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Article

The Optimal Fix-Free Code for Anti-Uniform Sources

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Abstract: An n symbol source which has a Huffman code with codelength vector $L_n = (1, 2, 3, \cdots, n-2, n-1, n-1)$ is called an anti-uniform source. In this paper, it is shown that for this class of sources, the optimal fix-free code and symmetric fix-free code is $C_n^* = (0, 11, 101, 1001, \cdots, 1\overbrace{0\cdots0}^{n-2}1)$.

Keywords: prefix code; fix-free code; anti-uniform sources

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1. Introduction

One of the basic problems in the context of source coding is to assign a code $C_n = (c_1, c_2, \cdots, c_n)$ with codelength vector $L_n = (\ell_1, \ell_2, \cdots, \ell_n)$ to a memoryless source with probability vector $P_n = (p_1, p_2, \cdots, p_n)$. Decoding requirements often constrain us to choose a code C_n from a specific class of codes, such as prefix-free codes, fix-free codes or symmetric fix-free codes. With a prefix-free code, no codeword is the prefix of another codeword. This property ensures that decoding in the forward direction can be done without any delay (instantaneously). Alternatively, with a fix-free code no codeword is the prefix or suffix of any other codeword [1,2]. Therefore, decoding of a fix-free code in both the forward and backward directions can be done without any delay. The ability to decode a fix-free code in both directions makes them more robust to transmission errors and faster decoding can be achieved compared to prefix-free codes. As a result, fix-free codes are used in video standards such as H.263+ and MPEG-4 [3]. A symmetric fix-free code is a fix-free code whose codewords are symmetric. In general, decoder implementation for a fix-free code requires more memory compared to that for a prefix-free code. Although the decoder for a symmetric fix-free code is the same as for a fix-free code [4], symmetric codes have greater redundancy in general.

Let $S(L_n) = \sum_{i=1}^{n} 2^{-\ell_i}$ denote the Kraft sum of the codelength vector L_n. A well-known necessary and sufficient condition for the existence of a prefix-free code with codelength vector L_n is the Kraft inequality, *i.e.*, $S(L_n) \leq 1$ [5]. However, this inequality is *only* a necessary condition on the existence of a fix-free code. Some sufficient conditions on the existence of a fix-free code were introduced in [6–11].

The optimal code for a specific class of codes is defined as the code with the minimum average codelength, *i.e.*, $\sum_{i=1}^{n} p_i \ell_i$ among all codes in that class. The optimal prefix-free code can easily be obtained using the Huffman algorithm [12]. Recently, two methods for finding the optimal fix-free code have been developed. One is based on the A^* algorithm [13], while the other is based on the concept of dominant sequences [14]. Compared to the Huffman algorithm, these methods are very complex.

A source with n symbols having Huffman code with codelength vector $L_n = (1, 2, 3, \cdots, n-2, n-1, n-1)$ is called an anti-uniform source [15,16]. Such sources have been shown to correspond to particular probability distributions. For example, it was shown in [17] and [18], respectively, that the normalized tail of the Poisson distribution and the geometric distribution with success probability greater than some critical value are anti-uniform sources. It was demonstrated in [15,16] that a source with probability vector $P_n = (p_1, p_2, \cdots, p_n)$ where $p_1 \geq p_2 \geq \cdots \geq p_n$, is anti-uniform if and only if

$$\sum_{j=i+2}^{n} p_j \leq p_i \qquad \text{for} \quad 1 \leq i \leq n-3. \tag{1}$$

As mentioned above, finding an optimal fix-free or symmetric fix-free code is complex. Thus, in this paper optimal fix-free and symmetric fix-free codes are determined for anti-uniform sources. In particular, it is proven that

$$C_n^* = (0, 11, 101, 1001, \cdots, 1\overbrace{0\cdots0}^{n-2}1),$$

is an optimal fix-free code for this class of sources. Since C_n^* is symmetric, this code is also an optimal symmetric fix-free code. Although for an anti-uniform source, the difference between the average codelength of the optimal prefix-free code and C_n^* is small (it is exactly equal to p_n), it is not straightforward to prove that C_n^* is the optimal fix-free code. In [19], the optimality of C_n^* among symmetric fix-free codes for a family of exponential probability distributions, which is an anti-uniform source, was discussed.

In [20], another class of fix-free codes called weakly symmetric fix-free codes was examined. A fix-free code is weakly-symmetric if the reverse of each codeword is also a codeword. In fact, every symmetric fix-free code is a weakly symmetric fix-free code, and every weakly symmetric fix-free code is a fix-free code. Thus, since the optimal code among fix-free codes and symmetric fix-free codes for anti-uniform sources is C_n^*, this code is also optimal for weakly symmetric fix-free codes.

The remainder of this paper is organized as follows. In Section 2, a sketch of the proofs of the main theorems, *i.e.*, Theorems 1 and 3, is provided, followed by the main results of the paper. Then detailed proofs of these results are given in Section 3.

2. A Sketch of the Proofs

Since a fix-free code is also a prefix-free code, the Kraft sum of an optimal fix-free code is not greater than 1. Therefore, the Kraft sum of this code is either equal to 1 or smaller than 1.

Proposition 1. *If*

$$\left(\ell_1^*, \cdots, \ell_{n-1}^*, \ell_n^*\right) = \arg \min_{L_n : S(L_n) \leq 1} \sum_{i=1}^{n} p_i \ell_i,$$

then we have

$$\left(\ell_1^*, \cdots, \ell_{n-1}^*, \ell_n^* + 1\right) = \arg \min_{L_n : S(L_n) < 1} \sum_{i=1}^{n} p_i \ell_i \, .$$

It can be inferred from Proposition 1 that if the Kraft sum of an optimal fix-free code is smaller than 1, then the average codelength of this code is not better than the codelength vector $\left(\ell_1^*, \cdots, \ell_{n-1}^*, \ell_n^* + 1\right)$. The optimal prefix-free code for an anti-uniform source has codelength vector $(1, 2, \cdots, n-1, n-1)$. Therefore, the optimal codelength vector with Kraft sum smaller than 1 for an anti-uniform source is the codelength vector $(1, 2, \cdots, n-1, n)$. Further, the codelength vector of C_n^* is $(1, 2, \cdots, n-1, n)$. Thus, if the Kraft sum of the optimal fix-free code for an anti-uniform source is smaller than 1, then the code C_n^* is optimal.

Proposition 2. *There is no symmetric fix-free code with Kraft sum 1 for $n > 2$.*

According to Proposition 2, the Kraft sum for an optimal symmetric fix-free code is smaller than 1. Thus, Propositions 1 and 2 prove the following theorem.

Theorem 1. *The optimal symmetric fix-free code for an anti-uniform source P_n is the code C_n^*.*

There exist fix-free codes with Kraft sum 1, for example $(00, 01, 10, 11)$ and $(01, 000, 100, 110, 111, 0010, 0011, 1010, 1011)$ [21]. Therefore, proving that the code C_n^* is the optimal fix-free code for anti-uniform sources requires that the average codelength for this code be better than every possible codelength for a fix-free code. To achieve this, we use the following theorem which was proven in [21].

Theorem 2. [21] *Let $L_n = (\ell_1, \cdots, \ell_n)$, $M_i(L_n) = |\{j | \ell_j = i\}|$ for $1 \leq i \leq \max_{1 \leq j \leq n} \ell_j$ and $H_i = 2i - \frac{1}{i} \sum_{j=1}^{i} 2^{(i,j)}$ where (i, j) denotes the greatest common divisor of i and j. If $S(L_n) = 1$, $M_i(L_n) > H_i$ for some i and $|\{\ell_1, \cdots, \ell_n\}| > 1$, then no fix-free code exists with codelength vector L_n.*

According to the definition of H_i, we have that $H_1 = 0$ and $H_2 = 1$. Therefore from Theorem 2, for L_n with Kraft sum 1 and

$$M_1(L_n) > 0 \text{ and } L_n \neq (1, 1),$$

or

$$M_2(L_n) > 1 \text{ and } L_n \neq (2,2,2,2),$$

there is no fix-free code.

Definition 1. *For a given n, let*

$$\mathbb{L}_n = \{L_n | S(L_n) = 1, M_1(L_n) = 0 \textit{ and } M_2(L_n) \leq 1\}.$$

From Theorem 2, if the Kraft sum of the optimal fix-free code is equal to 1, then the average codelength for this code is not smaller than that of the optimal codelength vector among those in $\mathbb{L}_n$ for $n > 4$. It can easily be verified that $|\mathbb{L}_n| = 0$ for $n < 7$. For anti-uniform sources, the following proposition characterizes the optimal codelength vector in $\mathbb{L}_n$ for $n \geq 7$.

Proposition 3. *Let $P_n = (p_1, \cdots, p_{n-1}, p_n)$ be the probability vector of an anti-uniform source with $p_1 \geq \cdots \geq p_{n-1} \geq p_n$. Then we have*

$$\arg \min_{L_n \in \mathbb{L}_n} \sum_{i=1}^{n} p_i \ell_i = \begin{cases} (2,3,3,3,3,3,3), & \textit{if } n = 7 \\ \\ (2,3,3,3,3,3,4,5,6,\cdots,n-6,n-5,n-4,n-4), & \textit{if } n > 7 \end{cases}$$

The last step requires that the average codelength of C_n^* is better than that of the given codelength vector in Proposition 3. This is given in the proof of the following theorem.

Theorem 3. *The optimal fix-free code for an anti-uniform source P_n is C_n^* for $n > 4$.*

Note that Theorem 3 is not true for $n = 4$. For example, for $P_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$ which is the probability vector of an anti-uniform source, the average codelength of the fix-free code $(00, 01, 10, 11)$ is better than that of C_4^*.

3. Proofs of the Results in Section 2

Proof of Proposition 1: Let $L_n = (\ell_1, \cdots, \ell_{n-1}, \ell_n)$ with $\ell_1 \leq \cdots \leq \ell_{n-1} \leq \ell_n$ and $1 > S(L_n)$. Thus, we have $2^{\ell_n} > 2^{\ell_n} S(L_n) = \sum_{i=1}^{n} 2^{\ell_n - \ell_i}$, and consequently

$$2^{\ell_n} \geq 1 + \sum_{i=1}^{n} 2^{\ell_n - \ell_i}.$$

Therefore, we can write

$$\begin{aligned} 1 &\geq \sum_{i=1}^{n} 2^{-\ell_i} + 2^{-\ell_n} \\ &= \sum_{i=1}^{n-1} 2^{-\ell_i} + 2^{-(\ell_n - 1)}. \end{aligned} \quad (2)$$

Let $L'_n = \left(\ell'_1, \cdots, \ell'_{n-1}, \ell'_n\right)$ such that $\ell'_i = \ell_i$ for $1 \leq i \leq n-1$ and $\ell'_n = \ell_n - 1$. From (2), we have that $S(L'_n) \leq 1$. According to the definition of $\left(\ell_1^*, \cdots, \ell_{n-1}^*, \ell_n^*\right)$, we can write

$$\sum_{i=1}^{n} p_i \ell_i^* \leq \sum_{i=1}^{n} p_i \ell'_i,$$

and consequently

$$\begin{aligned}\sum_{i=1}^{n-1} p_i \ell_i^* + (\ell_n^* + 1)\, p_n &= \sum_{i=1}^{n} p_i \ell_i^* + p_n \\ &\leq \sum_{i=1}^{n} p_i \ell_i' + p_n \\ &= \sum_{i=1}^{n} p_i \ell_i.\end{aligned}$$

This shows that the average codelength of $\left(\ell_1^*, \cdots, \ell_{n-1}^*, \ell_n^* + 1\right)$ is better than any other codelength vector, say L_n, with Kraft sum smaller than 1.

Proof of Proposition 2: Suppose that the Kraft sum of L_n, which is the codelength vector of the code C_n, is equal to 1. Let codeword $c = x_1 x_2 \cdots x_{\ell-1} 0$ (resp. $c = x_1 x_2 \cdots x_{\ell-1} 1$) with length ℓ ($\ell > 1$), be the longest codeword of C_n. Since the Kraft sum of L_n is equal to 1, the codeword $c' = x_1 x_2 \cdots x_{\ell-1} 1$ (resp. $c' = x_1 x_2 \cdots x_{\ell-1} 0$) belongs to C_n. However, both c and c' cannot be symmetric because $x_1 = 0$ and $x_1 = 1$ cannot both be true. Thus, C_n is not a symmetric fix-free code.

The following lemma will be used in the proof of Proposition 3.

Lemma 1. *For $n \geq 7$, let $P_n = (p_1, \cdots, p_{n-1}, p_n)$ with $p_1 \geq \cdots \geq p_{n-1} \geq p_n$ and*

$$P'_{n-1} = (p'_1, \cdots, p'_{n-2}, p'_{n-1}) = (p_1, \cdots, p_{n-2}, p_{n-1} + p_n)\,. \tag{3}$$

Further, suppose that

$$L_n^* = \left(\ell_1^*, \cdots, \ell_{n-1}^*, \ell_n^*\right) = \arg \min_{L_n \in \mathbb{L}_n} \sum_{i=1}^{n} p_i \ell_i,$$

and for $n > 7$

$$L'_{n-1} = \left(\ell'_1, \cdots, \ell'_{n-2}, \ell'_{n-1}\right) = \arg \min_{L_{n-1} \in \mathbb{L}_{n-1}} \sum_{i=1}^{n-1} p'_i \ell_i. \tag{4}$$

Then we have

$$\left(\ell_1^*, \cdots, \ell_{n-2}^*, \ell_{n-1}^*, \ell_n^*\right) = \begin{cases} (2,3,3,3,3,3,3), & \text{if } n = 7 \\ \\ \left(\ell'_1, \cdots, \ell'_{n-2}, \ell'_{n-1} + 1, \ell'_{n-1} + 1\right), & \text{if } n > 7 \end{cases} \tag{5}$$

Proof. It can easily be verified that $\mathbb{L}_7$ consists of all permutations of $2,3,3,3,3,3,3$. Thus, $p_1 \geq \cdots \geq p_6 \geq p_7$ completes the proof for $n = 7$. To prove the lemma for $n > 7$, we consider two cases: (1) $\ell_n^* = 3$, and (2) $\ell_n^* > 3$. First, note that $\ell_n^* = \max_{1 \leq i \leq n} \ell_i$, because $p_1 \geq \cdots \geq p_{n-1} \geq p_n$.

(1) $\ell_n^* = 3$: It can easily be shown that $(3,3,3,3,3,3,3,3)$ is the only codelength vector in $\cup_{n>7} \mathbb{L}_n$ with maximum codelength which is a not greater than 3. Therefore, to prove the lemma in this case it is enough to show that $(\ell'_1, \cdots, \ell'_7) = (3,3,3,3,3,3,2)$. According to the first argument in this proof, the codelength vector $(\ell'_1, \cdots, \ell'_7)$ is a permutation of $2,3,3,3,3,3,3$. Thus, proving that $p_7 + p_8$ is maximum over all probabilities in P'_7, *i.e.*, $p_7 + p_8 \geq p_1$, completes the proof for this case. If

$p_7 + p_8 < p_1$, then the average codelength of $(\ell''_1, \cdots, \ell''_8) = (2,3,3,3,3,3,4,4)$, is better than that of $(\ell^*_1, \cdots, \ell^*_8) = (3,3,3,3,3,3,3,3)$, because

$$\begin{aligned} \sum_{i=1}^{8} p_i \ell^*_i &= 3p_1 + 3(1-p_1) \\ &\overset{(a)}{>} 2p_1 + 3(1-p_1) + p_7 + p_8 \\ &= \sum_{i=1}^{8} p_i \ell''_i, \end{aligned}$$

where (a) follows from $p_7 + p_8 < p_1$. Thus, the codelength vector $(3,3,3,3,3,3,3)$ is not optimal, which is a contradiction. Therefore, $p_7 + p_8 \geq p_1$ and the proof for this case is complete.

(2) $\ell^*_n > 3$: The proof for this case is similar to the proof of the Huffman algorithm. Let $L_n = (\ell_1, \cdots, \ell_n) \in \mathbb{L}_n$, *i.e.*, $S(L_n) = 1$, $M_1(L_n) = 0$ and $M_2(L_n) \leq 1$, with $\ell_1 \leq \cdots \leq \ell_n$ and $3 < \ell_n$, and let $L''_{n-1} = (\ell''_1, \cdots, \ell''_{n-1}) = (\ell_1, \cdots, \ell_{n-2}, \ell_{n-1} - 1)$. It can easily be shown that $S(L_n) = 1$ implies that $\ell_n = \ell_{n-1}$. Since $\ell_n = \ell_{n-1}$, we have $S(L''_{n-1}) = S(L_n)$, and consequently $S(L''_{n-1}) = 1$. Further, since $\ell_n = \max_{1 \leq i \leq n} \ell_i$ and $\ell_n > 3$, $M_1(L_n) = 0$ and $M_2(L_n) \leq 1$ imply that $M_1(L''_{n-1}) = 0$ and $M_2(L''_{n-1}) \leq 1$, which gives $L''_{n-1} \in \mathbb{L}_{n-1}$, and we can write

$$\begin{aligned} \sum_{i=1}^{n} p_i \ell^*_i &= \min_{L_n \in \mathbb{L}_n} \sum_{i=1}^{n} p_i \ell_i \\ &\overset{(a)}{=} \left[\min_{L_n \in \mathbb{L}_n} \sum_{i=1}^{n-2} p_i \ell_i + (p_{n-1} + p_n)(\ell_{n-1} - 1) \right] + p_{n-1} + p_n \\ &\overset{(b)}{=} \left[\min_{L''_{n-1} \in \mathbb{L}_{n-1}} \sum_{i=1}^{n-1} p'_i \ell''_i \right] + p_{n-1} + p_n \\ &\overset{(c)}{=} \sum_{i=1}^{n-1} p'_i \ell'_i + p_{n-1} + p_n, \end{aligned}$$

where (a) follows from $\ell^*_n = \ell^*_{n-1}$, (b) follows from the definition of L''_{n-1} and (3), and (c) follows from (4). Therefore, for $n > 7$, since the average codelength of the given codelength vector in (5) is equal to $\sum_{i=1}^{n-1} p'_i \ell'_i + p_{n-1} + p_n$, this codelength vector is optimal and the proof is complete.

□

Proof of Proposition 3. The proposition is proved by induction on n. According to Lemma 1, the base of induction, *i.e.*, $n = 7$, is true. Assume that the proposition is true for all anti-uniform sources with $n-1$ symbols. Let $P_n = (p_1, \cdots, p_n)$ be the probability vector of an anti-uniform source. Also, suppose that $P'_{n-1} = (p'_1, \cdots, p'_{n-2}, p'_{n-1}) = (p_1, \cdots, p_{n-2}, p_{n-1} + p_n)$. From (1), it is obvious that P'_{n-1} is the probability vector of an anti-uniform source and $p'_1 \geq p'_2 \geq \cdots \geq p'_{n-3} \geq p'_{n-2}, p'_{n-1}$. Since we have $\ell'_{n-1} = \ell'_n$, where $(\ell'_1, \cdots, \ell'_{n-2}, \ell'_{n-1}) = \arg\min_{L_{n-1} \in \mathbb{L}_{n-1}} \sum_{i=1}^{n-1} p'_i \ell_i$, from the induction assumption we can write

$$(\ell'_1, \cdots, \ell'_{n-2}, \ell'_{n-1}) = \begin{cases} (2,3,3,3,3,3,3), & \text{if } n-1 = 7 \\ (2,3,3,3,3,3,4,5,\cdots,n-5,n-5), & \text{if } n-1 > 7 \end{cases}.$$

Therefore, we have $\ell'_{n-1} = n - 5$, and Lemma 1 completes the proof. $\square$

Proof of Theorem 3. We have that $|\mathbb{L}_5| = |\mathbb{L}_6| = 0$. Therefore, for $n = 5, 6$ the proof is complete. The proof for $n = 7$ is the same as the proof for $n > 7$, and so is omitted. Now suppose that $n > 7$. In the following, it is proven that the average codelength vector of $(\ell'_1, \ldots, \ell'_n) = (2, 3, 3, 3, 3, 3, 4, 5, 6, \cdots, n-4, n-4)$ is greater than or equal to that of C^*_n, *i.e.*, $\sum_{i=1}^{n} ip_i$, which completes the proof.

$$\begin{aligned}
\sum_{i=1}^{n} p_i \ell'_i &= 2p_1 + 3\left(p_2 + p_3 + p_4 + p_5 + p_6\right) + \sum_{i=7}^{n-1} (i-3)p_i + (n-4)p_n \\
&= \sum_{i=1}^{n} ip_i + p_1 + p_2 - p_4 - 2p_5 - 3p_6 - 3\sum_{i=7}^{n} p_i - p_n \\
&= \sum_{i=1}^{n} ip_i + (p_1 - p_3 - p_4 - p_n) + \left(p_2 - \sum_{i=4}^{n} p_i\right) + \left(p_3 - \sum_{i=5}^{n} p_i\right) + \left(p_4 - \sum_{i=6}^{n} p_i\right) \\
&\overset{\text{(a)}}{\geq} \sum_{i=1}^{n} ip_i,
\end{aligned}$$

where (a) follows from the fact that P_n is the probability vector of an anti-uniform source, *i.e.*, $p_i \geq \sum_{j=i+2}^{n} p_j$ for $n - 3 \geq i \geq 1$. $\square$

Author Contributions

All authors were involved in developing the problem, interpreting the results, and writing the paper. Ali Zaghian and Adel Aghajan conducted the analysis. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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