 n_3 A_1^2 A_3^2 \left(s_3 - s_1\right)^2 \left[B_1 B_3 \left(n_1 A_1 + n_2 A_2 + n_3 A_3\right) - \left(n_1 B_3 + n_3 B_1\right)\right]$$
$$+ n_2 n_3 A_2^2 A_3^2 \left(s_3 - s_2\right)^2 \left[B_2 B_3 \left(n_1 A_1 + n_2 A_2 + n_3 A_3\right) - \left(n_2 B_3 + n_3 B_2\right)\right]$$
$$- 2 n_1 n_2 n_3 A_1 A_2 A_3 \left[A_1 B_1 \left(s_2 - s_1\right)\left(s_3 - s_1\right) - A_2 B_2 \left(s_2 - s_1\right)\left(s_3 - s_2\right) + A_3 B_3 \left(s_3 - s_1\right)\left(s_3 - s_2\right)\right],$$
$$P^c = \left(n_1 A_1^2 B_1 s_1^2 + n_2 A_2^2 B_2 s_2^2 + n_3 A_3^2 B_3 s_3^2\right)\left(n_1 A_1 + n_2 A_2 + n_3 A_3\right) - \left(n_1 A_1 s_1 + n_2 A_2 s_2 + n_3 A_3 s_3\right)^2,$$
$$Q^c = \frac{y_c}{\beta_c}\left[\begin{array}{l} n_1 n_2 A_1 A_2 \left(s_2 - s_1\right)\left(A_1 B_1 s_1 - A_2 B_2 s_2\right) + n_1 n_3 A_1 A_3 \left(s_3 - s_1\right)\left(A_1 B_1 s_1 - A_3 B_3 s_3\right) \\ + n_2 n_3 A_2 A_3 \left(s_3 - s_2\right)\left(A_2 B_2 s_2 - A_3 B_3 s_3\right) \end{array}\right],$$
$$R^c = \frac{y_c^2}{\beta_c^2}\left[n_1 n_2 A_1^2 A_2^2 B_1 B_2 \left(s_2 - s_1\right)^2 + n_1 n_3 A_1^2 A_3^2 B_1 B_3 \left(s_3 - s_1\right)^2 + n_2 n_3 A_2^2 A_3^2 B_2 B_3 \left(s_3 - s_2\right)^2\right].$$

Then, a compromise plan with three stress levels is developed as follows:

1. The middle stress level s_2 and the high stress level s_3 are set to $\left(s_1 + s_3\right)/2$ and 1, respectively.
2. The proportion (π_2) of test units allocated to s_2 is predetermined ($0 < \pi_2 \leq 0.3$) and the number of test units allocated to s_2 is determined as $n_2 = \lfloor \pi_2 \cdot n \rfloor$.
3. For given π_2, the decision variables n_1, n_3, s_1, Δt and m are determined such that v^c is minimized.

6. Example and Sensitivity Analysis

LEDs are widely used as a light source for optical fiber transmission systems and consumer electronics due to their high brightness, low power consumption and high reliability. An ADT is employed in order to estimate the 0.1-th quantile of the lifetime distribution of the LEDs at the use condition. A failure-related degradation characteristic $y(t)$ of the LED is the percent decrease of its light intensity over time. It is assumed that the degradation characteristic follows a GP since the degradation of LED progresses monotonically as shown as the real degradation data in Liao and Elsayed [14]. The current is considered as the accelerating stress variable in the ADT, and the maximum and use stress levels are specified as 40 mA and 10 mA, respectively. In addition, the power model is assumed between the shape coefficient of the GP and current. The failure time of the LED is defined as the time when its light intensity degrades below 50% from its initial value. In other words, $y_c = 0.5$.

In order to pre-estimate the unknown parameters for optimally designing the ADT, part of the data in Liao and Elsayed [14] is regarded as the data obtained from a preliminary experiment. Liao and Elsayed [14] tested 20 units for 250 h using two stress variables, temperature and current. In this paper, only the current is considered as a stress variable, and the data obtained at 40 mA (and 413 K) for 50, 100 and 150 h from 5 units are regarded as the preliminary experimental data. These data are analyzed using the MLE method, and the estimates of $\delta_1 + \delta_2$ and β_c are respectively obtained as −2.74 and 7.17. In addition, the failure probability until 4 months at the use condition is estimated as 5×10^{-5}. Then, the pre-estimate of δ_1 is numerically obtained as −9.32 from Equation (7), and δ_2 is estimated as $6.58 \left(= \left(\delta_1 + \delta_2\right) - \delta_1 = -2.74 - 9.32\right)$. Suppose that the cost coefficients of operation, measurement and sample are respectively $C_{op} = 2.7$/hour, $C_m = 1.9$/measurement and $C_s = 30$/unit. Under several budgets, the optimal ADT plans for the pre-estimates are shown in Table 1. When $C_b = 2000$, the optimal ADT plan is given by:

$$\Delta t = 7, \ m = 26, \ n_1 = 6, \ n_2 = 13, \ s_1 = 0 \ \text{and} \ s_2 = 1. \tag{8}$$

That is, the optimal plan is to allocate 6 units to the use current (= 10 mA) and 13 units to the high current (= 40 mA) levels, and measure 26 times every 7 h the degradation characteristic for each unit.

Table 1. The ADT plans under several budgets.

C_b	Δt	m	n_1	n_2	s_1	s_2	v	Cost
1000	6	18	3	8	0	1	7.28×10^{-3}	997.8
2000	7	26	6	13	0	1	2.74×10^{-3}	2000.0
3000	9	30	8	18	0	1	1.58×10^{-3}	2991.0
4000	9	38	9	21	0	1	1.08×10^{-3}	3989.4

The pre-estimated values of δ_1, δ_2 and β_c used to design the above optimal plan may be different from the true values. It is therefore desirable to assess the sensitivity of the optimal plan to the uncertainties in δ_1, δ_2 and β_c. In this example, sensitivity analyses are conducted for misspecifications of ±10% in the pre-estimated values of δ_1, δ_2 and β_c, and the results are summarized in Table 2 where v_0 and v^* respectively denote v values for the plan in (8) and for the optimal plan obtained using the true values of δ_1, δ_2 and β_c. The ratios $\left(v_0/v^*\right)$ in Table 2 indicate that the plan in (8) is insensitive to the plausible departures of the true δ_1, δ_2 and β_c valules from their pre-estimated ones.

A compromise plan with three stress levels is needed if we want to check the adequacy of the assumed acceleration model. The compromise plans under various budgets for $\delta_1 = -9.32$, $\delta_2 = 6.58$, $\beta_c = 7.17$ and $\pi_2 = 0.2$ are shown in Table 3. When $C_b = 2000$, the compromise plan with three stress levels is given by:

$$\Delta t = 7, \ m = 26, \ n_1 = 5, \ n_2 = 3, \ n_3 = 11, \ s_1 = 0, \ s_2 = 0.5 \ \text{and} \ s_2 = 1. \tag{9}$$

That is, the compromise plan with three stress levels is that $n_1 = 5$, $n_2 = 3$ and $n_3 = 11$ are allocated to the use (= 10 mA), middle (= 20 mA) and high (= 40 mA) current levels, respectively, and measure 26 times every 7 h the degradation characteristics for each unit.

Sensitivity analyses for the compromise plan are conducted in a similar manner as for the case of the optimal plan and the results are summarized in Table 4 where v_0^c and v^{c*} are respectively v^c values for the plan in (9) and for the compromise plan obtained using the true values of δ_1, δ_2 and β_c. The ratios (v_0^c/v^{c*}) in Table 4 indicate that the plan in (9) is not sensitive to the plausible departures of the true δ_1, δ_2 and β_c values from their pre-estimated ones.

Table 2. Sensitivity Analysis of the optimal ADT plan for the example.

β_c	δ_1	δ_2	v_0	v^*	Ratio
6.453	−8.388	5.922	3.88×10^6	4.10×10^6	1.0544
		6.58	3.16×10^6	3.37×10^6	1.0670
		7.238	2.63×10^6	3.07×10^6	1.1677
	−9.32	5.922	1.39×10^3	1.50×10^3	1.0792
		6.58	1.06×10^3	1.07×10^3	1.0035
		7.238	8.56×10^4	8.65×10^4	1.0098
	−10.252	5.922	2.89×10^2	3.39×10^2	1.1734
		6.58	2.11×10^2	2.21×10^2	1.0505
		7.238	1.61×10^2	1.62×10^2	1.0097
7.17	−8.388	5.922	6.40×10^6	6.66×10^6	1.0412
		6.58	5.20×10^6	5.51×10^6	1.0580
		7.238	4.33×10^6	5.03×10^6	1.1597
	−9.32	5.922	3.58×10^3	3.80×10^3	1.0605
		6.58	2.74×10^3	2.74×10^3	1.0000
		7.238	2.22×10^3	2.25×10^3	1.0132
	−10.252	5.922	1.09×10^2	1.25×10^2	1.1524
		6.58	7.97×10^3	8.32×10^3	1.0436
		7.238	6.09×10^3	6.20×10^3	1.0185
7.887	−8.388	5.922	9.45×10^6	9.75×10^6	1.0308
		6.58	7.69×10^6	8.08×10^6	1.0507
		7.238	6.41×10^6	7.39×10^6	1.1531
	−9.32	5.922	7.32×10^3	7.69×10^3	1.0509
		6.58	5.60×10^3	5.60×10^3	1.0000
		7.238	4.55×10^3	4.63×10^3	1.0158
	−10.252	5.922	2.15×10^3	2.44×10^3	1.1373
		6.58	1.58×10^3	1.64×10^3	1.0384
		7.238	1.21×10^3	1.24×10^3	1.0252

Table 3. The compromise plans under several budgets when $\pi_2 = 0.2$.

C_b	Δt	m	n_1	n_2	n_3	s_1	s_2	s_3	v^c	Cost
1000	4	26	2	1	6	0	0.5	1	8.31×10^{-3}	995.4
2000	7	26	5	3	11	0	0.5	1	3.20×10^{-3}	2000.0
3000	8	42	5	3	11	0	0.5	1	1.88×10^{-3}	2993.4
4000	10	38	7	5	17	0	0.5	1	1.29×10^{-3}	3989.8

Table 4. Sensitivity Analysis of the compromise plan for the example.

β_c	δ_1	δ_2	v^c	v^{c*}	Ratio
6.453	−8.388	5.922	4.52×10^6	4.74×10^6	1.0500
		6.58	3.66×10^6	3.89×10^6	1.0640
		7.238	3.06×10^6	3.52×10^6	1.1508
	−9.32	5.922	1.63×10^3	1.74×10^3	1.0716
		6.58	1.24×10^3	1.25×10^3	1.0048
		7.238	9.93×10^4	1.01×10^3	1.0214
	−10.252	5.922	3.37×10^2	3.93×10^2	1.1655
		6.58	2.48×10^2	2.57×10^2	1.0372
		7.238	1.88×10^2	1.89×10^2	1.0028
7.17	−8.388	5.922	7.46×10^6	7.73×10^6	1.0362
		6.58	6.01×10^6	6.37×10^6	1.0604
		7.238	5.04×10^6	5.79×10^6	1.1490
	−9.32	5.922	4.20×10^3	4.42×10^3	1.0527
		6.58	3.20×10^3	3.20×10^3	1.0000
		7.238	2.56×10^3	2.63×10^3	1.0276
	−10.252	5.922	1.27×10^2	1.45×10^2	1.1415
		6.58	9.37×10^3	9.65×10^3	1.0297
		7.238	7.12×10^3	7.21×10^3	1.0117
7.887	−8.388	5.922	1.10×10^5	1.13×10^5	1.0278
		6.58	8.85×10^6	9.37×10^6	1.0592
		7.238	7.44×10^6	8.53×10^6	1.1474
	−9.32	5.922	8.58×10^3	8.95×10^3	1.0427
		6.58	6.54×10^3	6.54×10^3	1.0000
		7.238	5.25×10^3	5.42×10^3	1.0337
	−10.252	5.922	2.51×10^3	2.82×10^3	1.1242
		6.58	1.86×10^3	1.90×10^3	1.0241
		7.238	1.41×10^3	1.44×10^3	1.0184

7. Conclusions

In this paper, optimal ADT plans are developed based on the assumption that the degradation characteristica follow a GP. Under the constraint that the total experimental cost does not exceed a predetermined budget, the decision variables such as the number of measurements, the measurement times, the test stress levels and the number of units allocated to each stress level are optimally determined by minimizing the asymptotic variance of the MLE of the *q*-th quantile of the lifetime distribution at the use condition. Compromise plans are also developed to provide means to check the adequacy of the assumed acceleration model. Lastly, sensitivity analyses for assessing the effects of the uncertainties in the pre-estimates of unknown parameters on the optimal and compromise plans are illustrated with an example.

Unlike previous works on GDP-based ADT planning, the present SP-based approach is able to take into account the correlation among degradation measurements over time. Furthermore, compared to the WP- or GBM-based approach, the proposed GP-based approach can deal with more frequently encountered situations in which the degradation characteristic is always nonnegative and strictly increasing over time.

The constant-stress loading method is assumed in the present study. A fruitful area of future research would be to extend the present study to the cases of other stress loading methods (e.g., progressive-stress

loading), compare their relative performances in terms of the statistical efficiency, sample size, total experimental cost, *etc*, and implement the various algorithm to search the optimal solution.

Appendix

A. Derivation of the Fisher Information Matrix

The first partial derivatives of $\ln L$ with respect to each unknown parameter are given by:

$$\frac{\partial \ln L}{\partial \delta_1} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left[A_i \ln \Delta y_{ijk} - A_i \ln \beta - A_i \Psi_0(A_i) \right],$$

$$\frac{\partial \ln L}{\partial \delta_2} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left[A_i s_i \ln \Delta y_{ijk} - A_i s_i \ln \beta - A_i \Psi_0(A_i) s_i \right],$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left(-\frac{A_i}{\beta} + \frac{\Delta y_{ijk}}{\beta^2} \right)$$

where $\Psi_0(A_i)$ is the digamma function defined as $\Psi_0(A_i) = \partial \ln \Gamma(A_i)/\partial A_i = \Gamma'(A_i)/\Gamma(A_i)$. The second partial derivatives of $\ln L$ with respect to each parameter are obtained as follows:

$$\frac{\partial^2 \ln L}{\partial \delta_1^2} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left[A_i \ln \Delta y_{ijk} - A_i \ln \beta - A_i \Psi_0(A_i) - A_i^2 B_i \right]$$

$$\frac{\partial^2 \ln L}{\partial \delta_1 \partial \delta_2} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left[A_i s_i \ln \Delta y_{ijk} - A_i s_i \ln \beta - A_i \Psi_0(A_i) s_i - A_i^2 B_i s_i \right],$$

$$\frac{\partial^2 \ln L}{\partial \delta_1 \partial \beta} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left(-\frac{A_i}{\beta} \right)$$

$$\frac{\partial^2 \ln L}{\partial \delta_2^2} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left[A_i s_i^2 \ln \Delta y_{ijk} - A_i s_i^2 \ln \beta - A_i \Psi_0(A_i) s_i^2 - A_i^2 B_i s_i^2 \right]$$

$$\frac{\partial^2 \ln L}{\partial \delta_2 \partial \beta} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left(-\frac{A_i s_i}{\beta} \right)$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \sum_{k=1}^{m} \left(\frac{A_i}{\beta^2} - \frac{2\Delta y_{ijk}}{\beta^3} \right)$$

To evaluate the Fisher information matrix, $E(\ln \Delta y_{ijk})$ is first obtained as follows:

$$E(\ln \Delta y_{ijk}) = \int_0^{\infty} \ln \Delta y_{ijk} \frac{1}{\Gamma(A_i)} \frac{1}{\beta^{A_i}} \Delta y_{ijk}^{A_i - 1} e^{-\Delta y_{ijk}/\beta} d\Delta y_{ijk}$$

$$= \frac{1}{\Gamma(A_i)} \int_0^{\infty} \ln(\beta x) \cdot x^{A_i - 1} e^{-x} dx$$

$$= \frac{1}{\Gamma(A_i)} \left(\int_0^{\infty} \ln \beta \cdot x^{A_i - 1} e^{-x} dx + \int_0^{\infty} \ln x \cdot x^{A_i - 1} e^{-x} dx \right)$$

$$= \frac{1}{\Gamma(A_i)} \left[\ln \beta \cdot \Gamma(A_i) + \Gamma'(A_i) \right]$$

$$= \ln \beta + \Psi_0(A_i)$$

In addition, since $E\left(\Delta y_{ijk}\right) = A_i\beta$, the expectations of the negative second partial derivatives of $\ln L$ with respect to unknown parameters are given by:

$$E\left(-\frac{\partial^2 \ln L}{\partial \delta_1^2}\right) = m\sum_{i=1}^r n_i A_i^2 B_i, \quad E\left(-\frac{\partial^2 \ln L}{\partial \delta_1 \partial \delta_2}\right) = m\sum_{i=1}^r n_i A_i^2 B_i s_i,$$

$$E\left(-\frac{\partial^2 \ln L}{\partial \delta_1 \partial \beta}\right) = m\sum_{i=1}^r \frac{n_i A_i}{\beta}, \quad E\left(-\frac{\partial^2 \ln L}{\partial \delta_2^2}\right) = m\sum_{i=1}^r n_i A_i^2 B_i s_i^2,$$

$$E\left(-\frac{\partial^2 \ln L}{\partial \delta_2 \partial \beta}\right) = m\sum_{i=1}^r n_i \frac{A_i s_i}{\beta}, \quad E\left(-\frac{\partial^2 \ln L}{\partial \beta^2}\right) = m\sum_{i=1}^r \frac{n_i A_i}{\beta^2}$$

Then, the Fisher information matrix is given by (4).

B. Derivation of the Asymptotic Variance of $\hat{t}_{q,0}$ in the Case of Two Stress Levels

For the case of two stress levels, the Fisher information matrix is given by:

$$\mathbf{F} = m\begin{bmatrix} n_1 A_1^2 B_1 + n_2 A_2^2 B_2 & n_1 A_1^2 B_1 s_1 + n_2 A_2^2 B_2 s_2 & n_1 \frac{A_1}{\beta} + n_2 \frac{A_2}{\beta} \\ & n_1 A_1^2 B_1 s_1^2 + n_2 A_2^2 B_2 s_2^2 & n_1 \frac{A_1 s_1}{\beta} + n_2 \frac{A_2 s_2}{\beta} \\ symmetric & & n_1 \frac{A_1}{\beta^2} + n_2 \frac{A_2}{\beta^2} \end{bmatrix}$$

$$= m\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ & f_{22} & f_{23} \\ symmetric & & f_{33} \end{bmatrix}$$

$$= m\mathbf{F}'$$

Let $\mathbf{F}^{-1}$ be the inverse of the Fisher information matrix defined as:

$$\mathbf{F}^{-1} = \frac{1}{m}\begin{bmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \tilde{f}_{13} \\ & \tilde{f}_{22} & \tilde{f}_{23} \\ symmetric & & \tilde{f}_{33} \end{bmatrix}$$

Then, the asymptotic variance of $\hat{t}_{q,0}$ is obtained as follows.

$$\text{Avar}\left(\hat{t}_{q,0}\right) = \frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2} \mathbf{u}'\mathbf{F}^{-1}\mathbf{u} = \frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2} \frac{1}{m}\left(u_1^2\tilde{f}_{11} + 2u_1 u_3\tilde{f}_{13} + u_3^2\tilde{f}_{33}\right)$$

where

$$\tilde{f}_{11} = \frac{f_{22}f_{33} - f_{23}^2}{|\mathbf{F}'|} = \frac{1}{|\mathbf{F}'|}\frac{\beta_c^2}{y_c^2}\left[\left(n_1 A_1^2 B_1 s_1^2 + n_2 A_2^2 B_2 s_2^2\right)\left(n_1 A_1 + n_2 A_2\right) - \left(n_1 A_1 s_1 + n_2 A_2 s_2\right)^2\right],$$

$$\tilde{f}_{13} = \frac{f_{12}f_{23} - f_{13}f_{22}}{|\mathbf{F}'|} = \frac{1}{|\mathbf{F}'|}\frac{\beta_c}{y_c}n_1 n_2 A_1 A_2\left(s_2 - s_1\right)\left(A_1 B_1 s_1 - A_2 B_2 s_2\right),$$

$$\tilde{f}_{33} = \frac{f_{11}f_{22} - f_{12}^2}{|\mathbf{F}'|} = \frac{1}{|\mathbf{F}'|}n_1 n_2 A_1^2 A_2^2 B_1 B_2\left(s_2 - s_1\right)^2,$$

$$\left|\mathbf{F}'\right| = \left(f_{11}f_{22}f_{33} + 2f_{12}f_{13}f_{23} - f_{13}^2 f_{22} - f_{12}^2 f_{33} - f_{11}f_{23}^2\right)$$

$$= \frac{\beta_c^2}{y_c^2}\left\{n_1 n_2 A_1^2 A_2^2 \left(s_2 - s_1\right)^2 \left[n_1 B_2\left(A_1 B_1 - 1\right) + n_2 B_1\left(A_2 B_2 - 1\right)\right]\right\}.$$

Define:

$$K = n_1 n_2 A_1^2 A_2^2 \left(s_2 - s_1\right)^2 \left[n_1 B_2\left(A_1 B_1 - 1\right) + n_2 B_1\left(A_2 B_2 - 1\right)\right],$$

$$P = \left(n_1 A_1^2 B_1 s_1^2 + n_2 A_2^2 B_2 s_2^2\right)\left(n_1 A_1 + n_2 A_2\right) - \left(n_1 A_1 s_1 + n_2 A_2 s_2\right)^2,$$

$$Q = \frac{y_c}{\beta_c}\left[n_1 n_2 A_1 A_2 \left(s_2 - s_1\right)\left(A_1 B_1 s_1 - A_2 B_2 s_2\right)\right],$$

$$R = \frac{y_c^2}{\beta_c^2}\left[n_1 n_2 A_1^2 A_2^2 B_1 B_2 \left(s_2 - s_1\right)^2\right].$$

Then, $\mathrm{Avar}\left(\hat{t}_{q,0}\right)$ is reduced to:

$$\mathrm{Avar}\left(\hat{t}_{q,0}\right) = \frac{1}{\left\{g_0\left[G_0^{-1}(q)\right]\right\}^2} \frac{1}{m} \frac{u_1^2 P + 2u_1 u_3 Q + u_3^2 R}{K}.$$

Conflicts of Interest

The author declares no conflict of interest.

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