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Article

Adaptive Synchronization of Fractional Neural Networks with Unknown Parameters and Time Delays

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Abstract: In this paper, the parameters identification and synchronization problem of fractional-order neural networks with time delays are investigated. Based on some analytical techniques and an adaptive control method, a simple adaptive synchronization controller and parameter update laws are designed to synchronize two uncertain complex networks with time delays. Besides, the system parameters in the uncertain network can be identified in the process of synchronization. To demonstrate the validity of the proposed method, several illustrative examples are presented.

Keywords: fractional neural networks; synchronization; adaptive control; parameters identification; time delays

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1. Introduction

Since the pioneering work on Hopfield neural networks was reported in [1], the investigation of the dynamics of neurons has been receiving a lot of attention. In cellular neural networks [2], Petráš pointed out that fractional derivatives instead of the integer order one were a natural choice. Recently, the utilities of fractional-order neural networks have been extensively studied, since the networks provide neurons with a fundamental and general computation ability that can contribute to efficient information processing, stimulus anticipation and frequency-independent phase shifts of oscillatory neuronal firing [3]. It is well known that the uncertainty and time delay are unavoidable in many practical situations. For example, due to the finite switching speed of amplifier circuits in neural networks or dealing with motion-related problems, time delays exist in the information processing of neurons [4]. On the other hand, most studies have been available under the assumption that the system parameters are assumed to be known in advance. However, under some circumstances, it is difficult to determine the values of parameters. Therefore, the fractional neural networks with unknown parameters and time delay are more general and reasonable in the real world.

If the parameters and time delays are appropriately chosen, the neural networks can exhibit complicated behaviors even with strange chaotic attractors. Meanwhile, synchronization of coupled neural networks has been investigated due to its potential applications in various engineering, including chaos generators design, secure communications, chemical and biological systems, information processing, distributed computation, optics, social science, harmonic oscillation generation, human heartbeat regulation and power system protection (see, e.g., [4–11] and the references therein). However, there are very limited results on the synchronization of fractional-order neural networks.

In this paper, an identification method based on fractional adaptive synchronization is applied to parameter identification of fractional-order neural networks with time delays. Fractional adaptive synchronization was a generalization of the integer case [12,13]. The main contributions of this paper mainly include three aspects: (i) An adaptive controller is first proposed, which is more general than the nonlinear controller [14–16]. Furthermore, the adaptive controller can be used to identify the unknown parameters of nonlinear part, but the nonlinear one may not. (ii) Our model of fractional neural networks with time delays and unknown parameters is more general. (iii) Based on a novel Lyapunov-like function and the Gronwall–Bellman integral inequality, we derive synchronization criteria analytically.

As mentioned above, fractional-order neural networks are very effective at applications due to their infinite memory. Besides, the fractional-order parameter and time delays enrich the system performance by increasing freedom. Synchronization of fractional neural networks with time delays may be more useful in many applications, such as information, pattern recognition and image processing. Therefore, it is necessary and interesting to study time-delayed fractional neural networks both in theory and in applications. The outline of the paper is organized as follows: some preliminaries and fractional-order neural networks are introduced in Section 2. The adaptive controller is proposed for two coupling networks, such that they can be synchronized in the following section. Numerical experiments are presented to support the theoretical analysis in Section 4. The conclusions are given in the last section.

2. Preliminaries and Model Description

In the following, we introduce some basic definitions and the corresponding results, which will be used later on.

Definition 1 ([17]). *The fractional integral of order α for function f is defined as:*

$$D_{t_0,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \tag{1}$$

where $t \geq t_0$, $\alpha > 0$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2 ([17]). *The α-th-order Caputo fractional derivative of the given function $f(t)$ is defined as:*

$${}_C D_{t_0,t}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad m-1 < \alpha < m \in \mathbb{Z}^{+}. \tag{2}$$

In most situations, the initial time t_0 is often set to zero. Throughout the paper, $t_0 = 0$ and $D_{0,t}^{-\alpha}$ is simply denoted by I^{α} and ${}_C D_{0,t}^{\alpha}$ by D^{α} for brevity.

Lemma 1 ([18]). *If $x(t) \in C^1[0,b]$ and $0 < \alpha < 1$, then:*

$$(1) \quad D^{\alpha} I^{\alpha} x(t) = x(t), \quad 0 < t < b; \tag{3}$$

$$(2) \quad I^{\alpha} D^{\alpha} x(t) = x(t) - x(0), \quad 0 < t < b. \tag{4}$$

Lemma 2 ([19]). *Assume that $x(t) \in C^1[0,b]$ and satisfies:*

$$D^{\alpha} x(t) = f(t, x(t)) \geq 0, \quad 0 < \alpha < 1$$

for all $t \in [0,b]$, then $x(t)$ is monotonously non-decreasing. If

$$D^{\alpha} x(t) = f(t, x(t)) \leq 0, \quad 0 < \alpha < 1$$

then $x(t)$ is monotonously non-increasing.

Lemma 3 ([20]). *Let $x(t) \in \mathbb{R}$ be a continuous and differentiable function. Then, for any time instant $t \geq 0$:*

$$\frac{1}{2} D^{\alpha} [x^2(t)] \leq x(t) D^{\alpha} x(t), \quad 0 < \alpha < 1.$$

Lemma 4 ([21]). *For the given vectors x, y and a positive definite matrix $Q > 0$ with compatible dimensions, the following inequality holds,*

$$2x^T y \leq x^T Q x + y^T Q^{-1} y.$$

Lemma 5 (Gronwall–Bellman integral inequality [22]). *If $z(t)$ satisfies $z(t) \leq \int_0^t a(\tau) z(\tau) \mathrm{d}\tau + b(t)$ with $a(t)$ and $b(t)$ being known real functions, then*

$$z(t) \leq \int_0^t a(\tau) b(\tau) \exp\left(\int_{\tau}^{t} a(r) \mathrm{d}r \right) \mathrm{d}\tau + b(t).$$

If $b(t)$ is differentiable, then

$$z(t) \leq b(0) \exp\left(\int_0^t a(\tau) \mathrm{d}\tau \right) + \int_0^t \dot{b}(\tau) \exp\left(\int_\tau^t a(r) \mathrm{d}r \right) \mathrm{d}\tau.$$

In particular, if $b(t)$ is a constant, it immediately follows that

$$z(t) \leq b(0) \exp\left(\int_0^t a(\tau) \mathrm{d}\tau \right).$$

We study the following drive and response fractional neural networks with time delays:

$$D^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^m a_{ij} f_j(x_j(t)) + \sum_{j=1}^m b_{ij} g_j(x_j(t-\tau)) + I_i, \tag{5}$$

$$D^\alpha y_i(t) = -c_i y_i(t) + \sum_{j=1}^m \hat{a}_{ij} f_j(y_j(t)) + \sum_{j=1}^m \hat{b}_{ij} g_j(y_j(t-\tau)) + I_i + u_i(t). \tag{6}$$

Here, $0 < \alpha < 1, i = 1, 2, \cdots, m$; $x_i(t)$ denotes the state variable of the i-th neuron at time t; a_{ij} and b_{ij} denote the connection strengths and the time delay connection strengths, respectively; $\hat{a}_{ij}$ and $\hat{b}_{ij}$ are the estimations for the unknown connection strengths a_{ij} and b_{ij}; $f_j(x_j(t)), g_j(x_j(t))$ denote the activation functions of neurons; τ is the time delay; I_i is the input; $u_i(t)$ is a controller.

Hereafter, suppose that the activation functions $f_i(u)$ and $g_i(u)$ satisfy the Lipschitz conditions; that is, there exist constants $F_i > 0, G_i > 0$, such that:

$$|f_i(u) - f_i(v)| \leq F_i |u - v|, \quad |g_i(u) - g_i(v)| \leq G_i |u - v| \tag{7}$$

for $u, v \in \mathbb{R}$ and $i = 1, 2, \cdots, m$.

3. Adaptive Controller for Uncertain Fractional-Order Neural Networks

In this section, we study the adaptive synchronization of two coupled fractional neural networks with unknown parameters. By the adaptive control theory, a simple controller for synchronization is designed and parameters identification is realized.

Let $e_i(t) = y_i(t) - x_i(t)$ and $e(t) = y(t) - x(t) = (e_1(t), e_2(t), \cdots, e_m(t))^T$. Our goal is to design controller u, such that the trajectory of the response system (6) with initial condition y_0 can asymptotically approach that of the drive system (5) with initial condition x_0 and, finally, implement synchronization, in the sense that:

$$\lim_{t \longrightarrow \infty} \|e\| = \lim_{t \longrightarrow \infty} \|y(t) - x(t)\| = 0, \tag{8}$$

where $\|\cdot\|$ is the Euclidean norm.

Theorem 1. *The drive and response complex networks (5) and (6) can achieve synchronization, if the controller and the adaptive laws of parameters are taken as:*

$$\begin{cases} u_i(t) = -k_i(t) e_i(t), \\ D^\alpha \hat{a}_{ij} = -q_{ij} e_i(t) f_j(y_j(t)), \\ D^\alpha \hat{b}_{ij} = -r_{ij} e_i(t) g_j(y_j(t-\tau)), \end{cases} \tag{9}$$

where $q_{ij}, r_{ij}, \lambda_i$ are arbitrary positive constants and the feedback strength $k_i(t)$ is adapted according to the following update law:

$$D^{\alpha}k_i(t) = \lambda_i e_i^2(t). \tag{10}$$

Proof. From networks (5) and (6), one can get the following error system:

$$\begin{aligned} D^{\alpha}e_i(t) &= -c_i e_i(t) + \sum_{j=1}^{m} \hat{a}_{ij} f_j(y_j(t)) - \sum_{j=1}^{m} a_{ij} f_j(x_j(t)) \\ &\quad + \sum_{j=1}^{m} \hat{b}_{ij} g_j(y_j(t-\tau)) - \sum_{j=1}^{m} b_{ij} g_j(x_j(t-\tau)) + u_i(t) \\ &= -c_i e_i(t) + \sum_{j=1}^{m} a_{ij}[f_j(y_j(t)) - f_j(x_j(t))] + \sum_{j=1}^{m} b_{ij}[g_j(y_j(t-\tau)) - g_j(x_j(t-\tau))] \\ &\quad + \sum_{j=1}^{m} (\hat{a}_{ij} - a_{ij}) f_j(y_j(t)) + \sum_{j=1}^{m} (\hat{b}_{ij} - b_{ij}) g_j(y_j(t-\tau)) + u_i(t). \end{aligned} \tag{11}$$

Now, we introduce the following Lyapunov-like function for system (11):

$$\begin{aligned} V(t) &= \frac{1}{2}\sum_{i=1}^{m} e_i^2(t) + \frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{\lambda_i}(k_i(t)-\rho_i)^2 + \sum_{j=1}^{m}\frac{1}{q_{ij}}(\hat{a}_{ij}-a_{ij})^2 + \sum_{j=1}^{m}\frac{1}{r_{ij}}(\hat{b}_{ij}-b_{ij})^2\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} e^T(\xi) P e(\xi) \mathrm{d}\xi - \frac{1}{\Gamma(\alpha)}\int_0^{t-\tau} (t-\tau-\xi)^{\alpha-1} e^T(\xi) P e(\xi) \mathrm{d}\xi, \end{aligned} \tag{12}$$

where $\rho_i > 0, i = 1, 2, \cdots, m$ and P is a semi-positive definite matrix under determination.

Define:

$$w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} e^T(\xi) P e(\xi) \mathrm{d}\xi. \tag{13}$$

Applying Equation (3) to (13) yields:

$$D^{\alpha}w(t) = e^T(t) P e(t) \geq 0.$$

By Lemma 2, we have that $w(t)$ is monotonously non-decreasing, *i.e.*, $w(t) \geq w(t-\tau)$. Therefore, $V(t) \geq 0$.

From Lemma 3, Equations (9)–(11), the fractional derivatives of $V(t)$ along the solution can be derived as:

$$\begin{aligned} D^{\alpha}V(t) &\leq \sum_{i=1}^{m} e_i(t) D^{\alpha}e_i(t) + \sum_{i=1}^{m}\frac{1}{\lambda_i}(k_i(t)-\rho_i) D^{\alpha}k_i(t) + \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{1}{q_{ij}}(\hat{a}_{ij}-a_{ij}) D^{\alpha}\hat{a}_{ij} \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{1}{r_{ij}}(\hat{b}_{ij}-b_{ij}) D^{\alpha}\hat{b}_{ij} + e^T(t) P e(t) - e^T(t-\tau) P e(t-\tau) \\ &= -\sum_{i=1}^{m}(c_i+\rho_i) e_i^2(t) + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} e_i(t)[f_j(y_j(t)) - f_j(x_j(t))] \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{m} b_{ij} e_i(t)[g_j(y_j(t-\tau)) - g_j(x_j(t-\tau))] \\ &\quad + e^T(t) P e(t) - e^T(t-\tau) P e(t-\tau). \end{aligned} \tag{14}$$

It immediately follows from (14) that:

$$\begin{aligned} D^{\alpha}V(t) &\leq -e^T(t)(C+\rho)e(t)+e^T(t)A[f(y(t))-f(x(t))]+e^T(t)Pe(t) \\ &\quad +e^T(t)B[g(y(t-\tau))-g(x(t-\tau))]-e^T(t-\tau)Pe(t-\tau). \end{aligned} \tag{15}$$

Here, $C = diag(c_1, c_2, \cdots, c_m)$ and $\rho = diag(\rho_1, \rho_2, \cdots, \rho_m)$ are diagonal matrices; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ are the connection matrices, and:

$$\begin{aligned} f(x(t)) &= [f_1(x_1(t)), f_2(x_2(t)), \cdots, f_m(x_m(t))]^T, \\ g(x(t-\tau)) &= [g_1(x_1(t-\tau)), g_2(x_2(t-\tau)), \cdots, g_m(x_m(t-\tau))]^T. \end{aligned}$$

From Lemma 4 and assumption (7), we can get the following two inequalities:

$$\begin{aligned} & e^T(t)A[f(y(t))-f(x(t))] \\ \leq\ & \frac{1}{2}e^T(t)AA^Te(t)+\frac{1}{2}[f(y(t))-f(x(t))]^T[f(y(t))-f(x(t))] \\ \leq\ & \frac{1}{2}e^T(t)AA^Te(t)+\frac{1}{2}\sum_{i=1}^{m}F_ie_i^2(t) \\ =\ & \frac{1}{2}e^T(t)AA^Te(t)+\frac{1}{2}e^T(t)Fe(t), \end{aligned} \tag{16}$$

and:

$$\begin{aligned} & e^T(t)B[g(y(t-\tau))-g(x(t-\tau))] \\ \leq\ & \frac{1}{2}e^T(t)BB^Te(t)+\frac{1}{2}e^T(t-\tau)Ge(t-\tau), \end{aligned} \tag{17}$$

where $F = diag(F_1, F_2, \cdots, F_m)$ and $G = diag(G_1, G_2, \cdots, G_m)$.

It follows from Equations (16) and (17) that:

$$\begin{aligned} D^{\alpha}V(t) &\leq e^T(t)[-(C+\rho)+\frac{1}{2}(AA^T+BB^T+F)+P]e(t) \\ &\quad +e^T(t-\tau)(\frac{1}{2}G-P)e(t-\tau). \end{aligned}$$

Let:

$$\begin{aligned} P &= \frac{1}{2}G, \\ \rho &= -C+[\frac{1}{2}\lambda_{\max}(AA^T+BB^T+F)+\lambda_{\max}(P)+1]I. \end{aligned}$$

Then, one has:

$$D^{\alpha}V(t) \leq -e^T(t)e(t). \tag{18}$$

After fractional integration on both sides of the inequality (18), we have:

$$I^{\alpha}D^{\alpha}V(t) \leq -I^{\alpha}[e^T(t)e(t)].$$

Using Equation (4) leads to:

$$V(t) \leq V(0)-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^T(\tau)e(\tau)\mathrm{d}\tau.$$

Additionally, $e^T(t)e(t) \leq V(t)$ gives:

$$e^T(t)e(t) \leq V(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} e^T(\tau)e(\tau)\mathrm{d}\tau.$$

By Lemma 5, one has:

$$\begin{aligned} e^T(t)e(t) &\leq V(0)\exp(-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\mathrm{d}\tau) \\ &= V(0)\exp(-\frac{t^\alpha}{\Gamma(\alpha+1)}). \end{aligned}$$

Therefore, $e(t) \to 0$ as $t \to \infty$. □

Remark 1. *It should be noted that Theorem 1 only derives the local stability of the fractional error system (11). In order to successfully identify the unknown parameters, other conditions must be satisfied. When achieving the synchronization, i.e.,* $y_i(t) = x_i(t)$*, as* $t \to \infty$*, error system (11) can be simplified as follows:*

$$\sum_{j=1}^{m}(\hat{a}_{ij} - a_{ij})f_j(x_j(t)) + \sum_{j=1}^{m}(\hat{b}_{ij} - b_{ij})g_j(x_j(t-\tau)) = 0, \quad i = 1, 2, \cdots, m.$$

According to the linear independence [23]*, the correct identification of the unknown parameters* $\hat{a}_{ij}, \hat{b}_{ij}$ *is equivalent to the fact that the function elements* $\{f_j(x_j(t)), g_j(x_j(t-\tau)), j = 1, 2, \cdots, m\}$ *are linearly independent.*

Remark 2. *Assume* $\hat{a}_{ij} = a_{ij}, \hat{b}_{ij} = b_{ij}$*. The system (5) and (6) can be asymptotically synchronized under the following control strategy:*

$$\begin{cases} u_i(t) = -k_i(t)e_i(t), \\ D^\alpha k_i(t) = \lambda_i e_i^2(t), \end{cases}$$

where $\lambda_i > 0$*.*

Remark 3. *Assume* $b_{ij} = \hat{b}_{ij} = 0$*. The complete synchronization between system (5) and system (6) can be realized via the following control strategy:*

$$\begin{cases} u_i(t) = -k_i(t)e_i(t), \\ D^\alpha \hat{a}_{ij} = -q_{ij}e_i(t)f_j(y_j(t)), \\ D^\alpha k_i(t) = \lambda_i e_i^2(t), \end{cases}$$

where q_{ij}, λ_i *are positive.*

4. Numerical Simulation

4.1. Synchronization and Parameter Identification

In this section, numerical experiments are displayed to illustrate the effectiveness of the proposed method.

A group of fractional-order neural networks with time delay is considered, whose integer-order case was reported in [10],

$$D^{\alpha}x(t) = -Cx(t) + Af(x(t)) + Bg(x(t-\tau)), \tag{19}$$

where:

$$\alpha = 0.995, \quad C = I, \quad x(t) = (x_1(t), x_2(t))^T, \quad \tau = 1,$$
$$f(x(t)) = g(x(t)) = h(x(t)) = (\tanh(x_1), \tanh(x_2))^T.$$

The initial conditions for fractional neural networks (19) are given as follows:

$$x_1(s) = -0.4, \quad x_2(s) = -2, \quad \text{for all } s \in [-1, 0].$$

When we choose:

$$B = \begin{pmatrix} -2 & 0.2 \\ 0.3 & -2.5 \end{pmatrix}, \tag{20}$$

and A in the following three forms:

$$A_1 = \begin{pmatrix} 2 & 0.3 \\ 5 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0.5 \\ 5 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 0.45 \\ 5 & 3 \end{pmatrix}, \tag{21}$$

fractional system (19) is chaotic (corresponding to A_1, see Figure 1), has asymptotically stable equilibrium (corresponding to A_2, see Figure 2), and has stable periodic solution (corresponding to A_3, see Figure 3), respectively.

Figure 1. Chaotic attractor of the system (19).

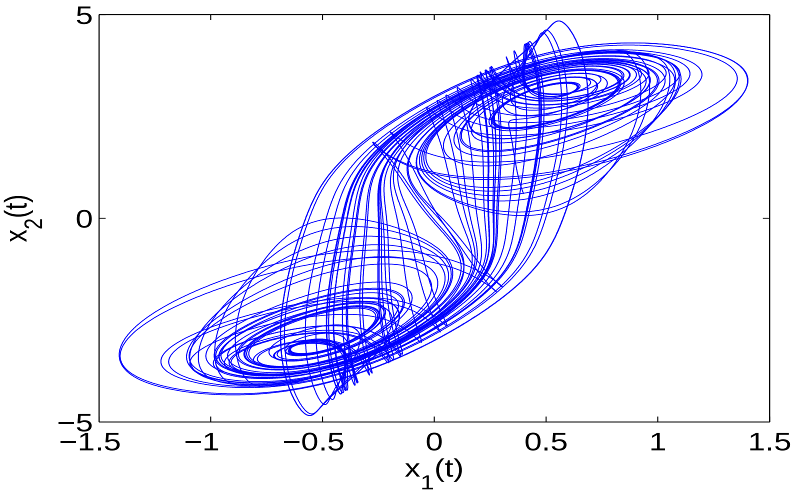


Figure 2. Asymptotically stable equilibrium of the system (19).

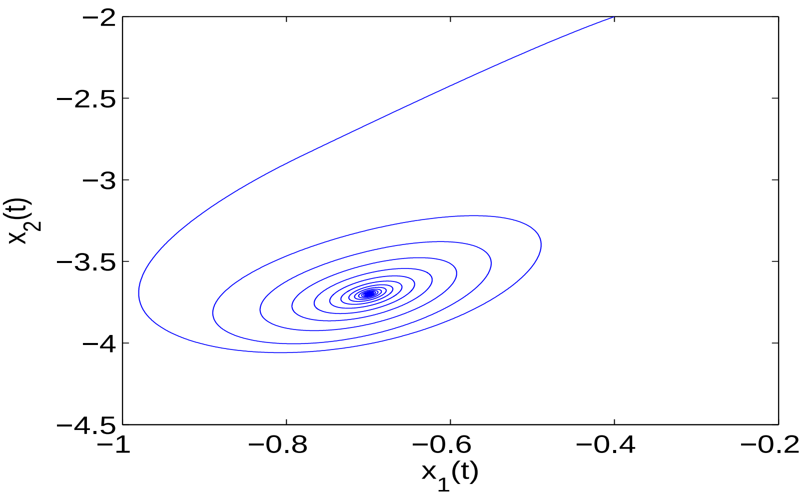
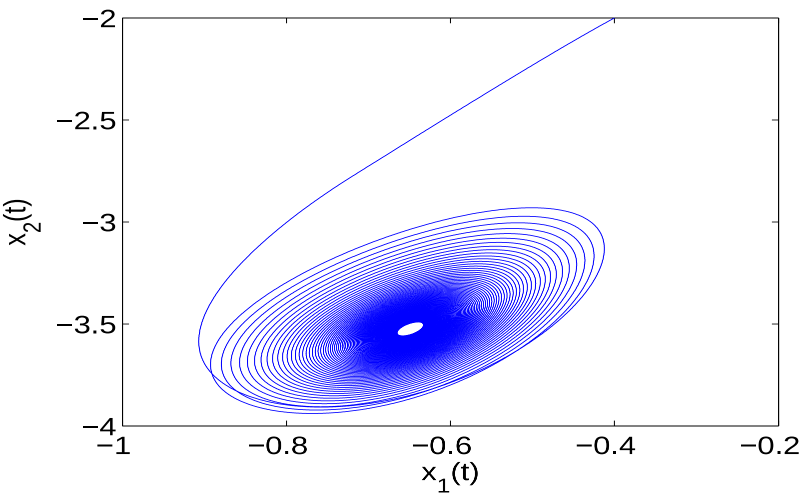


Figure 3. Asymptotically stable periodic solution of the system (19).



In the numerical simulations, the initial conditions and parameter values are given in the following:

$$\begin{cases} x(0) = (0.4, 0.6)^T, y(0) = (1, 1)^T, \\ \hat{a}_{11}(0) = 1, \hat{a}_{12}(0) = 1, \hat{a}_{21}(0) = 1, \hat{a}_{22}(0) = 1, \\ \hat{b}_{11}(0) = 1, \hat{b}_{12}(0) = 1, \hat{b}_{21}(0) = 1, \hat{b}_{22}(0) = 1, \\ k_1(0) = 1, k_2(0) = 1. \end{cases} \tag{22}$$

Now, we choose $A = A_1$, $q_{ij} = 5$, $r_{ij} = 5$ and $\lambda_1 = \lambda_2 = 5$ for the synchronization test.

In the following, the adaptive control strategy (9) and (10) is used to identify the uncertain system parameters. From Figures 4 and 5, it can be clearly seen that all the unknown system parameters $\hat{a}_{ij}$ and $\hat{b}_{ij}$ are successfully identified, respectively. Figure 6 shows that the synchronization error converges to zero asymptotically. From Figure 7, we can see that the adaptive control gains $k_i(t), i = 1, 2$ tend to some positive constants when system (5) and system (6) are synchronized.

Figure 4. Identification of uncertain parameters $\hat{A}$.

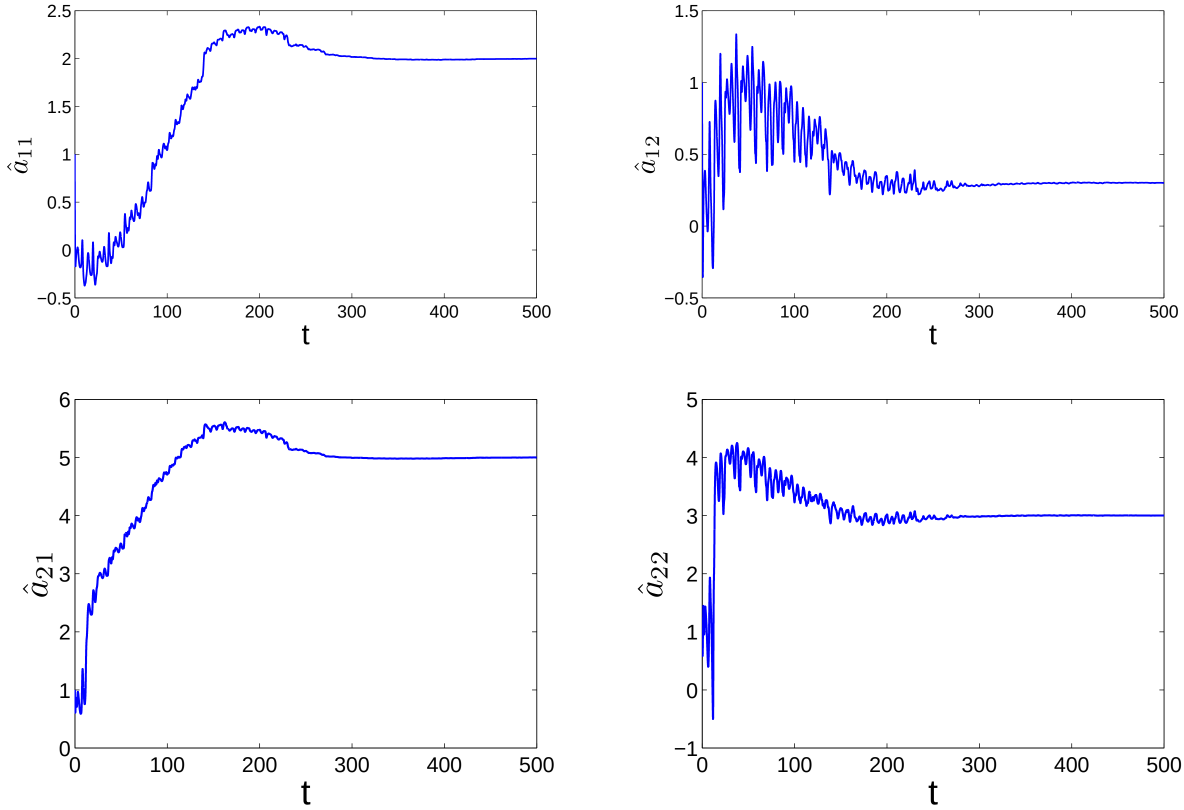


Figure 5. Identification of uncertain parameters $\hat{B}$.

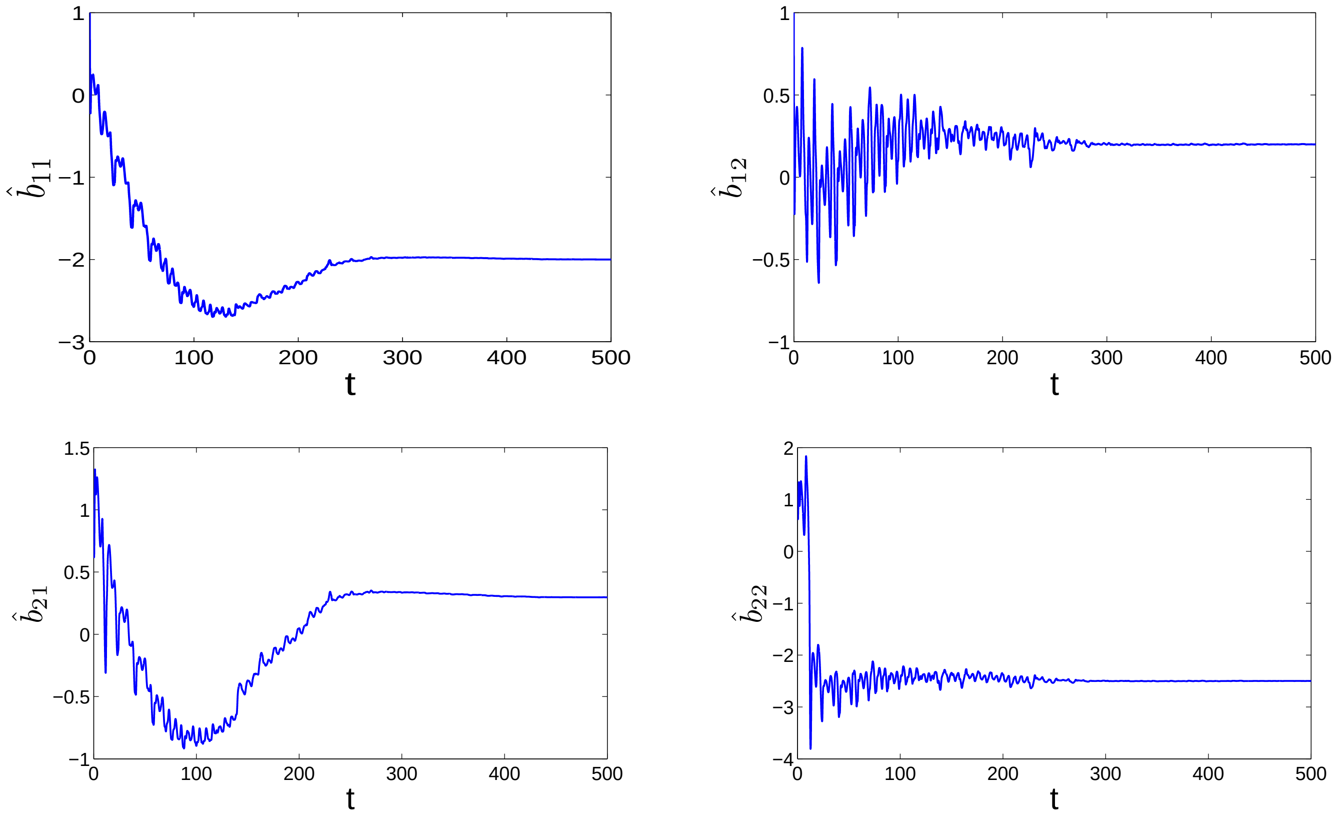


Figure 6. The time evolution of synchronization errors $e_1(t), e_2(t)$.

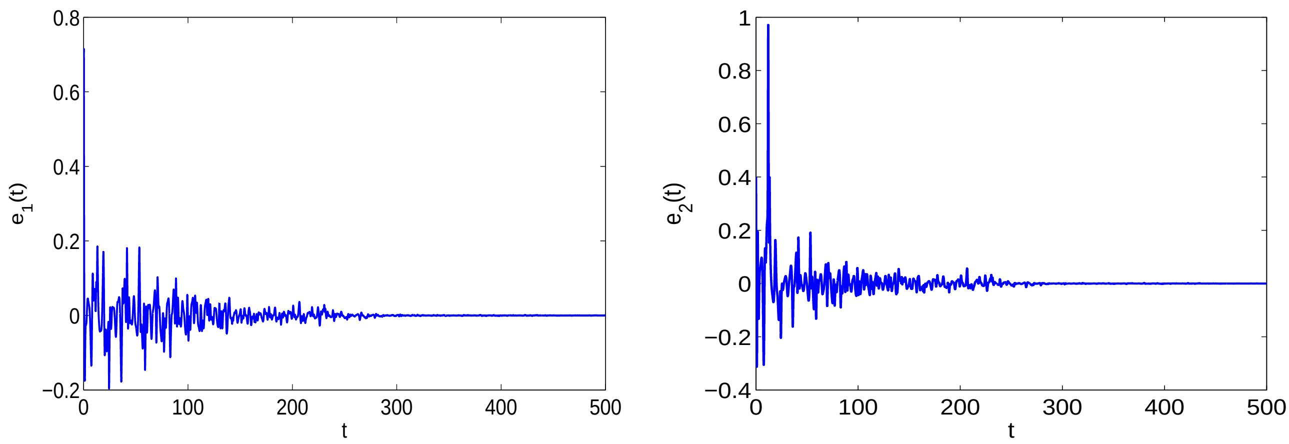
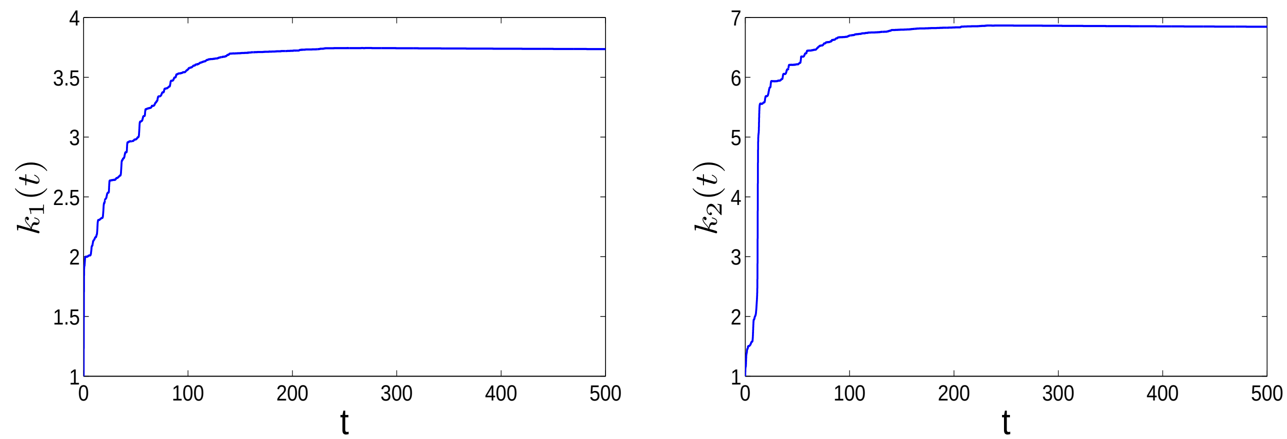


Figure 7. Time evolution of the controlling strength $k_1(t), k_2(t)$.



4.2. The Impact of Fractional Order on the Identification Process

The fractional order α_i has a direct effect on the chaotic behavior of the nonlinear dynamical systems. As a result, it also affects the synchronization behavior of fractional system. Bhalekar *et al.* observed that the synchronization error decreases as the order α increases [12,24]. However, the opposite phenomenon is also observed in [25]. In this section, a more visible method is used to examine its impact on the fractional order neural networks with unknown parameters and time delays.

In order to numerically examine and avoid the linear dependence of the related functions [12,26], we let $\alpha_1 = 0.6, \alpha_2 = 0.8, \alpha_3 = 0.99$. All of the other settings remain unchanged as stated in Section 4.1. From Figures 8–10, we can obtain that the small value of the fractional order α_i would be harmful not only for synchronization state, but also for identification of unknown parameters for fractional neural networks.

Figure 8. The system errors of different fractional.

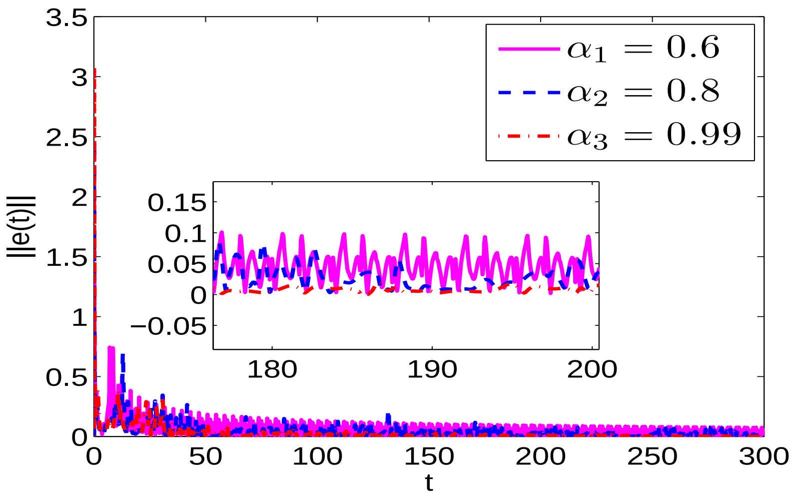


Figure 9. The error norms $\|A - \hat{A}\|$ of different fractional.

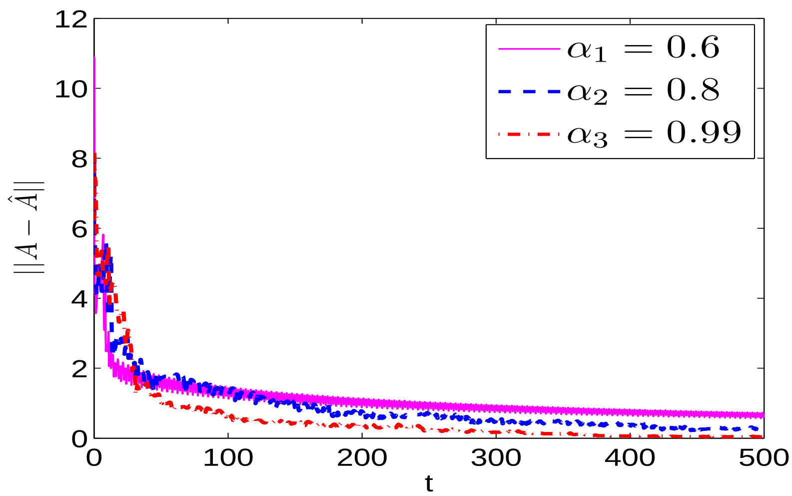
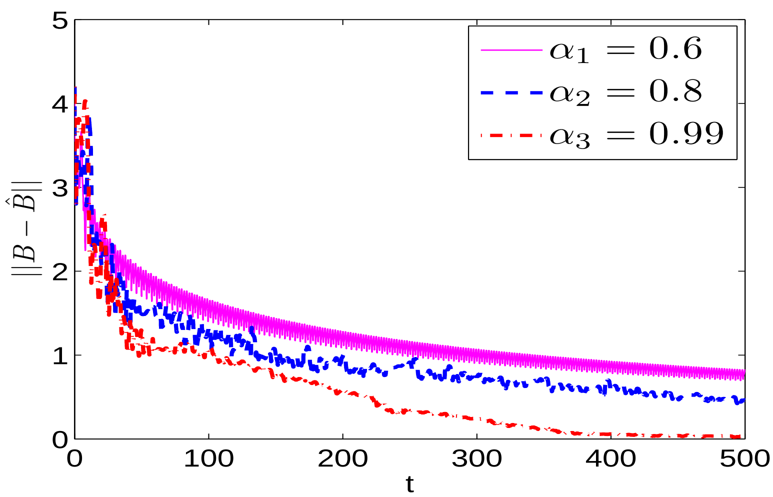


Figure 10. The error norms $\|B - \hat{B}\|$ of different fractional.



5. Conclusions

In this paper, the identification of parameters and synchronization of fractional-order neural networks with time delays were studied. Especially, our designed adaptive controllers for network synchronization are rather simple in form. Moreover, the method discussed in this work can be well applied to other fractional-order complex networks with time delays.

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Author Contributions

In this paper, Weiyuan Ma was in charge of the control theory and adaptive synchronization design. Changpin Li was in charge of the fractional calculus theory and paper writing. Yujiang Wu and Yongqing Wu were in charge of the discussion and the simulation. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Hopfield, J.J. Neural networks and physical systems with emergent collective computational abilities. *Proc. Natl. Acad. Sci. USA* **1982**, *79*, 2554–2558.
2. Petráš, I. A note on the fractional-order cellular neural networks. In Proceedings of International Joint Conference on Neural Networks, Vancouver, BC, Canada, 16–21 July 2006; pp. 1021–1024.
3. Lundstrom, B.; Higgs, M.; Spain, W.; Fairhall, A. Fractional differentiation by neocortical pyramidal neurons. *Nat. Neurosci.* **2008**, *11*, 1335–1342.
4. Gan, Q. Adaptive synchronization of Cohen–Grossberg neural networks with unknown parameters and mixed time-varying delays. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 3040–3049.
5. Gan, Q.; Xu, R.; Kang, X. Synchronization of unknown chaotic delayed competitive neural networks with different time scales based on adaptive control and parameter identification. *Nonlinear Dyn.* **2012**, *67*, 1893–1902.
6. Liao, T.L.; Tsai, S.H. Adaptive synchronization of chaotic systems and its application to secure communications. *Chaos Solitons Fractals* **2000**, *11*, 1387–1396.
7. Song, Q. Design of controller on synchronization of chaotic neural networks with mixed timevarying delays. *Neurocomputing* **2009**, *72*, 13–15.
8. Chen G.; Zhou, J.; Liu, Z. Global synchronization of coupled delayed neural networks and applications to chaotic CNN models. *Int. J. Bifurc. Chaos* **2004**, *14*, 2229–2240.

9. Gan, Q.; Dong, J.; Wu, M. Exponential synchronization for reaction-diffusion neural networks with mixed time-varying delays via periodically intermittent control. *Neurocomputing* **2014**, *9*, 1–25.
10. Wang, K.; Teng, Z.; Jiang, H. Adaptive synchronization of neural networks with time-varying delay and distributed delay. *Physica A* **2008**, *387*, 631–642.
11. Gan, Q.; Hu, R.; Liang, Y. Adaptive synchronization for stochastic competitive neural networks with mixed time-varying delays. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 3708–3718.
12. Si, G.; Sun, Z.; Zhang, H.; Zhang, Y. Parameter estimation and topology identification of uncertain fractional order complex networks. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 5158–5171.
13. Yang, L.; Jiang, J. Adaptive synchronization of driver-response fractional-order complex dynamical networks with uncertain parameters. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 1496–1506.
14. Wu, X.; Lu, H; Shen, S. Synchronization of a new fractional-order hyperchaotic system. *Phys. Lett. A* **2009**, *373*, 2329–2337.
15. Yang, L.; Jiang, J. Hybrid projective synchronization of fractional-order chaotic systems with time delay. *Discret. Dyn. Nat. Soc.* **2013**, *2013*, 459801.
16. Wang, S.; Yu, Y.; Wen, G. Hybrid projective synchronization of time-delayed fractional order chaotic systems. *Nonlinear Anal. Hybrid Syst.* **2014**, *11*, 129–138.
17. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*; Academic Press: San Diego, CA, USA, 1999.
18. Li, C.; Deng, W. Remarks on fractional derivative. *Appl. Math. Comput.* **2007**, *187*, 777–784.
19. Yu, J.; Hua, C.; Jiang, H.; Fan X. Projective synchronization for fractional neural networks. *Neural Netw.* **2014**, *49*, 87–95.
20. Aguila-Camacho, N.; Duarte-Mermoud, M.A.; Gallegos, J.A. Lyapunov functions for fractional order systems. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 2951–2957.
21. Boyd, S.; EI Ghaoui L.; Feron, E.; Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory*. SIAM: Philadelphia, PA, USA, 1994.
22. Slotine, J.J.E.; Li, W.P. *Applied Nonlinear Control*; Prentice Hall: Englewood Cliffs, NJ, USA, 1991.
23. Andrilli, S.; Hecker, D. *Elementary Linear Algebra*, 4th ed.; Academic Press: Burlington, MA, USA, 2010.
24. Bhalekar, S.; Daftardar-Gejji, V. Synchronization of different fractional order chaotic systems using active control. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 3536–3546.
25. Wu, X.; Lai, D; Lu, H. Generalized synchronization of the fractional-order chaos in weighted complex dynamical networks with nonidentical nodes. *Nonlinear Dyn.* **2012**, *69*, 667–683.
26. Tavazoei, M.S.; Haeri, M. Chaotic attractors in incommensurate fractional order systems. *Physica D* **2008**, *237*, 2628–2637.