Entropy **2013**, *15*, 5144-5143; doi:10.3390/e15125144

OPEN ACCESS

entropy

ISSN 1099-4300

www.mdpi.com/journal/entropy

Article

The κ-Generalizations of Stirling Approximation and Multinominal Coefficients

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Received: 20 August 2013; in revised form: 20 November 2013 / Accepted: 21 November 2013 / Published: 26 November 2013

Abstract: Stirling approximation of the factorials and multinominal coefficients are generalized based on the κ-generalized functions introduced by Kaniadakis. We have related the κ-generalized multinominal coefficients to the κ-entropy by introducing a new κ-product operation, which exists only when $\kappa \neq 0$.

Keywords: κ-entropy; κ-exponential; κ-logarithm; Stirling approximation

1. Introduction

As is well-known in the fields of statistical mechanics, Ludwig Boltzmann clarified the concept of entropy. He considered that a macroscopic system consists of a large number of particles. Each particle is assumed to be in one of the energy levels, E_i $(i = 1, \ldots, k)$, and the number of particles in the energy level, E_i, is denoted by n_i. The total number of particles is then $n = \sum_{i=1}^{k} n_i$, and the total energy of the macrostate is $E = \sum_i n_i E_i$. The number of ways of arranging n particles in k energy levels, such that each level, E_i, has n_i, $(i = 1, 2, \ldots, k)$ particles, are given by the multinominal coefficients:

$$\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix} \equiv \frac{n!}{n_1! \cdots n_k!} \tag{1}$$

which is proportional to the probability of the macrostates if all microscopic configurations are assumed to be equally likely. In the thermodynamic limit, in which n increases to infinity, we consider the relative number, $p_i \equiv n_i/n$, as the probability of the particle occupation of a certain energy level, E_i. In order to find the most probable macrostate, Equation (1) is maximized under a certain constraint. The Stirling approximation of the factorials leads to:

$$\ln \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix} \approx n\, S^{\mathrm{BGS}} \left(\frac{n_1}{n}, \ldots, \frac{n_k}{n} \right) \tag{2}$$

where

$$S^{\mathrm{BGS}} \left(\frac{n_1}{n}, \ldots, \frac{n_k}{n} \right) \equiv - \sum_{i=1}^{k} p_i \ln(p_i) \tag{3}$$

is the Boltzmann–Gibbs–Shannon (BGS) entropy.

Recently, one of the authors (H.S.) has shown one-parameter (q) generalizations of Gauss' law of error [1], Stirling approximation [2] and multinominal coefficients [3] based on the Tsallis q-deformed functions and associated multiplication operation (q-product) [4,5]. These mathematical structures are quite fundamental for the basis of any generalization of statistical physics as in the standard statistical physics. In particular, [3] has shown the one-to-one correspondence between the q-multinominal coefficient and Tsallis q-entropy [6–8], *i.e.*, the one-parameter (q) generalization of Equation (2). A combinatorial form of Tsallis entropy was derived in [9].

On the other hand, Kaniadakis [10–12] has proposed the κ-generalized statistical mechanics based on the κ-deformed functions, which are the different type of one-parameter deformations for the exponential and logarithmic functions. Based on the κ-deformed functions and the associated product operation (κ-product), we have already shown the κ-generalization of Gauss' law of error [13].

In this work, we show the κ-generalizations of the Stirling approximation and the relation between the κ-multinominal coefficient and κ-entropy. In the next section, the κ-factorial is introduced based on the κ-product. Then, the κ-generalization of the Stirling approximation is obtained. We see the failure of the naive approach relating the κ-multinominal coefficients with the κ-entropy. In order to overcome this difficulty, we introduce a new kind of κ-product in Section 3 and show the explicit relation between the κ-multinominal coefficients and the κ-entropy. As a complemental generalization, we explain another type of the κ-factorial, which is based on the κ-generalization of gamma function in Section 4. The final section is devoted to our conclusions.

2. κ-Stirling Approximation

Let us begin with the brief review of the κ-factorial and its Stirling approximation. The κ-generalized statistics [10–12] is based on the κ-entropy:

$$S_\kappa \equiv - \sum_i p_i \ln_{\{\kappa\}}(p_i) \tag{4}$$

where κ is a real parameter in the range $(-1, 1)$, and the κ-logarithmic function is defined by:

$$\ln_{\{\kappa\}}(x) \equiv \frac{x^{\kappa} - x^{-\kappa}}{2\kappa} \tag{5}$$

Its inverse function is the κ-exponential function defined by:

$$\exp_{\{\kappa\}}(x) \equiv \left[\sqrt{1 + \kappa^2 x^2} + \kappa x\right]^{\frac{1}{\kappa}} \tag{6}$$

In the limit of $\kappa \to 0$, both the κ-logarithmic and κ-exponential functions reduce to the standard logarithmic and exponential functions, respectively. We thus see that S_κ reduces to the BGS entropy of Equation (3) in the limit of $\kappa \to 0$.

Based on the above κ-deformed functions, the κ-product is defined by:

$$\begin{aligned} x \underset{\kappa}{\otimes} y &\equiv \exp_{\{\kappa\}}\left[\ln_{\{\kappa\}}(x) + \ln_{\{\kappa\}}(y)\right] \\ &= \left[\left(\frac{x^{\kappa} - x^{-\kappa}}{2}\right) + \left(\frac{y^{\kappa} - y^{-\kappa}}{2}\right) + \sqrt{1 + \left\{\left(\frac{x^{\kappa} - x^{-\kappa}}{2}\right) + \left(\frac{y^{\kappa} - y^{-\kappa}}{2}\right)\right\}^2}\,\right]^{\frac{1}{\kappa}} \end{aligned} \tag{7}$$

which reduces to the standard product, $x \cdot y$, in the limit of $\kappa \to 0$. From the definition, we readily see that this κ-product satisfies the associative and commutative laws [10–12].

Similarly, the κ-division is defined by:

$$\begin{aligned} x \underset{\kappa}{\oslash} y &\equiv \exp_{\{\kappa\}}\left[\ln_{\{\kappa\}}(x) - \ln_{\{\kappa\}}(y)\right] \\ &= \left[\left(\frac{x^{\kappa} - x^{-\kappa}}{2}\right) - \left(\frac{y^{\kappa} - y^{-\kappa}}{2}\right) + \sqrt{1 + \left\{\left(\frac{x^{\kappa} - x^{-\kappa}}{2}\right) - \left(\frac{y^{\kappa} - y^{-\kappa}}{2}\right)\right\}^2}\,\right]^{\frac{1}{\kappa}} \end{aligned} \tag{8}$$

which reduces to the standard division, x/y, in the limit of $\kappa \to 0$. By utilizing this κ-product, the κ-factorial, $n!_\kappa$, with $n \in \mathbb{N}$ is defined by:

$$\begin{aligned} n!_\kappa &\equiv 1 \underset{\kappa}{\otimes} 2 \underset{\kappa}{\otimes} \cdots \underset{\kappa}{\otimes} n \\ &= \left[\sum_{k=1}^{n}\left(\frac{k^{\kappa} - k^{-\kappa}}{2}\right) + \sqrt{\left\{\sum_{k=1}^{n}\left(\frac{k^{\kappa} - k^{-\kappa}}{2}\right)\right\}^2 + 1}\,\right]^{\frac{1}{\kappa}} = \exp_{\{\kappa\}}\left[\sum_{k=1}^{n} \ln_{\{\kappa\}}(k)\right] \end{aligned} \tag{9}$$

Now, we come to the Stirling approximation of the κ-factorials. For sufficiently large n, the summation is well approximated with the integral as follows:

$$\ln_{\{\kappa\}}(n!_\kappa) = \sum_{k=1}^{n} \ln_{\{\kappa\}}(k) \approx \int_0^n dx \ln_{\{\kappa\}}(x) = \frac{n^{1+\kappa}}{2\kappa(1+\kappa)} - \frac{n^{1-\kappa}}{2\kappa(1-\kappa)} \tag{10}$$

Clearly, Equation (10) reduces to the standard Stirling approximation in the limit of $\kappa \to 0$ as:

$$\lim_{\kappa \to 0} \ln_{\{\kappa\}}(n!_\kappa) \approx \lim_{\kappa \to 0} \frac{n}{1 - \kappa^2}\left\{\ln_{\{\kappa\}}(n) - \frac{n^{\kappa} + n^{-\kappa}}{2}\right\} = n\,(\ln n - 1) \tag{11}$$

Next, the κ-multinominal coefficient is defined by utilizing the κ-product and κ-division as follows:

$$\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_\kappa \equiv n!_\kappa \underset{\kappa}{\oslash} \left(n_1!_\kappa \underset{\kappa}{\otimes} \cdots \underset{\kappa}{\otimes} n_k!_\kappa \right) \tag{12}$$

where we assume:

$$n = \sum_{i=1}^{k} n_i \tag{13}$$

In the limit of $\kappa \to 0$, Equation (12) reduces to the standard multinominal coefficient of Equation (1).

Now, let us try to relate the κ-multinominal coefficients with the κ-entropy. Taking the κ-logarithm of Equation (12) and applying the κ-Stirling approximation leads to:

$$\ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_\kappa = \ln_{\{\kappa\}} (n!_\kappa) - \sum_{i=1}^{k} \ln_{\{\kappa\}} (n_i!_\kappa) \tag{14}$$

we then obtain:

$$\ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_\kappa \approx \frac{n^{1+\kappa}}{2\kappa(1+\kappa)} \left\{ 1 - \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1+\kappa} \right\} + \frac{n^{1-\kappa}}{2\kappa(1-\kappa)} \left\{ \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1-\kappa} - 1 \right\} \tag{15}$$

From this relation, we see that the above naive approach fails. Since the right hand side of Equation (15) consists of the two terms with different factors (one is proportional to $n^{1+\kappa}/(1+\kappa)$ and the other is proportional to $n^{1-\kappa}/(1-\kappa)$), this cannot be proportional to the κ-entropy, which can be written by:

$$\begin{aligned} S_\kappa \left(\frac{n_1}{n}, \ldots, \frac{n_k}{n} \right) &\equiv - \sum_{i=1}^{k} \left(\frac{n_i}{n} \right) \ln_{\{\kappa\}} \left(\frac{n_i}{n} \right) \\ &= \frac{1}{2\kappa} \left\{ 1 - \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1+\kappa} \right\} + \frac{1}{2\kappa} \left\{ \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1-\kappa} - 1 \right\} \end{aligned} \tag{16}$$

3. Introducing a New κ-Product

In order to overcome the above difficulty, we introduce another kind of κ-product based on the following function defined by:

$$u_{\{\kappa\}}(x) \equiv \frac{x^\kappa + x^{-\kappa}}{2} = \cosh \left(\kappa \ln(x) \right) \tag{17}$$

We here call it the κ-generalized unit function, since in the limit of $\kappa \to 0$, it reduces to the constant function $u_{\{\kappa\}}(x) = 1$. The basic properties of $u_{\{\kappa\}}(x)$ are as follows.

$$u_{\{\kappa\}}(x) = u_{\{\kappa\}} \left(\frac{1}{x} \right) \tag{18}$$

$$u_{\{\kappa\}}(x) \geq 1, \; \text{because } u_{\{\kappa\}}(x) - 1 = \frac{(x^{\frac{\kappa}{2}} - x^{-\frac{\kappa}{2}})^2}{2} \geq 0 \tag{19}$$

When $\kappa \neq 0$, the inverse function of $u_{\{\kappa\}}(x)$ can be defined by:

$$u_{\{\kappa\}}^{-1}(x) \equiv \left[\sqrt{x^2-1}+x\right]^{\frac{1}{\kappa}}, \qquad (x \geq 1) \tag{20}$$

In [14], the canonical partition function associated with the κ-entropy is obtained in terms of this $u_{\{\kappa\}}$ function. Note that:

$$\ln_{\{\kappa\}}(x) = \frac{x^{\kappa}-x^{-\kappa}}{2\kappa} = \frac{1}{\kappa}\,\sinh\big(\kappa\ln(x)\big) \tag{21}$$

the two kinds of the κ-deformed functions ($\ln_{\{\kappa\}}(x)$ and $\exp_{\{\kappa\}}(x)$; $u_{\{\kappa\}}(x)$ and $u_{\{\kappa\}}^{-1}(x)$) are thus associated with each other. This can be seen from the following relations:

$$\sqrt{1+\kappa^2\ln_{\{\kappa\}}^2(x)} = u_{\{\kappa\}}(x) \tag{22}$$

$$u_{\{\kappa\}}^{-1}\left(\sqrt{1+\kappa^2x^2}\right) = \exp_{\{\kappa\}}(x), \quad \text{for } x \geq 0 \tag{23}$$

Now, by utilizing these functions, a new κ-product is defined by:

$$\begin{aligned} x \underset{\kappa}{\odot} y &\equiv u_{\{\kappa\}}^{-1}\left[u_{\{\kappa\}}(x)+u_{\{\kappa\}}(y)\right] \\ &= \left[\left(\frac{x^{\kappa}+x^{-\kappa}}{2}\right)+\left(\frac{y^{\kappa}+y^{-\kappa}}{2}\right)+\sqrt{\left(\frac{x^{\kappa}+x^{-\kappa}}{2}+\frac{y^{\kappa}+y^{-\kappa}}{2}\right)^2-1}\;\right]^{\frac{1}{\kappa}} \end{aligned} \tag{24}$$

Similarly, the corresponding κ-division is defined by:

$$\begin{aligned} x \underset{\kappa}{\oslash} y &\equiv u_{\{\kappa\}}^{-1}\left[u_{\{\kappa\}}(x)-u_{\{\kappa\}}(y)\right] \\ &= \left[\left(\frac{x^{\kappa}+x^{-\kappa}}{2}\right)-\left(\frac{y^{\kappa}+y^{-\kappa}}{2}\right)+\sqrt{\left(\frac{x^{\kappa}+x^{-\kappa}}{2}-\frac{y^{\kappa}+y^{-\kappa}}{2}\right)^2-1}\;\right]^{\frac{1}{\kappa}} \end{aligned} \tag{25}$$

for any pair of the real numbers, x and y, such that $\left(\frac{x^{\kappa}+x^{-\kappa}}{2}\right)-\left(\frac{y^{\kappa}+y^{-\kappa}}{2}\right) \geq 1$, and $\kappa \neq 0$.

It is worth noting that the new κ-product operator, $\odot_{\kappa}$ (the κ-division operator, $\oslash_{\kappa}$), has no corresponding product (division) operator in the standard case of $\kappa = 0$. In other words, neither the product operator, $\odot_{\kappa}$, nor the division operator, $\oslash_{\kappa}$, is a deformed (or generalized) one, and it has meaning only when $\kappa \neq 0$. In addition, unless $\kappa = 0$, this new product satisfies:

$$x \underset{\kappa}{\odot} y = y \underset{\kappa}{\odot} x, \qquad \text{commutativity} \tag{26}$$

$$(x \underset{\kappa}{\odot} y) \underset{\kappa}{\odot} z = x \underset{\kappa}{\odot} (y \underset{\kappa}{\odot} z), \qquad \text{associativity} \tag{27}$$

There exists no real unit element on $\odot_{\kappa}$. Therefore, real numbers and this product constitute a semi-group. Note also that $x \oslash_{\kappa} x \neq 1$, because $u_{\{\kappa\}}^{-1}(0)$ does not exist by the definition of Equation (20). However, we see that the following identity holds.

$$x \underset{\kappa}{\odot} y \underset{\kappa}{\oslash} x = y, \quad (x, y > 1, \quad \text{and} \quad \kappa \neq 0) \tag{28}$$

Using the new κ-product, the associated κ-factorial can be introduced as:

$$\begin{aligned} n!^{\kappa} &\equiv 1 \underset{\kappa}{\odot} 2 \underset{\kappa}{\odot} \cdots \underset{\kappa}{\odot} n \\ &= \left[\sum_{k=1}^{n} \left(\frac{k^{\kappa} + k^{-\kappa}}{2} \right) + \sqrt{ \left\{ \sum_{k=1}^{n} \left(\frac{k^{\kappa} + k^{-\kappa}}{2} \right) \right\}^2 - 1 } \right]^{\frac{1}{\kappa}} \end{aligned} \tag{29}$$

Similar to Equation (10), the κ-Stirling approximation can be obtained as:

$$\begin{aligned} u_{\{\kappa\}}(n!^{\kappa}) &= \sum_{k=1}^{n} u_{\{\kappa\}}(k) \approx \int_0^n dx\, u_{\{\kappa\}}(x) \\ &= \frac{n^{1+\kappa}}{2(1+\kappa)} + \frac{n^{1-\kappa}}{2(1-\kappa)} \end{aligned} \tag{30}$$

Furthermore, the corresponding κ-multinominal coefficient can be defined by:

$$\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^{\kappa} \equiv n!^{\kappa} \underset{\kappa}{\oslash} \left(n_1!^{\kappa} \underset{\kappa}{\odot} \cdots \underset{\kappa}{\odot} n_k!^{\kappa} \right) \tag{31}$$

Applying the above Stirling approximation to Equation (31), we obtain:

$$u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^{\kappa} \approx \frac{n^{1+\kappa}}{2(1+\kappa)} \left\{ 1 - \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1+\kappa} \right\} - \frac{n^{1-\kappa}}{2(1-\kappa)} \left\{ \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1-\kappa} - 1 \right\} \tag{32}$$

This is the complemental relation to Equation (15), and by combining Equations (15) and (32), we have:

$$\ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{\kappa} \pm \frac{1}{\kappa}\, u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^{\kappa} \approx \pm \frac{n^{1\pm\kappa}}{\kappa(1\pm\kappa)} \left\{ 1 - \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1\pm\kappa} \right\} \tag{33}$$

We thus obtain the final result:

$$\begin{aligned} &\left(\frac{1-\kappa}{2n^{1-\kappa}} \right) \left(\ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{\kappa} - \frac{1}{\kappa} u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^{\kappa} \right) \\ &\qquad + \left(\frac{1+\kappa}{2n^{1+\kappa}} \right) \left(\ln_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{\kappa} + \frac{1}{\kappa} u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^{\kappa} \right) \\ &= \frac{1}{2\kappa} \left\{ 1 - \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1+\kappa} \right\} + \frac{1}{2\kappa} \left\{ \sum_{i=1}^{k} \left(\frac{n_i}{n} \right)^{1-\kappa} - 1 \right\} = S_{\kappa} \left(\frac{n_1}{n}, \ldots, \frac{n_k}{n} \right) \end{aligned} \tag{34}$$

Note that since:

$$\lim_{\kappa \to 0} \frac{1}{\kappa}\, u_{\{\kappa\}} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}^{\kappa} = 0 \tag{35}$$

as shown in the following, Equation (34) reduces to the standard case of Equation (2) in the limit of $\kappa \to 0$.

To prove Equation (35), we consider:

$$\frac{1}{\kappa}\, u_{\{\kappa\}} \left[\begin{matrix} n \\ n_1 \cdots n_k \end{matrix} \right]^{\kappa} = \frac{1}{\kappa} \left(\sum_{\ell=1}^{n} u_{\{\kappa\}}(\ell) - \sum_{i=1}^{k} \sum_{j=1}^{n_i} u_{\{\kappa\}}(j) \right) \tag{36}$$

In the limit of $\kappa \to 0$, the numerator of Equation (36) reduces to $n - \sum_{i=1}^{k} n_i$, since $\lim_{\kappa \to 0} u_{\{\kappa\}} = 1$. Consequently, both the denominator, κ, and the numerator become zero as $\kappa \to 0$. Then, applying l'Hopital's rule and using the relation:

$$\frac{d}{d\kappa}\, u_{\{\kappa\}}(x) \; = \ln x \left(\frac{x^{\kappa} - x^{-\kappa}}{2} \right) \tag{37}$$

we obtain Equation (35).

We would like to comment that our approach is not a generalization of the traditional combinatorial approach, since the new κ-product and κ-division are not deformed operators, which are ingredients of the associated κ-factorial Equation (29) and κ-multinominal Equation (31). There are no corresponding factorials and multinominals in the standard case of $\kappa = 0$. We think this fact is not a drawback, but will provide us with a different view. One of the possible reasons why these new operators play an important role only when $\kappa \neq 0$ is that there exits a new degree of freedom only when $\kappa \neq 0$.

4. Another κ-Generalization of the Factorial

In general, some different types of generalization can be introduced. As another κ-generalization of a factorial, we here explain a different type of the κ-generalized factorial, $n_{\kappa}!$, concerning the κ-generalized gamma function, $\Gamma_{\kappa}(x)$. In other words, neither $n!_{\kappa}$ Equation (9) nor $n!^{\kappa}$ Equation (29) match this $n_{\kappa}!$.

As is well known, the standard factorial, $n!$, is related with the gamma function:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \exp(-t)\, dt \tag{38}$$

as $n! = \Gamma(n+1)$, which is readily shown by using the property $\Gamma(x+1) = x\Gamma(x)$ of the gamma function and setting $x = n \in \mathbb{N}$. In addition, the Taylor expansion of the standard exponential function can be written in the form:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} \tag{39}$$

Based on these facts, Kaniadakis [15] introduced the following κ-generalized factorial as:

$$n_{\kappa}! = \Gamma_{\kappa}(n+1) \tag{40}$$

where the κ-generalized gamma function is defined by:

$$\Gamma_{\kappa}(x) \equiv \left(1 - \kappa^2 (x-1)^2 \right) \int_0^{\infty} t^{x-1} \frac{\exp_{\{\kappa\}}(-t)}{\sqrt{1+\kappa^2 t^2}}\, dt \tag{41}$$

and the κ-exponential function can be expanded in the form:

$$\exp_{\{\kappa\}}(x) = \sum_{n}^{\infty} \frac{x^n}{\Gamma_\kappa(n+1)} \tag{42}$$

Of course, in the limit of $\kappa = 0$, the κ-generalized gamma function, $\Gamma_\kappa(x)$, reduces to the standard gamma function, $\Gamma(x)$; consequently, the κ-generalized factorial, $n_\kappa!$, reduces to the standard factorial, $n!$. Since:

$$\frac{d}{dt}\exp_{\{\kappa\}}(t) = \frac{\exp_{\{\kappa\}}(t)}{\sqrt{1+\kappa^2 t^2}} \tag{43}$$

by using integration by parts, the $\Gamma_\kappa(x)$ can be also expressed as:

$$\Gamma_\kappa(x) = \Big(1-\kappa^2(x-1)^2\Big)(x-1)\int_0^\infty t^{x-2}\exp_{\{\kappa\}}(-t)\,dt \tag{44}$$

and by using the standard $\Gamma(x)$, as:

$$\Gamma_\kappa(x) = \frac{1-|\kappa|(x-1)}{|2\kappa|^{x-1}} \frac{\Gamma\left(\frac{1}{|2\kappa|}-\frac{x-1}{2}\right)}{\Gamma\left(\frac{1}{|2\kappa|}+\frac{x-1}{2}\right)}\Gamma(x) \tag{45}$$

By integrating by parts, we obtain the property:

$$\Gamma_\kappa(x+1) = \frac{x(x-1)}{1-\kappa^2(x-2)^2}\,\Gamma_\kappa(x-1) \tag{46}$$

which is equivalent with:

$$n_\kappa! = \frac{n(n-1)}{1-\kappa^2(n-2)^2}\,(n-2)_\kappa! \tag{47}$$

The κ-generalized factorial also can be expressed as:

$$n_\kappa! = \frac{n!}{\prod_{j=1}^{n-1}\Big(1-(2j-n)\kappa\Big)} \tag{48}$$

Recently, Díaz and Pariguan [16] introduced the k-generalized gamma function and the Pochhammer k-symbol $(x)_{n,k}$, which is defined by:

$$(x)_{n,k} \equiv x(x+k)(x+2k)\cdots(x+(n-1)k) \tag{49}$$

for $x \in \mathbb{C}, k \in \mathbb{R}$ and $n \in \mathbb{N}$. By setting $k = 1$, one obtains the standard Pochhammer symbol, $(x)_n$, which is also known as the raising factorial:

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \tag{50}$$

For $\mathrm{Re}(x) > 0$, the k-generalized gamma function is given by:

$$\Gamma_k(x) = \int_0^\infty t^{x-1}\exp\left(-\frac{t^k}{k}\right)\,dt \tag{51}$$

and satisfies the following properties:

$$\Gamma_k(x+k) = x\,\Gamma_k(x) \tag{52}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \tag{53}$$

$$\Gamma_k(k) = 1 \tag{54}$$

and the $\Gamma_k(x)$ is logarithmically convex for $x \in \mathbb{R}$.

Following Díaz and Pariguan, we can introduce the Pochhammer κ-symbol as:

$$(x)_{n,\kappa} = \frac{\Gamma_\kappa(x+n)}{\Gamma_\kappa(x)} \tag{55}$$

After straightforward calculations, we see that:

$$(x)_{2m,\kappa} = \frac{(x)_{2m}}{\prod_{j=0}^{m-1}\left(1-\kappa^2(x+2j-1)^2\right)} \tag{56}$$

$$(x)_{2m-1,\kappa} = \frac{(x)_{2m-1}}{\prod_{j=0}^{m-1}\left(1-\kappa^2(x+2j-2)^2\right)} \tag{57}$$

for $m \in \mathbb{N}$.

5. Conclusions

We have generalized Stirling approximation of the factorials and multinominal coefficients based on the κ-deformed functions, which are different one-parameter generalizations from the q-deformed functions by Tsallis. In order to relate the κ-generalized multinominal coefficients to the κ-entropy, we showed the failure of a naive approach, which is similar to that in [3] for the q-generalization. In order to overcome this difficulty, we have introduced a new kind of κ-product operation, which never existed in the standard case of $\kappa = 0$ and have obtained the relation between the κ-generalized multinominal coefficients and the κ-entropy. We hope that our approach will shed some light on a future study.

The final result Equation (34) clearly states that the maximizing κ-entropy is equivalent to maximizing the left hand side of Equation (34) the same as in the standard case of Equation (2).

Acknowledgments

We would like to thank the critical comments of anonymous reviewers. This research was partially supported by JSPS KAKENHI Grant Number 25400188.

Conflicts of Interest

The authors declare no conflict of interest.

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