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Article

Entropy of Quantum Measurement

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Abstract: A notion of entropy of a normal state on a finite von Neumann algebra in Segal's sense is considered, and its superadditivity is proven together with a necessary and sufficient condition for its additivity. Bounds on the entropy of the state after measurement are obtained, and it is shown that a weakly repeatable measurement gives minimal entropy and that a minimal state entropy measurement satisfying some natural additional conditions is repeatable.

Keywords: entropy; von Neumann algebra; instrument

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1. Introduction

The notion of the entropy of a state of a physical system was introduced by John von Neumann (see [1]) in the setup that is now classical for quantum mechanics. In this approach, the observables of a physical system are identified with self-adjoint operators on a separable Hilbert space, and the states of the system, with the positive operators of trace one on this space. This setting has been generalized in more modern theories, in particular in the so-called algebraic approach to quantum physics in which the bounded observables of a physical system form the self-adjoint part of a C*-, or von Neumann, algebra (see [2–5]). The origin of this approach goes back to I. Segal [6], who first indicated the basic features of such an algebraic formalism. However, despite its obvious importance, the unique notion of the entropy of a state on an arbitrary C*-, or von Neumann, algebra has not been unambiguously established. On the other hand, a lot of work has been done in this field, and an interested reader may consult, e.g., [7–10]. In our considerations, we adopt a definition of entropy due to I. Segal, which is similar to the classical Boltzmann–Gibbs entropy and applies to normal states on a finite von Neumann algebra.

In the paper, we show the superadditivity of the entropy considered, together with a necessary and sufficient condition of its additivity and give bounds on the entropy of the state after measurement.

Moreover, we show that a weakly repeatable measurement gives minimal entropy and that a minimal state entropy measurement satisfying some natural additional conditions is repeatable.

2. Preliminaries and Notation

Let $\mathcal{M}$ be a von Neumann algebra, *i.e.*, an algebra of bounded operators on a Hilbert space $\mathcal{H}$ with identity $\mathbb{1}$ being the identity operator, closed in the weak operator topology given by the family of seminorms:

$$\mathcal{M} \ni x \mapsto |\langle \xi | x\eta \rangle|, \qquad \xi, \eta \in \mathcal{H},$$

and such that $x^* \in \mathcal{M}$ whenever $x \in \mathcal{M}$. For a projection $p \in \mathcal{M}$, we set $p^\perp = \mathbb{1} - p$. By $\mathcal{M}_*$ is denoted the predual of $\mathcal{M}$, which is a Banach space of bounded linear functionals on $\mathcal{M}$, such that $(\mathcal{M}_*)^* = \mathcal{M}$. The elements of $\mathcal{M}_*$ are called *normal*. The positive elements φ of $\mathcal{M}_*$ having norm one, *i.e.*, such that $\varphi(\mathbb{1}) = 1$, are called *normal states*. $\mathcal{M}_*^+$ will stand for the positive elements of $\mathcal{M}_*$; its elements, which are not states, bear sometimes the name of *non-normalized states*. For $\varphi \in \mathcal{M}_*^+$, we define its *support*, denoted by $\mathbf{s}(\varphi)$, as the smallest projection p in $\mathcal{M}$, such that:

$$\varphi(p) = \varphi(\mathbb{1}).$$

The following formula holds true:

$$\mathbf{s}(\varphi) = \big(\sup\{q \in \mathcal{M} : q \text{ — projection},\ \varphi(q) = 0\}\big)^\perp.$$

A linear map $\Phi\colon \mathcal{M} \to \mathcal{M}$ is said to be *normal* if it is continuous in the $\sigma(\mathcal{M}, \mathcal{M}_*)$ topology. For a linear normal positive map Φ, we define its *support* $\mathbf{s}(\Phi)$ in the same way as for normal positive functionals, *i.e.*, as the smallest projection p in $\mathcal{M}$, such that:

$$\Phi(p) = \Phi(\mathbb{1}).$$

For the support, the following relation holds true:

$$\Phi(\mathbf{s}(\Phi)x) = \Phi(x\,\mathbf{s}(\Phi)) = \Phi(x), \qquad x \in \mathcal{M};$$

moreover, if:

$$\Phi(\mathbf{s}(\Phi)x\mathbf{s}(\Phi)) = 0$$

and $\mathbf{s}(\Phi)x\mathbf{s}(\Phi) \geq 0$, then $\mathbf{s}(\Phi)x\mathbf{s}(\Phi) = 0$. The same relations hold true for the normal positive functionals.

Lemma 1. *Let $\Phi\colon \mathcal{M} \to \mathcal{M}$ be a linear normal positive map, and let $0 \leq a \leq \mathbb{1}$, $a \in \mathcal{M}$, be such that:*

$$\Phi(a) = \Phi(\mathbb{1}).$$

Then:

$$\boldsymbol{s}(\Phi) = \boldsymbol{s}(\Phi)a = a\,\boldsymbol{s}(\Phi).$$

Proof. We have $\mathbb{1} - a \geq 0$, and:

$$\Phi(\mathbb{1} - a) = 0,$$

so:

$$\mathbf{s}(\Phi)(\mathbb{1} - a)\mathbf{s}(\Phi) = 0,$$

which yields:

$$\mathbf{s}(\Phi)(\mathbb{1} - a) = 0,$$

showing the claim. □

3. Instruments in Quantum Measurement Theory

In this chapter, we briefly recall the theory of instruments by E. Davies and J. Lewis (see [11,12]), which serves as a mathematical tool for a description of the process of quantum measurement.

Let $(\Omega, \mathfrak{F})$ be a measurable space of values of an observable of a physical system, *i.e.*, Ω is an arbitrary set, and $\mathfrak{F}$ is a σ-field of subsets of Ω (usually, we have as Ω the set $\mathbb{R}$ of all real numbers, and $\mathfrak{F}$ is the Borel subsets $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$). Let $\mathcal{M}$ be a von Neumann algebra. An instrument on $(\Omega, \mathfrak{F})$ is a map $\mathcal{E}\colon \mathfrak{F} \to \mathfrak{L}^+(\mathcal{M}_*)$ from the σ-field $\mathfrak{F}$ into the set of all positive linear transformations on the predual $\mathcal{M}_*$, such that:

(i) $(\mathcal{E}_\Omega \varphi)(\mathbb{1}) = \varphi(\mathbb{1})$ for all $\varphi \in \mathcal{M}_*$,

(ii) $\mathcal{E}_{\bigcup_{n=1}^\infty \Delta_n} \varphi = \sum_{n=1}^{\infty} \mathcal{E}_{\Delta_n} \varphi$
for any $\varphi \in \mathcal{M}_*$ and pairwise disjoint sets Δ_n from $\mathfrak{F}$, where the series on the right-hand side is convergent in the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology on $\mathcal{M}$.

In measurement theory, $\mathcal{E}_\Omega \varphi$ represents the state of the system after measurement, provided that before measurement, the system was in the state φ. The map $\mathcal{E}_\Omega$ sends states to states; thus, it is a *quantum channel* (in the terminology of quantum information theory). Accordingly, the maps $\mathcal{E}_\Delta$ could be called *deficient channels*, since they send states to "almost states" in the sense that $\mathcal{E}_\Delta \varphi$ are positive normal functional,s but there may be $(\mathcal{E}_\Delta \varphi)(\mathbb{1}) \neq 1$. In particular, in von Neumann's measurement theory, if observable T with the spectral decomposition:

$$T = \sum_i \lambda_i e_i$$

is measured in a system being in the state φ, we have:

$$\mathcal{E}_\Omega \varphi = \sum_i e_i \varphi e_i, \tag{1}$$

where $(e_i \varphi e_i)(a) = \varphi(e_i a e_i)$. In the language of density matrices, equality Equation (1) reads:

$$\mathcal{E}_\Omega (D_\varphi) = \sum_i e_i D_\varphi e_i, \tag{2}$$

where D_φ is the density matrix corresponding to the state φ, *i.e.*,

$$\varphi(a) = \operatorname{tr} a D_\varphi.$$

It is worth noting that channels of the form of Equation (2) are objects of intensive investigations; in the theory of instruments, they constitute the class of so-called *Lüders instruments* (*cf.* the remarks after Theorem 3).

Consider now for each $\mathcal{E}_\Delta$ its dual map $\mathcal{E}^*_\Delta : \mathcal{M} \to \mathcal{M}$ defined by:

$$\varphi(\mathcal{E}^*_\Delta(x)) = (\mathcal{E}_\Delta\varphi)(x), \quad \varphi \in \mathcal{M}_*,\, x \in \mathcal{M}.$$

The dual instrument is then defined as a map $\mathcal{E}^* : \mathfrak{F} \to \mathfrak{L}^+_n(\mathcal{M})$ from $\mathfrak{F}$ into the set of all positive normal linear transformations on $\mathcal{M}$, such that:

(i*) $\mathcal{E}^*_\Omega(\mathbb{1}) = \mathbb{1}$,

(ii*) $\mathcal{E}^*_{\bigcup_{n=1}^\infty \Delta_n}(x) = \sum_{n=1}^{\infty} \mathcal{E}^*_{\Delta_n}(x)$
for any $x \in \mathcal{M}$ and pairwise disjoint sets Δ_n from $\mathfrak{F}$, where the series on the right-hand side is convergent in the $\sigma(\mathcal{M}, \mathcal{M}_*)$-topology on $\mathcal{M}$.

For an instrument $\mathcal{E}$, its associated observable is defined as a map $e : \mathfrak{F} \to \mathcal{M}$ by the formula:

$$e(\Delta) = \mathcal{E}^*_\Delta(\mathbb{1}). \tag{3}$$

Thus, e is a positive operator valued measure ($\equiv$ POVM, semi-spectral measure). If for any Δ, $e(\Delta)$ is a projection, then e is a projection-valued measure ($\equiv$ PVM, spectral measure).

Suppose that the measured system is in state φ. Then, for observable $e(\Delta)$, we want $\varphi(e(\Delta))$ to be the probability that the observed value is in set Δ, which should be equal to $(\mathcal{E}_\Delta\varphi)(\mathbb{1})$. This leads to the equality:

$$\varphi(e(\Delta)) = (\mathcal{E}_\Delta\varphi)(\mathbb{1}) = \varphi(\mathcal{E}^*_\Delta(\mathbb{1})),$$

which justifies the definition of observable adopted earlier.

Among many important classes of instruments, there are weakly repeatable and repeatable ones, which express the celebrated von Neumann repeatability hypothesis: *if the physical quantity is measured twice in succession in a system, then we get the same value each time* (*cf.* [1,12]). Their definitions are as follows.

Definition 1. *An instrument $\mathcal{E}$ associated with observable e is called weakly repeatable if the following condition holds:*

$$(\mathcal{E}_{\Delta_1}(\mathcal{E}_{\Delta_2}\varphi))(\mathbb{1}) = (\mathcal{E}_{\Delta_1\cap\Delta_2}\varphi)(\mathbb{1})$$

for all sets $\Delta_1, \Delta_2 \in \mathfrak{F}$ and any $\varphi \in \mathcal{M}_*$, or equivalently,

$$\mathcal{E}^*_{\Delta_1}(\mathcal{E}^*_{\Delta_2}(\mathbb{1})) = \mathcal{E}^*_{\Delta_1\cap\Delta_2}(\mathbb{1}), \qquad \Delta_1, \Delta_2 \in \mathfrak{F},$$

which in terms of observable reads:

$$\mathcal{E}^*_{\Delta_1}(e(\Delta_2)) = e(\Delta_1 \cap \Delta_2).$$

The weak repeatability of an instrument may be characterized in the following way.

Lemma 2. *Let $\mathcal{E}$ be an instrument. The following are equivalent:*

(i) $\mathcal{E}$ is weakly repeatable,
(ii) for any $\Delta, \Theta \in \mathfrak{F}$, such that $\Delta \cap \Theta = \emptyset$, we have $\mathcal{E}^*_\Delta \mathcal{E}^*_\Theta = 0$,
(iii) for any $\Delta, \Theta \in \mathfrak{F}$, we have $\mathcal{E}^*_\Delta \mathcal{E}^*_\Theta = \mathcal{E}^{*2}_{\Delta\cap\Theta}$,
(iv) for any $\Delta \in \mathfrak{F}$, we have $\mathcal{E}^*_\Delta \mathcal{E}^*_{\Delta'} = 0$, where $\Delta' = \Omega \setminus \Delta$,
(v) for any $\Delta \in \mathfrak{F}$, we have $\mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Delta(\mathbb{1})\big) = \mathcal{E}^*_\Delta(\mathbb{1})$,
(vi) for any $\Delta, \Theta \in \mathfrak{F}$, such that $\Delta \subset \Theta$, we have $\mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Theta(\mathbb{1})\big) = \mathcal{E}^*_\Theta\big(\mathcal{E}^*_\Delta(\mathbb{1})\big) = \mathcal{E}^*_\Delta(\mathbb{1})$.

Proof. First, we shall show the equivalence of Conditions (ii)–(iv).

(ii) $\Longleftrightarrow$ (iii): Suppose that (ii) holds. For any $\Delta, \Theta \in \mathfrak{F}$, we have:

$$\mathcal{E}^*_\Delta \mathcal{E}^*_\Theta = \mathcal{E}^*_\Delta \mathcal{E}^*_{\Delta\cap\Theta} + \mathcal{E}^*_\Delta \mathcal{E}^*_{\Delta'\cap\Theta} = \mathcal{E}^*_\Delta \mathcal{E}^*_{\Delta\cap\Theta} = \big(\mathcal{E}^*_{\Delta\cap\Theta} + \mathcal{E}^*_{\Delta\cap\Theta'}\big)\mathcal{E}^*_{\Delta\cap\Theta} = \mathcal{E}^{*2}_{\Delta\cap\Theta},$$

showing the implication (ii) $\Longrightarrow$ (iii). The converse implication is obvious.

(iv) $\Longleftrightarrow$ (v): For any $\Delta \in \mathfrak{F}$, we have:

$$\mathbb{1} = \mathcal{E}^*_\Delta(\mathbb{1}) + \mathcal{E}^*_{\Delta'}(\mathbb{1}),$$

hence:

$$\mathcal{E}^*_\Delta(\mathbb{1}) = \mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Delta(\mathbb{1})\big) + \mathcal{E}^*_{\Delta'}\big(\mathcal{E}^*_\Delta(\mathbb{1})\big).$$

Thus:

$$\mathcal{E}^*_\Delta(\mathbb{1}) = \mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Delta(\mathbb{1})\big)$$

if and only if:

$$\mathcal{E}^*_\Delta\big(\mathcal{E}^*_{\Delta'}(\mathbb{1})\big) = 0,$$

which, since the map $\mathcal{E}^*_\Delta \mathcal{E}^*_{\Delta'}$ is positive, holds if and only if $\mathcal{E}^*_\Delta \mathcal{E}^*_{\Delta'} = 0$.

(ii) $\Longrightarrow$ (vi): For $\Delta \subset \Theta$, we have $\Delta \cap \Theta' = \emptyset$, and thus, $\mathcal{E}^*_\Delta \mathcal{E}^*_{\Theta'} = \mathcal{E}^*_{\Theta'} \mathcal{E}^*_\Delta = 0$. Consequently,

$$\mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Theta(\mathbb{1})\big) = \mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Theta(\mathbb{1})\big) + \mathcal{E}^*_\Delta\big(\mathcal{E}^*_{\Theta'}(\mathbb{1})\big) = \mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Omega(\mathbb{1})\big) = \mathcal{E}^*_\Delta(\mathbb{1}),$$

and, analogously, $\mathcal{E}^*_\Theta\big(\mathcal{E}^*_\Delta(\mathbb{1})\big) = \mathcal{E}^*_\Delta(\mathbb{1})$.

(vi) $\Longrightarrow$ (v): Obvious.

(iv) $\Longrightarrow$ (ii). Let $\Delta \cap \Theta = \emptyset$. Then, $\Theta \subset \Delta'$, and from the additivity of $\mathcal{E}^*$, we get:

$$\mathcal{E}^*_{\Delta'} = \mathcal{E}^*_\Theta + \mathcal{E}^*_{\Delta'\cap\Theta'} \geq \mathcal{E}^*_\Theta.$$

Consequently, for each $x \in \mathcal{M}$, $x \geq 0$, we obtain on account of the positivity of $\mathcal{E}^*_\Delta$ and the inequality:

$$\mathcal{E}^*_\Theta(x) \leq \mathcal{E}^*_{\Delta'}(x),$$

the relation:

$$0 \leq \mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Theta(x)\big) \leq \mathcal{E}^*_\Delta\big(\mathcal{E}^*_{\Delta'}(x)\big) = 0,$$

showing that $\mathcal{E}^*_\Delta \mathcal{E}^*_\Theta = 0$.

Thus, Conditions (ii)–(iv) are equivalent. Clearly, (i) $\Longrightarrow$ (v). We shall show that:

(ii) and (iii) $\Longrightarrow$ (i). For arbitrary $\Delta_1, \Delta_2 \in \mathfrak{F}$, we have:

$$\mathcal{E}^*_{\Delta_1}\big(\mathcal{E}^*_{\Delta_2}(\mathbb{1})\big) = \mathcal{E}^*_{\Delta_1}\big(\mathcal{E}^*_{\Delta_1\cap\Delta_2}(\mathbb{1}) + \mathcal{E}^*_{\Delta_1'\cap\Delta_2}(\mathbb{1})\big) = \mathcal{E}^*_{\Delta_1\cap\Delta_2}(\mathbb{1}),$$

showing the weak repeatability of $\mathcal{E}$. $\square$

Definition 2. *An instrument $\mathcal{E}$ is called repeatable if for any $\Delta_1, \Delta_2 \in \mathfrak{F}$:*

$$\mathcal{E}_{\Delta_1}\mathcal{E}_{\Delta_2} = \mathcal{E}_{\Delta_1\cap\Delta_2},$$

or equivalently,

$$\mathcal{E}^*_{\Delta_1}\mathcal{E}^*_{\Delta_2} = \mathcal{E}^*_{\Delta_1\cap\Delta_2}.$$

It is obvious that a repeatable instrument is weakly repeatable. We have the following characterization of repeatability.

Lemma 3. *Let $\mathcal{E}$ be an instrument. The following are equivalent:*

(i) $\mathcal{E}$ is repeatable,
(ii) for any $\Delta \in \mathfrak{F}$, we have $\mathcal{E}^{*2}_\Delta = \mathcal{E}^*_\Delta$,
(iii) for any $\Delta, \Theta \in \mathfrak{F}$, such that $\Delta \subset \Theta$, we have $\mathcal{E}^*_\Delta\mathcal{E}^*_\Theta = \mathcal{E}^*_\Theta\mathcal{E}^*_\Delta = \mathcal{E}^*_\Delta$.

Proof. First, observe that each of the conditions above implies, on account of Lemma 2, the weak repeatability of $\mathcal{E}$. Now, we have:

(i) $\Longrightarrow$ (ii): Obvious.

(ii) $\Longrightarrow$ (iii): Let $\Delta \subset \Theta$. Then, $\Delta \cap \Theta' = \emptyset$, and from Lemma 2, we obtain $\mathcal{E}^*_\Delta\mathcal{E}^*_{\Theta'} = 0$, so:

$$\mathcal{E}^*_\Delta\mathcal{E}^*_\Theta = \mathcal{E}^*_\Delta\mathcal{E}^*_\Theta + \mathcal{E}^*_\Delta\mathcal{E}^*_{\Theta'} = \mathcal{E}^*_\Delta\mathcal{E}^*_\Omega = \mathcal{E}^*_\Delta\left(\mathcal{E}^*_\Delta + \mathcal{E}^*_{\Delta'}\right) = \mathcal{E}^{*2}_\Delta = \mathcal{E}^*_\Delta.$$

(iii) $\Longrightarrow$ (i). For any $\Delta, \Theta \in \mathfrak{F}$, we have, employing the weak repeatability of $\mathcal{E}$,

$$\mathcal{E}^*_\Delta\mathcal{E}^*_\Theta = \mathcal{E}^*_\Delta\left(\mathcal{E}^*_{\Delta\cap\Theta} + \mathcal{E}^*_{\Delta'\cap\Theta}\right) = \mathcal{E}^*_{\Delta\cap\Theta}. \quad \square$$

For weakly repeatable instruments, we have yet another remarkable property.

Lemma 4. *Let $\mathcal{E}$ be a weakly repeatable instrument. Then, for any $\Delta, \Theta \in \mathfrak{F}$, such that $\Delta \cap \Theta = \emptyset$, we have:*

$$\mathbf{s}(\mathcal{E}^*_\Delta)\mathbf{s}(\mathcal{E}^*_\Theta) = 0.$$

Proof. From Lemma 2 (ii), we obtain:

$$\mathcal{E}_\Delta(\mathbf{s}(\mathcal{E}^*_\Delta)e(\Theta)\mathbf{s}(\mathcal{E}^*_\Delta)) = \mathcal{E}^*_\Delta(e(\Theta)) = \mathcal{E}^*_\Delta\left(\mathcal{E}^*_\Theta(\mathbb{1})\right) = 0,$$

which yields:

$$\mathbf{s}(\mathcal{E}^*_\Delta)e(\Theta)\mathbf{s}(\mathcal{E}^*_\Delta) = 0,$$

and thus:

$$\mathbf{s}(\mathcal{E}^*_\Delta)e(\Theta) = 0.$$

From the weak repeatability of $\mathcal{E}$, it follows that:

$$\mathcal{E}^*_\Theta(e(\Theta)) = e(\Theta) = \mathcal{E}^*_\Theta(\mathbb{1}),$$

so on account of Lemma 1, we get:

$$\mathbf{s}(\mathcal{E}^*_\Theta) = e(\Theta)\mathbf{s}(\mathcal{E}^*_\Theta),$$

and hence:

$$\mathbf{s}(\mathcal{E}^*_\Delta)\mathbf{s}(\mathcal{E}^*_\Theta) = \mathbf{s}(\mathcal{E}^*_\Delta)e(\Theta)\mathbf{s}(\mathcal{E}^*_\Theta) = 0. \quad \square$$

4. Concept of Entropy

In the case of the full algebra $\mathbb{B}(\mathcal{H})$, a well-established concept of entropy goes back to John von Neumann [1], who defined entropy of state ρ as:

$$S(\rho) = -\operatorname{tr} D_\rho \log D_\rho,$$

where D_ρ is the density matrix of ρ, *i.e.*, a positive operator of trace one, such that:

$$\rho(a) = \operatorname{tr} aD_\rho, \qquad a \in \mathbb{B}(\mathcal{H}).$$

Unfortunately, when we are dealing with arbitrary von Neumann algebras, a satisfactory general definition of entropy is lacking. However, an interesting attempt at such a definition in the case of a semi-finite algebra, being at the same time a natural straightforward generalization of von Neumann's idea, is due to I. Segal [13] and goes as follows.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a normal semi-finite faithful trace τ. For any normal state ρ, there exists a unique nonnegative self-adjoint operator D_ρ affiliated with $\mathcal{M}$ (see the Appendix), called the *density* of ρ, such that for each $a \in \mathcal{M}$, we have:

$$\rho(a) = \tau(aD_\rho).$$

In particular, if D_ρ is bounded, then $D_\rho \in \mathcal{M}$ (as a matter of fact, this will be the only case of our interest). The Segal entropy of ρ, denoted by $S(\rho)$, is defined just for bounded D_ρ as:

$$S(\rho) = -\tau(D_\rho \log D_\rho)$$

(*cf.* [13]). Now, the definition above still requires some involved arguments concerning the trace of operator $D_\rho \log D_\rho$. Namely, $D_\rho \log D_\rho$ is bounded, but it is not defined on the whole of $\mathcal{H}$ (instead, it is defined on the domain of $\log D_\rho$, so only densely defined). In the case of the full algebra $\mathbb{B}(\mathcal{H})$, the customary procedure is to take its closure and obtain a bounded operator defined on $\mathcal{H}$. It turns out that the same is possible in von Neumann algebra $\mathcal{M}$, namely closure of $D_\rho \log D_\rho$ belongs to $\mathcal{M}$, so we may apply trace τ to it. This procedure is described in the Appendix, where the closure of a product of two operators A and B is denoted by $A \cdot B$. Thus, strictly speaking, we should write $D_\rho \cdot \log D_\rho$, rather than $D_\rho \log D_\rho$, but for the sake of simplicity, we shall stick to the simpler notation for the product without the central dot in the middle. However, it should be remembered that all of the products AB in the remainder of the paper are to be understood as $A \cdot B$, *i.e.*, $\overline{AB}$, especially, when we are dealing with unbounded operators. If A and B are bounded, then $A \cdot B$ means that we have a bounded closed operator; thus, $A \cdot B \in \mathcal{M}$ (see the Appendix).

Remark 1. *Despite being a seemingly straightforward generalization of von Neumann's entropy, the Segal definition exhibits fundamental differences in many respects from that of von Neumann. For example, while the density operator in the von Neumann definition is trace-class and, thus, has a discrete spectrum with the eigenvalues summing up to one, this is not the case in the Segal definition. Furthermore, the von Neumann entropy of a state is nonnegative (which is a consequence of the above property of the density operator), while the Segal entropy of a state need not be such. In addition, there* *are some technical problems while dealing with a semi-finite trace, etc. For these reasons, we shall consider the case of a finite von Neumann algebra and adopt a definition of entropy more in the spirit of the classical Boltzmann–Gibbs notion, where for a density function f on a probability space $(\Omega, \mathfrak{F}, \mu)$, its entropy is defined as:*

$$H(f) = \int_\Omega f \log f \, d\mu.$$

As will be seen, our definition, which is just that of Segal up to a minus sign, assigns a finite nonnegative entropy to a state, and more generally, for each non-normalized state in $\mathcal{M}_*^+$ with bounded density, its entropy is finite.

It should be noted that some fundamental investigations concerning entropy and related notions in the above setup were carried out in [14].

Thus, let $\mathcal{M}$ be a von Neumann algebra with a normal faithful finite trace τ, $\tau(\mathbb{1}) = 1$. For each $\rho \in \mathcal{M}_*^+$ with bounded density D_ρ, we define its entropy $H(\rho)$ as:

$$H(\rho) = \tau(D_\rho \log D_\rho).$$

Let:

$$D_\rho = \int_0^\infty \lambda \, e(d\lambda)$$

be the spectral decomposition of D_ρ. Since $\lambda \log \lambda \geq \lambda - 1$, we have:

$$\begin{aligned} H(\rho) &= \tau\Big(\int_0^\infty \lambda \log \lambda \, e(d\lambda) \Big) = \int_0^\infty \lambda \log \lambda \, \tau(e(d\lambda)) \geq \int_0^\infty (\lambda - 1) \, \tau(e(d\lambda)) \\ &= \int_0^\infty \lambda \, \tau(e(d\lambda)) - \int_0^\infty \tau(e(d\lambda)) = \tau(D_\rho) - \tau(\mathbb{1}) = \rho(\mathbb{1}) - 1, \end{aligned} \tag{4}$$

showing that entropy is bounded from below, and in particular, it is nonnegative for states. Moreover, since D_ρ is bounded, its spectrum is a bounded set; thus, the function $\lambda \mapsto \lambda \log \lambda$ is bounded on the spectrum, which yields that entropy is bounded from above.

Proposition 1. *Let $a, b \in \mathcal{M}$ be such that $0 \leq a \leq b$. Then:*

$$\tau(a \log b - a \log a) \geq 0, \tag{5}$$

with equality if and only if $ab = ba = a^2$. Moreover, the numbers $\tau(a \log b)$ and $\tau(a \log a)$ are finite.

Proof. Since:

$$0 \leq a \leq b,$$

we have:

$$0 \leq (\log b) a (\log b) \leq (\log b) b (\log b) = b \log^2 b.$$

The operator on the right-hand of the inequality above is bounded (belongs to $\mathcal{M}$); hence, $(\log b) a (\log b)$ is also bounded (belongs to $\mathcal{M}$). Moreover,

$$(\log b) a (\log b) = \big(a^{1/2} \log b\big)^* a^{1/2} \log b;$$

thus, $a^{1/2}\log b$ is bounded (belongs to $\mathcal{M}$). Consequently, $a^{1/2}(\log b - \log a)$ and $a^{1/2}$ belong to $\mathcal{M}$; so, from the properties of trace, we obtain:

$$\tau(a(\log b - \log a)) = \tau\big(a^{1/2}\big(a^{1/2}(\log b - \log a)\big)\big) = \tau\big(a^{1/2}(\log b - \log a)a^{1/2}\big). \tag{6}$$

Since the logarithm is an operator monotone function, we have:

$$\log b - \log a \geq 0,$$

yielding:

$$a^{1/2}(\log b - \log a)a^{1/2} \geq 0,$$

and finally, on account of Equation (6):

$$0 \leq \tau\big(a^{1/2}(\log b - \log a)a^{1/2}\big) = \tau(a(\log b - \log a)).$$

Assume first that:

$$\tau(a\log b - a\log a) = 0. \tag{7}$$

Then, as was seen above,

$$\tau\big(a^{1/2}(\log b - \log a)a^{1/2}\big) = 0,$$

and from the faithfulness of τ, we get:

$$a^{1/2}(\log b - \log a)a^{1/2} = 0,$$

i.e.,

$$\big(a^{1/2}(\log b - \log a)^{1/2}\big)\big(a^{1/2}(\log b - \log a)^{1/2}\big)^* = 0.$$

This gives:

$$a^{1/2}(\log b - \log a)^{1/2} = 0,$$

yielding:

$$a(\log a - \log b) = 0,$$

i.e.,

$$a\log a = a\log b.$$

Taking adjoints, we get:

$$a\log a = (\log b)a.$$

In particular, $\log b$ commutes with a, leaves the range of a invariant and coincides with $\log a$ on the range of a. Thus, on the range of a, we have:

$$a|\operatorname{Range} a = e^{\log a}|\operatorname{Range} a = e^{\log b}|\operatorname{Range} a = b|\operatorname{Range} a,$$

which is equivalent to the equalities:

$$ab = ba = a^2.$$

Conversely, let the equality above hold. Then, a and b commute, so we get, after taking logarithms of both sides:

$$2\log a = \log a + \log b,$$

that is:

$$\log a = \log b,$$

giving the equality:

$$a \log a = a \log b,$$

and, thus, Equation (7). $\square$

Now, we are in a position to show the superadditivity of entropy.

Theorem 1. *Let $\rho, \varphi \in \mathcal{M}_*^+$ have bounded densities D_ρ and D_φ, respectively. Then:*

$$H(\rho) + H(\varphi) \leq H(\rho + \varphi), \tag{8}$$

with equality if and only if $D_\rho D_\varphi = 0$.

Proof. On account of inequality Equation (5), we have:

$$\tau(D_\rho \log D_\rho) \leq \tau(D_\rho \log(D_\rho + D_\varphi)) \qquad \text{and} \qquad \tau(D_\varphi \log D_\varphi) \leq \tau(D_\varphi \log(D_\rho + D_\varphi)),$$

moreover, all of the numbers above are finite. Summing up both sides, we obtain, taking into account a rather obvious formula $D_{\rho+\varphi} = D_\rho + D_\varphi$,

$$H(\rho) + H(\varphi) \leq H(\rho + \varphi).$$

From Proposition 1, it follows that we have equality in Equation (8) if and only if:

$$D_\rho(D_\rho + D_\varphi) = D_\rho^2,$$

which amounts to the relation $D_\rho D_\varphi = 0$. $\square$

For any positive $a \in \mathcal{M}$, by $\mathbf{s}(a)$ is denoted its support, *i.e.*, the projection onto the closure of the range of a. We have:

$$a = \mathbf{s}(a)a = a\mathbf{s}(a).$$

The following simple lemma shows a relation between the support of a normal state and the support of its density.

Lemma 5. *Let $\rho \in \mathcal{M}_*^+$ have density D_ρ. Then, $\mathbf{s}(\rho) = \mathbf{s}(D_\rho)$.*

Proof. We have:

$$\rho(\mathbf{s}(D_\rho)) = \tau(\mathbf{s}(D_\rho)D_\rho) = \tau(D_\rho) = \rho(\mathbb{1}),$$

showing that:

$$\mathbf{s}(D_\rho) \geq \mathbf{s}(\rho).$$

On the other hand, for each projection $q \in \mathcal{M}$, such that $\rho(q) = 0$, we have:

$$0 = \tau(qD_\rho) = \tau(qD_\rho q),$$

and the faithfulness of τ yields:

$$qD_\rho q = 0,$$

i.e.,

$$qD_\rho = 0,$$

hence:

$$q\mathbf{s}(D_\rho) = 0.$$

Consequently,

$$q \leq \mathbf{s}(D_\rho)^\perp,$$

and thus:

$$\sup\{q \in \mathcal{M} : q \text{ — projection},\ \rho(q) = 0\} \leq \mathbf{s}(D_\rho)^\perp,$$

giving:

$$\mathbf{s}(\rho) = \big(\sup\{q \in \mathcal{M} : q \text{ — projection},\ \rho(q) = 0\}\big)^\perp \geq \mathbf{s}(D_\rho). \quad \square$$

Now, as an immediate corollary to Theorem 1, we obtain:

Corollary 1. *Let $\rho, \varphi \in \mathcal{M}_*^+$ have bounded densities. Then:*

$$H(\rho) + H(\varphi) = H(\rho + \varphi)$$

if and only if:

$$\mathbf{s}(\rho)\mathbf{s}(\varphi) = 0.$$

Indeed, from Theorem 1, it follows that the equality for the entropies holds if and only if $D_\rho D_\varphi = 0$, which is equivalent to the equality:

$$\mathbf{s}(D_\rho)\mathbf{s}(D_\varphi) = 0,$$

and now, Lemma 5 gives the claim.

5. Entropy of Measurement

Following [15], we adopt the following definition.

Definition 3. *A reading scale is a finite partition of the value space of the measured observable:*

$$\Omega = \bigcup_{i=1}^{n} \Delta_i,$$

where $\Delta_i \in \mathfrak{F}$ for any $i = 1, 2, \ldots, n$ and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Such a reading scale will be denoted by $\mathcal{R}$.

Let us consider now the measurement represented by instrument $\mathcal{E}$. Let the system be in the initial state $\rho \in \mathcal{M}_*^+$. Then, the final state of the system is $\mathcal{E}_\Omega\rho$. For any reading scale $\mathcal{R} = \{\Delta_i : i = 1, 2, \ldots, n\}$, we have:

$$\mathcal{E}_\Omega\rho = \sum_{i=1}^{n} \mathcal{E}_{\Delta_i}\rho.$$

Considering only the non-zero summands, denote:

$$\frac{\mathcal{E}_{\Delta_i}\rho}{(\mathcal{E}_{\Delta_i}\rho)(\mathbb{1})} = \rho_i, \qquad (\mathcal{E}_{\Delta_i}\rho)(\mathbb{1}) = \alpha_i.$$

ρ_i are normal states, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

Theorem 2. *For every normal state ρ of the system, such that $\mathcal{E}_\Omega \rho$ has bounded density, we have:*

$$\sum_i H(\rho_i) + H((\alpha_i)) \leq H(\mathcal{E}_\Omega \rho) \leq \sum_i \alpha_i H(\rho_i), \tag{9}$$

where $H((\alpha_i))$ stands for the (minus) classical entropy of the sequence (α_i):

$$H((\alpha_i)) = \sum_i \alpha_i \log \alpha_i.$$

Proof. As the entropy is a convex function, which is an immediate consequence of the operator convexity of the function $\lambda \mapsto \lambda \log \lambda$, and:

$$\mathcal{E}_\Omega \rho = \sum_i \alpha_i \rho_i,$$

we obtain:

$$H(\mathcal{E}_\Omega \rho) \leq \sum_i \alpha_i H(\rho_i).$$

On the other hand, the superadditivity of entropy yields:

$$H(\mathcal{E}_\Omega \rho) = H\Big(\sum_i \alpha_i \rho_i\Big) \geq \sum_i H(\alpha_i \rho_i).$$

Furthermore, for $0 < \alpha \leq 1$, and a normal state φ with density D_φ having the spectral decomposition:

$$D_\varphi = \int_0^\infty \lambda\, e(d\lambda),$$

we have:

$$\begin{aligned} H(\alpha\varphi) &= \int_0^\infty \alpha\lambda \log(\alpha\lambda)\, \tau(e(d\lambda)) \\ &= \alpha\Big(\log\alpha \int_0^\infty \lambda\, \tau(e(d\lambda)) + \int_0^\infty \lambda \log\lambda\, \tau(e(d\lambda))\Big) \\ &= (\alpha\log\alpha)\tau(D_\varphi) + H(\varphi) = \alpha\log\alpha + H(\varphi). \end{aligned}$$

Hence:

$$H(\mathcal{E}_\Omega \rho) \geq \sum_i H(\alpha_i \rho_i) = \sum_i (\alpha_i \log \alpha_i + H(\rho_i)) = H((\alpha_i)) + \sum_i H(\rho_i). \quad \square$$

Definition 4. *The measurement associated with instrument $\mathcal{E}$ is called a minimal state entropy one if, for any normal state ρ and any reading scale $\mathcal{R}$, it attains the lower bound of Equation (9).*

Now, we are in a position to show connections between minimal state entropy measurements and repeatable measurements. First, as a corollary to our earlier considerations, we obtain a generalization of a result proven in [15] for the full algebra $\mathbb{B}(\mathcal{H})$ and repeatable measurements.

Theorem 3. *The measurement associated with a weakly repeatable instrument $\mathcal{E}$ is a minimal state entropy one.*

Proof. Let $\{\Delta_i : i = 1, \ldots, n\}$ be an arbitrary reading scale. From the weak repeatability of $\mathcal{E}$, it follows, by virtue of Lemma 4, that for any positive ρ in $\mathcal{M}_*$, the supports of $\mathcal{E}_{\Delta_i}\rho$ are pairwise orthogonal, and Corollary 1 gives the claim. □

An interesting class of instruments is the one for which $\mathcal{E}^*_\Omega$ is the so-called Lüders operation, *i.e.*,

$$\mathcal{E}^*_\Omega(x) = \sum_i e_i x e_i, \qquad x \in \mathcal{M},$$

where e_i are projections and $\sum_i e_i = \mathbb{1}$. This class contains, in particular, Lüders instruments considered in [16] and von Neumann instruments considered in [17]. As for the Lüders operation, it was introduced by G. Lüders [18] in 1951 and afterwards investigated, together with its various generalizations, in [19–21]. One important feature of the Lüders operation is that it is a conditional expectation, in particular the relation $\mathcal{E}^*_\Omega = \mathcal{E}^{*2}_\Omega$ holds. Considering instruments with spectral measures as their observables, we have:

Theorem 4. *Let $\mathcal{E}$ be an instrument having as its observable a spectral measure. The following are equivalent:*

(i) $\mathcal{E}^*_\Omega = \mathcal{E}^{*2}_\Omega$, and $\mathcal{E}$ is of minimal state entropy;
(ii) $\mathcal{E}$ is repeatable.

Proof. (i) $\Longrightarrow$ (ii): Let e be the observable of $\mathcal{E}$. For arbitrary $\Delta \in \mathfrak{F}$, we have on account of the additivity of $\mathcal{E}$ and Lemma 4:

$$\begin{aligned}\mathcal{E}^*_\Omega(\mathbf{s}(\mathcal{E}^*_\Delta)) &= \mathcal{E}^*_\Delta(\mathbf{s}(\mathcal{E}^*_\Delta)) + \mathcal{E}^*_{\Delta'}(\mathbf{s}(\mathcal{E}^*_\Delta)) \\ &= \mathcal{E}^*_\Delta(\mathbf{s}(\mathcal{E}^*_\Delta)) + \mathcal{E}^*_{\Delta'}(\mathbf{s}(\mathcal{E}^*_\Delta)\mathbf{s}(\mathcal{E}^*_\Delta)) = \mathcal{E}^*_\Delta(\mathbf{s}(\mathcal{E}^*_\Delta)) = \mathcal{E}^*_\Delta(\mathbb{1}) = e(\Delta),\end{aligned}$$

and thus:

$$\mathcal{E}^*_\Omega(e(\Delta)) = \mathcal{E}^*_\Omega(\mathcal{E}^*_\Omega(\mathbf{s}(\mathcal{E}^*_\Delta))) = \mathcal{E}^*_\Omega(\mathbf{s}(\mathcal{E}^*_\Delta)) = e(\Delta).$$

By virtue of ([22], Theorem 1), for every instrument $\mathcal{E}$ whose observable is a spectral measure e, we have the representation:

$$\mathcal{E}^*_\Delta(x) = e(\Delta)\mathcal{E}^*_\Omega(x), \qquad \Delta \in \mathfrak{F},\, x \in \mathcal{M}, \tag{10}$$

which yields:

$$\mathcal{E}^*_\Delta(e(\Delta)) = e(\Delta)\mathcal{E}^*_\Omega(e(\Delta)) = e(\Delta)^2 = e(\Delta),$$

showing that $\mathcal{E}$ is weakly repeatable.

For any $\Delta \in \mathfrak{F}$, set:

$$\mathcal{F}^*_\Delta = \mathcal{E}^{*2}_\Delta.$$

Let Δ_n be arbitrary pairwise disjoint sets from $\mathfrak{F}$. For each $x \in \mathcal{M}$, we have on account of the continuity of $\mathcal{E}^*_{\Delta_n}$ in the $\sigma(\mathcal{M}, \mathcal{M}_*)$-topology and Lemma 2:

$$\begin{aligned}\mathcal{F}^*_{\bigcup_{n=1}^\infty \Delta_n}(x) &= \mathcal{E}^*_{\bigcup_{n=1}^\infty \Delta_n}\big(\mathcal{E}^*_{\bigcup_{n=1}^\infty \Delta_n}(x)\big) = \sum_{n=1}^\infty \mathcal{E}^*_{\Delta_n}\left(\sum_{k=1}^\infty \mathcal{E}^*_{\Delta_k}(x)\right) \\ &= \sum_{n=1}^\infty \sum_{k=1}^\infty \mathcal{E}^*_{\Delta_n}\big(\mathcal{E}^*_{\Delta_k}(x)\big) = \sum_{n=1}^\infty \mathcal{E}^*_{\Delta_n}\big(\mathcal{E}^*_{\Delta_n}(x)\big) = \sum_{n=1}^\infty \mathcal{F}^*_{\Delta_n}(x),\end{aligned}$$

showing the σ-additivity of the map $\mathcal{F}^* \colon \mathfrak{F} \to \mathcal{L}^+_n(\mathcal{M})$. Moreover,

$$\mathcal{F}^*_\Omega(\mathbb{1}) = \mathcal{E}^*_\Omega\big(\mathcal{E}^*_\Omega(\mathbb{1})\big) = \mathcal{E}^*_\Omega(\mathbb{1}) = \mathbb{1};$$

thus, $\mathcal{F}^*$ is a dual instrument. For its observable f, we have by virtue of the weak repeatability of $\mathcal{E}$:

$$f(\Delta) = \mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Delta(\mathbb{1})\big) = \mathcal{E}^*_\Delta(e(\Delta)) = e(\Delta).$$

Hence, for each $x \in \mathcal{M}$, we get, taking into account the fact that the observable of $\mathcal{F}$ is a spectral measure and using representation Equation (10) for $\mathcal{F}^*$:

$$\mathcal{E}^*_\Delta\big(\mathcal{E}^*_\Delta(x)\big) = \mathcal{F}^*_\Delta(x) = e(\Delta)\mathcal{F}^*_\Omega(x) = e(\Delta)\mathcal{E}^{*2}_\Omega(x) = e(\Delta)\mathcal{E}^*_\Omega(x) = \mathcal{E}^*_\Delta(x),$$

i.e., by virtue of Lemma 2, $\mathcal{E}$ is repeatable.

(ii) $\Longrightarrow$ (i): Obvious, by virtue of Theorem 3 and the definition of repeatability. □

6. Conclusions

We have investigated properties of entropy in Segal's sense for measurements represented by instruments on finite von Neumann algebras. Bounds for the entropy of the state after measurement have been found, and minimal state entropy measurements have been analyzed in some detail. In the course of our analysis, we have also obtained conditions for superadditivity and the additivity of entropy.

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Appendix

Let $\mathcal{M}$ be a von Neumann algebra acting in a Hilbert space $\mathcal{H}$ with a normal faithful finite trace τ. The algebra of *measurable operators* $\widetilde{\mathcal{M}}$ is defined as a topological *-algebra of densely defined closed operators on $\mathcal{H}$ affiliated (see below) with $\mathcal{M}$ with strong addition "$+$" and strong multiplication "$\cdot$", *i.e.*,

$$A + B = \overline{A + B}, \qquad A \cdot B = \overline{AB}, \qquad A, B \in \widetilde{\mathcal{M}}.$$

In particular, there exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ contained in the domain of every operator from $\widetilde{\mathcal{M}}$, which is left invariant by the elements from $\widetilde{\mathcal{M}}$, so the sum and product above are closed densely defined operators. Moreover, if A is measurable and bounded, then $A \in \mathcal{M}$.

An operator A is said to be *affiliated* with a von Neumann algebra $\mathcal{M}$ if, for every unitary $u' \in \mathcal{M}'$, we have $u'A = Au'$. Here, $\mathcal{M}'$ stands for the *commutant* of $\mathcal{M}$, *i.e.*, the set of all bounded operators z' on $\mathcal{H}$, such that $z'x = xz'$ for every $x \in \mathcal{M}$. A more appealing definition for self-adjoint positive operators says that for the spectral decomposition:

$$A = \int_0^\infty \lambda \, e(d\lambda)$$

of A, its spectral projections $e(\Delta)$, $\Delta \in \mathcal{B}(\mathbb{R})$, are in $\mathcal{M}$.

Conflicts of Interest

The author declares no conflict of interest.

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