

Review

Conformal Relativity *versus* Brans–Dicke and Superstring Theories

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Abstract: We show how conformal relativity is related to Brans–Dicke theory and to low-energy-effective superstring theory. Conformal relativity or the Hoyle–Narlikar theory is invariant with respect to conformal transformations of the metric. We show that the conformal relativity action is equivalent to the transformed Brans–Dicke action for $\omega = -3/2$ (which is the border between standard scalar field and ghost) in contrast to the reduced (graviton-dilaton) low-energy-effective superstring action which corresponds to the Brans–Dicke action with $\omega = -1$. We show that like in ekpyrotic/cyclic models, the transition through the singularity in conformal cosmology in the string frame takes place in the weak coupling regime. We also find interesting self-duality and duality relations for the graviton-dilaton actions.

Keywords: conformal relativity; Brans–Dicke theory; superstring theory

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1. Introduction

It is well-known that some of the fundamental equations of physics such as Maxwell equations or massless Dirac equation are invariant with respect to conformal transformations of the metric [1]. On the other hand, the Einstein equations and the massless Klein–Gordon equation are not invariant with respect to these transformations. However, a modification of these equations which involves both metric and scalar field (scalar-tensor gravity) may possess the property of conformal invariance. Such theories have been given some new interest and have been called conformal relativity theories. They are not exactly new since they were studied already and their fundamental version is well-known under the name of the Hoyle–Narlikar theory [2–8]. However, in [9,10] the geometrical evolution of the universe was reinterpreted as an evolution of the mass represented by a scalar field in a flat universe, which is a promising aspect of these models. The idea is quite interesting and it can help to resolve the problem of the dark energy in the universe [11–15]. Similar ideas have been developed in yet another modification of general relativity called Self Creation Cosmology [16,17], which also solves the dark energy problem together with a series of other cosmological problems including Pioneer spacecraft puzzle [18].

In this paper we show the connection between conformal relativity and other alternative gravity theories such as Brans–Dicke theory [19] and their overlap with unified gauge-gravity theories superstring and M-theory [20–22]. In fact, the Brans–Dicke theory (with Dirac’s Large Numbers Hypothesis as a precursor [23]) belongs to a larger class of the scalar-tensor theories of gravity [24] whose properties has also been studied in the context of supernovae data and density perturbations [25,26].

In Section 2 we discuss the basic properties of conformal invariance, present the field equations of conformal relativity and show how they behave under conformal transformations of the metric. In Section 3 we study the mutual relations between conformal relativity, Brans–Dicke theory and low-energy-effective superstring theories for a reduced graviton-dilaton spectrum. In Section 4 we present the basic cosmological solutions of conformal relativity for Friedmann universes in both the Einstein and the string frames. In Section 5 we make a summary of the results.

A signature of the metric convention $(-+++)$ is applied throughout the paper, together with the Riemann and Ricci tensor convention denoted $+++$ by Misner, Thorne and Wheeler [27] (same conventions as in Hawking and Ellis [28]).

2. Conformal Relativity

Suppose that we have two spacetime manifolds $\mathcal{M}, \tilde{\mathcal{M}}$ with metrics $g_{\mu\nu}, \tilde{g}_{\mu\nu}$ and *the same* coordinates x^μ . We say that the two manifolds are *conformal* to each other if they are related by the following *conformal transformation*:

$$\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \quad (1)$$

and the function Ω which is called a conformal factor must be a twice-differentiable function of coordinates x^μ and lie in the range $0 < \Omega < \infty$. The conformal transformations shrink or stretch the distances between the two points described by the same coordinate system x^μ on the manifolds $\mathcal{M}, \tilde{\mathcal{M}}$, respectively, but they preserve the angles between vectors (in particular null vectors which

define light cones), which leads to a conservation of the (global) causal structure of the manifold [28]. If we take a constant Ω , we then deal with the so-called *scale transformations* [24]. In fact, conformal transformations are *localized* scale transformations $\Omega = \Omega(x)$.

On the other hand, the *coordinate transformations* $x^\mu \rightarrow \tilde{x}^\mu$ only relabel the coordinates and do not change geometry, and they are entirely *different* from conformal transformations [7]. This is crucial since conformal transformations lead to a *different physics* on conformally related manifolds $\mathcal{M}, \tilde{\mathcal{M}}$ [24]. Since this will usually be related to a *different coupling* of a physical field to gravity, we will be talking about different *frames* in which the physics is studied (see also [29] for a slightly different view).

In $D = 4$ spacetime dimensions, the determinant of the metric $g = \det g_{\mu\nu}$ transforms as:

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g} \quad (2)$$

It is obvious from Equation (1) that the following relations for the inverse metrics and the spacetime intervals hold:

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu} \quad (3)$$

$$d\tilde{s}^2 = \Omega^2 ds^2 \quad (4)$$

Finally, the notion of conformal flatness means that:

$$\tilde{g}_{\mu\nu} \Omega^{-2}(x) = g_{\mu\nu} \equiv \eta_{\mu\nu} \quad (5)$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric.

The application of Equation (1) to the Christoffel connection coefficients gives [28]:

$$\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + \frac{1}{\Omega} (g_\mu^\lambda \Omega_{,\nu} + g_\nu^\lambda \Omega_{,\mu} - g_{\mu\nu} g^{\lambda\kappa} \Omega_{,\kappa}) \quad (6)$$

$$\Gamma_{\mu\nu}^\lambda = \tilde{\Gamma}_{\mu\nu}^\lambda - \frac{1}{\Omega} (\tilde{g}_\mu^\lambda \Omega_{,\nu} + \tilde{g}_\nu^\lambda \Omega_{,\mu} - \tilde{g}_{\mu\nu} \tilde{g}^{\lambda\kappa} \Omega_{,\kappa}) \quad (7)$$

The Ricci tensors and Ricci scalars in the two related frames $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ transform as:

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \Omega^{-2} [4\Omega_{,\mu} \Omega_{,\nu} - \Omega_{,\sigma} \Omega^{,\sigma} g_{\mu\nu}] - \Omega^{-1} [2\Omega_{;\mu\nu} + \square \Omega g_{\mu\nu}] \quad (8)$$

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - 3\Omega^{-2} \Omega_{,\rho} \Omega^{,\rho} \tilde{g}_{\mu\nu} + \Omega^{-1} [2\Omega_{;\mu\nu} + \tilde{g}_{\mu\nu} \tilde{\square} \Omega] \quad (9)$$

$$\tilde{R} = \Omega^{-2} \left[R - 6 \frac{\square \Omega}{\Omega} \right] \quad (10)$$

$$R = \Omega^2 \left[\tilde{R} + 6 \frac{\tilde{\square} \Omega}{\Omega} - 12 \tilde{g}^{\mu\nu} \frac{\Omega_{,\mu} \Omega_{,\nu}}{\Omega} \right] \quad (11)$$

and the appropriate d'Alambertian operators change under Equation (1) as:

$$\tilde{\square} \phi = \Omega^{-2} \left(\square \phi + 2g^{\mu\nu} \frac{\Omega_{,\mu}}{\Omega} \phi_{,\nu} \right) \quad (12)$$

$$\square \phi = \Omega^{-2} \left(\tilde{\square} \phi - 2\tilde{g}^{\mu\nu} \frac{\Omega_{,\mu}}{\Omega} \phi_{,\nu} \right) \quad (13)$$

In these formulas the d'Alembertian $\tilde{\square}$ taken with respect to the metric $\tilde{g}_{\mu\nu}$ is different from $\square$ which is taken with respect to a conformally rescaled metric $g_{\mu\nu}$.

An important feature of the conformal transformations is that they preserve Weyl conformal curvature tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{2}{D-2} (g_{\mu[\sigma} R_{\rho]\nu} + g_{\nu[\rho} R_{\sigma]\mu}) + \frac{2}{(D-1)(D-2)} R g_{\mu[\rho} g_{\sigma]\mu} \quad (14)$$

and we have (note that the first index is raised):

$$\tilde{C}^{\mu}_{\nu\rho\sigma} = C^{\mu}_{\nu\rho\sigma} \quad (15)$$

under Equation (1). The full set of conformal transformations of both geometrical and physical quantities can be found in [30,31] including curvature invariants useful in the studies of higher-order gravity theories [32].

Let us remind that the vacuum Einstein–Hilbert action of general relativity reads as:

$$S_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \tilde{R} \quad (16)$$

where

$$\kappa^2 = 8\pi G \quad (17)$$

Application of the conformal transformation Equation (1) together with the help of Equation (10) yields (we have assumed that $\kappa^2 = 6$ for a while):

$$S_{EH} = \frac{1}{2} \int d^4x \sqrt{-g} \Omega^2 \left(\frac{1}{6} R - \frac{\square \Omega}{\Omega} \right) \quad (18)$$

which means that the vacuum part of the Einstein–Hilbert action is not invariant under conformal transformation Equation (1), apart from the case of the global transformations of the trivial type $\tilde{g}_{\mu\nu} = \text{const.} \times g_{\mu\nu}$.

However, a modification of the Einstein–Hilbert action Equation (16) that allows for a scalar field $\tilde{\Phi}$, which reads as:

$$\tilde{S} = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \tilde{\Phi} \left(\frac{1}{6} \tilde{R} \tilde{\Phi} - \tilde{\square} \tilde{\Phi} \right) \quad (19)$$

together with the appropriate redefinition of the scalar field:

$$\tilde{\Phi} = \Omega^{-1} \Phi \quad (20)$$

is, in fact, conformally invariant since the conformally transformed action has the same form, *i.e.*,

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \Phi \left(\frac{1}{6} R \Phi - \square \Phi \right) \quad (21)$$

Now, one can see that the original form of the Einstein–Hilbert action can be recovered from Equation (19) (or, alternatively Equation (21) provided we assume that:

$$\kappa^2 = \frac{6}{\tilde{\Phi}^2} = \frac{6}{\varphi_0^2} = \text{const} \quad (22)$$

The action Equation (19) (or similarly Equation (21)) is usually represented in a different form by the application of the expression for a covariant d’Alambertian for a scalar field in general relativity:

$$\tilde{\square} \tilde{\Phi} = \frac{1}{\sqrt{-\tilde{g}}} \tilde{\partial}_\mu \left(\sqrt{-\tilde{g}} \tilde{\partial}^\mu \tilde{\Phi} \right) \quad (23)$$

which, after integrating out the boundary term, gives [24]:

$$\tilde{S} = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{6} \tilde{R} \tilde{\Phi}^2 + \tilde{\partial}_\mu \tilde{\Phi} \tilde{\partial}^\mu \tilde{\Phi} \right] \quad (24)$$

and the second term is just a kinetic term for the scalar field (cf. [1,28]). The Equation (24) is also conformally invariant since the application of the formulas (2), (10) and (20) together with the appropriate integration of the boundary term gives the same form of the equations:

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[\frac{1}{6} R \Phi^2 + \partial_\mu \Phi \partial^\mu \Phi \right] \quad (25)$$

Because of the type of non-minimal coupling of gravity to a scalar field $\tilde{\Phi}$ or Φ in Equations (24) and (25) and the relation to Brans–Dicke theory (see Section (3)), we say that these equations are presented in the *Jordan frame* [24,33].

The conformally invariant actions (19) and (21) are the basis to derive the equations of motion via the variational principle. The resulting equations of motion are conformally invariant, too. The equations of motion for scalar fields $\tilde{\Phi}$ and Φ are conformally invariant:

$$\left(\tilde{\square} - \frac{1}{6} \tilde{R} \right) \tilde{\Phi} = \Omega^{-3} \left(\square - \frac{1}{6} R \right) \Phi = 0 \quad (26)$$

and they have the structure of the Klein–Gordon equation with the mass term replaced by the curvature term [5]. The conformally invariant Einstein equations are obtained from variation of $\tilde{S}$ with respect to the metric $\tilde{g}_{\mu\nu}$ and read as:

$$\left(\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \right) \frac{1}{6} \tilde{\Phi}^2 + \frac{1}{6} \left[4 \tilde{\Phi}_{;\mu} \tilde{\Phi}_{;\nu} - \tilde{g}_{\mu\nu} \tilde{\Phi}_{;\alpha} \tilde{\Phi}^{;\alpha} \right] + \frac{1}{3} \left[\tilde{g}_{\mu\nu} \tilde{\square} \tilde{\Phi} - \tilde{\Phi} \tilde{\Phi}_{;\mu\nu} \right] = 0 \quad (27)$$

In order to prove the conformal invariance of the field Equation (27), it is necessary to know the rule of the conformal transformations for the double covariant derivative of a scalar field, *i.e.*,

$$\begin{aligned} \tilde{\Phi}_{;\mu\nu} &= \tilde{\Phi}_{;\mu\nu} - \tilde{\Gamma}_{\mu\nu}^\rho \tilde{\Phi}_{;\rho} = -\Omega^{-2} \Phi \Omega_{;\mu\nu} + \Omega^{-1} \Phi_{;\mu\nu} + 4\Omega^{-3} \Phi \Omega_{;\mu} \Omega_{;\nu} \\ &\quad - 2\Omega^{-2} (\Phi_{;\mu} \Omega_{;\nu} + \Omega_{;\mu} \Phi_{;\nu}) - \Omega^{-3} \Phi g_{\mu\nu} \Omega_{;\rho} \Omega^{;\rho} + \Omega^{-2} g_{\mu\nu} \Phi_{;\rho} \Omega^{;\rho} \end{aligned} \quad (28)$$

and

$$\Phi_{;\mu\nu} = \tilde{\Phi} \Omega_{;\mu\nu} + \Omega \Phi_{;\mu\nu} + \frac{2}{\Omega} \tilde{\Phi} \Omega_{;\mu} \Omega_{;\nu} + 2 \left(\Omega_{;\mu} \tilde{\Phi}_{;\nu} + \tilde{\Phi}_{;\mu} \Omega_{;\nu} \right) - \frac{1}{\Omega} \tilde{\Phi} \tilde{g}_{\mu\nu} \Omega_{;\rho} \Omega^{;\rho} - \frac{1}{\Omega} \tilde{g}_{\mu\nu} \tilde{\Phi}_{;\rho} \Omega^{;\rho} = 0 \quad (29)$$

($\tilde{}$ means the covariant derivative with respect to $\tilde{g}_{\mu\nu}$).

Inserting Equations (8), (10), (20) and (28) into (27) gives the same *conformally invariant* form of the field equations as:

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \frac{1}{6} \Phi^2 + \frac{1}{6} \left[4 \Phi_{;\mu} \Phi_{;\nu} - g_{\mu\nu} \Phi_{;\alpha} \Phi^{;\alpha} \right] + \frac{1}{3} \left[g_{\mu\nu} \square \Phi - \Phi \Phi_{;\mu\nu} \right] = 0 \quad (30)$$

These are exactly the same field equations as in the Hoyle–Narlikar theory [7]. Note that the scalar field equations of motion (26) can be obtained by the appropriate contraction of Equations (27) and (30) so that they are not independent and do not supply any additional information [34].

Another point is that the Equations (27) or (30) apparently could give directly the vacuum Einstein field equations for $\tilde{\Phi} = \varphi_0 = \sqrt{6}/\kappa = \sqrt{6/8\pi G} = \text{const.}$ (cf. Equation (22)). The same is obviously true for the field Equation (30) with the same value of $\Phi = \varphi_0 = \sqrt{6}/\kappa = \sqrt{6/8\pi G} = \text{const.}$ However, this limit is restricted to the case of vanishing Ricci curvature $R = 0$ or $\tilde{R} = 0$ (so only flat Minkowski space limit is allowed) which can be seen from the scalar field equations of motion (26).

The admission of the matter part:

$$S_{\text{Matter}} = \frac{1}{2} \int d^4x \sqrt{-g} \mathcal{L}_{\text{Matter}} \quad (31)$$

into the action (21), with the matter energy-momentum tensor:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{-g} \mathcal{L}_{\text{Matter}}) \quad (32)$$

allows to generalize the field Equation (30) to:

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \frac{1}{6} \Phi^2 + \frac{1}{6} [4\Phi_{;\mu}\Phi_{;\nu} - g_{\mu\nu}\Phi_{;\alpha}\Phi^{;\alpha}] + \frac{1}{3} [g_{\mu\nu}\Phi\Box\Phi - \Phi\Phi_{;\mu\nu}] = T_{\mu\nu} \quad (33)$$

Equation (33), after contraction, gives modified field Equation (26):

$$\left(\Box - \frac{1}{6} R \right) \Phi = \frac{T}{\Phi} \quad (34)$$

Note that putting $\Phi = \phi_0 = \sqrt{6/8\pi G}$ into Equation (33) gives Einstein field equations provided $T = -(1/6)R\Phi^2$.

In the conformal frame we add the matter term:

$$\tilde{S}_{\text{Matter}} = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \Omega^{-4} \tilde{\mathcal{L}}_{\text{Matter}} \quad (35)$$

with

$$\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}} \frac{\partial}{\partial \tilde{g}_{\mu\nu}} (\sqrt{-\tilde{g}} \Omega^{-4} \tilde{\mathcal{L}}_{\text{Matter}}) \quad (36)$$

Some manipulations with the help of Equations (1–3) show that:

$$\tilde{T}^{\mu\nu} = \Omega^{-6} T^{\mu\nu} \quad (37)$$

$$\tilde{T} = \Omega^{-4} T \quad (38)$$

The admission of Equation (35) gives the conformally invariant field Equation (21) as:

$$\left(\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \right) \frac{1}{6} \tilde{\Phi}^2 + \frac{1}{6} [4\tilde{\Phi}_{;\mu}\tilde{\Phi}_{;\nu} - \tilde{g}_{\mu\nu}\tilde{\Phi}_{;\alpha}\tilde{\Phi}^{;\alpha}] + \frac{1}{3} [\tilde{g}_{\mu\nu}\tilde{\Phi}\tilde{\Box}\tilde{\Phi} - \tilde{\Phi}\tilde{\Phi}_{;\mu\nu}] = \tilde{T}_{\mu\nu} \quad (39)$$

which after contraction gives a modified Equation (26):

$$\left(\tilde{\Box} - \frac{1}{6} \tilde{R} \right) \tilde{\Phi} = \frac{\tilde{T}}{\tilde{\Phi}} \quad (40)$$

Note again that putting $\tilde{\Phi} = \tilde{\phi}_0 = \sqrt{6/8\pi G}$ into Equation (39) gives the Einstein field equations, provided $\tilde{T} = -(1/6)\tilde{R}\tilde{\Phi}^2$. However, as we will see in a moment, the *trace* of the energy-momentum tensor *must be zero* in order to conserve the energy-momentum in a conformal frame.

Let us take the perfect fluid as a source of gravity with the four-velocity u^μ ($u_\mu u^\mu = -1$), the energy density ϱ and the pressure p , *i.e.*,

$$T^{\mu\nu} = (\varrho + p)u^\mu u^\nu + pg^{\mu\nu} \quad (41)$$

and transform it into a conformally related frame:

$$\tilde{T}^{\mu\nu} = (\tilde{\varrho} + \tilde{p})\tilde{u}^\mu \tilde{u}^\nu + \tilde{p}\tilde{g}^{\mu\nu} \quad (42)$$

where

$$\tilde{u}^\mu = \frac{dx^\mu}{d\tilde{s}} = \frac{1}{\Omega} \frac{dx^\mu}{ds} = \Omega^{-1}u^\mu \quad (43)$$

It is easy to note that the imposition of the conservation law in the first frame:

$$T^{\mu\nu}_{;\nu} = 0 \quad (44)$$

gives in the conformally related frame:

$$\tilde{T}^{\mu\nu}_{;\nu} = -\frac{\Omega^{,\mu}}{\Omega}\tilde{T} \quad (45)$$

From Equation (45) it appears transparent that the conformally transformed energy-momentum tensor is conserved only if the trace of it vanishes ($\tilde{T} = 0$) [33,35]. For example, in the case of barotropic fluid with:

$$p = (\gamma - 1)\varrho \quad \gamma = \text{const.}, \quad (46)$$

it vanishes only for radiation $p = (1/3)\varrho$. This means that only the photons may obey the equivalence principle and this is not the case for other types of matter, since with non-vanishing trace in Equation (45) we deal with creation of matter process (compare with Self Creation Cosmology of [16,17] which has the same field Equations (39) and (45), but the Equation (40) is the same only for a vanishing curvature scalar $\tilde{R}$).

A more general example of a conformally invariant theory is the Weyl conformal gravity [36]:

$$S_W = -\frac{\gamma}{4\kappa^2} \int d^4x \sqrt{-g} C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda} = -\frac{\gamma}{2\kappa^2} \int d^4x \left[R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right] \quad (47)$$

where γ is a dimensionless constant and $C_{\mu\nu\rho\lambda}$ is the Weyl tensor given by Equation (14). If one includes the scalar field and Einstein–Hilbert action, then the conformally invariant Weyl term plays the role of a quantum correction [37], *i.e.*,

$$S_{RW} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R - \frac{1}{2} \int d^4x \sqrt{-g} [\partial_\mu S \partial^\mu S + 2V(S)] - \frac{\gamma}{4\kappa^2} \int d^4x \sqrt{-g} C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda} \quad (48)$$

where S is a scalar field. The field equations are:

$$G_{\mu\nu} - \gamma B_{\mu\nu} = \kappa^2 T_{\mu\nu} \quad (49)$$

with $G_{\mu\nu}$ being the Einstein tensor, $T_{\mu\nu}$ the energy-momentum tensor for the scalar field S , and the Bach tensor $B_{\mu\nu}$ defined as:

$$B_{\mu\nu} \equiv 2C_{\mu\rho\nu\sigma}{}^{;\rho;\sigma} + G^{\rho\sigma}C_{\mu\rho\nu\sigma} \quad (50)$$

The conformally invariant fourth-order Weyl gravity, which includes the scalar field S , the vector field A_μ and the tensor field ψ , is given by the action [38]:

$$\begin{aligned} S_{RWS} = & \int d^4x \sqrt{-g} \left[-\alpha C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda} - \frac{1}{2} \partial_\mu S \partial^\mu S - \frac{1}{12} S^2 R - \lambda S^4 \right. \\ & \left. - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \gamma^\mu \psi_{;\mu} - \frac{i}{2} \overline{\psi_{;\mu}} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu A_\mu \psi - h S \bar{\psi} \psi \right] \end{aligned} \quad (51)$$

where α , h and λ are dimensionless constants; γ^μ are the Dirac matrices.

3. Relation to Brans–Dicke and Low-energy-effective Superstring Theories

Due to the admission of a non-minimally coupled scalar field, conformal relativity suggests its close relation to some other scalar-tensor theories of gravity such as Brans–Dicke theory and the reduced low-energy-effective superstring theories. It is widely known that Brans–Dicke theory (BD) [19] was based on the ideas of Jordan [33] and Mach [39]. According to them, the inertial masses of the elementary particles are not fundamental constants but the result of the particles' interaction with the rest of the universe represented by some cosmic field [35]. In physical terms it can be formulated by the fact that gravitational “constant” G is related to an average value of a scalar field Φ_{BD} which is not constant. Due to some estimations concerning a simple form of a covariant field equation, the size and mean mass density of the universe, the expectation value of the scalar field Φ_{BD} is normalized, such as $\langle \Phi_{BD} \rangle \simeq 1/G$. These assumptions led Brans and Dicke to replace G by the inverse of the scalar field $1/\Phi_{BD}$ and to include an extra energy-momentum tensor for the scalar field.

The resulting Brans–Dicke action reads as [7]:

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\Phi_{BD} R - \frac{\omega}{\Phi_{BD}} \partial_\mu \Phi_{BD} \partial^\mu \Phi_{BD} \right] + S_m \quad (52)$$

where ω is the Brans–Dicke parameter.

Varying the action (52) (with matter Lagrangian included) one gets the field equations of the Brans–Dicke theory in the form:

$$\square \Phi_{BD} = \frac{8\pi T}{3 + 2\omega} \quad (53)$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = & \frac{8\pi}{\Phi_{BD}} T_{\mu\nu} + \frac{\omega}{\Phi_{BD}^2} \left(\Phi_{BD,\mu} \Phi_{BD,\nu} - \frac{1}{2} g_{\mu\nu} \Phi_{BD,\rho} \Phi_{BD,\rho} \right) \\ & + \frac{1}{\Phi_{BD}} (\Phi_{BD;\mu\nu} - g_{\mu\nu} \square \Phi_{BD}) \end{aligned} \quad (54)$$

Independently, the conservation law for matter energy-momentum tensor:

$$T^{\mu\nu}_{;\nu} = 0 \quad (55)$$

may be imposed [35].

In fact, the Equation (53) was obtained by subtracting the equation of motion of the Brans–Dicke field obtained from the action (52):

$$2\Phi_{BD}\square\Phi_{BD} - \Phi_{BD,\rho}\Phi^{BD,\rho} + \frac{R}{\omega}\Phi_{BD}^2 = 0 \quad (56)$$

and the contracted Equation (54). The Einstein limit of Equations (53)–(54) is recovered for $\omega \rightarrow \infty$ [35].

In order to relate conformal relativity with Brans–Dicke theory we refer to conformally invariant actions (24) and (25) in Jordan frame and define:

$$\frac{1}{12}\Phi^2 = e^{-\phi} \quad (57)$$

$$\frac{1}{12}\tilde{\Phi}^2 = e^{-\tilde{\phi}} \quad (58)$$

which gives these actions in the form:

$$S = \int d^4x \sqrt{-g} e^{-\phi} \left[\frac{1}{6}R + \frac{3}{2}\partial_\mu\phi\partial^\mu\phi \right] \quad (59)$$

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} e^{-\tilde{\phi}} \left[\frac{1}{6}\tilde{R} + \frac{3}{2}\tilde{\partial}_\mu\tilde{\phi}\tilde{\partial}^\mu\tilde{\phi} \right] \quad (60)$$

These actions, however, are special cases of the Brans–Dicke action which can be realized once one defines:

$$\Phi_{BD} = e^{-\phi} \quad (61)$$

in Equation (52) together with putting that $16\pi = 1$, which gives Equation (52) in the form:

$$S = \int d^4x \sqrt{-g} e^{-\phi} [R - \omega\partial_\mu\phi\partial^\mu\phi] \quad (62)$$

so that one can immediately see that Equations (59) and (62) are the same provided that the Brans–Dicke parameter:

$$\omega = -\frac{3}{2} \quad (63)$$

On the other hand, if one takes:

$$\omega = -1 \quad (64)$$

in Equation (62), then one obtains the low-energy-effective superstring action for only graviton and dilaton in the spectrum [21,22,40]:

$$S = \int d^4x \sqrt{-g} e^{-\phi} [R + \partial_\mu\phi\partial^\mu\phi] \quad (65)$$

In fact, the action (62) represents Brans–Dicke theory in a special frame which is known as *string frame*. It is because in superstring theory the coupling constant g_s is related to the vacuum expectation value of the dilaton by [21,22]:

$$g_s \propto e^{\phi/2} \quad (66)$$

Now, the field equations which are obtain by the variation of Equation (62) with respect to the dilaton ϕ and the graviton $g_{\mu\nu}$, respectively, are [41]:

$$R + \omega \partial_\mu \phi \partial^\mu \phi - 2\omega \square \phi = 0 \quad (67)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi e^\phi T_{\mu\nu} + (\omega + 1) \partial_\mu \phi \partial_\nu \phi - \left(\frac{\omega}{2} + 1\right) g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + g_{\mu\nu} \square \phi - \phi_{;\mu\nu} \quad (68)$$

Following the same track as for Brans–Dicke field equations, now we can contract Equation (68) and use Equation (67) to get a similar Equation to (53) in string frame, *i.e.*,

$$\partial_\rho \phi \partial^\rho \phi - \square \phi = \frac{8\pi}{2\omega + 3} e^\phi T \quad (69)$$

Notice that putting $T = 0$ in Equation (69) (which is the case for dilaton-graviton theory of superstring cosmology) we have that $\square \phi = \partial_\rho \phi \partial^\rho \phi$, which gives exactly the form of pre-big-bang field equations as presented in [41] with an arbitrary value of ω . On the other hand, the limit $\omega \rightarrow -\frac{3}{2}$ of the Equation (69) is singular unless we take the trace of the energy-momentum tensor $T = 0$.

In fact, it can be just obtained by the application of Equation (61) into Equation (53) since:

$$\square \Phi_{BD} = e^{-\phi} (\partial_\rho \phi \partial^\rho \phi - \square \phi) \quad (70)$$

It is interesting to note that the Ricci scalar can be calculated from Equations (67)–(68) as:

$$R_{\mu\nu} = 8\pi e^\phi T_{\mu\nu} - \phi_{;\mu\nu} + (\omega + 1) (\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + g_{\mu\nu} \square \phi) \quad (71)$$

and for low-energy-effective superstring theory $\omega = -1$ the whole lot of its terms vanish. However, this is not the case in conformal relativity $\omega = -(3/2)$ for which this expression is not so simple, and of course from Equation (69) one sees that the trace T of the energy momentum tensor must be zero in this limit.

In the context of superstring theory (with no matter fields) one often starts looking for the exact solutions starting from the two equations [42–44]:

$$R + \omega \partial_\mu \phi \partial^\mu \phi - 2\omega \square \phi = 0 \quad (72)$$

$$R_{\mu\nu} + \phi_{;\mu\nu} - (\omega + 1) (\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + g_{\mu\nu} \square \phi) = 0 \quad (73)$$

In the next section we will try to use these equations to present cosmological solutions.

Now we come to an important remark. Namely, taking the limit $\omega = -3/2$ in Brans–Dicke field Equation (53) is singular unless we assume that the trace of the energy-momentum tensor of matter *vanishes*. The point is that, in fact, the conformal relativity field Equation (34) is not an independent equation from Equation (33), as it is the case in Brans–Dicke theory, but that it is exactly a contraction of Equation (33)! Then, it may suggest that in order to get the proper limit of Brans–Dicke theory from conformal relativity one should also assume that in Brans–Dicke theory the trace of the energy-momentum of matter should vanish. As we have mentioned, vanishing of the energy-momentum for the perfect fluid requires its equation of state for radiation.

Now let us move to the problem of frames. The different frames are defined by the *coupling properties* of the scalar field (dilaton) to gravity in the theory. In all three cases we have discussed

so far (conformal relativity, Brans–Dicke, superstrings) we deal with *non-minimal* coupling of the scalar field to gravity. For Brans–Dicke we call it *Jordan frame* and for superstrings we call it *string frame*. For conformal relativity we will also call it *Jordan frame* although we deal with non-minimal coupling in both conformally related frames (conformal invariance).

However, for all three theories, one is usually interested in the question of what is going on in the *Einstein frame*, which is defined as the frame in which the scalar field is *minimally* coupled to gravity.

Let us now begin with the action Equation (62) which admits an arbitrary value of the parameter ω . It is easy to show [40,42,43] that under a choice of a conformal factor:

$$\Omega = e^{-\frac{\phi}{2}} \quad (74)$$

the action (62) transforms into:

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \left(\omega + \frac{3}{2} \right) \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi \right] \quad (75)$$

which for $\omega = -3/2$ gives exactly the Einstein–Hilbert action with no matter Lagrangian:

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \tilde{R} \quad (76)$$

However, for low-energy-effective theory with $\omega = -1$ this is not Einstein–Hilbert action, but the Einstein gravity coupled minimally to a non-vanishing scalar field ϕ . Notice that in terms of the Φ field defined in conformal relativity according to Equation (57) the relation (74) reads as:

$$\Omega = e^{-\frac{\phi}{2}} = \frac{\Phi}{\sqrt{12}} \quad (77)$$

or, basically, $\tilde{\Phi} = \text{const.}$ in Equation (20) which has already been mentioned gives the Einstein limit of conformal relativity. This means that the choice of the Einstein frame is *unique* in conformal relativity and requires a constant $\tilde{\Phi}$.

The appropriate field equations that come from Equation (75) are:

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \tilde{T}_{\mu\nu} \equiv \left(\omega + \frac{3}{2} \right) \left[2 \tilde{\partial}_\mu \phi \tilde{\partial}_\nu \phi - g_{\mu\nu} \tilde{\partial}_\rho \phi \tilde{\partial}^\rho \phi \right] \quad (78)$$

$$\left(\omega + \frac{3}{2} \right) \tilde{\square} \phi = 0 \quad (79)$$

From Equations (78) and (79) it follows that $\omega = -3/2$ is exactly the border between a standard scalar field and a ghost (negative kinetic energy) which is required in order to have an expansion minimum in varying constant cosmologies [45] and to mimic the phantom field [46] which violates the weak energy condition $\tilde{\rho} + \tilde{p} < 0$. This is easily seen after calculating the energy-momentum tensor and expressing the energy density and pressure in terms of scalar field, *i.e.*,

$$\tilde{\rho} = \left(\omega + \frac{3}{2} \right) \dot{\phi}^2 \quad (80)$$

$$\tilde{p} = \left(\omega + \frac{3}{2} \right) \dot{\phi}^2 \quad (81)$$

This also means that, formally, the energy density and pressure vanish due to a special choice of Brans–Dicke parameter ω and not necessarily due to a vanishing of the scalar field in the Einstein frame.

Obviously, a more general choice of the conformal factor which leaves the action (62) invariant reads as an appropriate generalization of the choice Equation (20):

$$\Omega = \frac{e^{-\frac{\tilde{\Phi}}{2}}}{e^{-\frac{\Phi}{2}}} = \frac{\Phi}{\tilde{\Phi}} \quad (82)$$

and this is exactly the conformally invariant transformation (20) of the scalar field in conformal relativity. Having applied Equation (82) to Equation (62) shows its conformal invariance, *i.e.*,

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} e^{-\tilde{\Phi}} \left[\tilde{R} + \frac{3}{2} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} \right] \quad (83)$$

In fact it works for any value of ω ! Notice that conformal invariance leads to a *self-duality transformation* [40] known from superstring theory between the scalar fields, which leaves the action invariant and reads as:

$$\tilde{\Phi} \rightarrow \Phi \quad (84)$$

or

$$\tilde{\phi} \rightarrow \phi \quad (85)$$

or

$$\Omega \rightarrow \frac{1}{\Omega} \quad (86)$$

A slightly different way of getting conformal invariance of the action (59) is the following conformal transformation [40,47]:

$$\Omega = e^{-\phi} \quad (87)$$

which brings it to the form:

$$S = \int d^4x \sqrt{-\tilde{g}} e^{\phi} \left[\tilde{R} + \frac{3}{2} \partial_\mu \phi \partial^\mu \phi \right] \quad (88)$$

and it remains conformally invariant provided we replace:

$$\phi \rightarrow -\tilde{\phi} \quad (89)$$

which gives:

$$S = \int d^4x \sqrt{-\tilde{g}} e^{-\tilde{\phi}} \left[\tilde{R} + \frac{3}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \right] \quad (90)$$

and this is another example of duality symmetry known from superstring theory and may relate weak coupling regime with a strong coupling regime of various superstring actions [47].

It is worth mentioning that an interesting example of the duality-like symmetry for a metric-dilaton model:

$$S = \int d^4x \sqrt{-g} [A(\phi) g^{ab} \partial_a \phi \partial_b \phi + B(\phi) R + C(\phi)] \quad (91)$$

where $A(\phi)$, $B(\phi)$, and $C(\phi)$ are the functions of the dilaton, was considered in [48–50], including some generalizations [51].

4. Conformal Cosmology in Einstein and String Frames

We discuss Friedmann cosmology in the two conformally related frames as given in Equation (1), *i.e.*,

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (92)$$

$$ds^2 = -dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (93)$$

From Equations (1), (92) and (93) one can easily see that the time coordinates and scale factors are related by [26,41,43]:

$$d\tilde{t} = \Omega dt \quad (94)$$

$$\tilde{a} = \Omega a \quad (95)$$

where for the full conformal invariance one has to apply the definition of conformal factor Equation (20). However, in the case of studying the Einstein limit, we must take one of the scalar fields constant, so that the conformal factor has the form given by Equation (74). This requires a replacement of Equations (94) and (95) into:

$$d\tilde{t} = e^{-\frac{\phi}{2}} dt \quad (96)$$

$$\tilde{a} = e^{-\frac{\phi}{2}} a \quad (97)$$

where the quantities with tildes are in the Einstein frame while those without tildes are in the string frame. In fact, the transformations (94) and (96) are coordinate transformations rather than conformal transformations, so they are responsible for a difference in the time measurements/scales in both frames. Historically these scales were called atomic and cosmological, for example [7,23].

Imposing Friedmann metric Equation (93) with in the Einstein frame the Equations (78) and (79) give:

$$\left(\omega + \frac{3}{2} \right) \left[\ddot{\phi} + 3 \frac{\dot{\tilde{a}}}{\tilde{a}} \dot{\phi} \right] = 0 \quad (98)$$

$$3 \frac{\dot{\tilde{a}}^2 + k}{\tilde{a}^2} = \left(\omega + \frac{3}{2} \right) \dot{\phi}^2 \quad (99)$$

$$-2 \frac{\ddot{\tilde{a}}}{\tilde{a}} - \frac{\dot{\tilde{a}}^2 + k}{\tilde{a}^2} = \left(\omega + \frac{3}{2} \right) \dot{\phi}^2 \quad (100)$$

The solutions of Equations (98)–(100) for an arbitrary value of the parameter $\omega \neq 3/2$ and $k = 0$ read as:

$$\tilde{a} = |\tilde{t}|^{\frac{1}{3}} \quad (101)$$

$$\phi = \phi_0 + \frac{1}{\sqrt{3(\omega + \frac{3}{2})}} \ln |\tilde{t}| \quad (102)$$

On the other hand, it is clear from Equations (98)–(100) that for $\omega = -3/2$ and $k = 0$ the unique solution gives:

$$\dot{\tilde{a}} = 0 \quad (103)$$

which is just a flat Minkowski universe. If we assume $k \neq 0$, then we get from Equation (99) that:

$$\tilde{a} = \sqrt{-k} \tilde{t} + \tilde{t}_0 \quad (104)$$

which is admissible only for $k = -1$ and this solution represents Milne universe [7] (in which there is no acceleration of the expansion since $\ddot{\tilde{a}} = 0$). However, its relation to Minkowski spacetime requires coordinate transformations that involves the two time scales—a dynamical one and an atomic one [7], which may be responsible for cosmological redshift effect. On the other hand, the solution for $k = +1$ would be possible only if the cosmological constant was admitted—again, cosmological redshift in this Static Einstein model would be the result of a different time scaling [7].

In the string frame we use the Friedmann metric Equation (92) that, imposed into the Equations (72) and (73) for an arbitrary value of the parameter ω , gives the following set of equations:

$$\dot{\phi} - 3\frac{\dot{a}}{a} = \frac{\ddot{\phi}}{\dot{\phi}} \quad (105)$$

$$-3\frac{\dot{a}^2 + k}{a^2} = -\left(\frac{\omega}{2} + 1\right)\dot{\phi}^2 + \ddot{\phi} \quad (106)$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = \frac{\omega}{2}\dot{\phi}^2 + \frac{\dot{a}}{a}\dot{\phi} \quad (107)$$

These Equations (105)–(107) give the following solutions [52]:

$$a(t) = |t|^{\frac{3(\omega+1) \pm \sqrt{3(2\omega+3)}}{3(3\omega+4)}} \quad (108)$$

$$\phi(t) = \frac{-1 \pm \sqrt{3(2\omega+3)}}{3\omega+4} \ln |t| \quad (109)$$

where following pre-big-bang/ekpyrotic scenario the solutions for negative times are also admitted. From Equations (110) and (111) one can first find the pre-big-bang solutions for $\omega = -1$ [21] which are very well-known and read:

$$a(t) = |t|^{\pm \frac{1}{\sqrt{3}}} \quad (110)$$

$$\phi(t) = (\pm\sqrt{3} - 1) \ln |t| \quad (111)$$

However, the conformal relativistic solutions for $\omega = -\frac{3}{2}$ are:

$$a(t) = |t| \quad (112)$$

$$\phi(t) = 2 \ln |t| \quad (113)$$

and show that they do not allow for two branches “+” and “−” and so they do not seem to allow scale factor duality [53,54]:

$$a(t) \rightarrow \frac{1}{a(-t)}, \quad \phi \rightarrow \phi - 6 \ln a \quad (114)$$

which is a cosmological consequence of string duality symmetries [20]. However, unlike pre-big-bang solutions (110) and (111) which must be regularized at Big Bang singularity because both the curvature

and the string coupling Equation (66) diverge there, the solutions (112) and (113) do not lead to strong coupling singularity in the sense of string theory, since:

$$g_s = e^{\frac{\phi}{2}} = |t| \quad (115)$$

and is regular for $t = 0$. This has an interesting analogy with the ekpyrotic/cyclic universe scenario where, in fact, the transition through Big Bang singularity takes place in the weak coupling regime [55–58]. Note that from Equations (96) and (97) we have:

$$\tilde{t} = \tilde{t}_0 + \ln t \quad (116)$$

which according to Equation (112) gives:

$$\tilde{a} = 1 \quad (117)$$

This is consistent with what we have obtained in Equation (103) and reflects the fact that in the Einstein frame the limit of an expanding flat Friedmann metric is Minkowski universe.

5. Discussion

We have shown that there is a simple relation between conformal relativity (Hoyle–Narlikar theory) and a reduced (graviton-dilaton) low-energy-effective superstring theory to Brans–Dicke theory. This relation shows that the former is recovered from Brans–Dicke theory if one takes $\omega = -3/2$ while the latter is recovered if one takes $\omega = -1$. This may allow to study the exact cosmological solutions of conformal relativity and its properties appealing to the well-studied properties of the Brans–Dicke theory and, in particular, to low-energy-effective superstring effective theory.

In fact, Brans–Dicke parameter $\omega = -3/2$ gives a border between a standard scalar field and a ghost (negative kinetic energy) which is required in order to have an expansion minimum in varying constant cosmologies [45] and to mimic the phantom field [46] which violates the weak energy condition $\rho + p < 0$.

The conformally transformed energy-momentum tensor is conserved in conformal relativity only if its trace vanishes. In the case of barotropic fluid it vanishes only for radiation. This means that only the photons may obey the equivalence principle and this is not the case for other types of matter, since with non-vanishing trace we deal with creation of matter process. We have compared that conformal relativity has lots of analogy with Self Creation Cosmology of [16,17], which has the same field equations provided we deal with cosmological models of vanishing curvature scalar.

We have presented basic cosmological solutions of conformal relativity in both Einstein (minimally coupled) and string (non-minimally coupled) frames. The Einstein limit of flat conformal cosmological solutions is unique and it is flat Minkowski space, which requires the scalar field/mass evolution instead of the scale factor evolution to explain cosmological redshift. An interesting observation is that like in ekpyrotic/cyclic models, a possible transition through a singularity in conformal cosmology in the string frame takes place in the weak coupling regime.

Some other interesting points can be made about the vanishing of some of the PPN (parametrized-post-Newtonian) parameters [27] in the theories under consideration. It appears that in

conformal relativity only one of them (β_1) vanishes while for superstring-effective theory two of them vanish (γ and β_4).

Despite its restricted generality, conformally invariant theory of gravity still seems to be attractive, for instance with respect to the problem of a unique choice of vacuum in quantum field theory in curved spaces [1]. There is no doubt that studies of the conformally invariant theories are useful in the context of superstring and M-theory (see, e.g., [47,59,60]).

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