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Mathematics Teacher Educators' Practices to Support Teachers in the Design of Mathematical Tasks

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Abstract: Since teachers have the greatest impact on student learning, it is crucial to consider how professional development programs (PDP) for teachers can enhance their contribution, especially in designing mathematical tasks for teaching. This paper focuses on identifying patterns of practices of mathematics teacher educators related to crucial aspects of two teacher PDPs: one conducted face-to-face and the other using a Massive Open Online Course (MOOC). The Meta-Didactical Transposition is employed as the theoretical framework for comparing the two PDPs and for identifying patterns of practices. The findings suggest that educators, both in face-to-face and online settings, consider certain practices to guide teachers in designing mathematical tasks. This paper aims to share experiences of good practices that can be implemented by other researchers seeking to guide teachers in task design, either alone or in small groups.

Keywords: task design; teachers' professional development; mathematics teacher educators; MOOC; digital technology; Meta-Didactical Transposition

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1. Introduction

Among the many stakeholders in mathematics education, including students, teachers, teacher educators, and researchers, "it is the teacher who *can affect to the greatest extent the achievement of one of the main purposes of the research enterprise, that is, the improvement of students' learning of mathematics*" (emphasis in the original) [1] (p. 365). This statement aligns with the findings of [2], who, through a statistical meta-analysis and synthesis of 500,000 studies, confirmed that, aside from students, teachers have the most significant influence on student achievement. Therefore, the teachers need professional support, not only to enhance their own professional development but also to support students in mathematical understanding. The evidence shows that involving teachers in all aspects of classroom activities, including task development, is crucial. This involvement is not only beneficial for teachers' professional growth, "but without their involvement some aspects of task design would most likely be neglected" [3] (p. 105). Further, supporting teachers in creating mathematical tasks can reduce the need to adopt and modify existing tasks [4] and enhance the implementation of the task [5]. In this paper we present the outcomes of two studies conducted in different countries, both focusing on supporting teachers in the process of mathematical task design. Our objective is to identify common patterns of practices that emerge when developing professional development programs (PDPs) to aid teachers in designing mathematical tasks. We also aim to pinpoint similarities and differences in methods and results when a PDP is carried out face-to-face compared to a Massive Open Online Course (MOOC).

Between 2014 and 2018, separate doctoral studies in mathematics education were conducted: Ratnayake conducted a study in New Zealand, while Taranto conducted a study in Italy. Neither researcher was aware of the other's research. The PDP developed in New Zealand took place in the researcher's home country, Sri Lanka, in a face-to-face format, focusing on algebra content and digital technology (DT). Conversely, the PDP in Italy involved Italian teachers from across Italy, enrolled in a MOOC with arithmetic and algebra content. We outline the details of both studies in the following sections. However, it is important to highlight that despite the variations in the development of activities (i.e., different countries, different contents and face-to-face vs. online), and the differing primary goals of the two studies, they shared a common objective: designing and conducting a supportive PDP that would help teachers to design mathematical tasks. In this paper we compare the two PDPs and investigate the practices of mathematics teacher educators (MTE) in planning and conducting these PDPs. The research questions that guide this study are as follows:

- What pattern of practices emerges when mathematics teacher educators aim to develop a PDP to support teacher design of mathematical tasks in two different contexts?
- What are the similarities and differences, in terms of practices adopted and results obtained, when a PDP is conducted face-to-face versus through a MOOC?

The paper is divided into six main sections, as follows. Section 2 illustrates the literature review, followed by the theoretical framework in Section 3. Section 4 illustrates the research method employed, explaining the design and implementation of our PDPs, with details on participants, methodological choices, and data collection. Section 5 presents and discusses the results obtained from the analyses, focusing on specific case studies, and includes a comparison between the two experiences. Finally, Section 6 encompasses discussions and conclusions.

2. Literature Review

Within the literature on teacher PD, "one constant finding [. . .] is that notable improvements in education almost never take place in the absence of professional development" [6] (p. 4). Numerous studies have investigated the nature of mathematics teachers' knowledge and how to support them to improve their mathematical knowledge for teaching (e.g., [7,8]). Although limited research has investigated the teacher as a designer of tools (e.g., [9]), there is increasing involvement of teachers "in collaborative design of curricular materials" [10] (p. 259) in practice. Nonetheless, teachers often find themselves needing to create support structures for implementing curriculum or activities in their classrooms [9,11].

Instructional design stands as a crucial facet of teacher education. Some investigations into the purpose of such design have revealed that teachers use it to structure activities and streamline their actions within the classroom [12]. For instance, they might seek to enhance the efficiency of actions taking place within tight timeframes [13] or to reduce uncertainty levels [14]. Maher [15] investigated both the tools developed by teachers and their conceptual learning related to their role as a designer. He observed, first, that materials and products may be revised, modified, or supplemented by teachers in order to fit their needs and contexts. Second, this modification is a cyclical process of design, testing, and revision, as teachers learn more about what works and what does not work according to their perceptions of classroom needs. To address the challenges of being designers in a DT era, teachers need to adopt a designer's mindset, to see themselves as designers. Kirschner [16] argues that to be competent as designers, "teachers need to be seen as, and become, accomplished at least in three fields, namely: the domain in which they teach. . ., the art and science of learning and teaching. . . and the science of research and design (p. 321)". The question of how to help teachers to develop these skills is a challenge that educators face. From the results of a case study conducted with 29 teachers in George and Sanders [17] suggest that "the task analysis could be used as a diagnostic tool to identify needs, either to provide feedback to teachers and/or as a way for them to self-assess tasks and their reasons for setting a particular type of task, to identify possible needs" (p. 2890). They further suggest that PDPs could be designed to address a range of professional support and should provide opportunities for teachers to choose whether and to which they want to attend based on their individual needs.

Many studies have considered the formats of effective PDPs to assist teachers in the design of mathematical tasks using DT (e.g., [18,19]). Although teacher PDPs have been recognized as a crucial element in fostering teachers' skills as task designers, questions surrounding when and how to organize them, the content to be included, the areas to cover, and the practical and theoretical factors to consider in designing such programs remain unanswered [20]. As a result, it is valuable to invest time and resources in teacher education programs that aim to support teachers in the design of instructional tasks. To address this need, we compared two successful PDPs conducted to support teachers in designing mathematical tasks and sought to identify the patterns of teacher educators in conducting such PDPs.

3. Theoretical Framework

The theoretical framework that we considered to compare our PDPs is the Meta-Didactical Transposition (MDT). This framework can be used to describe and analyze mathematics teachers' and educators' practices within institutional contexts [21]. In the case examined in this paper, the context pertains to PDPs designed by educators and aimed at teachers, with the goal of collaboratively working on mathematics task design.

The MDT model is grounded in the Anthropological Theory of the Didactic (ATD, [22], extending Chevallard's notions of praxeology and didactical transposition. A praxeology is structured in terms of two main levels [23]: (a) The "know-how" (praxis), which includes a family of similar tasks to be studied, as well as the techniques available to solve them; (b) The "knowledge" (logos), which includes the "discourses" that describe, explain, and justify the techniques that are used within a more or less sophisticated frame and may even produce new techniques. Note that the "knowledge level" can be further decomposed into two components, i.e., Technologies and Theories. The description provided is sufficient for our purposes. In the following, we use the term "argument" to mean both).

A praxeology consists of a task, one or more techniques, and a more or less structured argument that justifies or frames the technique(s) for that task. Therefore, it encompasses both the know-how and the knowledge with respect to a family of tasks.

The MDT model distinguishes between didactical and meta-didactical praxeologies. The didactical praxeologies aim to model the mathematical activity when solving a didactical task, such as to teach a particular mathematical topic. The meta-didactical praxeologies concern meta-didactical tasks, such as those to reflect on possible didactical praxeologies for teaching that particular concept [24].

Referring to teachers as learners in PDP, the MDT considers their didactical praxeologies in a situation of learning: for this reason, they are called 'meta-didactical' [25]. At both levels (didactical and meta-didactical) a praxeology is made up of the previously mentioned components, according to [22]: task, technique, and argument. An example of the components of a didactical praxeology (of a teacher in class) could be as follows: introducing students to the type of task; how to organize such an approach; why one has to, and knows how to, organize it like that. Meta-didactical praxeology components could refer to educators or teachers and have the same structure as the didactical ones. An example of components of a meta-didactical praxeology (of a teacher in PDP) could be as follows: solving an assigned didactical task; how to solve it; why one has to, and knows how to, solve in such a way. Concretely, educators' praxeologies are meta-didactical in the sense that they deal with a discourse about the didactical issues given as tasks to the teachers, who from their side, have their didactical praxeologies.

When educators prepare PDPs, it is necessary to consider the importance of institutions (schools, educational programmes, . . .) so as to consider not only the educational programmes for teachers but also the work of teachers in classrooms and to value the work that teachers do in their communities. Likewise, consider the community of educators involved in PDPs who take on not only the role of designers of the tasks for teachers but also that of teacher educators and academics managing research. Thus, once working on the design of the PDP that will be offered to teachers, all educators involved sharing the same meta-didactical praxeologies. When teachers and educators work together, mutual influences can arise. Thus, on the one hand, teachers encounter new teaching paradigms and receive stimulating stimuli that can evolve their practice, embracing new ideas, points of view, practices or simply gaining awareness of the content they have encountered during the programmes. On the other hand, researchers' practices may also evolve, leading to changes and/or awareness of their practices.

This paper specifically focuses on the educators' meta-didactical praxeologies related to crucial steps for supporting teachers in the design of mathematical tasks.

4. Methodology and Data Analysis

In this section, first, we describe the two research contexts in which the PDPs were conducted, the Sri Lankan and the Italian, respectively. Next, we briefly compare the two, outlining the similarities and differences that set them apart. Finally, we describe how we collected data and what kinds of analysis we performed.

4.1. Research Contexts

4.1.1. PDP Conducted with Sri Lankan Teachers

One objective of the study was to investigate how a supportive PD intervention can be designed to improve teacher production of rich tasks using DT. The first iteration of data collection was carried out in Sri Lanka, with 12 Advanced Level (AL) mathematics teachers representing six educational zones. Four cases, comprising groups of three teachers [19], were closely observed.

This case study intervention was conducted in three stages. In the first stage, the teachers designed a DT algebra task in a group of three. In the second stage, the first author conducted a PDP with the teachers. In addition to the features of a rich DT task, the design of the PDP also considered how to support the teachers in planning a lesson to implement the task that they had designed. This included considering what decisions teachers may need to make and the role of resources, orientations, and goals (ROG) in making those decisions, based on Schoenfeld's [26] theory and an understanding of the three-point FOCUS framework [27] for planning, delivering, and reviewing a lesson. In the third stage, the teachers were given an opportunity to modify their preliminary task or to design a new task. After this modification stage, a teacher from each group implemented the task with her/his students in the classroom. They then had an opportunity to modify their task, followed by a post-implementation discussion where they reflected on their work.

Data were collected using various tools, including pre- and post-questionnaires, individual and group interviews, and post-implementation focused interviews. The questionnaires comprised a Likert-style attitude scale, as well as both closed and open-ended questions. These questions were designed to gather data about participants' prior experience in teaching mathematics with DT, task design, and attendance at PD programmes. During the semi-structured interviews, teachers reflected on their experiences in task design and implementation processes and shared their expectations for future PDs in these areas. The processes of task development and implementation were observed and video recorded, whereas the two interviews were audio-recorded. All documents, including the initially designed and modified tasks by teachers, were collected for analysis. The analysis involved both qualitative and quantitative methods (for more details see [19]). For instance, two tasks designed by each group were examined using the Task Richness Framework, which was developed during this project. Following that, one-tailed paired t-test was used to find the significance of changes of the richness of tasks designed by each group before and after the PD intervention. All video and audio recordings were translated from Singhalese to English, transcribed, coded, and analyzed to identify patterns in the practices of Mathematics teacher educators.

4.1.2. PDP Conducted with the Italian Teachers

This PDP was part of a PhD study at the Department of Mathematics 'G. Peano', University of Turin, Italy. The MathMOOCUniTo project had been in place since 2015 in the aforementioned department [28]. It is focused on designing and delivering MOOCs for Italian in-service mathematics teachers at all school levels, to increase teachers' professional competencies and improve their classroom practices. These MOOCs are designed by university researchers in collaboration with researcher–teachers (i.e., mathematics teacher who has been in service for several years and therefore has some experience in his/her profession as a school teacher and who collaborates with a university research group) and are free and available online for teachers through a Moodle platform (https://difima.i-learn.unito.it/ (accessed on 28 July 2023)). Every MOOC is subdivided into modules lasting one or two weeks. In addition to resources (videos, Sways (Sway (https://sway.office.com/ (accessed on 28 July 2023)): Microsoft tool that allows users to combine text and media to sustain the showing of online content), useful links), the modules contain weekly tasks, along with a final task for teachers.

In the following, we concentrate our attention on the MOOC Numeri (delivered from November 2016 to February 2017). There were 278 teachers enrolled in MOOC Numeri. They were all Italian in-service mathematics teachers (Grades 1–13). The MOOC Numeri consists of five thematic modules, each lasting one week. In these thematic modules, educators show examples of activities to MOOC teachers, which also contain mathematical tasks: activities on number sense (module 1), on formative evaluation methodologies to be adopted with the school students (module 2 and 3), on recursion and iteration (module 4) and the transition from arithmetic to algebra (module 5). In each module, the MOOC teachers are asked to discuss and comment on these activities on communication message boards, special spaces set up for online communication within the MOOC. Some activities concern mathematical content that they know very well, so they can also comment on the experiences they have had in class with their students, whereas the content of other activities is not usually part of the curriculum (Link to the Italian curriculum: http://www.indire.it/lucabas/lkmw_file/licei2010/indicazioni_nuovo_impaginato/_decreto_indicazioni_nazionali.pdf (accessed on 28 July 2023)). The tasks required in the various modules are: viewing the proposed materials, active participation (interacting on the communication message boards) and executing simple requests (such as filling in a questionnaire). After the thematic modules, there always follows a final module in which the teachers are called upon to put into practice the education received. Precisely, they have to design a project—a teaching activity—on the mathematical core on which MOOC is based (in this case arithmetic and algebra), and this project must contain one or more mathematical tasks. The design work is individual. It had to be conducted using specific software (Learning Designer: a tool to help teachers design teaching and learning activities and share their learning designs with each other. In fact, each production is associated with a link. Anyone who has the link can access the design created; more details in §6.2) in 2 weeks. After that, one week was devoted to peer review of the project work of another MOOC teacher.

The study objective of the PhD project was to analyze whether and how mathematics teachers benefit from PD in an asynchronous learning environment such as a MOOC. For further details on the thesis see [29].

The Moodle platform records all actions carried out by individual MOOC teachers on the platform. Interactions on the communication message boards in the various MOOC modules were collected, as well as all MOOC teachers' project work and peer reviews for analysis.

4.1.3. A Brief Comparison between the Two PDPs

We briefly showed similarities and differences between our educational experiences.

We recognize two similarities: (i) Without being in touch, we were both involved in the management of PDPs for in-service secondary mathematics teachers; (ii) Each project supported teachers to design mathematical tasks themselves. Neither of the projects wanted to assess the mathematical sophistication of the content of the tasks, but rather the researchers supported teachers by encouraging and motivating them to design tasks either individually or in small groups.

Differences are shown in Table 1.

Table 1. A design comparison of Sri Lankan and Italian PDPs.

	Sri Lanka	Italy
Nature of PDPs	Face-to-face	MOOC (online)
Moment of design	Before and after the PD intervention	At the end of the MOOC, after examining a set of examples provided by the educators and after benefiting from other MOOC teachers' interactions in communication message boards
Modality of the design	Teachers wrote their tasks on paper	Teachers had to use Learning Designer
Type of design	DT algebra task and planning a lesson to implement such a task	Design a lesson that contains DT or non-DT tasks (arithmetic, algebra)
Time devoted to design	2 days for task design (2 months for the whole process of task design, implementation and modification)	2 weeks
Revision of the design	Teachers design collaboratively (implementation, reflection, and modification)	MOOC teachers design alone but afterwards they received feedback from another teacher (peer review)

Both PhD projects focused mainly on teachers and the possibility of producing PDPs for them. It was worth focusing on the mathematics teacher educators who had developed such PDPs to analyze what practices they had considered in order to produce them. The theoretical framework that fits with this intention is precisely the Meta-Didactical Transposition that we show in the following section.

4.2. Data Collection and Analysis

Identification of the meta-didactical praxeologies was made possible by reflecting on and comparing the design and implementation phases in which we, as mathematics teacher educators, were involved during our educational experiences. We knew that the Sri Lankan and Italian praxeologies might not coincide because of the different nature of the courses (e.g., face-to-face vs. online; institutional context), but they were similar in their purpose.

In comparing the two PDPs from the mathematics teacher educators' point of view, a pattern of recurrent practices emerged, although not in the same order in the two experiences, which could be summarized in the following three practices: Examples, Discussions, and Design. These are shown and explored in the next section.

For each of these practices, we identified the educators' meta-didactical praxeologies by selecting the tasks that were essential to support, encourage, and motivate the teachers to design mathematical tasks. In particular, we describe the tasks, as well as the techniques adopted by the educators to solve such tasks, and the related justification (argument), in both the Sri Lankan face-to-face course and the Italian MOOC. For the argument, we particularly wondered how the chosen techniques were justified and supported by theories in mathematics education, or more generally in the educational field.

5. Results

In this Section, we proceed as follows. First, the Sri Lankan PDP is presented and then the Italian one is presented. Each PDP is illustrated as follows: the meta-didactical praxeologies of the educators are described with a detailed textual part and its synthesis in a table, taking into account the praxeological components (task, techniques, argument). Subsequently, the application of the meta-didactical praxeology is illustrated by employing examples. Finally, there is an annotated illustration of a mathematical task design made by one of the teachers who took part in each PDP in light of the educators' meta-didactical praxeologies.

5.1. Sri Lankan Experience

5.1.1. The Meta-Didactical Praxeologies of the Educators

The main focus of the PDP was to support teachers to design DT tasks themselves. The task design process was conducted in two stages: before and after the PDP. Firstly, the teachers designed a DT task in groups of three. There was no formal guidance from the researcher at this stage and teachers were free to choose an algebra topic from the AL syllabus, preferably from Grade 12. Three out of four groups chose graphs of a quadratic function, whereas the fourth group chose domain and range of a function. The DT used was GeoGebra.

As seen in the praxeology in Table 2, here the researchers wanted teachers to design a DT task themselves without intervention from the researcher. In addition, we also wanted to motivate them to design tasks in small groups of three. Thus, the teachers were encouraged to form groups with other teachers, either from the same school or from the schools in the same educational zone.

Table 2. Educators' meta-didactical praxeology to encourage discussion among teachers.

	Teachers' Discussion during the Initial Task Design Process
Task	To enable teachers to exchange ideas in order to design a task with their existing knowledge
Techniques	Dividing teachers in small communities of inquiry (four groups of three) Each group to decide a topic for which they would design a task To exchange teachers' prior knowledge to design a DT task for their students
Argument	To understand what existing knowledge they had in task design

After this initial design process, the first-named author conducted a PDP with the teachers based on theoretical principles of rich DT tasks taken from the literature (e.g., [30–32]). These features were discussed along with an exemplary task. The idea was to give some theoretical knowledge to teachers on what a rich DT task comprises, since constructing a DT task was a novel experience for the teachers. Further, it was clear, based on the answers provided at the first interview, that these teachers did not have clear ideas about what constitutes a mathematical task. The 12 features of a rich DT task presented to the teachers later formed the Task Richness Framework (TRF), which was used to examine the tasks during data analysis. Table 3 presents the researcher praxeology.

Table 3. Educators' meta-didactical praxeology to provide an example from which the teachers can draw inspiration.

	Exemplar Task Prepared by the Researchers
Task	To support teachers to design a rich DT task
Techniques	Introducing and discussing features of a rich DT mathematical task Showing an exemplar task with these features
Argument	To provide teachers with theoretical knowledge on developing DT tasks

Discussion among the teachers was encouraged during the task design processes, both before and after the PDP, at the post-implementation discussion and in the final modification. These discussions helped each teacher to add ideas from their existing knowledge and experience. They valued working in groups [19,33]. In addition to promoting working in groups, we also wanted to motivate teachers to undertake self-reflection. When teachers produced the tasks for the researcher, she directed them to examine them using the features of a rich DT task discussed during the PD. After modification, the teachers described how they had designed and modified the task based on their discussions. Another reflection took place after the implementation. Here, the teachers reflected on their work, how they planned the lesson, how it worked with the students, and how they were going to modify it further. The researchers' praxeology in encouraging discussions among teachers is shown in Table 4.

Table 4. Educators' meta-didactical praxeology to encourage further discussion among teachers.

	Teachers' Discussion during the Task Design Process and after Implementation
Task	To enable teachers to exchange ideas to design a rich task and to reflect on their own work
Techniques	Grouping teachers in small communities of inquiry (four groups of three) Each group was to discuss whether to modify the task designed at the beginning or to create a new one in the light of the stimuli received during the PDP Each group was to reflect on the features of a rich task discussed at the PDP when modifying the tasks Supporting teachers to reflect on the tasks themselves Directing teachers to reflect on their task implementation lesson
Argument	To have communities of inquiry [34,35]

After the PDP, the teachers had an opportunity to modify their preliminary task or to design a new task. Each group decided to modify its first task. In this modification stage, the teachers were directed to examine their task themselves in order to give them ownership of the task and allow them to decide whether their task was good enough to implement with their students. Further, the researchers wanted the teachers to design a task that all the members of the group would be satisfied with and could walk into their classes confident they were implementing a rich task. As a result, the teachers modified their tasks a couple of times until all the teachers in each group were satisfied with them. The researcher praxeology in directing teachers to design tasks in groups is illustrated in Table 5.

Table 5. Educators' meta-didactical praxeology to support teachers' design.

	Task Design Carried Out by the Teachers
Task	Allowing teachers to design a DT algebra task
Techniques	Suggested they form small groups to work in Directed teachers to design a task based on their existing knowledge Recommended use of GeoGebra as their DT Suggested they either design a new task based on the points discussed at the PD or modify one Suggested further modifications, if necessary, after the implementation
Argument	To support teachers to design DT mathematical tasks themselves

5.1.2. Application of the Meta-Didactical Praxeologies

We described the meta-didactical praxeologies that educators followed in the previous section. In this section, we show the results obtained during the task design session. We begin by providing evidence of the improvements of designing rich tasks with a quantitative analysis of the richness of the tasks followed by an example of a rich task designed by one group.

During the PD intervention, we discussed the features of a rich DT mathematics task with an example. The teachers in groups of three modified their tasks based on the points discussed during the PD intervention. During the group interview after the task design stage, the teachers appreciated the PD design and the benefits of working collaboratively in groups. For example, the following utterances are responses of two teachers to the interview question on the nature of the PD program. These responses also imply an improvement in their confidence in task design.

Nimali: We didn't have an idea about how to write a task. So through this we had a good idea about that

Malka: We can now prepare tasks ourselves for the lessons that students may find difficult.

In addition to qualitative data, we examined the tasks designed by each group before and after the PD intervention. The tasks designed by each group were scored individually by two researchers using the TRF developed in this study, followed by a discussion to ensure validity. A scale of 0–3, 0–4, or 0–5 was used to score for each principle. These scores were chosen to enable the sufficient differentiation of each factor in a task. For instance, the factors were categorized in three categories—important, very important, and most important—and scored out of 3, 4, or 5, respectively. There are 12 principles, and the scales are given in Table 6.

Table 6. The DT Task Richness Framework (TRF).

Principles of Rich Tasks	Scale	First Task		Second Task	
		Evidence	Score	Evidence	Score
Focuses on mathematical ideas, e.g., epistemological obstacles	0–4				
Considers the role of language and discourse	0–3				
Students give written interpretations and reflections	0–5				
Goes beyond routine methods	0–4				
Encourages student investigation	0–5				
Has multi-representational aspects	0–4				
Appropriate for student instrumental genesis	0–3				
Provides opportunities for instrumental feedback	0–3				
Integration of DT and by-hand techniques	0–3				
Aims for generalization	0–5				
Students think about proof	0–4				
Develops mathematical theory	0–3				

Although some groups produced more than one modification of their task, we compared two tasks from each group, namely the preliminary task and the final task, which they implemented, to maintain consistency among the groups. We used a one-tailed paired *t*-test to understand whether there were any significant changes in the richness of the tasks designed by each group, and Table 7 demonstrates that the changes in the richness of the tasks designed by each group were statistically significant.

Table 7. Analysis of pre-and post-intervention TRF scores for the tasks.

Group	Pre-Intervention (Max 46)	Post-Intervention (Max 46)	t	p
A	18	29	3.11	<0.005
B	12	25	4.03	<0.001
C	12	32	4.44	<0.0005
D	21	27	1.80	<0.05

The analysis in Table 7 supports the idea that the nature of the PDP conducted in this study had the potential to make a positive influence on teacher task design. In summary, educators' meta-didactical praxeologies to encourage discussion among teachers, to provide an example from which the teachers can draw inspiration, to encourage further discussion among teachers, and to support teachers' task design were exhibited in the Sri Lankan experience.

5.1.3. A Mathematical Task Design Made by a Group of Teachers within the PDP

We now share two tasks designed by one of the four groups (Group C), with their scores as an example.

The teachers of Group C were all in the age group 31–40 years, having less than five years of teaching experience, and holding a bachelor's degree with a substantial component of mathematics. The main goal of the task they chose was to help students to understand the variation of the graph of a quadratic function in the form of $f(x) = ax^2 + bx + c$ when the discriminant (Δ) is negative (see Figure 1). It is more like a set of teacher notes rather than a task designed for students. The task provides the steps needed to rearrange the function using the method of completing the square. Following that, the argument that determines how the sign of the given expression depends on the sign of a when $\Delta < 0$ is given. Two sets of functions for each condition, $a > 0$ and $a < 0$, are given for students to draw the graphs. There is no mention of using DT, no direction to find a generalisation, and no evidence that students were guided to investigate and interpret their findings.

<u>Quadratic functions</u>

Change of the graph when $\Delta < 0$

1. $f(x) = ax^2 + bx + c, \quad a \neq 0$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

2. For all real values of x; $\left(x + \frac{b}{2a}\right)^2 \geq 0$

$\therefore$ When $b^2 - 4ac < 0$ $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$

is always positive.

3. Then the sign of $f(x)$ will be the sign of a.
4. Divide the class in 2 (*it was 6 here, they cut it and put 2*) groups and guide them to observe the behaviour of the graph. Each group will get one condition given below.

i. $a > 0$ *and* $b > 0$
ii. $a > 0$ *and* $b < 0$
iii. $a > 0$ *and* $b = 0$
iv. $a < 0$ *and* $b < 0$
v. $a < 0$ *and* $b > 0$
vi. $a < 0$ *and* $b = 0$

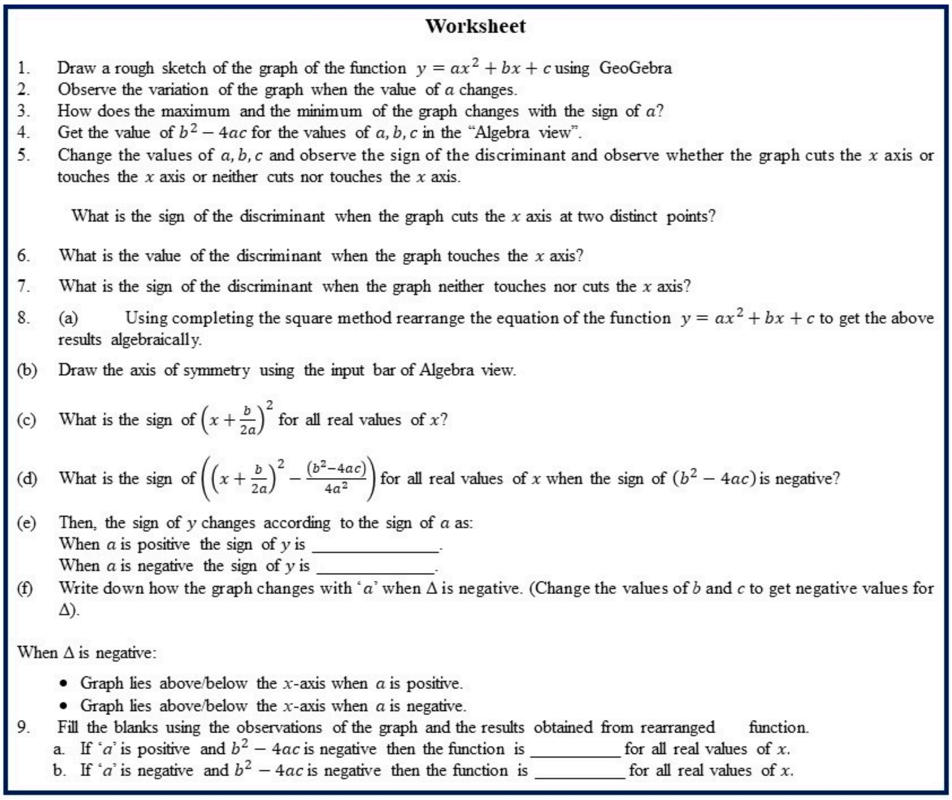
(Later they have cut these 6 conditions)

$a > 0$	$a < 0$
$y = x^2 + x + 1$	$y = -x^2 + x - 2$
$y = x^2 - x + 2$	$y = -2x^2 - 2x - 3$
$y = x^2 + 4$	$y = -2x^2 - 4$

Figure 1. The first task designed by Group C in the Sri Lankan study.

The resources for this first task were the AL syllabus, teacher instructional manual, discussion among the group members of the group, and GeoGebra as the DT. Before having an opportunity to implement this task, teachers participated in the PDP led by the first author, followed by an opportunity to modify the task. Figure 2 gives this modified task.

In addition to the mathematical concepts focused on Task 1, the modified task focused on the effect of the sign of a on the concavity of the graph, the axis of symmetry, and completing the square. Both DT and by-hand techniques were carefully integrated, allowing students to use sliders to obtain different graphs using GeoGebra and the use of paper-and-pencil work to complete the square of the general form of the function. While guiding students to investigate themselves, opportunities were also provided for them to interpret their results. Moreover, the task directs students to use GeoGebra as the DT and provides opportunities for student instrumental genesis and instrumental feedback. Students were able to use multiple representations, such as graphs, algebra, numbers, and natural language, during both DT and by-hand techniques. The teachers expected students to draw graphs using GeoGebra and observe how the variation of the graphs depended on the sign of *a* and the discriminant, and to use by-hand techniques to complete the square of the function. Further, the task guided students to generalize the effect of *a* on the relative position of the graph when Δ is negative. Thus, according to the scores given for each principle of the TRF, this task scored 32/46 according to the TRF (see Table 8 for details).



Worksheet

1. Draw a rough sketch of the graph of the function $y = ax^2 + bx + c$ using GeoGebra
2. Observe the variation of the graph when the value of a changes.
3. How does the maximum and the minimum of the graph changes with the sign of a?
4. Get the value of $b^2 - 4ac$ for the values of a, b, c in the "Algebra view".
5. Change the values of a, b, c and observe the sign of the discriminant and observe whether the graph cuts the x axis or touches the x axis or neither cuts nor touches the x axis.

   What is the sign of the discriminant when the graph cuts the x axis at two distinct points?

6. What is the value of the discriminant when the graph touches the x axis?
7. What is the sign of the discriminant when the graph neither touches nor cuts the x axis?
8. (a) Using completing the square method rearrange the equation of the function $y = ax^2 + bx + c$ to get the above results algebraically.

(b) Draw the axis of symmetry using the input bar of Algebra view.

(c) What is the sign of $\left(x + \frac{b}{2a}\right)^2$ for all real values of x?

(d) What is the sign of $\left(\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2-4ac)}{4a^2}\right)$ for all real values of x when the sign of $(b^2 - 4ac)$ is negative?

(e) Then, the sign of y changes according to the sign of a as:
When a is positive the sign of y is ______________.
When a is negative the sign of y is ______________.

(f) Write down how the graph changes with 'a' when Δ is negative. (Change the values of b and c to get negative values for Δ).

When Δ is negative:

- Graph lies above/below the x-axis when a is positive.
- Graph lies above/below the x-axis when a is negative.

9. Fill the blanks using the observations of the graph and the results obtained from rearranged function.
   a. If 'a' is positive and $b^2 - 4ac$ is negative then the function is __________ for all real values of x.
   b. If 'a' is negative and $b^2 - 4ac$ is negative then the function is __________ for all real values of x.

Figure 2. The modified task of Group C in the Sri Lankan study.

Table 8. Using the TRF to assess the richness of Group C's tasks.

Principles of Rich Tasks	First Task		Final Task	
	Evidence	Score	Evidence	Score
Focuses on mathematical ideas, e.g., epistemological obstacles	Behaviour of the graph when delta is negative, completing the square, sign of the function	3	Good: Variation of the graph with the sign of *a*. Sign of the graph when delta is negative. Completing the square.	4
Considers the role of language & discourse	Words such as behaviour, real values of *x*, positive and symbols like Δ > 0, *a* < 0, etc but none of them aimed at the students	1	Many mathematical words and symbols such as discriminant, variation, maximum and minimum, touches, sketch, the axis of symmetry, completing the square. All in the context of student direction	3

Table 8. *Cont.*

Principles of Rich Tasks	First Task		Final Task	
	Evidence	Score	Evidence	Score
Students give written interpretations and reflections	No evidence for students' interpretations	0	Students' are asked a number of 'what' questions and are to fill in blanks but are only required to explain in one question 'How does the maximum . . . change?'	3
Goes beyond routine methods	Considers the relationship between the sign of a and the sign of $f(x)$. No standard solution methods.	2	Students are guided to think logically about the sign of $f(x)$ when delta is negative and when $a > 0$ and $a < 0$. Students are guided with given steps in the task.	2
Encourages student investigation	Students 'observe the behaviour' of the graph and investigate the effect of a	1	The whole worksheet is structured around student investigation using GeoGebra. Students are asked to observe and answer questions and to find values of b and c that make delta negative. Very directed investigation.	3
Has multi-representational aspects	Involves mathematical language, graphs and algebra	2	Use graphs, algebra and values obtained from the algebra view, along with extensive natural language use. Link the graphs with algebra and graphs with numbers.	4
Appropriate for student instrumental genesis	Unclear. No mention of how they will observe the graph	0	Students need function entry, variation of a, obtain values of $b2 - 4ac$, draw the axis of symmetry and find the sign of expressions. These seem appropriate and had been covered.	3
Provides opportunity for instrumental feedback	Students observe the graphs to identify the effect of a	1	Graph shape and position relative to axes, sign of discriminant, sign of a, sign of	3
Integration of DT and by-hand techniques	Not mention of DT techniques present	0	Good. Use GeoGebra to draw the graphs and observe the changes of the discriminant. Complete the square and fill the blanks by-hand.	3
Aims for generalisation	Completing the square for a general quadratic function, but given. Aims to generalise effect of a.	2	Completing the square for a general quadratic function. Considers the general effect of a, b and c on the discriminant and the relationship to the graph. Aims to generalise effects of the discriminant and a on the function's graph.	4
Students think about proof	No evidence	0	No evidence	0
Develops mathematical theory	No evidence	0	No evidence	0
Totals		**12/46**		**32/46**

5.2. The Italian Experience

5.2.1. The Meta-Didactical Praxeologies of the Educators

In MOOC Numeri, five modules on arithmetic and algebra contents were created. After an introductory module (the first week), the module activities were offered weekly, with a duration of one week. Module 6 was related to the final task of the MOOC (which we focus on in the following discussion). Thus, MOOC Numeri was six weeks long (with the addition of three weeks to complete the final task).

The educators in MOOC Numeri were three university researchers (including the second author) and 10 researcher–teachers. All of them were involved in the design (see Table 9), in the course delivery, and in monitoring its evolution in terms of the interaction among participants and the educational resources made available. The MOOC activities did not cover all the topics of the curricula but aimed to provide detailed methodological indications on how to deal with some topics of particular importance for the mathematical education of the students. The activities offer concrete examples to be carried out in the classroom through a laboratory-based methodology [36] and technologies.

Table 9. Educators' meta-didactical praxeology to provide some examples from which the teachers could draw inspiration.

	Examples of Activities Prepared by Educators
Task	To propose to MOOC teachers' activities on the number core (based on arithmetic and algebra)
Techniques	To subdivide the activities into one-week modules To choose activities based on laboratory-based methodology and on the use of technology To transpose, in a digital format, materials, and didactical resources for teacher education
Argument	To innovate methodology and strategies of teaching mathematics as highlighted in the Italian curriculum and give the MOOC teachers ideas for drawing up their final task design

All activities were defined before the start of MOOC and were digitally transposed to make them available in the MOOC environment. In each module, there were the following resources: short videos (3–5 min) where experts introduced the conceptual node of the week and illustrate some guidelines to explain what teachers had to do to get the digital badge of the module. Note that each module is associated with a digital badge (digital badges are a validated indicator of accomplishment, skill, quality or interest that can be earned in various learning environments [37]. In the case of our MOOC, they are created by a course administrator and then they can be issued automatically by the platform each time the MOOC teacher accomplishes the tasks in the module).

The activities of the module were illustrated through Sway. In fact, the activities alternated textual parts with images and audio files where the author(s) deepened certain aspects verbally, and with Word and Excel files and GeoGebra applets that could be viewed directly from Sway, which were also downloaded for use with school students.

The educators took care to enable asynchronous interactions among teachers. In fact, in each module, they inserted suitable communication message boards to allow the teachers to express opinions about the content of the course, exchange experiences with colleagues, and benefit from other participants' ways of thinking. In the communication message boards, there were title or stimulus questions used to initiate peer discussions on the topics dealt with in the module. In this way, the educators accompanied the teachers in reading the materials and identifying their focus. Moreover, the educators chose to limit their interventions in these communication message boards to initiate an online community among teachers. The praxeologies generated are presented in Table 10 (For more information, see [28]).

Table 10. Educators' meta-didactical praxeology to encourage teachers' discussion on the activities examined.

	MOOC Teachers' Discussion on the Activities Examined
Task	To enable MOOC teachers to exchange opinions, reflections, ideas on the MOOC activities
Technique	Inserting specific communication message boards in each module Entering a stimulus question or title in order to accompany MOOC teachers in their reading of the materials and identifying their focus Reducing educators' interventions, but monitoring behind the scenes
Argument	To support the establishment of a community made up of only MOOC teachers

MOOC Numeri included, as its final module, two production activities on a project-based methodology [38]: designing a project work using specific software and reviewing the project work designed by a colleague (or peer review)—See Table 11.

Table 11. Educators' meta-didactical praxeology to support teachers' design.

	Task Design Carried Out by MOOC Teachers
Task	To allow MOOC teachers to design a project with arithmetic and algebra content
Techniques	MOOC teachers were asked to carry out an individual project work, using Learning Designer software Each project was to be reviewed by another MOOC teacher
Argument	Project-based learning [38]

The project work consisted of designing a teaching activity that contained one or more mathematical tasks together with a description and a prior analysis of the potential of the project work for the school students' learning. The project work comprised an individual design and each teacher knew from the very beginning that s/he would have to deal with this final production activity. During the previous weeks, each teacher had had an opportunity to view the educators' activities based on arithmetic and algebra. Moreover, through the communication message boards, they had occasion to compare thoughts on these topics with the other teachers. At the opening of the final module, they had two weeks to define their design. The teachers were free to choose the content of their project work as long as they were focused on arithmetic and algebra, the thematic core of the MOOC. The educators gave a lot of freedom to the teachers since they did not want to influence them or to restrain their creativity. To accomplish this task, the teachers had to use a web-based tool, the Learning Designer designed by Laurillard [39]. It allows us to describe the activity in a textual way and it is also possible to upload materials (images, word, excel, GeoGebra files). In addition, Learning Designer associates a link to each design created so anyone who has the link can view the design. This was a convenient feature for managing online peer reviews.

The peer review activity was proposed to stimulate collaboration among the teachers and to foster formative assessment among peers [40]. It was a one-to-one peer review whereby each teacher had to review a colleague's project work from an educational point of view, without any assessment intention. The teachers were divided into groups by the educators. The peer review started after the second week devoted to the project work. A grid containing the review criteria was given. The reviewers were asked to indicate how much that certain aspect was present (Note that there was no evaluation. It was only required to indicate from 1 to 5 how much that certain aspect was present. It did not mean that if that aspect was absent, the project work was of little value. In fact, there was no final score (as the sum of the individual scores awarded)) by using a Likert scale from 1 (aspect little present) to 5 (aspect highly present). The final request was to leave a comment highlighting the strengths of the project work, the parts that could be improved and possible reviewer's curiosities. The educators gave them one week to accomplish this task.

5.2.2. Application of the Meta-Didactical Praxeologies

We have illustrated the meta-didactical praxeologies followed by the educators and now show the results obtained through their implementation. We began by considering a MOOC module and describing one example of the activities it contains. Following that, we illustrated some discussions of the MOOC teachers on the communication message boards of the module and some quantitative data related to the project work designed by the teachers. Finally, we examined an example of project work to show the mathematical task design carried out in the light of the praxeologies implemented by the educators.

Module 4 of MOOC Numeri is dedicated to the concept of mathematical recursion and iteration. It is a concept that is little dealt with in Italian schools but is of fundamental importance. In fact, knowing how to produce iterative or recursive reasoning is the basis of mathematical induction. We chose to focus on this module because, as mentioned, it addresses topics that are not very common in the Italian school curriculum, but this did not discourage the MOOC teachers. On the contrary, some of them took inspiration from this module to design their project work, as we show in the following example. In module 4, the educators proposed six examples of activities. We describe one called "The Sierpinski's triangle".

The activity begins by describing Sierpinski's triangle (Figure 3).

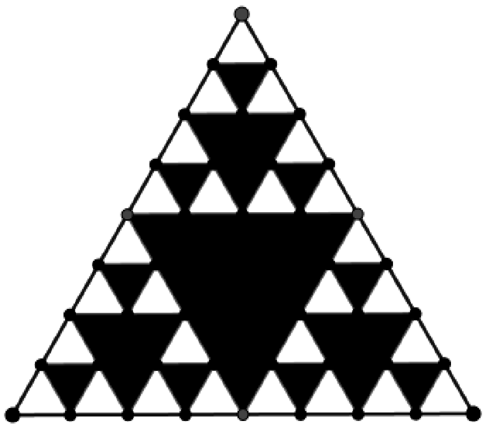
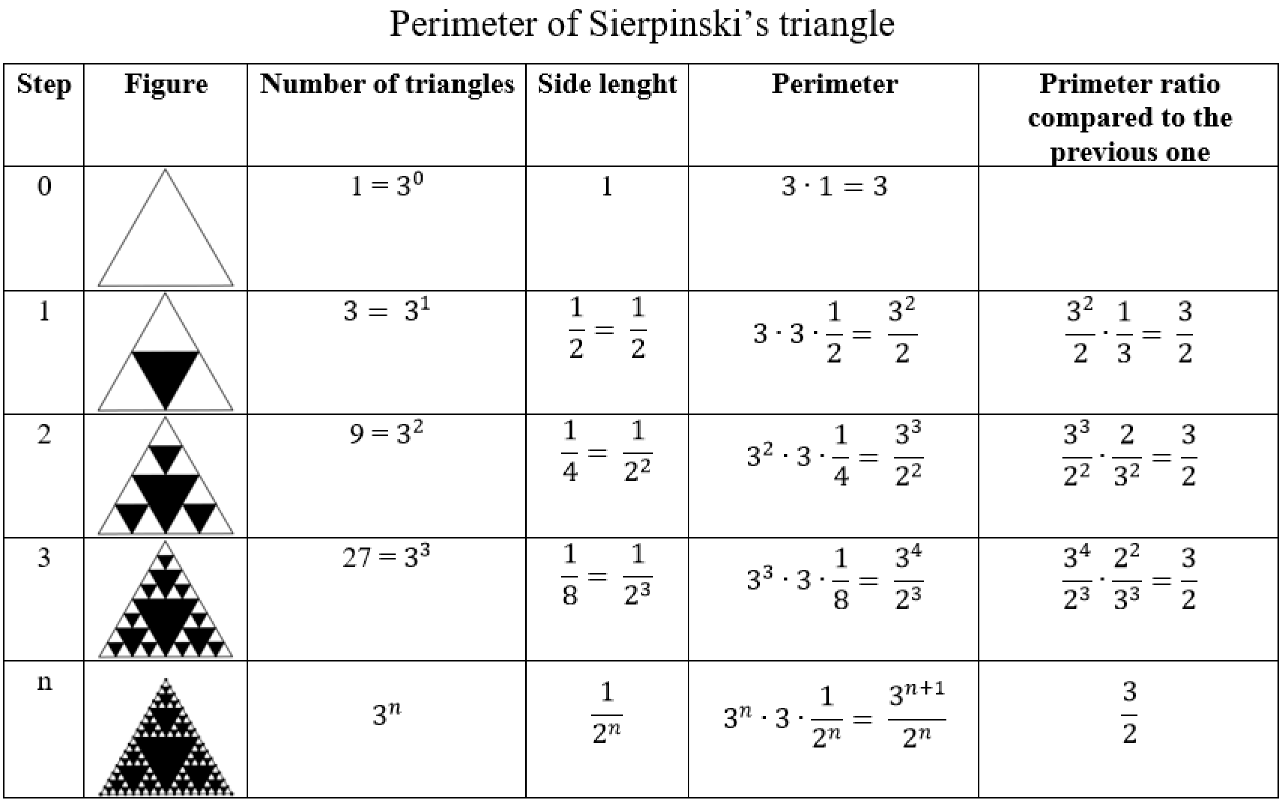


Figure 3. Sierpinski's triangle.

It is one of the first fractal objects in the history of mathematics. The figure is obtained by removing the medial triangle (i.e., the triangle that has the vertices on the midpoints of the three sides). Each triangle obtained at a given step of the construction is reduced by a homothetic factor 1/2, compared to the triangle in the previous step. The educators gave the teachers some ideas on how to work with this fractal with the students. In particular, a video illustrated how to create Sierpinski's triangle in GeoGebra (by generating a tool to obtain the construction iteratively). The educators also shared a file on which two worksheets to be solved by the students were provided, one on the perimeter of a Sierpinski's triangle (Figure 4) and another on its area. Specific reflection questions accompany the worksheets (e.g., (i) Looking at the "number of triangles" column, which regularity can you deduce? (ii) By looking at the "side length" column, what regularity can you observe? (iii) Write down the formula that allows you to calculate the measurement of the side after n steps, etc.).



Perimeter of Sierpinski's triangle

Step	Figure	Number of triangles	Side lenght	Perimeter	Primeter ratio compared to the previous one
0		$1 = 3^0$	1	$3 \cdot 1 = 3$	
1		$3 = 3^1$	$\frac{1}{2} = \frac{1}{2}$	$3 \cdot 3 \cdot \frac{1}{2} = \frac{3^2}{2}$	$\frac{3^2}{2} \cdot \frac{1}{3} = \frac{3}{2}$
2		$9 = 3^2$	$\frac{1}{4} = \frac{1}{2^2}$	$3^2 \cdot 3 \cdot \frac{1}{4} = \frac{3^3}{2^2}$	$\frac{3^3}{2^2} \cdot \frac{2}{3^2} = \frac{3}{2}$
3		$27 = 3^3$	$\frac{1}{8} = \frac{1}{2^3}$	$3^3 \cdot 3 \cdot \frac{1}{8} = \frac{3^4}{2^3}$	$\frac{3^4}{2^3} \cdot \frac{2^2}{3^3} = \frac{3}{2}$
n		3^n	$\frac{1}{2^n}$	$3^n \cdot 3 \cdot \frac{1}{2^n} = \frac{3^{n+1}}{2^n}$	$\frac{3}{2}$

Figure 4. Worksheet on the Sierpinski's triangle.

In this module, the padlet (https://it.padlet.com/ (accessed on 28 July 2023)) was inserted as a communication message board. Here are some comments posted by the teachers related to this activity that show how it had been positively valued by them. They found it interesting and informative and thought that their students would too.

> M.C. (high school teacher): I found the proposed activities particularly interesting and especially the way they are presented. [. . .] this year I will propose [. . .] in grade 10 [the activity of] the Sierpinski's triangle. I think I get the attention and interest of my students, since I have never experienced laboratory activities in class.

> V.M. (middle school teacher): I find the proposed activities are very interesting and stimulating. Introducing the principle of induction or recursion is not easy at all, but these ideas are a good starting point. Very nice the activity on the Sierpinski's triangle.

The MOOC Numeri has had a total of 278 teachers enrolled, and out of them, 116 (42% of members) completed all the MOOC tasks. Table 12 shows which MOOC modules inspired the teachers to design their project work, taking into account that 48% chose to consider a different topic from those considered in the five modules.

Table 12. Project work topics that have been inspired by the MOOC modules.

Module 1	Module 2	Module 3	Module 4	Module 5
15%	4%	3%	8%	22%

5.2.3. A Mathematical Task Design Made by a Teacher within the PDP

In the following, we present an example of project work carried out by a primary school teacher (we call her Kelly) on Sierpinski's triangle. We chose it for the following reasons:

- It is an example of project work that has taken its cue from the examples proposed by the educators;
- It allows us to observe how a MOOC teacher was able to adapt to her scholastic level the proposals of the educators, even those that could be more difficult to achieve;
- The proposed project work was a DT task design, in line with the intentions of the teachers from Sri Lanka.

Kelly's project work was called "A Christmas Card". It was aimed at primary school students (Grade 4) with the idea to create a Christmas card using Sierpinski's triangle as an image. Kelly articulates her project in six phases. In phase 1 the teacher proposes that her students create a Christmas card with a particular image, that of Sierpinski's triangle. She shows them an image of it and talks about the mathematician Sierpinski. In phase 2 the students, working in groups, are invited to solve a manipulative mathematical task, namely to build the Sierpinski's triangle using plastic strips of different colors and lengths, with the help of the clasps. At their disposal are various types of strips, and the pupils have to choose three strips of 1, three of 1/2, and three of 1/4 to build the triangle (Figure 5). In this phase, the students explain what they did to create the triangle, the difficulties encountered, and what type of strips were most appropriate.

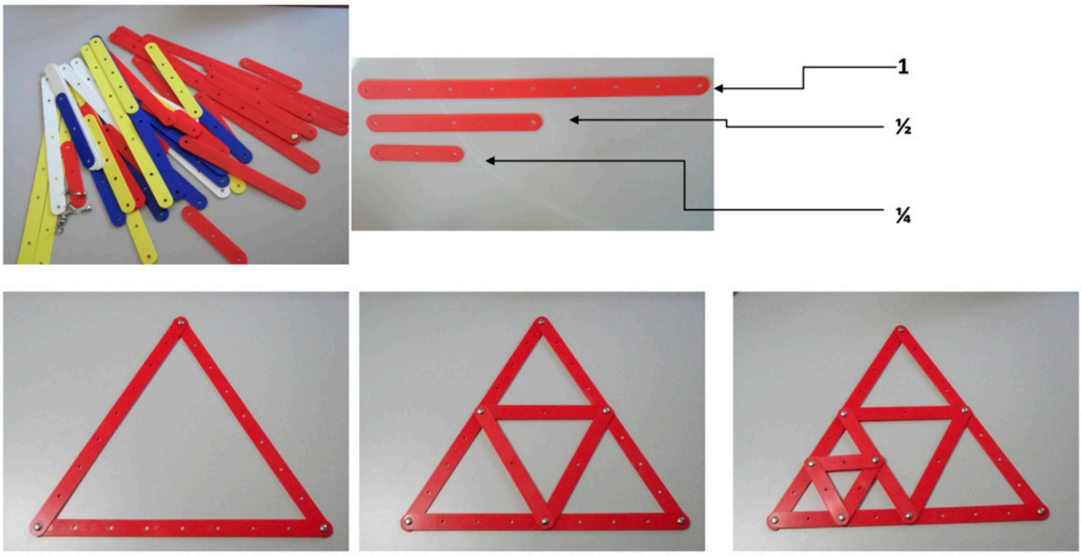
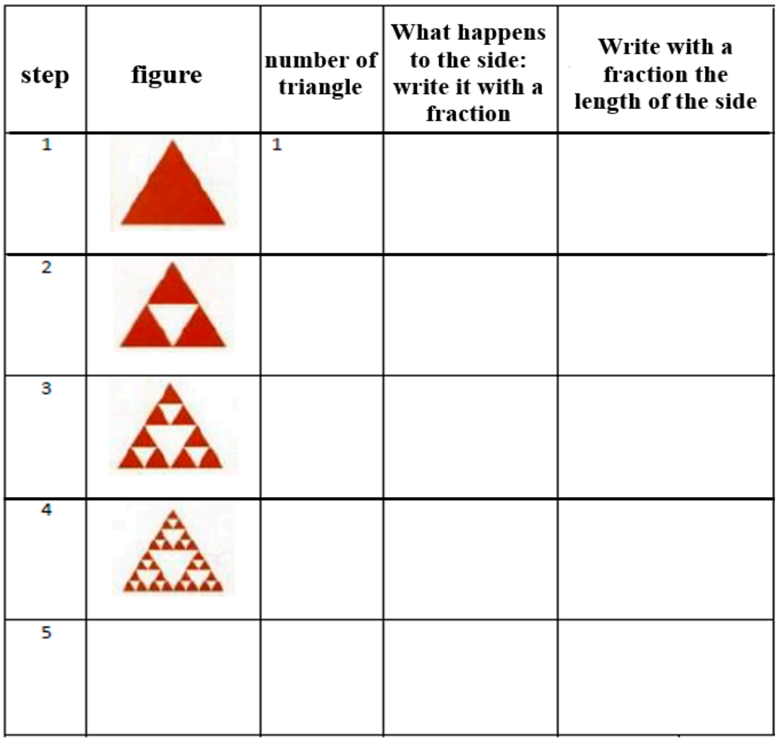


Figure 5. Plastic strips used to make the Sierpinski's triangle.

Then, in phase 3, the teacher makes use of technology, and, as proposed in the MOOC, the teacher wants to build the triangle tool of Sierpinski using GeoGebra. The teacher helps the pupils to achieve this goal, giving instructions and explaining the necessary commands. In phase 4, each pupil creates their own Sierpinski's triangle. Then, at the interactive whiteboard, they share the work they have performed. The triangles are printed to make the Christmas card together with the assistance of an art teacher. In phase 5 the pupils reflect together and write down what Sierpinski's tool on GeoGebra does. In addition, the teacher helps them to discuss the relationship between the side and the perimeter of the triangle and how the colored surface changes as smaller triangles are created. In the last phase, the teacher opens a discussion to reflect with the pupils on this activity. Finally, a test, another mathematical task designed by Kelly taking inspiration from a worksheet on the perimeter of a triangle presented by the educators (See Figure 6), is given to the pupils.

We can see how Kelly simplified the worksheet for her students. She also added four questions to accompany it, comprising a simplified version of those proposed by the educators: (i) Can you complete the table in step 5? (ii) If you look at the column of the number of triangles, what do you see? (iii) If you look at the measurement column on a side, what do you notice? (iv) Does the perimeter increase or decrease with each step? Motivate your answer.



step	figure	number of triangle	What happens to the side: write it with a fraction	Write with a fraction the length of the side
1		1		
2				
3				
4				
5				

Figure 6. Kelly's worksheet.

Kelly's project work was based on one of the activities proposed by educators in MOOC Numeri. In doing so, she adapted a subject that is generally dealt with in secondary school for primary school, and even then, not very frequently. She has been able to calibrate different methodologies (frontal lesson; laboratory activities with plastic strips; use of technology with GeoGebra; argumentation, reflection, and verification activities). We do not know if and when Kelly implemented her project work, i.e., we do not know if this design would be effective in practice. The goal of the MOOC was not to monitor the classroom testing of designs; rather, it provided resources for consciously designing activities on the number core.

We would like to stress that the aim of the analysis in this section is not to check if/how the activities were effectively improved in the school but how the materials and suggestions proposed in the MOOC influenced teachers' practices. Here, the fact that Kelly is a teacher from primary school emphasizes, even more, the relevance of her proposal from this standpoint. In fact, if the topic of mathematical recursion/iteration is not generally addressed in secondary school, it is certainly not the subject of primary school programs. However, the fact that Kelly has not only focused on this topic but has also chosen it as the topic of her project work, illustrating how much the educators' meta-didactical praxeologies have affected her practices.

Moving on to the feedback from Kelly's reviewer, who was another MOOC teacher. Table 13 shows how much that certain aspect of the categories that should have been considered in the design was present (from 1 = little present aspect to 5 = highly present aspect) in Kelly's project work in the reviewer's opinion.

Table 13. Presence of the categories in Kelly's project work.

Connections to the Real World	Creativity	Collaboration	Use of Technology	General Considerations
4	4	3	3	4

From these results, we can deduce that the reviewer judges the project work to be of good quality. This is also confirmed by the comment that he makes and that we report briefly below.

> *The activity proposes a very original and creative work that allows students to recognize geometric figures, properties and their significant elements. Balanced use of manual modelling using plastic strips and modelling using GeoGebra [. . .]. The strengths of the work are: (a) Presentation of a new mathematical concept in a real and original situation; (b) Use of technology; (c) Guided discussion; (d) Propaedeutic modelling for abstraction; (e) Well constructed test worksheet [. . .]. As weaknesses of the work I have identified: (a) The valuation criteria of the asset are not indicated; (b) I would suggest the introduction of a final questionnaire to reflect on the activity.*

In general, the reviewer's comment is positive and leaves some suggestions that Kelly might consider.

5.3. A comparison between Sri Lankan Italian Educators' Meta-Didactical Praxeologies

Through an analysis of the educators' meta-didactical praxeologies in the two PDPs, the issues that emerge are three kinds of common and essential elements that the educators' meta-didactical praxeologies are based on:

i. *Examples*: the examples of activities prepared by educators and proposed to teachers;
ii. *Discussions*: the teachers' discussions, (in terms of reflection, modification, change of opinion and ideas), orchestrated by the educators, on the activities;
iii. *Design*: the task design carried out by the teachers based on (i) and (ii);

A comparison between the two sets of Sri Lankan and Italian educators' meta-didactical praxeologies follows.

Example

In terms of educators' meta-didactical praxeologies, the two studies used two different approaches. For instance, the Sri Lankan design provided one exemplar task and guided the teachers to evaluate it in terms of features of a rich DT task. Therefore, the teachers evaluated the task but did not have an opportunity to try out the exemplar task. In the Italian design educators provided many tasks and the teachers had an opportunity to use exemplar tasks designed by educators. The use of examples in each context is compared in Table 14.

Table 14. Comparison of educators' meta-didactical praxeologies to provide examples.

	Sri Lanka	**Italy**
Task	One exemplar task	Few exemplar tasks
	DT task	DT tasks and non-DT tasks
	Theoretical	Practically used in their classrooms
Techniques	Teachers were directed to discuss the task in terms of features of a rich DT task.	Teachers were invited to use the tasks practically in their classrooms.
Argument	To provide teachers with theoretical knowledge on developing DT tasks that they can use in their task design.	To innovate methodology and strategies of teaching mathematics and give ideas for drawing up their final task design.

Discussions

The Italian educators' meta-didactical praxeology to encourage discussions appeared once—after the teachers used the tasks whereas Sri Lankan educators' happened in three places—at the beginning in planning and designing the task; modifying the task, and reflecting on the task implementation. Another difference is that the Sri Lankan teachers were encouraged to have a face-to-face discussion and the Italian teachers had online discussions. Although the meta-didactical praxeologies to encourage discussions were presented in different ways in the two contexts (face-to-face and online), each PDP had common intentions (see Tables 2, 4 and 9), such as:

1. To motivate teachers to discuss and exchange ideas to find answers to issues they face in task design;
2. To promote self and peer reflections, and to understand the importance of them in PD;
3. To establish communities (among MOOC teachers in the Italian context and having small groups either in the school or in the educational zone in Sri Lanka).

Design

Intending to support teachers to design mathematical tasks themselves, the educators' meta-didactical praxeology to support teacher design also took place in the two projects in two different ways. Table 15 shows the differences and similarities in each context at the level of tasks, techniques, and arguments.

Table 15. Comparison of educators' meta-didactical praxeologies to support teachers' design.

	Sri Lanka	Italy
Task	A DT algebra task	A project work (including tasks) on arithmetic and algebra content
Techniques	In small groups of three	Individual work
	Using GeoGebra	Using Learning Designer
	Directed to design a preliminary task with their existing knowledge and then to modify or re-design after the researcher intervention.	Directed to design tasks after using exemplar tasks.
	design, self-reflection as a group, modification, implementation, self-and peer-reflection, modification methodology was applied	*practical use of exemplar tasks, design, peer-reflection* methodology was applied.
Argument	To support teachers to design DT tasks themselves	Project-based learning

6. Discussion and Conclusions

In this paper, we present a meta-analysis of the practices employed by mathematics teacher educators during PDPs aimed at assisting teachers in the realm of task design. Essentially, our focus revolves around dissecting the meta-didactical praxeologies of mathematics teacher educators, an aspect that, as of our current understanding, has not been critically examined or emphasized in the existing literature.

In particular, in this paper we sought to address two research questions, which are:

- What patterns of practices emerge when mathematics teacher educators aim to develop a PDP to support teacher design of mathematical tasks in two different contexts?
- What are the similarities and differences in methods and results if a PDP is conducted face-to-face or through MOOC?

In addition to the similarities already highlighted in Section 4.1.3, we realized that the three practices of Example, Discussion, and Design were present in both studies. These practices were the starting point that led us to reflect on the fact that they are fundamental elements that we would suggest considering in every PDP. It is crucial to consider what examples of activities educators have to prepare to show to the teachers, to trigger productive discussions among them, to produce reflections on their use in class, to encourage an exchange of ideas and opinions, and to elicit proposals for modification of the activities. When a PDP is developed in this way, teachers benefit in a manner that can enable them to design mathematical tasks independently and in a more conscious way, as our data have shown.

The identification of these three practices in meta-didactical praxeologies, compared in Section 5.3, can inform other research. These comparisons represent the application of the three practices to the specific cases treated here. In particular, the examples discussed refer to PDPs devoted to training on mathematical task design and, as shown by the analysis, the results have been positive in both experiences. Thus, the findings suggest that more likely either using examples practically or studying with theoretical aspects would benefit teachers' design of tasks. A blend of the two approaches could also have a positive influence on teachers' task design.

The discussions among the teachers occurred in two approaches in the two contexts. For instance, Italian teachers had opportunities to comment on their peers' tasks. In addition, they also had opportunities to share their experience in implementing the tasks designed by researchers in the MOOC. In the Sri Lankan study, the teachers had opportunities to reflect on their tasks and on their implementation since they designed the tasks in groups. In doing so, in the Sri Lankan study, the objective was to design tasks that would then be inserted into a lesson, while in the Italian study the aim was to design an entire lesson that contains mathematical tasks. As Gimenez and colleagues [41] confirmed, with their professional task design with prospective teachers, reflecting and giving feedback in design-based research (DBR) cycles contributed positively to the redesign of tasks. In line with this finding, teachers of the projects present in this paper also benefited from their reflections. Thus, in the future, PDPs promoting either peer- or self-reflections on teacher productions would be an advantage to improve the quality of the tasks.

In addition to the knowledge and the experience of designing tasks, the Italian researchers had prepared some sample tasks that they developed as part of their initial praxeologies. On the other hand, in the Sri Lankan context, the researchers' initial praxeologies comprised the theoretical aspects, including an exemplar task. The idea of discussing such a task was to give a better understanding of a rich DT task for the teachers. In the Italian context, on the other hand, teachers practically used the exemplar tasks designed by the researchers and shared in the MOOC in their classrooms. Thus, each project provided exemplar tasks for teachers in two different approaches with the same intention: to support them to understand what a rich mathematical task is and to be able to begin the process of task design themselves. After having these experiences, teachers designed tasks themselves. In each situation, the ability to design mathematical tasks was becoming the teachers' new didactical praxeology.

Certainly, the teachers were not aware of the structure of the common patterns—examples, discussion, and design—which are the pillars of the PDP. However, having led the PDP by putting these patterns into practice has meant that teachers have been able to learn or refine the practice of designing a mathematical task for their students. The examples provided were fundamental. In the Italian context, they were given as a starting point to stimulate reflection and online discussion among teachers. In the Sri Lankan context, they were given after the first production of the teachers to make them reflect on how to modify what they designed. These are methodological differences: we cannot say that either is better, as the implementation contexts are different. However, we can say that these methodological choices have both produced positive results, allowing teachers to have a reflection on their didactical praxeologies, i.e., to have an impact on their meta-didactical praxeologies.

In each study, it was evident that the teachers were able to design tasks either individually or in groups after attending the PDPs. Although the tasks were not at a very high level, the teachers showed that, with continuous support from the educators, they were capable of developing tasks. As Lee and Özgün-Koca [42] suggest, exposure to more exemplar tasks is another possible support that educators could provide to teachers to assist them to be more effective in task design. In addition to that, providing continuous support for teachers in the design process is essential (e.g., [43]). Finally, we conclude that well-planned PDPs have the potential to support teachers in their design of mathematical tasks (including DT tasks) irrespective of the mode of instruction, either online or face-to-face, or perhaps in blended mode. In doing so, having a design comprising small groups or pairs coupled with the opportunity to reflect together on the tasks would be advantageous in improving the quality of the tasks, and hence for a teacher's professional learning.

The two studies discussed in this paper were conducted in two different contexts—in a South Asian country and a European country. The two contexts are different in available resources in schools, teachers' experience in using DT, and their experience in developing tasks. While the Sri Lankan teachers had no prior experience in developing DT tasks, the Italian teachers had had prior experience either with or without DT. Thus, we argue that irrespective of the context, the three aspects (examples, discussions, and design) are likely crucial patterns to be practiced in future PDPs that focus on teacher design of mathematical tasks. However, it would be worth investigating other patterns of practices of mathematics teacher educators in this regard in future research.

Last but not least, although it might be intuitive to assume that within a PDP dedicated to task design, all mathematics teacher educators adhere to certain recognized practices—such as providing examples, fostering discussions, and guiding the design process—it cannot be taken as given that they are consciously aware of employing these practices.

Our intentions in this paper extended beyond the mere identification of these practices. We also sought a suitable term to encapsulate them. Our choice fell upon the term "pattern". The choice of this word, which we already take for granted at the outset of the paper, was by no means trivial to identify. In fact, this word is also linked to the mode of execution of the practices themselves. We have discarded "flow of practices" because these practices may not necessarily be executed in a cyclical manner. They can be executed forwards or backwards, rearranged in permutations, and some may be iterated. This variability hinges upon the educators' responses to the proposed concepts or the trajectory of the mathematics teacher educator's instructional design. Likewise, we discarded the term "model" as our intent was not to present a rigid and inflexible model of practices, but rather a construct that was easily identifiable, albeit in its dynamism and freedom of execution. What remains uncertain is the applicability of this pattern not only in physical settings but also in remote learning environments (e.g., MOOCs), where its functionality endures despite asynchronous conditions.

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