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A $\bar{\partial}$-Dressing Method for the Kundu-Nonlinear Schrödinger Equation

Jiawei Hu [1] and Ning Zhang [1,2,*]

[1] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China; 202282150078@sdust.edu.cn

[2] Department of Fundamental Course, Shandong University of Science and Technology, Taian 271019, China

* Correspondence: skd991310@sdust.edu.cn

Abstract: In this paper, we employed the $\bar{\partial}$-dressing method to investigate the Kundu-nonlinear Schrödinger equation based on the local 2×2 matrix $\bar{\partial}$ problem. The Lax spectrum problem is used to derive a singular spectral problem of time and space associated with a Kundu-NLS equation. The N-solitions of the Kundu-NLS equation were obtained based on the $\bar{\partial}$ equation by choosing a special spectral transformation matrix, and a gradual analysis of the long-duration behavior of the equation was acquired. Subsequently, the one- and two-soliton solutions of Kundu-NLS equations were obtained explicitly. In optical fiber, due to the wide application of telecommunication and flow control routing systems, people are very interested in the propagation of femtosecond optical pulses, and a high-order, nonlinear Schrödinger equation is needed to build a model. In plasma physics, the soliton equation can predict the modulation instability of light waves in different media.

Keywords: $\bar{\partial}$-dressing method; spectral transform; soliton solution; Kundu-NLS equation

MSC: 35C08; 35Q51; 37K40

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1. Introduction

In recent years, the nonlinear Schrödinger equation (NLS) [1,2] has emerged as an important component of the soliton equation. It appears in many different physical applications, such as in plasma physics and nonlinear optics [3], which have a wide range of applications. With the advances of this research, the classical Schrödinger equation [4–6],

$$iu_t + u_{xx} + 2|u|^2 u = 0 \tag{1}$$

and its evolutionary form can represent a well-known integrable system in the field of mathematics. In Equation (1), u denotes the complex envelope of the waves, x and t denote propagation distance and scaled time, respectively, and i is the imaginary unit. The Schrödinger equation has a stable soliton solution, which can be properly denoted by the linear dispersion problems. However, in physics, particularly in optical fibers, this model needs to be described using the high-order Schrodinger equation because of interest in the propagation of femtosecond light pulses. In the field of mathematics, the study of nonlinear equations with variable coefficients has also led to the further development of integrable systems. The similar transformation technique, Wronskian technique, Jacobi elliptic approach, and direct algebraic method can solve the nonlinear Schrodinger equation with variable coefficients [7–10]. By using the similar transformation technique, we can solve the optically smooth position solution of the nonlinear Schrodinger equation with variable coefficients. In 1984, Kundu proved that the nonlinear Schrödinger equation leads to an integrable high-order nonlinear equation with variable coefficients under nonlinear transformation, which is called the Kundu-NLS equation. It can be associated with the nonlinear Schrödinger equation via the Lax equation, and it does not change the dispersion term.

In this article, we focus primarily on the Kundu-NLS equation [11–16],

$$iu_t + u_{xx} + 2\alpha^2|u|^2u - (\gamma_t + \gamma^2 - i\gamma_{xx})u + 2i\gamma_x u_x = 0, \tag{2}$$

where $\gamma(x,t)$ is a free gauge function, and α is a real constant. Using the $\bar{\partial}$-dressing method, we created one- and two-soliton solutions. The equation of the respirator and higher-order rouge wave solutions was obtained using the Darboux transform [15,16]. It is also possible to obtain the equation's soliton solution by using the Riemann–Hilbert method [11–14]. As far as we know, the soliton solution of the Kundu-NLS equations has not been solved by using the $\bar{\partial}$-dressing method.

The $\bar{\partial}$-dressing method suggested by Zakharov and Shabat [17,18] was later developed by Beals, Coifman, Manakov, Ablowitz, Fokas, and others [19–23]. A wide variety of equations have been successfully investigated using the $\bar{\partial}$-dressing method [24–30]. However, the equations with a normalization boundary condition, $\psi \to I$, at infinity are usually considered. Consequently, the spectral problems and hierarchies on some significant integrable equations, such as the KE equation, Kundu-NLS equations, and others, cannot be adequately deduced using the $\bar{\partial}$-dressing method. We are primarily concerned with the $\bar{\partial}$-equation for normalization, $\psi \to D$, where D is a nondegenerate matrix function of x and t. Using the $\bar{\partial}$-dressing method, the Lax pair and deriving the soliton solutions could be investigated.

The structure of this article is as follows: In Section 2, by using the $\bar{\partial}$-dressing method, we obtained the Lax pair by generalizing the properties of the Cauchy–Green operators. In Section 2.2, by using the $\bar{\partial}$-dressing method, we derived the Kundu-NLS hierarchy (with the source) from the relationship between the $\bar{\partial}$-dressing transformation matrix and the potential matrix. In Section 3, a formula for N-soliton solutions of the Kundu-NLS equation was constructed, and we give explicit one- and two-soliton solutions for the Kundu-NLS equation.

2. The $\bar{\partial}$-Dressing Method

2.1. Spectral Transform and Lax Pair

In this section, in order to analyze the Kundu-NLS equation, we focus on the Lax pair of Equation (2):

$$\begin{aligned} \psi_x &= -iz\sigma_3\psi + Q\psi, \\ \psi_t &= -2iz^2\sigma_3\psi + \tilde{Q}i\sigma_3\psi, \end{aligned} \tag{3}$$

where $\tilde{Q} = Q_x + Q^2$, with $Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Consider a matrix $\bar{\partial}$ problem with a non-normalization boundary condition. The $\bar{\partial}$-dressing method's objective is to create a system of linear equations for ψ that is compatible. We start from the 2×2 matrix $\bar{\partial}$-problem in the complex z-plane,

$$\bar{\partial}\psi(z) = \psi(z)R(z), \psi(z) \to D, z \to \infty, \tag{4}$$

where $D = D(x,t)$ is a nondegenerate matrix function of x and t, and $R(z,\bar{z})$ is the matrix of spectral transform connected to a nonlinear equation and $\bar{\partial} \equiv \partial/\partial z$. We obtained a formal solution of the $\bar{\partial}$-problem via Equation (4):

$$\psi(z) = D + \psi RC_z, \tag{5}$$

where C_z represents the Cauchy–Green integral operator on the left, and this is given by

$$\psi RC_z = \frac{1}{2\pi i}\int\int \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}\psi(\zeta)R(\zeta). \tag{6}$$

In this case, we suppressed the fact that the variable $\bar{z}$ depends on ψ and R. The expression in Equation (5) empowers us to officially compose an answer to the $\bar{\partial}$ issue (5) concerning the matrix R:

$$\psi(z) = D \cdot (I - RC_z)^{-1}, \tag{7}$$

where I is the 2×2 unit matrix.

Define

$$D = e^{i(zx+2z^2t)\sigma_3}, \tag{8}$$

where D satisfied $D_x = iz\sigma_3 D$, $D_t = 2iz^2\sigma_3 D$.

It is simple to demonstrate that the operator C_z satisfies (for a certain set of matrix functions) $f(z)$ and $g(z)$,

$$g(z)[f(z)C_z]C_z + [g(z)C_z]f(z)C_z = [g(z)C_z][f(z)C_z]. \tag{9}$$

Define a pair:

$$\begin{aligned} \langle f, g\rangle &= \tfrac{1}{2\pi i}\int\int dz \wedge d\bar{z} f(z) g^T(z), \\ \langle f, g\rangle^T &= \langle g, f\rangle, \end{aligned} \tag{10}$$

where the superscript T means transposition. Then, it is evident that the above pairing has the following features [24].

Proposition 1.

$$\begin{aligned} &\langle f, g\rangle^T = \langle g, f\rangle \\ &\langle fR, g\rangle = \langle f, gR^T\rangle \\ &\langle fC_z, g\rangle = -\langle f, gC_z\rangle \\ &\langle fRC_z, g\rangle = -\langle f, gR^TC_z\rangle \\ &\langle f(I - RC_z), g\rangle = \langle f, g(I + R^TC_z)\rangle \end{aligned} \tag{11}$$

In addition, it will be easy to test and verify the following prosperities:

$$\frac{1}{\mu - z} f(z)C_z = \frac{1}{\mu - z}\{[f(z)C_z] - [f(\mu)C_\mu]\} \tag{12}$$

and

$$\begin{aligned} zf(z)C_z &= z[f(z)C_z] + \langle f(z)\rangle, \\ z^2 f(z)C_z &= z^2[f(z)C_z] + z\langle f(z)\rangle + \langle zf(z)\rangle, \end{aligned} \tag{13}$$

or, in general, for some positive integer λ,

$$z^\lambda f(z)C_z = z^\lambda[f(z)C_z] + \Sigma_{j=0}^{\lambda-1} z^j \langle z^{\lambda-1-j} f(z)\rangle, \tag{14}$$

where $\langle f(z)\rangle$ is defined by putting $\langle f(z), I\rangle$.

We allow the x-dependence of the Kundu-NLS equation to be expressed as follows. It is significant that the form of the Lax pair for a particular equation is entirely determined by the space–time dependence of the transform matrix $R(x, t, z)$:

$$R_x = iz[R, \sigma_3]. \tag{15}$$

Then, Equation (7) can be utilized to compute

$$\begin{aligned} \psi_x &= D_x(I - RC_z)^{-1} + D[(I - RC_z)^{-1} R_x C_z (I - RC_z)^{-1}] \\ &= iz\sigma_3 D(I - RC_z)^{-1} + iz\psi[R, \sigma_3]C_z(I - RC_z)^{-1} \\ &= iz\sigma_3 D(I - RC_z)^{-1} + iz\psi R\sigma_3 C_z(I - RC_z)^{-1} - iz\psi\sigma_3 RC_z(I - RC_z)^{-1}. \end{aligned} \tag{16}$$

The first term on the right can be transformed as follows:

$$\begin{aligned} iz\psi RC_z &= \frac{1}{2\pi}\int\int \frac{\zeta d\zeta\wedge d\bar{\zeta}}{\zeta - z}\psi(\zeta)R(\zeta) \\ &= i\langle\psi R\rangle + iz(\psi RC_z) = i\langle\psi R\rangle + iz(\psi - D). \end{aligned} \tag{17}$$

Hence,

$$\psi_x = iz\sigma_3 D(I - RC_z)^{-1} + i\langle\psi R\rangle\sigma_3(I - RC_z)^{-1} - izD\sigma_3(I - RC_z)^{-1} + iz\psi\sigma_3.$$

Next, we calculate the second term, $izD\sigma_3(I - RC_z)^{-1}$, from Equation (17), where we have

$$\begin{aligned} zD &=< \psi R > + z\psi(I - RC_z) \\ zD(I - RC_z)^{-1} &=< \psi R > (I - RC_z)^{-1} + z\psi = (< \psi R > D^{-1} + z)\psi. \end{aligned} \tag{18}$$

As a result,

$$\psi_x + iz[\sigma_3, \psi] = iz\sigma_3\psi - i[\sigma_3, < \psi R >]D^{-1}\psi. \tag{19}$$

Next, we introduce the potential:

$$Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} = -i[\sigma_3, < \psi R >]D^{-1}, \tag{20}$$

then we can also obtain the spectral problem of Zakharov–Shabat:

$$\psi_x + iz[\sigma_3, \psi] = iz\sigma_3\psi + Q\psi. \tag{21}$$

In order to acquire the time dependency of R, we select a linear equation such that

$$R_t = [R, \Omega]. \tag{22}$$

We suppose

$$\Omega(z) = 2iz^2\sigma_3 + \frac{1}{2\pi i}\int\int \frac{\omega(\zeta)\sigma_3}{\zeta - z} d\zeta\wedge d\bar{\zeta}, \tag{23}$$

which comprises both a polynomial part, $\Omega_p(z)$, and a singular part, $\Omega_s(z)$, and $\omega(\zeta)$ is a scalar function.

First, we use the polynomial dispersion relation only, $\Omega = \Omega_p = 2iz^2\sigma_3$. Then, using Equations (5), (7), and (22), we obtain

$$\begin{aligned} \psi_t = [D\cdot(I - RC_z)]_t &= D_t(I - RC_z)^{-1} + D(I - RC_z)^{-1}R_tC_z(I - RC_z)^{-1} \\ &= 2iz^2\sigma_3 D(I - RC_z)^{-1} + \psi R\Omega C_z(I - RC_z)^{-1} - \psi\Omega RC_z(I - RC_z)^{-1} \\ &= 2iz^2\sigma_3\psi + 2iz^2\psi R\sigma_3 C_z(I - RC_z)^{-1} - 2iz^2\psi\sigma_3(I - RC_z)^{-1} + 2iz^2\psi\sigma_3. \end{aligned} \tag{24}$$

We form the definition of the left Cauchy operation:

$$\begin{aligned} z^2\psi RC_z &= \frac{1}{2\pi}\int\int \frac{\zeta^2 d\zeta\wedge d\bar{\zeta}}{\zeta - z}\psi(\zeta)R(\zeta) \\ &=< z\psi R > + z < \psi R > + z^2\psi RC_z \\ &=< z\psi R > + z < \psi R > + z^2(\psi - D), \end{aligned} \tag{25}$$

$$z^2 D =< z\psi R > + z < \psi R > + z^2\psi(I - RC_z), \tag{26}$$

and

$$z^2(I-RC_z)^{-1} = D^{-1}z^2\psi + D^{-1} < z\psi R > (I-RC_z)^{-1} \\ + D^{-1}z < \psi R > (I-RC_z)^{-1}, \tag{27}$$

$$z(I-RC_z)^{-1} = D^{-1} < \psi R > (I-RC_z)^{-1} + D^{-1}z\psi. \tag{28}$$

From Equations (24) and (27), we obtain

$$\psi_t + 2iz^2[\sigma_3, \psi] = 2iz^2\sigma_3\psi + (-2Q < \psi R >^{diag} + 2 < \psi R >_x^{off} \\ -2iz\sigma_3 < \psi R >^{off} + 2Q < \psi R >)D^{-1}\psi + 2zQ\psi. \tag{29}$$

We suppose $\bullet^{off}$ means the off-diagonal part of matrix $\bullet$, and $\bullet^{diag}$ means the diagonal part of matrix $\bullet$. Furthermore,

$$QD = -i[\sigma_3, < \psi R >], \tag{30}$$

$$< \psi R >= \frac{i}{2}\sigma_3 QD, \tag{31}$$

$$< \psi R >_x^{off} = \frac{i}{2}\sigma_3 Q_x D + \frac{i}{2}\sigma_3 Q D_x = \frac{i}{2}\sigma_3 Q_x D - \frac{1}{2}zQD. \tag{32}$$

So, the time-spectral problem is obtained:

$$\psi_t + 2iz^2[\sigma_3, \psi] = 2iz^2\sigma_3\psi + i\sigma_3 Q^2\psi + i\sigma_3 Q_x\psi + 2zQ\psi, \\ \psi_t + 2iz^2[\sigma_3, \psi] = (2z^2 + \tilde{Q})i\sigma_3\psi + 2zQ\psi. \tag{33}$$

In what follows, we consider the singular dispersion relation in Equation (23):

$$\psi_t = (\psi R\Omega_s C_z - \psi\Omega_s)(I-RC_z)^{-1} + \psi\Omega_s. \tag{34}$$

Resorting in Equations (23) and (23); $\psi R\Omega_s C_z$ in Equation (34) satisfies

$$\psi R\Omega_s C_z = \psi\Omega_s + \frac{1}{2\pi i}\int\int \frac{\omega(\zeta)\psi(\zeta)\sigma_3}{\zeta - z} d\zeta \wedge d\bar{\zeta}. \tag{35}$$

Hence, we have

$$\psi_t = -\frac{1}{2\pi i}\int\int \frac{\omega(\zeta)\psi(\zeta)\sigma_3}{\zeta - z} d\zeta \wedge d\bar{\zeta}(I-RC_z)^{-1} + \psi\Omega_s. \tag{36}$$

By using the relations

$$\frac{1}{\rho - z}\frac{1}{\zeta - \rho} = \frac{1}{\zeta - z}(\frac{1}{\rho - z} - \frac{1}{\rho - \zeta}), \tag{37}$$

we find that

$$\frac{1}{z-\zeta}(I-RC_z)^{-1} = \frac{1}{z-\zeta}\psi^{-1}(\zeta)\psi(z). \tag{38}$$

according to which Equation (36) then gives a time-dependent linear equation with the singular dispersion relation:

$$\psi_t = -\frac{1}{2\pi i}(\omega(z)C_z\psi\sigma_3\psi^{-1})\psi + \psi\Omega_s. \tag{39}$$

The time-spectral problem is

$$\psi_t + 2iz^2[\sigma_3, \psi] = 2iz^2\sigma_3\psi + i\sigma_3 Q^2\psi + i\sigma_3 Q_x\psi + 2zQ\psi + \\ -\frac{1}{2\pi i}(\omega(z)C_z\psi\sigma_3\psi^{-1})\psi + \psi\Omega_s. \tag{40}$$

2.2. Recursion Operator

In this section, we derive the Kundu-NLS equation with the source. In fact, if we want to work with the $\bar{\partial}$ method, we need the $\bar{\partial}$ problem from Equation (4) together with the linear Equations (15) and (22) controlling the space-time dependence of $R(x,t,z)$. From Equations (20) and (22), we are aware of the time evolution of the potential Q:

$$Q_t = -i[\sigma_3, < \psi R >_t]D^{-1} - i[\sigma_3, < \psi R >_t]D_t^{-1}. \tag{41}$$

By applying the obvious relation $\bar{\partial} f(z) C_k = f(z)$, we obtain

$$\begin{aligned} -i[\sigma_3, < \psi R >_t]D^{-1} &= -i[\sigma_3, 2iz^2\sigma_3 < \psi R > + < \psi R_t, I(I + R^T C_z)^{-1} >]D^{-1} \\ &= 2z^2[\sigma_3, \sigma_3 < \psi R >]D^{-1} - i[\sigma_3, < \psi R_t \psi^{-1} >], \end{aligned} \tag{42}$$

$$-i[\sigma_3, < \psi R >_t]D_t^{-1} = i[\sigma_3, < \psi R >]D^{-1}D_t D^{-1} = -2iz^2 Q\sigma_3. \tag{43}$$

Since $\bar{\partial}(\psi^{-1})^T = -(\psi^{-1})^T R^T$, it is demonstrable that $I \cdot (I + R^T C_z)^{-1} = (\psi^{-1})^T$. Hence,

$$Q_t = 2z^2[\sigma_3, \sigma_3 < \psi R >]D^{-1} - i[\sigma_3, < \psi R_t \psi^{-1} >] - 2iz^2 Q\sigma_3 \tag{44}$$

$$-i[\sigma_3, < \psi R_t \psi^{-1} >] = i[\sigma_3, < \psi(R\Omega - \Omega R)\psi^{-1}, I >], \tag{45}$$

$$\begin{aligned} Q_t = 2z^2[\sigma_3, \sigma_3 < \psi, R >]D^{-1} - 2iz^2 Q\sigma_3 - i\alpha_n[\sigma_3, < \bar{\partial}(z^n M(z) >] \\ + i[\sigma_3, < w(z)M(z) >]. \end{aligned} \tag{46}$$

where $M(x,t,z) = \psi(x,t,z)\sigma_3\psi^{-1}(x,t,z)$, and $M(x,t,z)$ satisfies the following equation:

$$M_x + 2iz[\sigma_3, M] = [Q, M]. \tag{47}$$

Let us decompose M as the total of the off-diagonal and diagonal parts,

$$M = \frac{1}{2}\sigma_3(\sigma_3 M + M\sigma_3) + \frac{1}{2}\sigma_3(\sigma_3 M - M\sigma_3) = M^{diag} + M^{off}. \tag{48}$$

Then, Equation (48) can be written as the following two equations:

$$\begin{aligned} M_x^{diag} &= -[Q, M^{off}], \\ M_x^{off} + 4iz\sigma_3 M^{off} &= [Q, M^{diag}]. \end{aligned} \tag{49}$$

Based on the asymptotic condition $\psi \to D$ when $x \to \infty$, we have $M^{diag} = \sigma_3 + \partial_x^{-1}[Q, M^{off}]$; then, we can rewrite the second equation of (45) as

$$M_x^{off} + 4iz\sigma_3 M^{off} = [Q, \sigma_3 + \partial_x^{-1}[Q, M^{off}]]. \tag{50}$$

It is challenging to determine the explicit solution for Equation (46). Hence, we introduce the operator for recursion in the form

$$\wedge \cdot = \frac{i}{4}\sigma_3(\partial_x - [Q, \partial_x^{-1}[Q, \cdot]], \tag{51}$$

which, evidently, does not depend on k. Then, Equation (50) gives

$$\begin{aligned} M^{off} &= -\tfrac{i}{2}(\wedge - z)^{-1}Q, \\ Q_t &= -2i\alpha_n\sigma_3 < \bar{\partial}(z^n M^{off}) > +2z^2[\sigma_3, \sigma_3 < \psi R >]D^{-1} \\ &\quad - 2iz^2 Q\sigma_3 + i[\sigma_3, < w(z)M(z) >]. \end{aligned} \tag{52}$$

The Kundu-NLS equation can be obtained; we expand $(\wedge - z)^{-1}$:

$$(\wedge - z)^{-1} = -\sum_{j=1}^{\infty} z^{-j} \wedge^{j-1},$$

then one possible rewrite for the second equation in Equation (52) is

$$Q_t = -\alpha_n \sigma_3 \sum_{j=1}^{\infty} < \bar{\partial} z^{n-j} > \wedge^{j-1} Q + i[\sigma_3, < w(z) M(z) >]. \tag{53}$$

By utilizing $\bar{\partial} z^{n-j} = \pi \delta(z) \delta_{j,n+1}, j = 1, 2, 3 \cdots$, and $\sum_{j=1}^{\infty} < \bar{\partial}(z^{n-j}) > \wedge^{j-1} Q = - \wedge^n Q$, we are able to derive the hierarchy of equations containing the Kundu-NLS that corresponds to the specific x-dependence of the spectral transform:

$$\begin{aligned} &M_x + iz[\sigma_3, M] = [Q, M], \\ &Q_t + \alpha_n \sigma_3 \wedge^n Q = 2z^2 [\sigma_3, \sigma_3 < \psi, R >] D^{-1} - 2iz^2 Q \sigma_3 + i[\sigma_3, < w(z) M(z) >]. \end{aligned} \tag{54}$$

3. Soliton Solution

In this section, we will provide the soliton solution of the Kundu-NLS equations within the $\bar{\partial}$-dressing method. First, we will construct the N-solitons of the Kundu-NLS Equation (1), which is still based on the $\bar{\partial}$-dressing method.

We choose a spectral transform matrix R as

$$R(z) = \sum_{j=1}^{\infty} \pi \begin{pmatrix} 0 & -c_j e^{-2i\theta(z)} \delta(z - z_j) \\ \bar{c}_j e^{2i\theta(z)} \delta(z - \bar{z}_j) & 0 \end{pmatrix}, \tag{55}$$

where c_j is constant and $\theta(z) = zx + 2z^2 t$. Let $\widetilde{Q} = QD$; then, we have

$$\widetilde{Q} = -i[\sigma 3, < \psi R >] = \begin{pmatrix} 0 & E \\ -\bar{E} & 0 \end{pmatrix}. \tag{56}$$

Substituting Equation (55) into Equation (56) leads to

$$\begin{aligned} E(x,t) = -2i < \psi R >_{12} &= -\sum_{j=1}^{\infty} c_j \int\int dz \wedge d\bar{z} \psi_{11}(z) R_{12}(z) \\ &= -\sum_{j=1}^{\infty} c_j \int\int dz \wedge d\bar{z} \psi_{11}(z) e^{-2i\theta(z)} \delta(z - z_j) \\ &= -2i \sum_{j=1}^{\infty} c_j e^{-2i\theta(z_j)} \psi_{11}(z_j). \end{aligned} \tag{57}$$

In order to obtain the $\psi_{11}(z_1)$, substituting Equation (57) into Equation (5), we have

$$\psi_{11}(z) = d_{11} + \sum_{j=1}^{\infty} \frac{\bar{c}_j}{z - \bar{z}_j} e^{2i\theta(\bar{z}_j)} \psi_{12}(\bar{z}_j), \tag{58}$$

$$\psi_{12}(z) = -\sum_{m=1}^{\infty} \frac{c_m}{z - z_m} e^{-2i\theta(z_m)} \psi_{11}(z_m), \tag{59}$$

where d_{11} represents the Equation (4) position element of the matrix D. By replacing z in Equation (58) with z_n and z in (59) with $\bar{z}_j$, we obtain a linear equation system for $\psi_{11}(z_n)$:

$$\psi_{11}(z_n) + \sum_{m=1}^{\infty} B_{n,m} \psi_{11}(z_m) = d_{11}, n = 1, 2, \ldots, N, \tag{60}$$

with

$$
\begin{aligned}
B_{n,m} &= \sum_{j=1}^{\infty} C_j(z_n)\overline{C_m(z_j)}, \\
C_j(z_n) &= \sum_{j=1}^{\infty} \frac{\bar{c}_j}{z-\bar{z}_j} e^{2i\theta(\bar{z}_j)},
\end{aligned}
\tag{61}
$$

where B is a square matrix of the order of N.

Furthermore, we introduce notations

$$
\begin{aligned}
V &= I + (B_{n,m}), \\
\hat{\psi}_{11} &= (\psi_{11}(z_1), \ldots, \psi_{11}(z_N))^T.
\end{aligned}
\tag{62}
$$

Then, Equation (62) will reduce to the linear system in the matrix form:

$$
V\hat{\psi}_{11} = D = (d_{11}, \ldots, d_{11})^T. \tag{63}
$$

From this, substituting $\hat{\psi}_{11}$ into Equation (57) gives the formula

$$
E(x,t) = 2id_{11}m, \tag{64}
$$

where

$$
m = \frac{detV^{aug}}{detV}, \tag{65}
$$

V are $N \times N$ matrices, and V^{aug} are $(N+1) \times (N+1)$ matrices, defined by

$$
V^{aug} = \begin{pmatrix} 0 & Y \\ I & V \end{pmatrix}, Y = (Y_1, Y_2, \ldots, Y_N), Y_j = -c_j e^{-2i\theta(z_j)}. \tag{66}
$$

By using Equations (64) and (65), we obtain the N-soliton solution of the Kundu-NLS equation:

$$
\begin{aligned}
u &= 2ime^{i(zx+2z^2t)} D_{11} \\
&= 2ime^{i(zx+2z^2t)} e^{i(zx+2z^2t)} = 2ime^{2i(zx+2z^2t)}.
\end{aligned}
\tag{67}
$$

For $N = 1$, set

$$
R(z) = \pi \begin{pmatrix} 0 & -ce^{-2izx}\delta(z-z_1) \\ \bar{c}e^{2izx}\delta(z-\bar{z_1}) & 0 \end{pmatrix}, \tag{68}
$$

where $c = c(t)$ can be found in Equation (22).

As follows from Equation (20), the soliton solution is given by

$$
\begin{aligned}
q &= -2i < \psi R >_{12} D_{22}^{-1} = -\frac{1}{\pi} D_{22}^{-1} \int\int dz \wedge d\bar{z} \psi_{11}(z) R_{12}(z) \\
&= -cD_{22}^{-1} \int\int dz \wedge d\bar{z} \psi_{11}(z) e^{-2izx} \delta(z-z_1) \\
&= -2icD_{22}^{-1} e^{-2iz_1x} \psi_{11}(z_1).
\end{aligned}
\tag{69}
$$

In order to confirm $\psi_{11}(z_1)$, by substituting Equation (68) into Equation (5), we obtain

$$
\psi_{11}(z) = D_{11} + \frac{\bar{c}}{z-\bar{z}_1} \psi_{12}(\bar{z}_1) e^{2i\bar{z}_1x}, \tag{70}
$$

$$
\psi_{12}(z) = \frac{-c}{z-z_1} \psi_{11}(z_1) e^{2iz_1x}. \tag{71}
$$

We set $z = z_1$ in Equation (70) and set $z = \bar{z}_1$ in Equation (71),

$$
\psi_{11}(z_1) = (D_{11} - \frac{|c|^2}{|z-\bar{z}_1|^2} e^{2i(\bar{z}_1-z_1)x})^{-1} = D_{11} - e^{4\eta(x-a)} \psi_{11}(z_1), \tag{72}
$$

where $z_1 = \xi + i\eta$. The function $c(t)$ has the following representation:

$$c = -2\eta e^{-2\eta a + i\phi}, \tag{73}$$

where a and ϕ are the undetermined functions about t. We rewrite the Equation (23) as

$$\Omega(z) = 2iz^2\sigma_3 + \frac{1}{2\pi i}\int\int \frac{\omega(\zeta)\sigma_3}{\zeta - z} d\zeta \wedge d\bar{\zeta} = [2iz_1^2 + (\omega_0 - i\omega_1)]\sigma_3. \tag{74}$$

On the one hand, by using Equation (68), we obtain

$$c_t = -4icz_1^2 - 2c\omega_0 + 2ci\omega_1. \tag{75}$$

On the other hand, using Equation (72),

$$c_t = c(-2\eta a_t + i\phi_t). \tag{76}$$

By comparing the two Equations (74) and (75), we find that

$$\begin{aligned} a &= (4\xi + \tfrac{\omega_1}{\eta})t + \xi_0, \\ \phi &= -4(\xi^2 - \eta^2)t + 2\omega_0 t + \phi_0, \end{aligned} \tag{77}$$

where ξ_0 and ϕ_0 are constants.

By substituting these results into Equation (69), we obtain the following soliton solution:

$$u = 4i\eta e^{-2i\xi x + i\phi + 2zx + 4z^2 t}\mathrm{sech}2\eta(x - a). \tag{78}$$

For N = 2, the formula Equation (55) gives the two-soliton solution of the Kundu-NLS Equation (2), which is given by

$$u = 2ime^{2i(zx + 2z^2 t)}, \tag{79}$$

where $m = \frac{detV^{aug}}{detV}$, and

$$V = \begin{pmatrix} 1 + B_{11} & B_{12} \\ B_{21} & 1 + B_{22} \end{pmatrix}, \tag{80}$$

$$V^{aug} = \begin{pmatrix} 0 & -c_1 e^{-2ixz_1 - 4itz_1^2} & -c_2 e^{-2ixz_2 - 4itz_2^2} \\ 1 & 1 + B_{11} & B_{12} \\ 1 & B_{21} & 1 + B_{22} \end{pmatrix}, \tag{81}$$

$$\begin{aligned} B_{ij} &= -\frac{F_{j,1}}{(z_i - \bar{z_1})(z_j - \bar{z_1})} - \frac{F_{j,2}}{(z_i - \bar{z_2})(z_j - \bar{z_2})}, \\ c_j\bar{c}_i &= e^{f_i + f_j}, F_{i,j} = e^{[2i(\bar{z}_i - z_j)x + 4i(\bar{z}_i^2 - z_j^2)t + f_i + f_j]}, i, j = 1, 2, \end{aligned} \tag{82}$$

where f_i,f_j are two arbitrary constants.

According to Formula (78), one-soliton solution of the Kundu-NLS Equation (2) is shown in Figure 1.

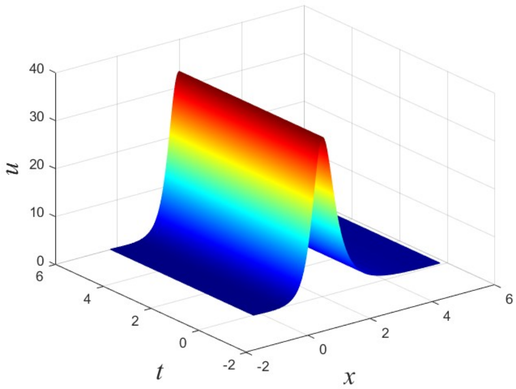

Figure 1. One-soliton solution of (65) with $\eta = 1, a = 1, \xi = 1, \phi = 1, z = 1 + i$.

4. Conclusions and Remarks

In this study, we systematically investigate the Kundu-NLS equation by using the $\bar{\partial}$-dressing method. By employing matrix spectral analysis, spectral problems regarding time and space were obtained, which were reduced to Lax pairs of Kundu-NLS equations. In order to obtain the solution, matrix transformation was applied. The soliton solution is obtained by using the $\bar{\partial}$-dressing method. In short, the $\bar{\partial}$-dressing method is an effective method for solving equations in integrable systems, and this $\bar{\partial}$-dressing method shows great potential to address equations in integrable systems in the future.

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