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Full Research Paper

Bell-Boole Inequality: Nonlocality or Probabilistic Incompatibility of Random Variables?

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Abstract: The main aim of this report is to inform the quantum information community about investigations on the problem of probabilistic compatibility of a family of random variables: a possibility to realize such a family on the basis of a single probability measure (to construct a single Kolmogorov probability space). These investigations were started hundred of years ago by J. Boole (who invented Boolean algebras). The complete solution of the problem was obtained by Soviet mathematician Vorobjev in 60th. Surprisingly probabilists and statisticians obtained inequalities for probabilities and correlations among which one can find the famous Bell's inequality and its generalizations. Such inequalities appeared simply as constraints for probabilistic compatibility. In this framework one can not see a priori any link to such problems as nonlocality and "death of reality" which are typically linked to Bell's type inequalities in physical literature. We analyze the difference between positions of mathematicians and quantum physicists. In particular, we found that one of the most reasonable explanations of probabilistic incompatibility is mixing in Bell's type inequalities statistical data from a number of experiments performed under different experimental contexts.

Keywords: Bell's inequality, nonlocality, "death of reality", probabilistic incompatibility of random variables, Boole's necessary condition, Vorobjev theorem, contextual description of the EPR-Bohm experiment, parameters of measurement devices, fluctuations of hidden variables of source.

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1. Introduction

The aim of this report is to present to the physical community (especially, its quantum information part) results of purely probabilistic studies on the problem of *probabilistic compatibility of a family of random variables*. They were done during last hundred years. And they have the direct relation to Bell's inequality. A priori studies on probabilistic compatibility have no direct relation to the well known fundamental problems which are typically discussed by physicists, namely, *realism and locality* [1]–[15], see e.g. [16]–[21] for recent debates.

We remark that our considerations would not imply that the conventional interpretation of Bell's inequality [1]–[15] should be rejected. In principle, Bell's conditions (nonlocality, "death of reality") could also be taken into account. Our aim is to show that Bell's conditions are only *sufficient, but not necessary* for violation of Bell's inequality. Therefore other interpretations of violation of this inequality are also possible. Bell's alternative – either quantum mechanics or local realism – can be extended – either existence of a single probability measure* for incompatible experimental contexts or quantum mechanics. We notice that existence of such a single probability was never assumed in classical (Kolmogorov) probability space, but it was used by J. Bell to derive his inequality (it was denoted by ρ in Bell's derivation). Therefore if one wants to use Bell's inequality, he should find reasonable arguments supporting Bell's derivation. Roughly speaking: Why do we use such an assumption in quantum physics, although we have never used it in classical probability theory?

This paper is based on the results of research of mathematicians interested in the probabilistic structure of Bell's inequality, Accardi [24]–[26], Fine [27], Pitowsky [28], [29], Rastal [30], Hess and Philipp [31]–[33] and the author [34]–[36]. On one hand, it is amazing that so many people came to the same conclusion practically independently. On the other hand, it is also amazing that this conclusion is not so much known by physicists (even mathematically interested researchers working in quantum information theory). There is definitely a problem of communication between the physical and mathematical communities. I hope that this report would inform physicists about some general ideas of mathematicians on Bell's inequality.

Since in this paper we shall discuss Bell's proof of its inequality and its versions, we present (for reader's convenience) these proofs in the appendix.

2. Sufficient conditions of probabilistic compatibility

Consider a system of three random variables $a_i, i = 1, 2, 3$. Suppose for simplicity that they take discrete values and moreover they are dichotomous: $a_i = \pm 1$. Suppose that these variables as well as their pairs can be measured and hence joint probabilities for pairs are well defined: $P_{a_i,a_j}(\alpha_i, \alpha_j) \geq 0$ and $\sum_{\alpha_i,\alpha_j=\pm 1} P_{a_i,a_j}(\alpha_i, \alpha_j) = 1$.

Question: *Is it possible to construct the joint probability distribution,* $P_{a_1,a_2,a_3}(\alpha_1, \alpha_2, \alpha_3)$, *for any triple of random variables?*

Surprisingly this question was asked and answered for hundred years ago by Boole (who invented

*By using the terminology of modern probability theory one should speak about existence of a single Kolmogorov probability space [22], [23].

Boolean algebras). This was found by Itamar Pitowsky [37], [38], see also preface [18]. To study this problem, Boole derived inequality which coincides with the well known in physics Bell's inequality. Violation of this Boole-Bell inequality implies that for such a system of three random variables the joint probability distribution $P_{a_1,a_2,a_3}(\alpha_1, \alpha_2, \alpha_3)$ does not exist.

Thus Bell's inequality was known in probability theory. It was derived as a constraint which violation implies nonexistence of the joint probability distribution.

Different generalizations of this problem were studied in probability theory. The final solution (for a system of n random variable) was obtained by Soviet mathematician Vorobjev [39] (as was found by Hess and Philipp [33]). His result was applied in purely macroscopic situations – in game theory and optimization theory.

We emphasize that for mathematicians consideration of Bell's type inequalities did not induce revolutionary reconsideration of laws of nature. The joint probability distribution does not exist just because those observables could not be measured simultaneously.

3. Statistics of Polarization Projections

We consider now one special application of Boole's theorem the EPR-Bohm experiment for measurements of spin projections for pairs of entangled photons.[†]Denote corresponding random variables by a^1_θ and a^2_θ, respectively (the upper index $k = 1, 2$ denotes observables for corresponding particles in a pair of entangled photons). Here θ is the angle parameter determining the setting of polarization beam splitter. For our purpose it is sufficient to consider three different angles: $\theta_1, \theta_2, \theta_3$. (In fact, for real experimental tests we should consider four angles, but it does not change anything in our considerations).

By using the condition of precise correlation for the singlet state we can identify observables

$$a_\theta(\lambda) \equiv a^1_\theta(\lambda) = a^2_\theta(\lambda).$$

The following discrete probability distributions are well defined: $P_{a_\theta}(\alpha)$ and $P_{a_{\theta_i},a_{\theta_j}}(\alpha, \beta)$. Here $\alpha, \beta = \pm 1$. We remark that in standard derivations of Bell's type inequality for probabilities (and not correlations), see appendix, there are typically used the following symbolic expressions of probabilities: $P(a_\theta(\lambda) = \alpha)$ and $P(a_{\theta_i}(\lambda) = \alpha, a_{\theta_j}(\lambda) = \beta)$. However, by starting with a single probability P (defined on a single space of "hidden variables" Λ) we repeat Bell's schema (which we would not like to repeat in this paper).

Thus we are precisely in the situation which was considered in probability theory. Boole (and Vorobjev) would ask: Do polarization-projections for any triple of angles have the joint probability distribution? Can one use a single probability measure P? The answer is negative – because the Boole-Bell inequality is violated (or because necessary condition of Vorobjev theorem is violated). Thus it is impossible to introduce the joint probability distribution for an arbitrary triple of angles.

On the other hand, Bell started his considerations with the assumption that such a single probability measure exists, see appendix. He represented all correlations as integrals with respect to the same

[†]Although both Boole's and Bell's theorems are based on the same inequality, the conclusions are totally different. These are "nonexistence of the joint probability distribution" and "either local realism or quantum mechanics", respectively. Thus we would like to analyze the EPR-Bohm experiment from the viewpoint of Boole (Vorobjev, Accardi, Fine, Pitowsky, Rastal, Hess and Philipp and the author).

probability measure ρ :

$$\langle a_{\theta_i}, a_{\theta_j} \rangle = \int_\Lambda a_{\theta_i}(\lambda) a_{\theta_j}(\lambda) dP(\lambda).$$

(We shall use the symbol P, instead of Bell's ρ to denote probability).

In opposite to Bell, Boole would not be so much excited by evidence of violation of Bell's inequality in the EPR-Bohm experiment. The situation when pairwise probability distributions exist. but a single probability measure P could not be constructed is rather standard. What would be a reason for existence of P in the case when the simultaneous measurement of three projections of polarization is impossible?

A priori nonexistence of P has nothing to do with nonlocality or "death of reality." The main problem is not the assumption that polarization projections are represented in the "local form":

$$a^1_{\theta_i}(\lambda), a^2_{\theta_j}(\lambda)$$

and not in the "nonlocal form"

$$a^1_{\theta_i}(\lambda | a^2_{\theta_j} = \beta), a^2_{\theta_j}(\lambda | a^1_{\theta_i} = \alpha),$$

where $\alpha, \beta = \pm 1$. The problem is nor assigning to each λ the definite value of the random variable – "realism."

The problem is impossibility to realize three random variables

$$a_{\theta_1}(\lambda), a_{\theta_2}(\lambda), a_{\theta_3}(\lambda)$$

on the same space of parameters Λ with same probability measure P. By using the modern terminology we say that it is impossible to construct a Kolmogorov probability space for such three random variables.

In this situation it would be reasonable to find sources of nonexistence of a Kolmogorov probability space. We remark that up to now we work in purely classical framework– neither the ψ-function nor noncommutative operators were considered. We have just seen [7], [14], [15] that experimental statistical data violates the necessary condition for the existence of a single probability P. Therefore it would be useful to try to proceed purely classically in the probabilistic analysis of the EPR-Bohm experiment. We shall do this in the next section.

4. Contexts and Probabilities

As was already emphasized in my book [40], the crucial point is that in this experiment one combines statistical data collected on the basis of three different complexes of physical conditions (contexts). We consider context C_1 – setting θ_1, θ_2, context C_2 – setting θ_1, θ_3, and finally context C_3 – setting θ_2, θ_3. We recall that already in Kolmogorov's book [22] (where the modern axiomatics of probability theory was presented) it was pointed out that each experimental context determines its own probability space. By Kolmogorov in general three contexts $C_j, j = 1, 2, 3$, should generate three Kolmogorov spaces: with sets of parameters Ω_j and probabilities P_j.

The most natural way to see the source of appearance of such spaces is to pay attention to the fact that (as it was underlined by Bohr) the result of measurement is determined not only by the initial state of a system (before measurement), but also by the *whole measurement arrangement.* Thus states of measurement devices are definitely involved. We should introduce not only space Λ of states of a system

(a pair of photons), but also spaces of states of polarization beam splitters – Λ_θ. (We proceed under the assumption that the state of polarization beam splitter depends only on the orientation θ. In principle, we should consider two spaces for each θ for the first and the second splitters. In reality they are not identical.) Thus, see [40], for the context C_1 the space of parameters ("hidden variables") is given by

$$\Lambda_1 = \Lambda \times \Lambda_{\theta_1} \times \Lambda_{\theta_2},$$

for the context C_2 it is

$$\Lambda_2 = \Lambda \times \Lambda_{\theta_1} \times \Lambda_{\theta_3},$$

for the context C_3 it is

$$\Lambda_3 = \Lambda \times \Lambda_{\theta_2} \times \Lambda_{\theta_3}.$$

And, of course, we should consider three probability measures

$$dP_1(\lambda, \lambda_{\theta_1}, \lambda_{\theta_2}), dP_2(\lambda, \lambda_{\theta_1}, \lambda_{\theta_3}), dP_3(\lambda, \lambda_{\theta_2}, \lambda_{\theta_3}).$$

Random variables are functions on corresponding spaces

$$a_{\theta_1}(\lambda, \lambda_{\theta_1}), a_{\theta_2}(\lambda, \lambda_{\theta_2}), a_{\theta_3}(\lambda, \lambda_{\theta_3}).$$

Of course, Bell's "condition of locality" is satisfied (otherwise we would have e.g.

$$a_{\theta_1}(\lambda, \lambda_{\theta_1}, \lambda_{\theta_2}), a_{\theta_2}(\lambda, \lambda_{\theta_2}, \lambda_{\theta_1})$$

for the context C_1).

In this situation one should have strong arguments to assume that these three probability distributions could be obtained from a single probability measure

$$dP_1(\lambda, \lambda_{\theta_1}, \lambda_{\theta_2}, \lambda_{\theta_3})$$

on the space

$$\Lambda = \Lambda \times \Lambda_{\theta_1} \times \times \Lambda_{\theta_2} \times \Lambda_{\theta_3}.$$

5. Wave Function and Probability

Finally, we come to quantum mechanics. Our contextual analysis of the EPR-Bohm experiment implies that the most natural explanation of nonexistence of a single probability space is that *the wave function does not determine probability in quantum mechanics* (in contrast to Bell's assumption). We recall that Born's rule contains not only the ψ-function but also spectral families of commutative operators which are measured simultaneously. Hence, the probability distribution is determined by the ψ-function as well as spectral families, i.e., observables.

Such an interpretation of mathematical symbols of the quantum formalism does not imply neither nonlocality nor "death of reality."[‡]

[‡]One should not accuse the author in critique of J. Bell. J. Bell by himself did a similar thing with the von Neumann no-go theorem, see [1], by pointing out that some assumptions of von Neumann were nonphysical.

6. Bell's Inequality and Negative and P-adic Probabilities

By looking for a trace in physics of the Boole-Vorobjev conclusion on nonexistence of probability one can find that this problem was intensively discussed, but in rather unusual form (at least from the mathematical viewpoint). During our conversations on the probabilistic structure of Bell's inequality Alain Aspect permanently pointed out to a probabilistic possibility to escape Bell's alternative: either local realism or quantum mechanics. This possibility mentioned by Alain Aaspect is consideration of negative valued probabilities. A complete review on solving "Bell's paradox" with the aid of negative probabilities was done by Muckenheim [41]. Although negative probabilities are meaningless from the mathematical viewpoint (however, see [42]- [49] for an attempt to define them mathematically by using p-adic analysis), there is some point in consideration of negative probabilities by physicists. In the light of our previous studies this activity can be interpreted as a sign of understanding that "normal probability distribution" does not exist. Surprisingly, but negative probability approach to Bell's inequality can be considered as a link to Boole-Vorobjev's viewpoint on violation of Bell's inequality.

7. Detectors Efficiency (Fair sampling)

Another trace of nonexistence of probability can be found in physical literature on detectors efficiency [59]– [61] or more generally unfair sampling [62],[63]. People are well aware about the fact that the real experiments induce huge losses of photons. A priori there are no reasons to assume that ensembles of entangled photons which pass polarization beam splitters for different choices of orientations have identical statistical properties (hypothesis on fair sampling). Such identity of statistical properties is a consequence of the existence of a single probability P serving all experimental setting at the same time. Thus unfair sampling implies that such a probability does not exist. However, in general nonexistence of probability is not equivalent to unfair sampling. Contextuality (dependence on the context of experiment) might be (but need not be!) exhibited via unfair sampling.

8. Eberhard-Bell Theorem

In quantum information community rather common opinion is that one could completely exclude probability distributions from derivation of Bell's inequality and proceed by operating with frequencies. One typically refers to the result of works [10]–[12] which we shall call the Eberhard-Bell theorem (in fact, the first frequency derivation of Bell's inequality was done by Stapp[9], thus it may be better to speak about Bell-Stapp-Eberhard theorem). By this theorem Bell's inequality can be obtain only under assumptions of realism – the maps $\lambda \to a_\theta(\lambda)$ is well defined – and locality – the random variable $a_\theta(\lambda)$ does not depend on other variables which are measured simultaneously with it. Thus (in opposite to the original Bell derivation) existence of the probability measure P serving for all polarization (or spin) projections is not assumed.

At the first sight it seems that our previous considerations have no relation to the Eberhard-Bell theorem. One might say: "Yes, Bell proceeded wrongly, but his arguments are still true, because they were justified by Eberhard in the frequency framework."

As was shown [40], the use of frequencies, instead of probabilities, does not improve Bell's consid-

erations, see also Hess and Philipp [33]. The contextual structure of the EPR-Bohm experiment plays again the crucial role. If we go into details of Eberhard's proof, we shall immediately see that he operated with statistical data obtained from three different experimental contexts, C_1, C_2, C_3, in such a way as it was obtained on the basis of a single context. He took results belonging to one experimental setup and add or substract them from results belonging to another experimental setup. These are not proper manipulations from the viewpoint of statistics. One never performs algebraic mixing of data obtained for totally different sample. Thus if one wants to proceed in Eberhard's framework, he should find some strong reasons that the situation in the EPR-Bohm experiment differs crucially from the general situation in statistical experiments. I do not see such reasons. Moreover, the EPR-Bohm experimental setup is very common from the general statistical viewpoint.

Moreover, Eberhard's framework pointed to an additional source of nonexistence of a single probability distribution, see De Baere [50] and also [51]–[53]. Even if we ignore the contribution of measurement devices, then the ψ-function still need not determine a single probability distribution. In Eberhard's framework we should operate with results which are obtained in different runs. One could ask: Is it possible to guarantee that different runs of experiment produce the same probability distribution of hidden parameters? It seems that there are no reasons for such an assumption. We are not able to control the source on the level of hidden variables. It may be that the ψ-function is just a symbolic representation of the source, but it represents a huge ensemble of probability distributions of hidden variables. If e.g. hidden variables are given by classical fields, see e.g. [54]–[56], then a finite run of realizations (emissions of entangled photons) may be, but may be not representative for the ensemble of hidden variables produced by the source.

9. Comparing of the EPR and the EPR-Bohm experiment

Typically the original EPR experiment [57] for correlations of coordinates and momenta and the EPR-Bohm experiment for spin (or polarization) projections are not sharply distinguished. People are almost sure that it is the same story, but the experimental setup was modified to move from "gedanken experiment" to real physical experiment. However, it was not the case! We should sharply distinguish these two experimental frameworks.

The crucial difference between the original EPR experiment and a new experiment which was proposed by Bohm is that these experiments are based on quantum states having essentially different properties. The original EPR state

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\hbar}(x_1 - x_2 + x_0)p\right\} dp,$$

and the singlet state

$$\psi = \frac{1}{\sqrt{2}}(|+>|->-|->|+>)$$

which is used in the EPR-Bohm experiment have in common only one thing: they describe correlated (or by using the modern terminology entangled) systems. But, in contrast to the EPR-Bohm state, one can really (as EPR claimed) associate with the original EPR state a single probability measure describing incompatible quantum observables (position and momentum). The rigorous proof in probabilistic terms

was proposed by the author and Igor Volovich in [58]. On the other hand, as we have seen for the singlet state one could not construct a probabilistic model describing elements of reality corresponding to incompatible observables.

Thus the original EPR state is really exceptional from the general viewpoint of statistical analysis. But the EPR-Bohm state behaves "normally." In fact, there is no clear physical explanation why statistical data for incompatible contexts can be based on a single Kolmogorov space in one case and not in another. One possible explanation is that "nice probabilistic features of the original EPR-experiment" arise only due to the fact that it is "gedanken experiment."

10. Appendix: Proofs

10.1. Bell's inequality

Let $\mathcal{P} = (\Lambda, \mathcal{F}, P)$ be a Kolmogorov probability space: Λ is the set of parameters, $\mathcal{F}$ is a σ-algebra of its subsets (used to define a probability measure), P is a probability measure. For any pair of random variables $u(\lambda), v(\lambda)$, their covariation is defined by

$$< u, v >= \mathrm{cov(u, v)} = \int_\Lambda \mathrm{u}(\lambda)\ \mathrm{v}(\lambda)\ \mathrm{d}\mathbf{P}(\lambda).$$

We reproduce the proof of Bell's inequality in the measure-theoretic framework.

Theorem. (Bell inequality for covariations) *Let $a, b, c = \pm 1$ be random variables on $\mathcal{P}$. Then Bell's inequality*

$$| < a, b > - < c, b > | \leq 1 - < a, c > \tag{1}$$

holds.

Proof. Set $\Delta =< a, b > - < c, b >$. By linearity of Lebesgue integral we obtain

$$\Delta = \int_\Lambda a(\lambda)b(\lambda)d\mathbf{P}(\lambda) - \int_\Lambda c(\lambda)b(\lambda)dP(\lambda) \tag{2}$$

$$= \int_\Lambda [a(\lambda) - c(\lambda)]b(\lambda)dP(\lambda).$$

As

$$a(\lambda)^2 = 1, \tag{3}$$

we have:

$$|\Delta| = |\int_\Lambda [1 - a(\lambda)c(\lambda)]a(\lambda)b(\lambda)dP(\lambda)| \tag{4}$$

$$\leq \int_\Lambda [1 - a(\lambda)c(\lambda)]dP(\lambda).$$

It is evident that "hidden Bell's postulate" on the existence of a single probability measure P serving for three different experimental contexts (probabilistic compatibility of three random variables) plays the crucial role in derivation of Bell's inequality.

10.2. Wigner inequality

We recall the following simple mathematical result, see Wigner [6]:

Theorem 1.2. (Wigner inequality) *Let $a, b, c = \pm 1$ be arbitrary random variables on a Kolmogorov space $\mathcal{P}$. Then the following inequality holds:*

$$P(a = +1, b = +1) + P(b = -1, c = +1) \tag{5}$$

$$\geq \mathbf{P}(a = +1, c = +1).$$

Proof. We have:

$$P(a(\lambda) = +1, b(\lambda) = +1)$$

$$= P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = +1) \tag{6}$$

$$+P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = -1),$$

$$P(b(\lambda) = -1, c(\lambda) = +1)$$

$$= P(a(\lambda) = +1, b(\lambda) = -1, c(\lambda) = +1) \tag{7}$$

$$+P(\lambda \in \Lambda : a(\lambda) = -1, b(\lambda) = -1, c(\lambda) = +1),$$

and

$$P(a(\lambda) = +1, c(\lambda) = +1)$$

$$= P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = +1) \tag{8}$$

$$+P(a(\lambda) = +1, b(\lambda) = -1, c(\lambda) = +1).$$

If we add together the equations (6) and (7) we obtain

$$P(a(\lambda) = +1, b(\lambda) = +1) + P(b(\lambda) = -1, c(\lambda) = +1)$$

$$= P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = +1)$$

$$+P(a(\lambda) = +1, b(\lambda) = +1, c(\lambda) = -1) \tag{9}$$

$$+P(a(\lambda) = +1, b(\lambda) = -1, c(\lambda) = +1)$$

$$+P(a(\lambda) = -1, b(\lambda) = -1, c(\lambda) = +1).$$

But the first and the third terms on the right hand side of this equation are just those which when added together make up the term $P(a(\lambda) = +1, c(\lambda) = +1)$ (Kolmogorov probability is additive). It therefore

follows that:

$$P(a(\lambda)=+1,b(\lambda)=+1)+\mathbf{P}(b(\lambda)=-1,c(\lambda)=+1)$$
$$=P(a(\lambda)=+1,c(\lambda)=+1)$$
$$+P(a(\lambda)=+1,b(\lambda)=+1,c(\lambda)=-1) \quad (10)$$
$$+P(a(\lambda)=-1,b(\lambda)=-1,c(\lambda)=+1)$$

By using non negativity of probability we obtain the inequality:

$$P(a(\lambda)=+1,b(\lambda)=+1)+\mathbf{P}(b(\lambda)=-1,c(\lambda)=+1)$$
$$\geq P(a(\lambda)=+1,c(\lambda)=+1) \quad (11)$$

It is evident that "hidden Bell's postulate" on the existence of a single probability measure P serving for three different experimental contexts (probabilistic compatibility of three random variables) plays the crucial role in derivation of Wigner's inequality.

11. Conclusion

In probability theory Bell's type inequalities were studied during last hundred years as constraints for probabilistic compatibility of families of random variables – possibility to realize them on a single probability space. In opposite to quantum physics, such arguments as nonlocality and "death of reality" were not involved in considerations. In particular, nonexistence of a single probability space does not imply that the realistic description (a map $\lambda \to a(\lambda)$) is impossible to construct. Bell's type inequalities were considered as signs (sufficient conditions) of impossibility to perform simultaneous measurement *all random variables* from a family under consideration. Such an interpretation can be used for statistical data obtained in the EPR-Bohm experiment for entangled photons.

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