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Article

Existence of Entropy Solutions for Nonsymmetric Fractional Systems

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Abstract: The present work focuses on entropy solutions for the fractional Cauchy problem of nonsymmetric systems. We impose sufficient conditions on the parameters to obtain bounded solutions of L^{∞}. The solutions attained are unique and exclusive. Performance is established by utilizing the maximum principle for certain generalized time and space-fractional diffusion equations. The fractional differential operator is inspected based on the interpretation of the Riemann–Liouville differential operator. Fractional entropy inequalities are imposed.

Keywords: fractional calculus; fractional differential equation; entropy solution

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1. Introduction

Fractional order differential equations have been positively engaged in modeling of various different procedures and schemes in engineering, physics, chemistry, biology, medicine, and food processing [1–4]. In these requests, reflecting boundary value problems such as the existence and uniqueness of solutions for space-time fractional diffusion equations on bounded domains is a significant procedure. The existence and uniqueness of solutions for linear and nonlinear fractional differential equations has fascinated many investigators [5–13].

Fractional calculus created from the Riemann–Liouville description of fractional integral of order $\wp$ is in the form

$$ {}_aI_t^{\wp} f(t) = \int_a^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\phi(\tau)d\tau. $$

The fractional order differential of the function ϕ of order $\wp > 0$ is given by

$$ {}_aD_t^{\wp}\phi(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\wp}}{\Gamma(1-\wp)}\phi(\tau)d\tau. $$

When $a = 0$, we shall denote ${}_0D_t^{\wp}\phi(t) := D_t^{\wp} f(t)$ and ${}_0I_t^{\wp}\phi(t) := I_t^{\wp}\phi(t)$ in the follow-up. From above, for $a = 0$, we accomplish that

$$ D_t^{\wp} t^{\ell} = \frac{\Gamma(\ell+1)}{\Gamma(\ell-\wp+1)} t^{\ell-\wp}, \quad \ell > -1;\ 0 < \wp < 1 $$

and

$$ I_t^{\wp} t^{\ell} = \frac{\Gamma(\ell+1)}{\Gamma(\ell+\wp+1)} t^{\ell+\wp}, \quad \ell > -1;\ \wp > 0. $$

The Leibniz rule for arbitrary differentiations of smooth functions (with continuous derivatives for all orders) $\phi(t)$ and $\psi(t)$, $t \in [a,b]$ is formulated as (see p. 96 in [14]):

$$ \begin{aligned} {}_aD_t^{\wp}[\phi(t)\psi(t)] &= \sum_{n=0}^{k} \begin{pmatrix} \wp \\ n \end{pmatrix} {}_aD_t^{\wp-n}\phi(t)\, {}_aD_t^{n}\psi(t) - R_k^{\wp} \\ &= \sum_{n=0}^{k} \begin{pmatrix} \wp \\ n \end{pmatrix} {}_aD_t^{\wp-n}\psi(t)\, {}_aD_t^{n}\phi(t) - R_k^{\wp}, \end{aligned} $$

where $\wp \le k-1$,

$$ \begin{pmatrix} \wp \\ n \end{pmatrix} = \frac{\Gamma(\wp+1)}{\Gamma(n+1)\Gamma(\wp+1-n)} $$

and $R_k^{\wp}$ is the remainder of the series, which can be defined as follows:

$$ R_k^{\wp} = \Big(\frac{1}{k!\Gamma(-\wp)}\int_a^t (t-\tau)^{-\wp-1}\phi(\tau)d\tau\Big)\Big(\int_\tau^t {}_aD_t^{k+1}\psi(\theta)\,(\tau-\theta)^k d\theta\Big). $$

Additionally, the fractional differential operator achieves linearity (see p. 90 in [14])

$$ {}_aD_t^{\wp}[\rho\phi(t) + \sigma\psi(t)] = \rho\, {}_aD_t^{\wp}[\phi(t)] + \sigma\, {}_aD_t^{\wp}[\psi(t)]. $$

Recently, Alsaedi *et al.* [15] presented an inequality for fractional derivatives related to the Leibniz rule, as follows:

Lemma 1. *Let one of the following conditions be satisfied*

- $\mu \in C([0,T])$ *and* $\nu \in C^{\beta}([0,T]), \wp < \beta \le 1$
- $\nu \in C([0,T])$ *and* $\mu \in C^{\beta}([0,T]), \wp < \beta \le 1$
- $\mu \in C^{\beta}([0,T])$ *and* $\nu \in C^{\delta}([0,T]), \wp < \beta \le \beta + \delta,\ \beta, \delta \in (0,1),$

where

$$C^{\gamma}([0,T]) = \{\mu : [0,T] \to \mathbb{R} / |\mu(t) - \mu(t-h)| = O(h^{\gamma}) \textit{ uniformly for } 0 < t-h < t \leq T\}.$$

Then we have

$$D_t^{\wp}(\mu\nu)(t) = \mu(t) D_t^{\wp}\nu(t) + \nu(t) D_t^{\wp}\mu(t) - \frac{\wp}{\Gamma(1-\wp)} \int_0^t \frac{(\mu(s)-\mu(t))(\nu(s)-\nu(t))}{(t-s)^{\wp+1}} ds - \frac{\mu(t)\nu(t)}{\Gamma(1-\wp)t^{-\wp}}$$

point-wise.

If μ and ν have the same sign and are both increasing or both decreasing, then

$$D_t^{\wp}(\mu\nu)(t) \leq \mu(t) D_t^{\wp}\nu(t) + \nu(t) D_t^{\wp}\mu(t)$$

and for $\mu = \nu$,

$$D_t^{\wp}(\mu^2)(t) \leq 2\mu(t) D_t^{\wp}\mu(t). \tag{1}$$

Lemma 1 aims to confirm a conjecture by J. I. Diaz *et al.* [16]. They conjectured that for $\wp \in (0,1)$, inequality (1) that includes the Riemann–Liouville fractional derivative holds true.

We focus on entropy solutions for the fractional Cauchy problem of nonsymmetric systems. We execute sufficient conditions on the parameters to obtain a bounded solutions of L^{∞}. The solution is unique and exclusive. Performance is established by applying Lemma 1. The fractional differential operator is inspected according to the interpretation of the Riemann–Liouville differential operator. Various studies have discussed the fractional Cauchy problem [17,18] and entropy analysis [19–21].

2. Proposed Fractional System

We introduce the proposed nonsymmetric fractional system. The Cauchy problem for nonsymmetric system of Keyfitz–Kranzer type is given by the formula [22]

$$\begin{aligned} &\mu_t + \Big(\mu\,\theta(\mu, \omega_1, ..., \omega_n)\Big)_{\chi} = 0 \\ &\big(\mu\,\omega_j\,(t,\chi)\big)_t + \Big(\mu\,\omega_j\;\theta(\mu, \omega_1, ..., \omega_n)\Big)_{\chi} = 0, \quad j = 1, ..., n. \end{aligned}$$

The generalization of the system can be written by virtue of the Riemann–Liouville fractional calculus:

$$\begin{aligned} &D_t^{\wp}\mu(t,\chi) + \Big(\mu\,\theta(\mu, \omega_1, ..., \omega_n)\Big)_{\chi} = 0 \\ &D_t^{\wp}\big(\mu\,\omega_j\,(t,\chi)\big) + \Big(\mu\,\omega_j\;\theta(\mu, \omega_1, ..., \omega_n)\Big)_{\chi} = 0, \quad j = 1, ..., n \end{aligned} \tag{2}$$

with bounded measurable initial condition

$$\Big(\mu(0,\chi), \omega_j(0,\chi)\Big) = \Big(\mu_0(\chi), \omega_{j0}(\chi)\Big), \quad \mu_0(\chi) \geq 0,\ j = 1, ..., n, \tag{3}$$

and

$$\theta(\mu, \omega) := \Theta(\omega) - \Lambda(\mu) \tag{4}$$

is a nonlinear function, μ, ω are the density and the velocity of vehicles, while the function Λ is smooth and strictly increasing. The symmetric fractional system of (2) can be viewed as

$$D_t^{\wp}\omega_j(t,\chi) + \Big(\omega_j\, \theta(\mu, \omega_1, ..., \omega_n)\Big)_{\chi} = 0, \quad j = 1, ..., n,$$

where

$$\theta(\omega) = \sum_{j=1}^{n} \omega_j^k, \quad k > 1.$$

When $n = 1$ and $\Theta(\omega) = \omega$ in (4), System (2) reduces to the non symmetric form

$$\begin{aligned} D_t^{\wp}\mu(t,\chi) + \Big(\mu\,(\omega - \Lambda(\mu))\Big)_{\chi} &= 0 \\ D_t^{\wp}(\mu\omega)(t,\chi) + \Big(\mu\omega(\omega - \Lambda(\mu))\Big)_{\chi} &= 0. \end{aligned} \tag{5}$$

If we let $\nu := \mu\omega$, then we obtain the system

$$\begin{aligned} D_t^{\wp}\mu(t,\chi) + \Big(\nu - \mu\Lambda(\mu)\Big)_{\chi} &= 0 \\ D_t^{\wp}\nu(t,\chi) + \Big(\frac{\nu^2}{\mu} - \nu\Lambda(\mu)\Big)_{\chi} &= 0, \end{aligned} \tag{6}$$

with the bounded initial condition

$$\Big(\mu(0,\chi), \nu(0,\chi)\Big) = \Big(\mu_0(\chi), \nu_0(\chi)\Big), \quad \mu_0(\chi) \geq 0.$$

For $\Lambda(\mu) = \mu$, system (6) can be viewed as

$$\begin{aligned} D_t^{\wp}\mu(t,\chi) + \Big(\nu - \mu^2\Big)_{\chi} &= 0 \\ D_t^{\wp}\nu(t,\chi) + \Big(\frac{\nu^2}{\mu} - \nu\mu\Big)_{\chi} &= 0. \end{aligned} \tag{7}$$

System (2), for an integer case, was addressed by Keyfitz and Krranzer [22] as a model for an elastic string. System (5) was imposed by Aw and Rascle [23] as a macroscopic model for traffic flow, where μ, ω are the density and velocity of vehicles on the road, respectively. Systems (6) and (7) are pressure-less gas dynamic system models [24].

3. Solutions and Entropy Solutions

We study the following fractional system based on the above mentioned construction fractional dynamic systems:

$$\begin{aligned} D_t^{\wp}\mu(t,\chi) + \nabla\Big(\nu - \mu\Lambda(\mu)\Big) &= 0 \\ D_t^{\wp}\nu(t,\chi) + \nabla\Big(\frac{\nu^2}{\mu} - \nu\Lambda(\mu)\Big) &= 0, \end{aligned} \tag{8}$$

with the bounded initial condition

$$\Big(\mu(0,\chi), \nu(0,\chi)\Big) = \Big(\mu_0(\chi), \nu_0(\chi)\Big), \quad \mu_0(\chi) \geq 0,$$

where $t \in J := (0,T]$, $T < \infty$, $\Omega \in \mathbb{R}^2$ is a bounded domain, and the couple $(\mu,\nu) \in \big(C[J,\Omega], C[J,\Omega]\big)$ denotes the solution of system (8). Moreover, it achieves

$$\frac{\partial\mu}{\partial\zeta} = \frac{\partial\nu}{\partial\zeta} = 0, \quad \zeta \in \partial\Omega,$$

when μ,ν are smooth in J.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Assume that*

$$(\mu_0,\nu_0) \in H^1(\Omega) \times H^1(\Omega), \quad \mu_0 > 0,\ \nu_0 \geq 0,\ \text{in } \overline{\Omega}$$

where $H^1(\Omega) = \{u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega)\}$. If $\nu^2 \leq \mu^2$ and $\frac{CT^\wp}{\Gamma(\wp+1)} < 1$, $C > 0$, then there exists a unique bounded solution (μ,ν) for system (8).

Proof. The first three steps of the proof describe priori estimates whereas Step 4 addresses uniqueness.

Step 1. First estimate. We aim to prove that $(\mu,\nu) \in \big(L^2(\Omega), L^2(\Omega)\big)$. By expanding the first equation in (8) by μ, utilizing (1) and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}D_t^\wp \int_\Omega \mu^2(t,\chi) &= \frac{1}{2}\int_\Omega D_t^\wp \mu^2(t,\chi)\\ &\leq \int_\Omega \mu(t,\chi) D_t^\wp \mu(t,\chi)\\ &= -\int_\Omega \mu\nabla\Big(\nu - \mu\Lambda(\mu)\Big).\end{aligned}$$

By applying the Cauchy-Schwartz and Young inequalities, we derive

$$\begin{aligned}\frac{1}{2}D_t^\wp \|\mu\|_{L^2}^2 &\leq \|\mu\|_{L^2}\ \|\nu - \mu\Lambda(\mu)\|_{L^2}\\ &\leq \frac{1}{2}\|\mu\|_{L^2} + \frac{1}{2}\|\nu - \mu\Lambda(\mu)\|_{L^2}\end{aligned}$$

Thus by using the triangle inequality, we obtain

$$\begin{aligned}D_t^\wp \|\mu\|_{L^2}^2 &\leq \|\mu\|_{L^2} + \|\nu - \mu\Lambda(\mu)\|_{L^2}\\ &\leq \frac{3}{2}\|\mu\|_{L^2} + \ \|\nu\|_{L^2} + \frac{1}{2}\|\Lambda(\mu)\|_{L^2}.\end{aligned} \tag{9}$$

Similarly, the product of second equation in (8) by ν yields

$$\frac{1}{2}D_t^\wp \int_\Omega \nu^2(t,\chi) = -\int_\Omega \nu\nabla\Big(\frac{\nu^2}{\mu} - \nu\Lambda(\mu)\Big).$$

The above equation implies

$$\begin{aligned}D_t^\wp \|\nu\|_{L^2}^2 &\leq \|\nu\|_{L^2} + \|\frac{\nu^2}{\mu} - \nu\Lambda(\mu)\|_{L^2}\\ &\leq \frac{3}{2}\|\nu\|_{L^2} + \|\frac{\nu^2}{\mu}\|_{L^2} + \frac{1}{2}\|\Lambda(\mu)\|_{L^2}\\ &\leq \frac{3}{2}\|\nu\|_{L^2} + \|\mu\|_{L^2} + \frac{1}{2}\|\Lambda(\mu)\|_{L^2}\end{aligned} \tag{10}$$

Combining (9) and (10) indicates

$$D_t^{\wp}\Big(\|\mu\|_{L^2}^2+\|\nu\|_{L^2}^2\Big)\le\frac{5}{2}\Big(\|\mu\|_{L^2}+\|\nu\|_{L^2}\Big)+\|\Lambda(\mu)\|_{L^2}. \tag{11}$$

By employing

$$\|\theta(.,t)\|_{L^2}\le\frac{\|\theta(.,t)\|_{L^2}^2+1}{2},$$

inequality (11) becomes

$$D_t^{\wp}\Big(\|\mu\|_{L^2}^2+\|\nu\|_{L^2}^2\Big)\le\frac{5}{4}\Big(\|\mu\|_{L^2}^2+\|\nu\|_{L^2}^2\Big)+\frac{1}{2}\|\Lambda(\mu)\|_{L^2}^2+\frac{7}{4}. \tag{12}$$

By applying the generalized Gronwall lemma, we achieve

$$\sup_{t\in J}\Big(\|\mu\|_{L^2}^2+\|\nu\|_{L^2}^2\Big)\le\kappa_1E_{\wp}(\kappa_2T^{\wp})+\kappa_3,$$

where κ_1,κ_2 and κ_3 are sufficient large positive constants and $E_{\wp}$ is the Mittag-Leffler function. Hence solution (μ,ν) is bounded in $L^2(\Omega)$.

Step 2. Second estimate. We intend to prove that $(\mu,\nu)\in\big(L^\infty(\Omega),L^\infty(\Omega)\big)$.
Accumulating the first equation in (8) by $\triangle\mu$ (Laplace operator) and integrating over Ω, by considering that μ vanishes on the boundary of Ω Lemma 1, leads to

$$\begin{aligned}\int_\Omega D_t^{\wp}(\mu.\triangle\mu)=D_t^{\wp}\int_\Omega(\mu.\triangle\mu)&\le\int_\Omega\triangle\mu.D_t^{\wp}\mu\\&=-\int_\Omega\triangle\mu.\nabla\Big(\nu-\mu\Lambda(\mu)\Big)\\&\le K_1\int_\Omega\triangle\mu+K_2\int_\Omega\mu\triangle\mu+K_3\int_\Omega\triangle\mu.\nabla\mu.\end{aligned}$$

Using this equation, along with the Sobolev embedding, for $\nabla\nu\in L^2(\Omega)$ and $\nabla\Lambda\in L^2(\Omega)$ implies that there are two positive constants, namely, K_1 and K_2 such that $\|\nabla\nu\|_{L^2}\le K_1$ and $\|\nabla\Lambda\|_{L^2}\le K_2$. Consequently, we let $\|\Lambda\|_{L^2}\le K_3,\ K_3>0$. Integration by part for the left hand side of the above inequality, which is based on the Cauchy- Schwartz inequality results in

$$D_t^{\wp}\|\nabla\mu(.,t)\|_{L^2}^2\le C_1\Big(\|\triangle\mu(.,t)\|_{L^2}^2+2\|\nabla\mu(.,t)\|_{L^2}^2+\|\mu(.,t)\|_{L^2}^2+\frac{1}{2}\Big), \tag{13}$$

where $C_1:=\max\{K_i,\ i=1,2,3\}$ is a positive constant. Similarly, by multiplying the second equation in (8) by $\triangle\nu$, and kipping in mind that ν vaporizes on the boundary of Ω, Lemma 1 implies that

$$\begin{aligned}\int_\Omega D_t^{\wp}(\nu.\triangle\nu)=D_t^{\wp}\int_\Omega(\nu.\triangle\nu)&\le\int_\Omega\triangle\nu.D_t^{\wp}\nu\\&=-\int_\Omega\triangle\nu.\nabla\Big(\frac{\nu^2}{\mu}-\nu\Lambda(\mu)\Big)\\&\le\epsilon_1\int_\Omega\triangle\nu+K_2\int_\Omega\nu\triangle\nu+K_3\int_\Omega\triangle\nu.\nabla\nu,\end{aligned}$$

which, together with the Sobolev embedding, yields positive value of constant ϵ_1 satisfying $\|\nabla\mu\|_{L^2}\le\ \epsilon_1$. Thus, we have

$$D_t^{\wp}\|\nabla\nu(.,t)\|_{L^2}^2\le C_2\Big(\|\triangle\nu(.,t)\|_{L^2}^2+2\|\nabla\nu(.,t)\|_{L^2}^2+\|\nu(.,t)\|_{L^2}^2+\frac{1}{2}\Big), \tag{14}$$

where $C_2 := \max\{\epsilon_1, K_2, K_3\}$ is a positive constant. Combining (13) and (14) implies that

$$D_t^{\wp}\Big(\|\nabla\mu(.,t)\|_{L^2}^2 + \|\nabla\nu(.,t)\|_{L^2}^2\Big) \leq C\Big(\|\triangle\mu(.,t)\|_{L^2}^2 + 2\|\nabla\mu(.,t)\|_{L^2}^2 + \|\mu(.,t)\|_{L^2}^2 \\ + \|\triangle\nu(.,t)\|_{L^2}^2 + 2\|\nabla\nu(.,t)\|_{L^2}^2 + \|\nu(.,t)\|_{L^2}^2 + 1\Big), \tag{15}$$

where $C := \max\{C_1, C_2\}$ is a positive constant. By exploiting the generalized Gronwall lemma and the condition $(u_0, \mu_0) \in H^1(\Omega) \times H^1(\Omega)$, we obtain $(\mu, \nu) \in \big(L^\infty(\Omega), L^\infty(\Omega)\big)$.

Step 3. Upper bound. We aim to determine the upper bound of the fractional derivative. Let

$$\Upsilon(t) := \|\triangle\mu(.,t)\|_{L^2}^2 + \|\triangle\nu(.,t)\|_{L^2}^2,$$

$$\curlywedge(t) := 2\Big(\|\nabla\mu(.,t)\|_{L^2}^2 + \|\nabla\nu(.,t)\|_{L^2}^2\Big)$$

and

$$\lambda(t) := \|\mu(.,t)\|_{L^2}^2 + \|\nu(.,t)\|_{L^2}^2.$$

Given that μ and ν vanish at the boundary of Ω, we conclude that

$$0 = \int_\Omega \mu \leq \int_\Omega \mu_0, \quad 0 = \int_\Omega \nu \leq \int_\Omega \nu_0,$$

thus from [15], Remark 2, we have

$$\|\mu(.,t)\|_{L^2}^2 \leq \|\mu_0\|_{L^2}^2, \quad \|\nu(.,t)\|_{L^2}^2 \leq \|\nu_0\|_{L^2}^2.$$

Operating (15) by I^α, we derive

$$\begin{aligned} \curlywedge(t) &\leq \curlywedge_0 + C\Big(\int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)} \curlywedge(\tau)d\tau + \int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)} \Upsilon(\tau)d\tau \\ &+ \int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\lambda(\tau)d\tau + \int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}d\tau\Big) \\ &\leq \curlywedge_0 + \frac{C\,T^\wp}{\Gamma(\wp+1)} \sup_{t\in(0,T]} \curlywedge(t) + C\int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\Big(\Upsilon(\tau) + \lambda(\tau) + 1\Big)d\tau \\ &:= \curlywedge_0 + \frac{C\,T^\wp}{\Gamma(\wp+1)} \sup_{t\in(0,T]} \curlywedge(t) + C\int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\Psi(\tau)d\tau, \end{aligned} \tag{16}$$

where $\Psi := \Upsilon(\tau) + \lambda(\tau) + 1.$ Simple calculation implies

$$\begin{aligned} \sup_{t\in(0,T]} \curlywedge(t) &\leq \frac{\curlywedge_0}{1 - \frac{CT^\wp}{\Gamma(\wp+1)}} + \frac{C}{1 - \frac{CT^\wp}{\Gamma(\wp+1)}} \int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\Psi(\tau)d\tau \\ &:= \alpha_0 + \alpha_1 \sup_{t\in(0,T]} \int_0^t \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\Psi(\tau)d\tau. \end{aligned} \tag{17}$$

Hence, for all $t \in J$, we realize that

$$\sup_{t\in(0,T]} \curlywedge(t) \leq \alpha, \tag{18}$$

where α is a positive constant depending on $\wp, C, \lambda_0$ and $\sup_{t\in J}\|\Psi\|$.

Step 4. Uniqueness. Let (υ_1, υ_2) and (ν_1, ν_2) be two solutions for system (8) under the identical initial condition $(\upsilon_1^0, \upsilon_2^0) \in H^1(\Omega)$. Set $\mu = \upsilon_1 - \nu_1$ and $\nu = \upsilon_2 - \nu_2$ to arrive at

$$\begin{aligned} D_t^{\wp}\mu(t,\chi) + \nabla\Big(\nu - \mu\Lambda(\mu)\Big) &= 0 \\ D_t^{\wp}\nu(t,\chi) + \nabla\Big(\frac{\nu^2}{\mu} - \nu\Lambda(\mu)\Big) &= 0, \end{aligned} \tag{19}$$

Multiply the first equation in (19) by μ and the second equation in (19) by ν and integrate over Ω to obtain relation (12). By employing the generalized Gronwall lemma, we conclude that

$$\sup_{t\in(0,T]}\Big(\|\mu(t,.)\|_{L^2}^2 + \|\nu(t,.)\|_{L^2}^2\Big) \le \sigma,$$

where σ is an arbitrary constant depending on $T, \wp$ and the initial condition. System (8) admits a unique bounded global solution (μ, ν) of arbitrary initial value, satisfying $\mu^2 \ge \nu^2$. This completes the proof. □

Subsequently, we discuss the solutions for system (8) when $\mu^2 \le \nu^2$. In this case, we only obtain entropy solutions.

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Assume that*

$$(\mu_0, \nu_0) \in H^1(\Omega) \times H^1(\Omega), \quad \mu_0 \ge 0,\ \nu_0 > 0,\ in\overline{\Omega}$$

If $\nu^2 \ge \mu^2$, then system (8) satisfies the entropy fractional inequality

$$D_t^{\wp}\int_\Omega(\mu\ln\mu + \nu\ln\nu) + \int_\Omega\Big(\ln\mu.\nabla\nu + \ln\nu.\nabla\mu\Big) \le 4K_3\Big(\|\mu\|_{L^2} + \|\nu\|_{L^2}\Big), \tag{20}$$

where $\|\Lambda\|_{L^2} \le K_3,\ K_3 > 0.$

Proof. Multiplying the first equation in (8) by $\ln\mu$, integrating over Ω and exploiting Lemma 1, we arrive at

$$D_t^{\wp}\int_\Omega \mu\ln\mu \le \int_\Omega \mu D_t^{\wp}\ln\mu + \int_\Omega \ln\mu\, D_t^{\wp}\mu.$$

By utilizing the Cauchy- Schwartz inequality and yielding that μ vanishes on Ω, we take out

$$D_t^{\wp}\int_\Omega \mu\ln\mu \le \int_\Omega \ln\mu\, D_t^{\alpha}\mu = -\int_\Omega \ln\mu.\nabla\Big(\nu - \mu\Lambda(\mu)\Big).$$

A calculation implies

$$\begin{aligned} D_t^{\wp}\int_\Omega \mu\ln\mu &\le -\int_\Omega \ln\mu.\nabla\nu + \int_\Omega \ln\mu.\mu\nabla\Lambda(\mu) + \int_\Omega \ln\mu.\Lambda(\mu)\nabla\mu \\ &= -\int_\Omega \ln\mu.\nabla\nu + \int_\Omega \ln\mu.\Lambda(\mu)\nabla\mu. \end{aligned}$$

Thus, by using (see [25])

$$\int_\Omega \ln\mu.\,\nabla\mu \le 4\|\mu\|_{L^2},$$

we obtain

$$D_t^{\wp}\int_{\Omega}\mu\ln\mu+\int_{\Omega}\ln\mu.\nabla\nu\leq\int_{\Omega}\ln\mu.\,\nabla\mu.\,\Lambda(\mu)\leq 4K_3\|\mu\|_{L^2}. \tag{21}$$

Based on our assumption ($\nu^2\geq\mu^2$), we conclude that

$$D_t^{\wp}\int_{\Omega}\nu\ln\nu+\int_{\Omega}\ln\nu.\nabla\mu\leq\int_{\Omega}\ln\nu.\,\nabla\nu.\,\Lambda(\mu)\leq 4K_3\|\nu\|_{L^2}. \tag{22}$$

We arrive at the desired assertion by combining (21) and (22). This step completes the proof. □

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Assume that*

$$(\mu_0,\nu_0)\in H^1(\Omega)\times H^1(\Omega),\quad \mu_0\geq 0,\ \nu_0>0,\ in\overline{\Omega}$$

If $\nu^2\geq\mu^2$ then system (8) admits a bounded entropy solution.

Proof. Consider the fractional Cauchy problem

$$\begin{aligned}D_t^{\wp}\mu(t,\chi)+\nabla\Big(\nu-\mu\Lambda(\mu)\Big)&=-\ell\,\triangle\mu\\ D_t^{\wp}\nu(t,\chi)+\nabla\Big(\frac{\nu^2}{\mu}-\nu\Lambda(\mu)\Big)&=-\ell\,\triangle\nu,\end{aligned} \tag{23}$$

where $\ell>0$, subjected to the initial condition

$$\Big(\mu^{\ell}(0,\chi)=\mu_0(\chi)+\ell,\nu^{\ell}(0,\chi)=\nu_0(\chi)+\ell\Big).$$

It suffices to show that the fractional operator $D_t^{\wp}$ in (23) is bounded. Multiplying the first equation in (23) by $\ln\mu$, integrating over Ω, exploiting Lemma 1, employing the Cauchy–Schwartz inequality and defining that μ vanishes on Ω, we deduce

$$D_t^{\wp}\int_{\Omega}\mu\ln\mu+\int_{\Omega}\ln\mu.\nabla\nu\leq\int_{\Omega}\ln\mu.\,\nabla\mu.\,\Lambda(\mu)-\ell\int_{\Omega}\ln\mu.\,\triangle\mu. \tag{24}$$

By considering the earlier observation [25]

$$\int_{\Omega}\ln\mu\,\triangle\mu=-4\int_{\Omega}|\nabla\mu^{1/2}|^2,$$

and since $\int_{\Omega}\mu\leq\int_{\Omega}\mu_0$, which leads to

$$\|\mu\|_{L^2}^2\leq\|\mu_0\|_{L^2}^2\leq\|\mu^{\ell}\|_{L^2}^2,$$

then the inequality (24) reduces to

$$\begin{aligned}D_t^{\wp}\int_{\Omega}\mu\ln\mu+\int_{\Omega}\ln\mu.\nabla\nu&\leq\int_{\Omega}\ln\mu.\,\nabla\mu.\,\Lambda(\mu)+4\ell\int_{\Omega}|\nabla\mu^{1/2}|^2\\ &\leq 2K_3\big(\|\mu^{\ell}\|_{L^2}^2+1\big)+4\ell\int_{\Omega}|\nabla\mu^{1/2}|^2.\end{aligned} \tag{25}$$

Similarly, we may infer

$$D_t^{\wp}\int_\Omega \nu\ln\nu + \int_\Omega \ln\nu.\nabla\mu \le \int_\Omega \ln\nu.\,\nabla\nu.\,\Lambda(\mu) + 4\ell\int_\Omega |\nabla\nu^{1/2}|^2$$
$$\le 2K_3\big(\|\nu^\ell\|_{L^2}^2 + 1\big) + 4\ell\int_\Omega |\nabla\nu^{1/2}|^2. \tag{26}$$

Combining (25) and (26) and letting $\ell \to 0$, we arrive at

$$D_t^{\wp}\int_\Omega (\mu\ln\mu + \nu\ln\nu) + \int_\Omega \big(\ln\mu.\nabla\nu + \ln\nu.\nabla\mu\big) \le 4K_3.$$

Hence, the proof is completed. □

Theorem 4. *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the system*

$$D_t^{\wp}\mu(t,\chi) + \nabla\Big(\nu - \mu\Lambda(\mu)\Big) = -\varepsilon\,\nabla(\mu\nabla\nu)$$
$$D_t^{\wp}\nu(t,\chi) + \nabla\Big(\frac{\nu^2}{\mu} - \nu\Lambda(\mu)\Big) = -\varepsilon\,\nabla(\nu\nabla\mu), \tag{27}$$

where $\varepsilon > 0$, subjected to the initial condition

$$\Big(\mu^\varepsilon(0,\chi) = \mu_0(\chi) + \varepsilon,\ \nu^\varepsilon(0,\chi) = \nu_0(\chi) + \varepsilon\Big),$$

where

$$(\mu_0,\nu_0) \in H^1(\Omega)\times H^1(\Omega), \quad \mu_0 \ge 0,\ \nu_0 > 0,\ in\ \overline{\Omega}$$

If $\nu^2 \ge \mu^2$ then system (27) admits a bounded entropy solution.

Proof. Again it be adequate to present that the fractional operator $D_t^{\wp}$ in (27) is bounded. Similar to the procedure in Theorem 3, we deduce that by multiplying the first equation in (27) by $\ln\mu$, integrating over Ω, exploiting Lemma 1, applying the Cauchy–Schwartz inequality and determining that μ vanishes on Ω, we conclude that

$$D_t^{\wp}\int_\Omega \mu\ln\mu + \int_\Omega \ln\mu.\nabla\nu \le \int_\Omega \ln\mu.\,\nabla\mu.\,\Lambda(\mu) - \varepsilon\int_\Omega \ln\mu.\,\nabla(\mu\nabla\nu). \tag{28}$$

Since (see [25])

$$\int_\Omega \ln\mu\,\nabla(\mu\nabla\nu) = -\int_\Omega \nabla\mu.\nabla\nu,$$

and

$$\|\mu\|_{L^2}^2 \le \|\mu_0\|_{L^2}^2 \le \|\mu^\varepsilon\|_{L^2}^2,$$

then the inequality (28) reduces to

$$D_t^{\wp}\int_\Omega \mu\ln\mu + \int_\Omega \ln\mu.\nabla\nu \le \int_\Omega \ln\mu.\,\nabla\mu.\,\Lambda(\mu) + \varepsilon\int_\Omega \nabla u.\nabla\mu$$
$$\le 2K_3\big(\|\mu^\varepsilon\|_{L^2}^2 + 1\big) + \varepsilon\int_\Omega \nabla\mu.\nabla\nu. \tag{29}$$

In the same manner, we may derive

$$\begin{aligned} D_t^{\wp}\int_{\Omega}\nu\ln\nu+\int_{\Omega}\ln\nu.\nabla\mu &\leq \int_{\Omega}\ln\nu.\,\nabla\nu.\,\Lambda(\mu)+4\ell\int_{\Omega}|\nabla\nu^{1/2}|^2 \\ &\leq 2K_3\big(\|\nu^{\varepsilon}\|_{L^2}^2+1\big)+\varepsilon\int_{\Omega}\nabla\nu.\nabla\mu. \end{aligned} \tag{30}$$

Summing (29) and (30), we arrive at

$$D_t^{\wp}\int_{\Omega}(\mu\ln\mu+\nu\ln\nu)+\int_{\Omega}\big(\ln\mu.\nabla\nu+\ln\nu.\nabla\mu\big)\leq 2K_3\big(\|\mu^{\varepsilon}\|_{L^2}^2+\|\nu^{\varepsilon}\|_{L^2}^2+2\big)+2\varepsilon\int_{\Omega}\nabla\mu.\nabla\nu.$$

Hence, the proof is completed. □

Corollary 1. *Let the hypotheses of Theorem 4 hold. Then for $\varepsilon \to 0$, system (8) has a bounded entropy solution.*

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Author Contributions

Both authors jointly worked on deriving the results and approved the final manuscript. Both authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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