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Article

Maximum Profit Configurations of Commercial Engines

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Abstract: An investigation of commercial engines with finite capacity low- and high-price economic subsystems and a generalized commodity transfer law [$n \propto \Delta(P^m)$] in commodity flow processes, in which effects of the price elasticities of supply and demand are introduced, is presented in this paper. Optimal cycle configurations of commercial engines for maximum profit are obtained by applying optimal control theory. In some special cases, the eventual state—market equilibrium—is solely determined by the initial conditions and the inherent characteristics of two subsystems; while the different ways of transfer affect the model in respects of the specific forms of the paths of prices and the instantaneous commodity flow, *i.e.*, the optimal configuration.

Keywords: finite time thermodynamics; commercial engine; optimal configuration

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1. Introduction

In the realm of finite time thermodynamics, two issues are of essential importance—one is to determine the extremum of objective function and study the interrelation of different objective functions, the other is to determine the optimal thermodynamic process for given optimization objectives [1–16]. Curzon and Ahlborn [17] demonstrated that the efficiency at maximum power point is $\eta_{CA} = 1 - \sqrt{T_L / T_H}$ for an endoreversible Carnot heat engine operating between two constant temperature reservoirs with Newtonian heat transfer law [$q \propto \Delta(T)$]. Procaccia and Ross [18] proved that in all acceptable cycles, an endoreversible Carnot cycle with larger compression ratio can produce maximum power, *i.e.*, the Curzon-Ahlborn cycle [17] is the optimal configuration with only First and Second Law constraints. Ondrechen *et al.* [19] studied the optimal cycle configuration of an endoreversible heat engine with a finite thermal capacity reservoir and Newtonian heat transfer law for maximum work output. Chen *et al.* [20] investigated effects of heat leakage on the optimal cycle configuration of a heat engine with a finite thermal capacity reservoir and Newtonian heat transfer law for maximum work output. Linetskii and Tsirlin [21], and Andresen and Gordon [22] considered the minimum entropy generation of heat transfer process with Newtonian heat transfer law in heat exchanger. Based on reference [22], Badescu [23] optimized the heat transfer process with Newtonian heat transfer law for minimum lost available work by choosing the hot bath side as referee environment. Xia *et al.* [24] optimized the heat transfer process with Newtonian heat transfer law in heat exchanger for entransy dissipation minimization. Nevertheless, generally, heat transfer does not necessarily obey Newtonian heat transfer law, and it may follow other laws. Heat transfer laws not only influence the performance of given thermodynamic processes [25–29], but also influence the optimal configurations of thermodynamic processes for given optimization objectives. Yan *et al.* [30] investigated the optimal cycle configuration of an endoreversible heat engine with a finite thermal capacity reservoir and the linear phenomenological heat transfer law [$q \propto \Delta(T^{-1})$] for maximum work output. Chen *et al.* [31] investigated effects of heat leakage on the optimal cycle configuration of a heat engine with a finite thermal capacity reservoir and the linear phenomenological heat transfer law for maximum work output. Some studies on the optimal configuration of variable- temperature heat reservoir heat engine for maximum power output were also performed, with the generalized radiative heat transfer law [$q \propto \Delta(T^n)$] [32], generalized convective heat transfer law [$q \propto (\Delta T)^m$] [33], mixed heat resistance [34], and generalized heat transfer law [$q \propto (\Delta(T^n))^m$] [35], respectively. Andresen and Gordon [36] and Badescu [37] further optimized a class of heat transfer processes, with generalized radiative heat transfer law for minimum entropy generation [36] and minimum lost available work [37], respectively. Based on the generalized heat transfer law [$q \propto (\Delta(T^n))^m$], Chen *et al.* [38] and Xia *et al.* [39] derived the optimal temperature configurations of heat transfer processes for minimum entropy generation [38] and minimum lost available work [39]. Xia *et al.* [40] further investigated the minimum entransy dissipation of heat transfer processes with the generalized radiative heat transfer law.

In the realm of thermodynamics, a thermodynamic system can be described by extensive variables (such as mass, volume, internal energy, and entropy) and intensive variables (such as temperature, and pressure); and heat flux is generated by temperature difference. Similarly, in the realm of economics, variables can also be classified into extensive ones (such as labor, capital, and good) and intensive ones (such as price); moreover, commodity flow is generated by price difference. The striking resemblance of thermodynamics and economics has drawn much attention [2,4,6,7,10,41–49]. Rozonoer [41–43] studied the analogies between reversible thermodynamics and economics in detail, and proposed the term "resource economics" for the analysis of economic system using a thermodynamic approach. Based on the analogies between economics and thermodynamics, Saslow [45] developed economic analogies to the free energy, Maxwell relations, and the Gibbs-Duhem relationship. Salamon *et al.* [46], Berry *et al.* [4], Tsirlin [7,10,14], and Mironova *et al.* [6] addressed the research lines and methods of finite-time thermodynamics into economic analyses. They considered the finite rate commodity flow, and investigated the minimal expenses of resource exchange processes with linear commodity transfer law [$n \propto \Delta(P)$] and maximal profit rates of constant flow and reciprocal commercial engines (which are analogous to constant flow and reciprocal heat engines operating between infinite heat reservoirs in thermodynamics). De Vos [48–50] investigated the analogies among endoreversible heat engines, chemical engines and commercial engines. Based on a generalized commodity transfer law [$n \propto \Delta(P^m)$], where the exponent m is closely related to the price elasticity of supply and demand, De Vos [49,50] further investigated the optimal performances of endoreversible commercial engines. Martinas [51] investigated the similarities and differences between irreversible thermodynamics and irreversible economics. Tsirlin [52], Tsirlin *et al.* [53–55], and Amelkin *et al.* [56] established an analogy between the processes in microeconomics and irreversible thermodynamics, and defined a physical quality in economics that could be used to measure the irreversibility of commodity exchange processes, *i.e.*, capital dissipation, which is analogous to the physical quality of entropy generation in thermodynamics. Amelkin [57] investigated limit performances of a class of resource exchange processes in complex open microeconomic systems including sequential structure and parallel structure. Tsirlin and Kazakov [58] investigated the optimal cycle configuration of a commercial engine with a finite capacity economic subsystem and the linear transfer law for maximum profit.

This paper will further discuss the issue of commercial engine with a more generalized model by relaxing the assumption of linear transfer law. Actually, commodity flow in this model is assumed to follow the generalized transfer law [$n \propto \Delta(P^m)$] [48–50], which in economics represents possibility of different preferences. By applying the methods of finite time thermodynamics, this paper will provide the optimal cycle configuration of the commercial engine and give a straightforward and intuitive demonstration of price convergence in the model.

2. Model Description

The model of the commercial engine with finite capacity low-price economic subsystem and finite capacity high-price economic subsystem is illustrated in Figure 1. Both commodity flow and money flow are present in the model; the former flows from the low-price side to the high-price side and the latter flows in the opposite direction. In this paper, commodity flow is considered and it is measured in monetary terms.

Taxation, in particular VAT (value added tax) would give similar loss terms for the monetary flows in the opposite direction. It can be seen as the heat leakage in an irreversible heat engine model [28]. In the endoreversible commercial engine model discussed herein, it is neglected just as did for the endoreversible heat engine model [26,27].

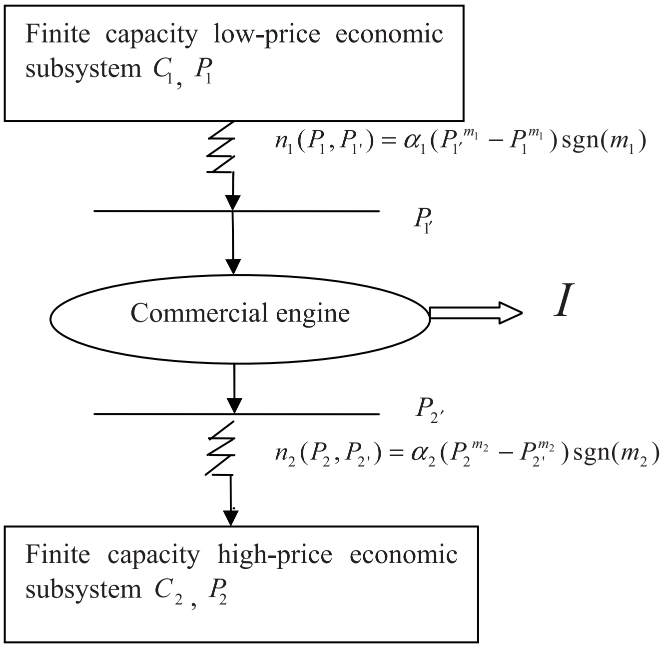
The capacity of the low-price economic subsystem is constant C_1. The commodity price in the subsystem is P_1, whose initial value is given by $P_1(0) = P_{10}$. In addition, the dynamics of P_1 satisfies the equation:

$$C_1 dP_1 / dt = -dN_1 / dt \tag{1}$$

which is an analogy to the dynamics of temperature of a heat reservoir with finite thermal capacity. It is a reasonable analogy because the behavior of P_1 described by Equation (1) is compatible with the common assumption in economics of diminishing marginal utility. Similarly, for the finite capacity high-price subsystem, the capacity is constant C_2, commodity price is P_2 with initial value $P_2(t_1) = P_{20}$ (t_1 is the initial time for selling) and dynamics:

$$C_2 dP_2 / dt = -dN_2 / dt \tag{2}$$

Figure 1. Commercial engine model.



To consider the cases where one side or both sides have infinite capacity, one merely needs to slightly modify the results of model in this paper by taking the limit $C \to \infty$. Furthermore, the commodity prices of the commercial engine corresponding to low-price and high-price sides are $P_{1'}$ and $P_{2'}$, respectively, with $P_1 < P_{1'} < P_{2'} < P_2$.

Different from the linear transfer law $n \propto \Delta(P)$ adopted in [58], commodity flows in the present model are generalized to follow the generalized transfer law $n \propto \Delta(P^m)$ [48–50]:

$$n_1(P_1, P_{1'}) = \alpha_1(P_{1'}^{m_1} - P_1^{m_1})\,\mathrm{sgn}(m_1), \quad n_2(P_{2'}, P_2) = \alpha_2(P_2^{m_2} - P_{2'}^{m_2})\,\mathrm{sgn}(m_2) \tag{3}$$

where $n_1(P_1, P_{1'})$ and $n_2(P_{2'}, P_2)$ are commodity flows corresponding to low-price and high-price sides of the commercial engine, $\alpha_1(t)$ and $\alpha_2(t)$ are the corresponding transfer coefficients, and exponents m_1 and m_2 are indicators of price elasticity of supply or demand. To elucidate, elasticity is a measure of the responsiveness of supply or demand to price changes, mathematically:

$$\varepsilon_1 = \frac{dn_1 / n_1}{dP_{1'} / P_{1'}}, \quad \varepsilon_2 = -\frac{dn_2 / n_2}{dP_{2'} / P_{2'}} \tag{4}$$

Substituting Equation (3) into Equation (4) yields:

$$\varepsilon_1 = \frac{m_1 P_{1'}^{m_1}}{P_{1'}^{m_1} - P_1^{m_1}}, \quad \varepsilon_2 = \frac{m_2 P_{2'}^{m_2}}{P_2^{m_2} - P_{2'}^{m_2}} \tag{5}$$

A one-to-one relationship between the exponent and elasticity is established above. It should be noted that m_1 and m_2 don't necessarily have to be the same, because different m's may represent different preferences of suppliers and demanders.

The amount of commodity exchange in the low-price side and high price side are denoted as ΔN_1

and ΔN_2, respectively. They are given by:

$$\Delta N_1 = \int_0^\tau n_1(P_1, P_{1'})dt = \int_0^\tau \alpha_1(t)[P_{1'}^{m_1}(t) - P_1^{m_1}(t)]\mathrm{sgn}(m_1)dt \tag{6}$$

$$\Delta N_2 = \int_0^\tau n_2(P_{2'}, P_2)dt = \int_0^\tau \alpha_2(t)[P_2^{m_2} - P_{2'}^{m_2}(t)]\mathrm{sgn}(m_2)dt \tag{7}$$

where τ is the given cycle period. Additionally, market equilibrium condition requires that:

$$\Delta N_1 = \Delta N_2 = \Delta N \tag{8}$$

It is further assumed that purchase and selling are separate and successive processes. At time t ($0 < t < t_1$), the commercial engine purchases commodity from the low-price subsystem; and at time t ($t_1 < t < \tau$), the commercial engine sells commodity to the high-price subsystem. Therefore, $\alpha_1(t)$ and $\alpha_2(t)$ have the following forms:

$$\alpha_1(t) = \begin{cases} \alpha_1, & 0 \le t \le t_1 \\ 0, & t_1 \le t \le \tau \end{cases}, \quad \alpha_2(t) = \begin{cases} 0, & 0 \le t \le t_1 \\ \alpha_2, & t_1 \le t \le \tau \end{cases} \tag{9}$$

where α_1 and α_2 are positive constants.

Finally, profit gained by the commercial engine is given by:

$$I = \int_0^\tau [P_{2'}(t)n_2(P_{2'}, P_2) - P_{1'}(t)n_1(P_1, P_{1'})]dt \tag{10}$$

3. Optimization

The optimization problem for the commercial engine is to maximize its profit with the constraints of market equilibrium conditions and the predetermined dynamics of prices in the two economic subsystems. Mathematically, the problem amounts to determine the optimal paths of $P_{1'}$ and $P_{2'}$, the optimal values of t_1 and ΔN to maximize Equation (10) subject to Equations (1), (2) and (8).

Following the method adopted in [58], optimization problem is decomposed into two sub-problems.

3.1. Problem 1

$$\max \quad I^- = \int_0^{t_1} -\alpha_1(P_{1'}^{m_1} - P_1^{m_1})\mathrm{sgn}(m_1)P_{1'}dt \tag{11}$$

$$\text{s.t } dP_1 / dt = \alpha_1(P_{1'}^{m_1} - P_1^{m_1})\mathrm{sgn}(m_1)/C_1 \tag{12}$$

$$\int_0^{t_1} \alpha_1(P_{1'}^{m_1} - P_1^{m_1})\mathrm{sgn}(m_1)dt = \Delta N_1 = \Delta N \tag{13}$$

Equation (12) is obtained by substituting Equation (6) into Equation (1). Substituting Equation (12) into Equations (11) and (13) yields:

$$\max \quad I^- = \int_{P_{10}}^{P_1(t_1)} -C_1 P_{1'} dP_1 \tag{14}$$

$$\int_{P_{10}}^{P_1(t_1)} C_1 dP_1 = \Delta N \tag{15}$$

Equation (12) itself can be transformed to:

$$\int_{P_{10}}^{P_1(t_1)} \frac{C_1}{\alpha_1(P_{1'}^{m_1} - P_1^{m_1})\operatorname{sgn}(m_1)} dP_1 = t_1 \tag{16}$$

The problem now becomes maximizing Equation (14) subject to Equations (15) and (16). The corresponding modified Lagrangian function is given by:

$$L_1 = C_1\left[-P_{1'} + \lambda_1 + \frac{\lambda_2}{\alpha_1(P_{1'}^{m_1} - P_1^{m_1})\operatorname{sgn}(m_1)}\right] \tag{17}$$

where λ_1 and λ_1 are Lagrangian multipliers.

First order condition with respect to $P_{1'}$ yields

$$P_{1'}^{m_1} - P_1^{m_1} = k_1 P_{1'}^{(m_1-1)/2} \tag{18}$$

where k_1 is a constant to be determined.

Combining Equation (18) with Equation (12) yields:

$$\frac{dP_{1'}}{dt} = \frac{m_1\alpha_1 k_1 \operatorname{sgn}(m_1)}{C_1} \cdot \frac{P_{1'}^{(m_1+1)/2}(P_{1'}^{m_1} - k_1 P_{1'}^{(m_1-1)/2})^{(m_1-1)/m_1}}{m_1 P_{1'}^{m_1} - (m_1-1)k_1 P_{1'}^{(m_1-1)/2}/2} \tag{19}$$

The dynamics of $P_{1'}$ are uniquely characterized by Equation (19), which, combined with Equations (18), (13) and the initial value of P_1, determines the paths of both $P_{1'}$ and P_1.

3.2. Problem 2

$$\max \quad I^+ = \int_{t_1}^{\tau} \alpha_2(P_2^{m_2} - P_{2'}^{m_2})\operatorname{sgn}(m_2)P_{2'}dt \tag{20}$$

$$\text{s.t. } dP_2/dt = -\alpha_2(P_2^{m_1} - P_{2'}^{m_1})\operatorname{sgn}(m_2)/C_2 \tag{21}$$

$$\int_{t_1}^{\tau} \alpha_2(P_2^{m_2} - P_{2'}^{m_2})\operatorname{sgn}(m_2)dt = \Delta N_2 = \Delta N \tag{22}$$

Equation (21) is obtained by substituting Equation (7) into Equation (2). Substituting Equation (21) into Equations (20) and (22) yields:

$$\max \quad I^+ = -\int_{P_{20}}^{P_2(\tau)} C_2 P_{2'} dP_2 \tag{23}$$

$$-\int_{P_{20}}^{P_2(\tau)} C_2 dP_2 = \Delta N \tag{24}$$

Equation (21) itself can be transformed to:

$$-\int_{P_{20}}^{P_2(\tau)} \frac{C_2}{\alpha_2(P_2^{m_2} - P_{2'}^{m_2})\operatorname{sgn}(m_2)} dP_2 = \tau - t_1 \tag{25}$$

The problem now becomes maximizing Equation (23) subject to Equations (24) and (25). The corresponding modified Lagrangian function is given by:

$$L_2 = -C_2[P_{2'} + \lambda_3 + \frac{\lambda_4}{\alpha_2(P_2^{m_2} - P_{2'}^{m_2})\operatorname{sgn}(m_2)}] \tag{26}$$

where λ_3 and λ_4 are Lagrangian multipliers. First order condition with respect to $P_{2'}$ yields:

$$P_2^{m_2} - P_{2'}^{m_2} = k_2 P_{2'}^{(m_2-1)/2} \tag{27}$$

where k_2 is a constant to be determined.

Combining Equation (27) with Equation (21) yields:

$$\frac{dP_{2'}}{dt} = -\frac{m_2\alpha_2 k_2 \operatorname{sgn}(m_2)}{C_2}\cdot\frac{P_{2'}^{(m_1+1)/2}(P_{2'}^{m_2} + k_2 P_{2'}^{(m_2-1)/2})^{(m_1-1)/m_1}}{m_2 P_{2'}^{m_2} + (m_2-1)k_2 P_{2'}^{(m_2-1)/2}/2} \tag{28}$$

The dynamics of $P_{2'}$ are uniquely characterized by Equation (28), which, combined with Equations (22), (27) and the initial value of P_2, determines the paths of both $P_{2'}$ and P_2. In sum, the optimal paths of $P_{1'}$ and $P_{2'}$ are described by Equations (19) and (28). However, analytical solutions to these differential equations exist only for a few exponents such as 1 and −1. For other exponents which do not admit analytical solutions, numerical method should be adopted.

To further determine the optimal values of ΔN, t_1, and I, one merely needs to substitute the paths of $P_{1'}$, P_1, $P_{2'}$ and P_2 into Equation (10) and solve the system of first order conditions.

4. Special case with $m_1 = 1$ and $m_2 = 1$

4.1. Analytical Solutions

For problem 1, Equations (19), (18) and (13) are simplified to $\frac{dP_{1'}}{dt} = \frac{\alpha_1 k_1}{C_1}$, $P_{1'} - P_1 = k_1$ and $\int_0^{t_1}\alpha_1(P_{1'} - P_1)dt = \Delta N$, respectively, in this case. Solving the system gives the paths of P_1 and $P_{1'}$, respectively:

$$P_1 = (\Delta N / C_1 t_1)t + P_{10} \quad (0 \le t \le t_1) \tag{29}$$

$$P_{1'} = (\Delta N / C_1 t_1)t + P_{10} + \Delta N / \alpha_1 t_1 \quad (0 \le t \le t_1) \tag{30}$$

For problem 2, Equations (28), (27) and (22) are simplified to $\frac{dP_{2'}}{dt} = \frac{\alpha_2 k_2}{C_2}$, $P_{2'} - P_2 = k_2$ and $\int_{t_1}^{\tau}\alpha_2(P_2 - P_{2'})dt = \Delta N$, respectively, in this case. Solving the system gives the paths of P_2 and $P_{2'}$, respectively:

$$P_2 = -[\Delta N / C_2(\tau - t_1)](t - t_1) + P_{20} \quad (t_1 \le t \le \tau) \tag{31}$$

$$P_{2'} = -[\Delta N / C_2(\tau - t_1)](t - t_1) + P_{20} - \Delta N / \alpha_2(\tau - t_1) \quad (t_1 \le t \le \tau) \tag{32}$$

Substituting Equations (29), (30), (31) and (32) into Equation (10) yields:

$$I = (P_{20} - P_{10})\Delta N - [(1/2C_2 + 1/2C_1 + 1/\alpha_2(\tau - t_1) + 1/\alpha_1 t_1]\Delta N \tag{33}$$

First order condition $\delta I / \delta t_1 = 0$ yields:

$$t_1^* = \frac{\sqrt{\alpha_2}\tau}{\sqrt{\alpha_1} + \sqrt{\alpha_2}} \tag{34}$$

Substituting Equation (29) into the First order condition $\delta I / \delta(\Delta N) = 0$ yields:

$$\Delta N^{*} = \frac{(P_{20} - P_{10})\tau}{\tau(1/C_1 + 1/C_2) + 2(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 / (\alpha_1 \alpha_2)} \tag{35}$$

Therefore:

$$I_{\max} = \frac{(P_{20} - P_{10})^2 \tau}{2\tau(1/C_1 + 1/C_2) + 4(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 / (\alpha_1 \alpha_2)} \tag{36}$$

Substituting Equations (34) and (35) into Equations (29), (30), (31) and (32) yields, respectively:

$$P_1^{*} = \frac{(P_{20} - P_{10})(\sqrt{\alpha_1} + \sqrt{\alpha_2})}{C_1\sqrt{\alpha_2}[\tau(1/C_1 + 1/C_2) + 2(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 / (\alpha_1\alpha_2)]} t + P_{10} \quad (0 \le t \le t_1) \tag{37}$$

$$P_{1'}^{*} = \frac{(P_{20} - P_{10})(\sqrt{\alpha_1} + \sqrt{\alpha_2})}{C_1\sqrt{\alpha_2}[\tau(1/C_1 + 1/C_2) + 2(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 / (\alpha_1\alpha_2)]} (t + \frac{C_1}{\alpha_1}) + P_{10} \quad (0 \le t \le t_1) \tag{38}$$

$$P_2^{*} = \frac{-(P_{20} - P_{10})(\sqrt{\alpha_1} + \sqrt{\alpha_2})}{C_2\sqrt{\alpha_1}[\tau(1/C_1 + 1/C_2) + 2(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 / (\alpha_1\alpha_2)]} (t - \frac{\sqrt{\alpha_2}\tau}{\sqrt{\alpha_1} + \sqrt{\alpha_2}}) + P_{20} \quad (t_1 \le t \le \tau) \tag{39}$$

$$P_{2'}^{*} = \frac{-(P_{20} - P_{10})(\sqrt{\alpha_1} + \sqrt{\alpha_2})}{C_2\sqrt{\alpha_1}[\tau(1/C_1 + 1/C_2) + 2(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 / (\alpha_1\alpha_2)]} (t - \frac{\sqrt{\alpha_2}\tau}{\sqrt{\alpha_1} + \sqrt{\alpha_2}} + \frac{C_2}{\alpha_2}) + P_{20} \quad (t_1 \le t \le \tau) \tag{40}$$

4.2. Results and Discussion

It is revealed above that both P_1^* and $P_{1'}^*$ increase linearly in the time, while both P_2^* and $P_{2'}^*$ decrease linearly in the time; commodity flows $n_1(P_1, P_{1'})$ and $n_2(P_{2'}, P_2)$ are constants over time; the optimal exchange time t_1^* is determined only by ratio of the transfer coefficients α_1 and α_2.

The most enlightening implication of the result is the convergence of the eventual values of P_1^*, $P_{1'}^*$, P_2^* and $P_{2'}^*$. Mathematically:

$$\lim_{\tau\to\infty} P_1^*(t_1) = \lim_{\tau\to\infty} P_{1'}^*(t_1) = \lim_{\tau\to\infty} P_{1'}^*(t_1) = \lim_{\tau\to\infty} P_{2'}^*(\tau) = \frac{C_1 P_{10} + C_2 P_{20}}{C_1 + C_2} \tag{41}$$

The common limit is exactly equilibrium price which completely clears the market, *i.e.*:

$$P_e = \frac{C_1 P_{10} + C_2 P_{20}}{C_1 + C_2} \tag{42}$$

which is a weighted average of the initial prices of two subsystems, and the weights are the corresponding capacities. Larger capacity indicates larger market power, therefore equilibrium price is more biased to the initial price of the party with larger capacity. Especially, if one side has infinite capacity, the equilibrium price will be the same as its initial price.

Additionally, The convergence of $P_{1'}$* and $P_{2'}$* also indicates that the instantaneous profit gained by the commercial engine diminishes to 0 as the cycle period approaches infinity, which further indicates that the total profit earned cannot be infinite. Equation (36) serves as an apt substantiation of this point:

$$\lim_{\tau \to \infty} I_{\max} = \frac{(P_{20} - P_{10})^2}{2(1/C_1 + 1/C_2)} \tag{43}$$

Finally, price convergence denies long-existing price discrepancy in a pure exchange market without exogenous interventions. Another simple but profound implication is that the profit-maximizing behavior of a commercial engine induces an optimal outcome for the market. In other words, the commercial engine, motivated by its own interest, acts as catalyst in the process of reducing price discrepancy and reaching market equilibrium. However, its existence cannot be permanent since its profit diminishes to 0 with time. In this perspective, the commercial engine can be viewed as an arbitrager whose profit-seeking action results in price parity; and the whole model simulates the dynamic process of the determination of equilibrium price.

5. Special Case with $m_1 = -1$ and $m_2 = -1$

5.1. Analytical Solutions

For problem 1, Equations (19), (18) and (13) are simplified to $\frac{dP_{1'}}{dt} = \frac{\alpha_1 k_1 (k_1 - 1)}{C_1} P_{1'}^{-1}$, $P_{1'}^{-1} - P_1^{-1} = k_1 P_{1'}^{-1}$ and $-\int_0^{t_1} \alpha_1 (P_{1'}^{-1} - P_1^{-1}) dt = \Delta N$, respectively. Solving the system gives the paths of P_1 and $P_{1'}$, respectively:

$$P_1 = \frac{\sqrt{\Delta N t_1 (\Delta N + 2CP_{10}) t + C^2 t_1^2 P_{10}^2}}{Ct_1} \tag{44}$$

$$P_{1'} = \frac{2\alpha_1 \sqrt{\Delta N t_1 (\Delta N + 2CP_{10}) t + C^2 t_1^2 P_{10}^2}}{2C\alpha_1 t_1 - (\Delta N^2 + 2\Delta N C P_{10})} \tag{45}$$

For problem 2, Equations (28), (27) and (22) are simplified to $\frac{dP_{2'}}{dt} = \frac{\alpha_1 k_2 (k_2 + 1)}{C_2} P_{2'}^{-1}$, $P_2^{-1} - P_{2'}^{-1} = k_2 P_{2'}^{-1}$ and $-\int_{t_1}^{\tau} \alpha_1 (P_2^{-1} - P_{2'}^{-1}) dt = \Delta N$, respectively. Solving the system gives the paths of P_2 and $P_{2'}$, respectively:

$$P_2 = \frac{\sqrt{(\Delta N^2 - 2\Delta N C_2 P_{20})(\tau - t_1)(t - t_1) + C_2^2 (\tau - t_1)^2 P_{20}^2}}{C_2 (\tau - t_1)} \tag{46}$$

$$P_{2'} = \frac{2\alpha_2 \sqrt{(\Delta N^2 - 2\Delta N C_2 P_{20})(\tau - t_1)(t - t_1) + C_2^2 (\tau - t_1)^2 P_{20}^2}}{-\Delta N^2 + 2\Delta N C_2 P_{20} + 2C_2 \alpha_2 (\tau - t_1)} \tag{47}$$

Substituting Equations (44), (45), (46) and (47) into Equation (10) yields:

$$I = \frac{\alpha_2 (\tau - t_1)(\Delta N^2 - 2\Delta N C_2 P_{20})}{\Delta N^2 - 2\Delta N C_2 P_{20} - 2C_2 \alpha_2 (\tau - t_1)} + \frac{\alpha_1 t_1 (\Delta N^2 + 2\Delta N C_1 P_{10})}{\Delta N^2 + 2\Delta N C_1 P_{10} - 2C_1 \alpha_1 t_1} \tag{48}$$

The optimal t_1^* and ΔN^* are jointly determined by first order conditions $\partial I / \partial t_1 = 0$ and $\partial I / \partial (\Delta N) = 0$. However, the system of polynomials cannot be solved explicitly.

5.2. Results and Discussion

This section focuses on discussing the behaviors of P_1^*, $P_{1'}^*$, P_2^* and $P_{2'}^*$ as $\tau \to \infty$. From Equations (44) and (46) one can obtain:

$$P_1^*(t_1) = \frac{\Delta N^* + C_1 P_{10}}{C_1} = \frac{\Delta N^*}{C_1} + P_{10} \tag{49}$$

$$P_2^*(\tau) = \frac{-\Delta N^* + C_2 P_{20}}{C_2} = \frac{-\Delta N^*}{C_2} + P_{20} \tag{50}$$

They depend on ΔN^* which cannot be solved explicitly.

To proceed, first suppose there does exist an optimal solution where P_1^*, $P_{1'}^*$, P_2^* and $P_{2'}^*$ converge eventually. Then ΔN^* is bounded; and thus as $\tau \to \infty$, Equation (48) becomes:

$$\lim_{\tau\to\infty} I = \frac{\Delta N^2 - 2\Delta N C_2 P_{20}}{-2C_2} + \frac{\Delta N^2 + 2\Delta N C_1 P_{10}}{-2C_1} \tag{51}$$

First order condition $\partial \lim_{\tau\to\infty} I / \partial(\Delta N) = 0$ yields:

$$\lim_{\tau\to\infty} \Delta N^* = \frac{P_{20} - P_{10}}{1/C_1 + 1/C_2} \tag{52}$$

To check whether the ΔN^* determined by Equation (52) supports such an optimal solution, substitute Equation (52) into Equations (49) and (50):

$$\lim_{\tau\to\infty} P_1^*(t_1) = \lim_{\tau\to\infty} P_2^*(\tau) = \frac{C_1 P_{10} + C_2 P_{20}}{C_1 + C_2} \tag{53}$$

Since $P_1 < P_{1'} < P_{2'} < P_2$, there must be:

$$P_e = \lim_{\tau\to\infty} P_1^*(t_1) = \lim_{\tau\to\infty} P_{1'}^*(t_1) = \lim_{\tau\to\infty} P_{1'}^*(t_1) = \lim_{\tau\to\infty} P_{2'}^*(\tau) = \frac{C_1 P_{10} + C_2 P_{20}}{C_1 + C_2} \tag{54}$$

It is revealed that all four prices share common limit; therefore convergence of prices is actually an optimal solution to this problem. The corresponding maximum profit is given by:

$$\lim_{\tau\to\infty} I_{\max} = \frac{(P_{20} - P_{10})^2}{2(1/C_1 + 1/C_2)} \tag{55}$$

Comparing the results of this case with $m_1 = m_2 = -1$ with those of the previous case with $m_1 = m_2 = 1$, one finds that the equilibrium price P_e, the optimal amount of commodity exchange ΔN^*, and the maximum profit $I_{\max}$ are the same. It should be noted that this phenomenon is not a coincidence. Actually, in this model, the eventual state—market equilibrium—is solely determined by the initial conditions and the inherent characteristics of two subsystems; while the different ways of transfer (reflected by different values of m_1 and m_2) affect the model in respects of the specific forms of the paths of prices and the instantaneous commodity flow, *i.e.*, the optimal configuration.

6. Conclusions

Commercial engines with finite capacity low-price economic subsystems and a generalized commodity transfer law [$n \propto \Delta(P^m)$] during commodity flow processes, in which the effects of the price elasticities of supply and demand are introduced, are investigated in this paper. The optimal cycle configurations of the commercial engines for maximum profit are obtained by applying optimal control theory. The optimal cycle configuration of the commercial engine with the linear transfer law [$n \propto \Delta P$] is that both the price estimation of finite capacity low-price economic subsystem and the commodity-buying price of the commercial engine change with time linearly and the difference between them is a constant, and the selling price of the commercial engine is a constant when it exchanges commodity with the infinite capacity high-price economic subsystem. The optimal cycle configuration of the commercial engine with the transfer law [$n \propto \Delta(P^{-1})$] is that both the price estimation of finite capacity low-price economic subsystem and the commodity-buying price of the commercial engine change with time non-linearly and the ratio between them is a constant, and the selling price of the commercial engine is a constant when it exchanges commodity with the infinite capacity high-price economic subsystem. The research in this paper further extends the research lines and methods of finite time thermodynamics to applications in fields of non-conventional thermodynamics. It is worthwhile to note that several authors [60–64] have criticized finite time thermodynamics (emphasis on the endoreversible model and the corresponding study results) in recent years. The responses to those articles can be seen in [65–69], especially, Chen *et al.* [67].

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