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Global Optimization and Common Best Proximity Points for Some Multivalued Contractive Pairs of Mappings

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Received: 14 July 2020; Accepted: 2 September 2020; Published: 7 September 2020

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Abstract: In this paper, we study a problem of global optimization using common best proximity point of a pair of multivalued mappings. First, we introduce a multivalued Banach-type contractive pair of mappings and establish criteria for the existence of their common best proximity point. Next, we put forward the concept of multivalued Kannan-type contractive pair and also the concept of weak Δ-property to determine the existence of common best proximity point for such a pair of maps.

Keywords: common best proximity point; fixed point; contraction map; complete metric space; multivalued map; optimization

JEL Classification: 47H10; 54H25; 54E50

1. Preliminaries

Let $(\Im, \rho)$ be a complete metric space and let $CB(\Im)$ denote the class of all nonempty closed and bounded subsets of the nonempty set $\Im$. For $\mathcal{A}, \mathcal{B} \in CB(\Im)$, the function $\mathcal{H} : CB(\Im) \times CB(\Im) \to [0, +\infty)$ defined by

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max\{\sup_{\xi \in \mathcal{B}} \Delta(\xi, \mathcal{A}), \sup_{\delta \in \mathcal{A}} \Delta(\delta, \mathcal{B})\},$$

where $\Delta(\delta, \mathcal{B}) = \inf_{\xi \in \mathcal{B}} \rho(\delta, \xi)$, is a metric on $CB(\Im)$.

For any two non-empty subsets $\mathcal{A}, \mathcal{B}$ of the metric space $(\Im, \rho)$, we shall use the following notations:

$$\mathcal{A}_{\mathcal{B}} = \{\theta \in \mathcal{A} : \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}) \text{ for some } \xi \in \mathcal{B}\},$$

$$\mathcal{B}_{\mathcal{A}} = \{\xi \in \mathcal{B} : \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}) \text{ for some } \theta \in \mathcal{A}\},$$

where $\rho(\mathcal{A}, \mathcal{B}) = \inf\{\rho(\theta, \xi) : \theta \in \mathcal{A}, \xi \in \mathcal{B}\}$.

For $\mathcal{A}, \mathcal{B} \in CB(\Im)$, we have

$$\rho(\mathcal{A}, \mathcal{B}) \leq H(\mathcal{A}, \mathcal{B}).$$

$\theta \in \Im$ is said to be a best proximity point (BPP, in short) of the multivalued map $\Gamma : \Im \to CB(\Im)$ if $\Delta(\theta, \Gamma\theta) = \rho(\mathcal{A}, \mathcal{B})$. $v \in \Im$ is called a fixed point of the multivalued map $\Gamma : \Im \to CB(\Im)$ if $v \in \Gamma v$.

Let $\Psi, \Omega : \mathcal{A} \to CB(\mathcal{B})$ be two multivalued maps. An element $\theta^* \in \mathcal{A}$ is said to be a common best proximity point (CBPP, in short) of Ψ and Ω if and only if

$$\Delta(\theta^*, \Psi\theta^*) = \rho(\mathcal{A}, \mathcal{B}) = \Delta(\theta^*, \Omega\theta^*).$$

Remark 1.

1. *In the metric space $(CB(\Im), \mathcal{H})$, $\theta \in \Im$ is a fixed point of Γ if and only if $\Delta(\theta, \Gamma\theta) = 0$. In general, $\theta \in \Gamma\xi$ if and only if $\Delta(\theta, \Gamma\xi) = 0$ for any $\theta, \xi \in \Im$.*
2. *For two closed sets $\mathcal{A}, \mathcal{B}$, when $\mathcal{A} \cap \mathcal{B} \neq \phi$, we have $\rho(\mathcal{A}, \mathcal{B}) = 0$. In that case, a fixed point and a BPP are identical.*
3. *The function Δ is continuous in the sense that if $\theta_n \to \theta$ as $n \to +\infty$, then $\Delta(\theta_n, \mathcal{A}) \to \Delta(\theta, \mathcal{A})$ as $n \to +\infty$ for any $\mathcal{A} \subseteq \Im$.*
4. *A CBPP is an element at which the functions $\theta \to \Delta(\theta, \Psi\theta)$ and $\theta \to \Delta(\theta, \Omega\theta)$ achieve a global minimum, for $\Delta(\theta, \Psi\theta) \geq \rho(\mathcal{A}, \mathcal{B})$ and $\Delta(\theta, \Omega\theta) \geq \rho(\mathcal{A}, \mathcal{B})$ for all $\theta \in \mathcal{A}$.*

The following lemmas are significant in the present context.

Lemma 1 ([1,2])**.** *Let $(\Im, \rho)$ be a metric space and $\mathcal{A}, \mathcal{B} \in CB(\Im)$. Then*

1. *$\Delta(\theta, \mathcal{B}) \leq \rho(\theta, \gamma)$ for any $\gamma \in \mathcal{B}$ and $\theta \in \Im$;*
2. *$\Delta(\theta, \mathcal{B}) \leq \mathcal{H}(\mathcal{A}, \mathcal{B})$ for any $\theta \in \mathcal{A}$.*

Lemma 2 ([3])**.** *Let $\mathcal{A}, \mathcal{B} \in CB(\Im)$ and let $\theta \in \mathcal{A}$. If $p > 0$, then there exists $\xi \in \mathcal{B}$ such that*

$$\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}) + p.$$

In general, we may not obtain a point $\xi \in \mathcal{B}$ such that

$$\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

But when $\mathcal{B}$ is compact, then such a point ξ exists, i.e., $\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B})$.

The notion of P-property was introduced by Sankar Raj [4]. Further, the idea of weak P property was put forward by Zhang et al. [5] to improve the results of Caballero et al. [6] on Geraghty-contractions.

Definition 1 ([4])**.** *Let $(\Im, \rho)$ be a metric space and $\mathcal{A}, \mathcal{B}$ be two non-empty subsets of $\Im$ such that $\mathcal{A}_{\mathcal{B}} \neq \phi$. The pair $(\mathcal{A}, \mathcal{B})$ satisfies the P-property if and only if $\rho(\theta_1, \xi_1) = \rho(\mathcal{A}, \mathcal{B}) = \rho(\theta_2, \xi_2)$ implies $\rho(\theta_1, \theta_2) = \rho(\xi_1, \xi_2)$, where $\theta_1, \theta_2 \in \mathcal{A}_{\mathcal{B}}$ and $\xi_1, \xi_2 \in \mathcal{B}_{\mathcal{A}}$.*

Definition 2 ([5])**.** *Let $(\Im, \rho)$ be a metric space and $\mathcal{A}, \mathcal{B}$ be two non-empty subsets of $\Im$ such that $\mathcal{A}_{\mathcal{B}} \neq \phi$. The pair $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property if and only if $\rho(\theta_1, \xi_1) = \rho(\mathcal{A}, \mathcal{B}) = \rho(\theta_2, \xi_2)$ implies $\rho(\theta_1, \theta_2) \leq \rho(\xi_1, \xi_2)$, where $\theta_1, \theta_2 \in \mathcal{A}$ and $\xi_1, \xi_2 \in \mathcal{B}$.*

The following well known lemma will be used in the sequel.

Lemma 3. *If $\{\theta_n\}$ is a sequence in a complete metric space $(\Im, \rho)$ such that $\rho(\theta_{n+1}, \theta_n) \leq \lambda\rho(\theta_n, \theta_{n-1})$ for all $n \in \mathbb{N}$, where $\lambda \in (0, 1)$, then $\{\theta_n\}$ is a Cauchy sequence.*

BPPs under different types of contractive conditions have been studied in [7–15]. Moreover, BPPs for different kinds of multivalued mappings have been studied in [16–19]. Some more relevant works may be found in [20–24].

In this paper, we put forward the idea of multivalued Banach-type contractive pair (MVBCP, in short) and with the help of weak P property, establish conditions under which such a pair admits a CBPP. Next, we define the notion of weak Δ-property and a multivalued Kannan-type contractive pair (MVKCP, in short) and prove an existence of CBPP result for that pair.

2. Common Best Proximity Point for MVBCP

In this section, first we define a MVBCP. The corresponding CBPP result follows.

Definition 3. *Let $(\Im, \rho)$ be a metric space and $\mathcal{A}, \mathcal{B}$ be two non-empty subsets of $\Im$. The pair of mappings $\Psi, \Omega : \mathcal{A} \to CB(\mathcal{B})$ is said to be a MVBCP if there exists $\tau \in [0, 1)$ such that*

$$\mathcal{H}(\Omega\theta, \Psi\xi) \leq \tau\rho(\theta, \xi)$$

for all $\theta, \xi \in \Im$.

Theorem 1. *Let $(\Im, \rho)$ be a complete metric space and $\mathcal{A}, \mathcal{B}$ be two non-empty closed subsets of $\Im$ such that $\mathcal{A}_{\mathcal{B}} \neq \phi$ and that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the weak P-property. Let the pair of mappings $\Psi, \Omega : \mathcal{A} \to CB(\mathcal{B})$ be a MVBCP such that $\Psi\theta$ and $\Omega\theta$ are compact for each $\theta \in \mathcal{A}$, and further $\Psi\theta \subseteq \mathcal{B}_{\mathcal{A}}$ and $\Omega\theta \subseteq \mathcal{B}_{\mathcal{A}}$ for all $\theta \in \mathcal{A}_{\mathcal{B}}$. Then Ψ and Ω have a CBPP.*

Proof. Fix $\theta_0 \in \mathcal{A}_{\mathcal{B}}$ and choose $\xi_0 \in \Omega\theta_0 \subseteq \mathcal{B}_{\mathcal{A}}$. By the definition of $\mathcal{B}_{\mathcal{A}}$, we choose $\theta_1 \in \mathcal{A}_{\mathcal{B}}$ such that

$$\rho(\theta_1, \xi_0) = \rho(\mathcal{A}, \mathcal{B}). \tag{1}$$

If $\xi_0 \in \Omega\theta_1 \cap \Psi\theta_1$, then we have

$$\rho(\mathcal{A}, \mathcal{B}) \leq \Delta(\theta_1, \Psi\theta_1) \leq \rho(\theta_1, \xi_0) = \rho(\mathcal{A}, \mathcal{B}), \text{ since } \xi_0 \in \Psi\theta_1,$$

and

$$\rho(\mathcal{A}, \mathcal{B}) \leq \Delta(\theta_1, \Omega\theta_1) \leq \rho(\theta_1, \xi_0) = \rho(\mathcal{A}, \mathcal{B}), \text{ since } \xi_0 \in \Omega\theta_1.$$

Thus $\rho(\mathcal{A}, \mathcal{B}) = \Delta(\theta_1, \Psi\theta_1) = \Delta(\theta_1, \Omega\theta_1)$, i.e., θ_1 is a CBPP of Ψ and Ω. Therefore, assume that $\xi_0 \notin \Omega\theta_1 \cap \Psi\theta_1$. Consider the case $\xi_0 \notin \Psi\theta_1$.

Since $\Psi\theta_1$ is compact, by Lemma 2 and the definition of MVBCP, there exist $\xi_1 \in \Psi\theta_1 \subseteq \mathcal{B}_{\mathcal{A}}$ and $\tau \in [0, 1)$ such that

$$0 < \Delta(\xi_0, \Psi\theta_1) < \rho(\xi_0, \xi_1) \leq \mathcal{H}(\Omega\theta_0, \Psi\theta_1) \leq \tau\rho(\theta_0, \theta_1). \tag{2}$$

Since $\xi_1 \in \mathcal{B}_{\mathcal{A}}$, there exists $\theta_2 \in \mathcal{A}_{\mathcal{B}}$ such that

$$\rho(\theta_2, \xi_1) = \rho(\mathcal{A}, \mathcal{B}). \tag{3}$$

From (1), (3) and weak P-property, we have that

$$\rho(\theta_1, \theta_2) \leq \rho(\xi_0, \xi_1). \tag{4}$$

From (2) and (4), we have that

$$\rho(\theta_1, \theta_2) \leq \rho(\xi_0, \xi_1) \leq \tau\rho(\theta_0, \theta_1). \tag{5}$$

If $\xi_1 \in \Omega\theta_2 \cap \Psi\theta_2$, then like earlier we can show that θ_2 is a CBPP of Ω and Ψ. Thus assume that $\xi_1 \notin \Omega\theta_2 \cap \Psi\theta_2$. Consider the case $\xi_1 \notin \Omega\theta_2$. Since $\Omega\theta_2$ is compact, there exists $\xi_2 \in \Omega\theta_2$ such that

$$0 < \Delta(\xi_1, \Omega\theta_2) < \rho(\xi_1, \xi_2) \leq \mathcal{H}(\Omega\theta_2, \Psi\theta_1)$$
$$\leq \tau\rho(\theta_1, \theta_2). \tag{6}$$

Since $\xi_2 \in \Omega\theta_2 \subseteq \mathcal{B}_{\mathcal{A}}$, there exists $\theta_3 \in \mathcal{A}_{\mathcal{B}}$ such that

$$\rho(\theta_3, \xi_2) = \rho(\mathcal{A}, \mathcal{B}). \tag{7}$$

From (3), (7) and weak P-property, we have that

$$\rho(\theta_2, \theta_3) \leq \rho(\xi_1, \xi_2). \tag{8}$$

Also, from (5) and (6),

$$\rho(\xi_1, \xi_2) \leq \tau\rho(\xi_0, \xi_1). \tag{9}$$

Continuing in this way, we obtain two sequences $\{\theta_n\}$ and $\{\xi_n\}$ in $\mathcal{A}_{\mathcal{B}}$ and $\mathcal{B}_{\mathcal{A}}$ respectively, satisfying

(B1) $\xi_{2n} \in \Omega\theta_{2n} \subseteq \mathcal{B}_{\mathcal{A}}$ and $\xi_{2n+1} \in \Psi\theta_{2n+1} \subseteq \mathcal{B}_{\mathcal{A}}$,

(B2) $\rho(\theta_{n+1}, \xi_n) = \rho(\mathcal{A}, \mathcal{B})$,

(B3) $\rho(\theta_n, \theta_{n+1}) \leq \tau\rho(\theta_{n-1}, \theta_n)$ and $\rho(\xi_n, \xi_{n+1}) \leq \tau\rho(\xi_{n-1}, \xi_n)$,

for each $n = 0, 1, 2, \ldots$.

From (**B3**) and Lemma 3, we observe that $\{\theta_n\}$ and $\{\xi_n\}$ both are Cauchy sequences. Since $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of a complete metric space, we conclude that $\mathcal{A}$ and $\mathcal{B}$ both are complete subspaces.

Hence, there exists $\theta \in \mathcal{A}$ and $\xi \in \mathcal{B}$ such that $\theta_n \to \theta$ and $\xi_n \to \xi$ as $n \to +\infty$.

We claim that $\Omega\theta_n$ converges to $\Omega\theta$. Indeed, if $m > n$, then

$$\begin{aligned}\mathcal{H}(\Omega\theta_n, \Omega\theta) &\leq \mathcal{H}(\Omega\theta_n, \Psi\theta_m) + \mathcal{H}(\Psi\theta_m, \Omega\theta) \\ &\leq \tau[\rho(\theta_n, \theta_m) + \rho(\theta_m, \theta)] \\ &\to 0 \text{ as } n \to +\infty.\end{aligned}$$

Similarly, we can show that $\Psi\theta_n$ converges to $\Psi\theta$.

From (**B2**) we have that

$$\rho(\theta_{n+1}, \xi_n) = \rho(\mathcal{A}, \mathcal{B})$$

for each $n = 0, 1, 2, \ldots$.

This implies

$$\lim_{n \to +\infty} \rho(\theta_{n+1}, \xi_n) = \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}). \tag{10}$$

Again, we claim that $\xi \in \Omega\theta \cap \Psi\theta$. Since $\xi_{2n} \in \Omega\theta_{2n}$, we have

$$\lim_{n \to +\infty} \Delta(\xi_{2n}, \Omega\theta) \leq \lim_{n \to +\infty} \mathcal{H}(\Omega\theta_{2n}, \Omega\theta) = 0, \text{(since } \Omega\theta_n \text{ converges to } \Omega\theta)$$
$$\implies \Delta(\xi, \Omega\theta) = 0.$$

Hence $\xi \in \Omega\theta$.

Also since $\xi_{2n+1} \in \Psi\theta_{2n+1}$, we have

$$\lim_{n\to+\infty} \Delta(\xi_{2n+1}, \Psi\theta) \leq \lim_{n\to+\infty} \mathcal{H}(\Psi\theta_{2n+1}, \Psi\theta) = 0, (\text{since } \Psi\theta_n \text{ converges to } \Psi\theta)$$
$$\Longrightarrow \Delta(\xi, \Psi\theta) = 0.$$

Hence $\xi \in \Psi\theta$. Therefore,

$$\xi \in \Omega\theta \cap \Psi\theta. \tag{11}$$

Finally, using (10) and (11) we have that

$$\rho(\mathcal{A}, \mathcal{B}) \leq \Delta(\theta, \Psi\theta) \leq \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B})$$
$$\Longrightarrow \Delta(\theta, \Psi\theta) = \rho(\mathcal{A}, \mathcal{B}),$$

and

$$\rho(\mathcal{A}, \mathcal{B}) \leq \Delta(\theta, \Omega\theta) \leq \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B})$$
$$\Longrightarrow \Delta(\theta, \Omega\theta) = \rho(\mathcal{A}, \mathcal{B}),$$

Hence θ is a CBPP of Ω and Ψ. $\square$

Next, we present an example in which the pair $(\mathcal{A}, \mathcal{B})$ satisfies only the weak P-property but not the P-property.

Example 1. *Consider $\Im = \mathbb{R}^2$ with the Euclidean metric ρ. Let $\mathcal{A} = \{(-5,0), (0,1), (5,0)\}$ and $\mathcal{B} = \{(\theta, \xi) : \xi = 2 + \sqrt{2-\theta^2}, \theta \in [-\sqrt{2}, \sqrt{2}]\}$. Then $\rho(\mathcal{A}, \mathcal{B}) = \sqrt{3}$ and $\mathcal{A}_{\mathcal{B}} = \{(0,1)\}$, $\mathcal{B}_{\mathcal{A}} = \{(\sqrt{2}, 2), (-\sqrt{2}, 2)\}$.*

Define a pair of multivalued maps $\Omega, \Psi : \mathcal{A} \to CB(\mathcal{B})$ in the following manner:

$$\Omega(-5,0) = \{(0, 2+\sqrt{2})\},\ \Omega(0,1) = \{(-\sqrt{2}, 2), (0, 2+\sqrt{2})\},\ \Omega(5,0) = \{(-1,3), (1,3)\},$$

and

$$\Psi(-5,0) = \{(-\sqrt{2}, 2), (-1,3)\},\ \Psi(0,1) = \{(\sqrt{2}, 2)\},\ \Psi(5,0) = \{(\sqrt{2}, 2), (1,3)\}.$$

By routine calculations, it is easy to check that the condition

$$\mathcal{H}(\Omega\theta, \Psi\xi) \leq \tau\rho(\theta, \xi)$$

is satisfied for all $\theta, \xi \in \Im$ and for $\tau = \frac{19}{20} \in [0,1)$.

Thus the pair Ψ, Ω is a MVBCP.

Finally, we observe that

$$\rho((0,1), (\sqrt{2}, 2)) = \rho((0,1), (-\sqrt{2}, 2)) = \sqrt{3} = \rho(\mathcal{A}, \mathcal{B}),$$

but

$$\rho((0,1), (0,1)) = 0 < \rho((\sqrt{2}, 2), (-\sqrt{2}, 2)) = 2\sqrt{2}.$$

Thus, $(\mathcal{A}, \mathcal{B})$ satisfies weak P-property, but not the P-property. Therefore, all conditions of Theorem 1 are satisfied and since $\Delta((0,1), \Psi(0,1)) = \Delta((0,1), \Omega(0,1)) = \sqrt{3} = \rho(\mathcal{A}, \mathcal{B})$, we conclude that $(0,1)$ is a CBPP of Ψ and Ω.

3. Common Best Proximity Point for MVKCP

In this section, we define the concepts of weak Δ-property and a MVKCP. Combining these two concepts, we establish a CBPP result.

Definition 4. *Consider the metric space $(CB(\Im), \mathcal{H})$ and let $\mathcal{A}, \mathcal{B}$ be two non-empty subsets in $CB(\Im)$ such that $\mathcal{A}_{\mathcal{B}} \neq \phi$. The pair $(\mathcal{A}, \mathcal{B})$ is said to have the weak Δ-property if and only if $\Delta(\theta, \mathcal{U}) = \rho(\mathcal{A}, \mathcal{B}) = \Delta(\xi, \mathcal{V}))$ implies $\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{U}, \mathcal{V})$, for all $\theta, \xi \in \mathcal{A}_{\mathcal{B}}$ and $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}_{\mathcal{A}}$.*

Definition 5. *Let $(\Im, \rho)$ be a metric space and $\mathcal{A}, \mathcal{B}$ be two non-empty subsets of $\Im$. The pair of mappings $\Psi, \Omega : \mathcal{A} \to CB(\mathcal{B})$ (Ψ and Ω may be identical) is said to be a multivalued Kannan-type contractive pair (MVKCP, in short) if there exists $\lambda \in [0, 1)$ such that*

$$\mathcal{H}(\Omega\theta, \Psi\xi) \leq \frac{\lambda}{2}[\Delta(\theta, \Omega\theta) + \Delta(\xi, \Psi\xi) - 2\rho(\mathcal{A}, \mathcal{B})] \tag{12}$$

for all $\theta, \xi \in \Im$.

Remark 2. *If Ψ, Ω is an MVKCP, the condition (12) is satisfied when $\Psi = \Omega$ as well.*

Definition 6 ([25])**.** *Let $(\Im, \rho)$ be a metric space and R be a self-map on $\Im$. R is said to be a Kannan mapping if there exists $0 \leq \lambda < \frac{1}{2}$ such that*

$$\rho(R\theta, R\xi) \leq \lambda\{\rho(\theta, R\theta) + \rho(\xi, R\xi)\},$$

for all $\theta, \xi \in \Im$.

Remark 3. *If $(\Im, \rho)$ is a complete metric space, then a Kannan mapping on $\Im$ possesses a unique fixed point.*

Now we present the main result of this section.

Theorem 2. *Let $(\Im, \rho)$ be a complete metric space and $\mathcal{A}, \mathcal{B}$ be two non-empty closed subsets of $\Im$ such that $\mathcal{A}_{\mathcal{B}} \neq \phi$ and that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the weak Δ-property. Let the pair of mappings $\Psi, \Omega : \mathcal{A} \to CB(\mathcal{B})$ be a MVKCP such that $\Psi\theta \subseteq \mathcal{B}_{\mathcal{A}}$ and $\Omega\theta \subseteq \mathcal{B}_{\mathcal{A}}$ for all $\theta \in \overline{\mathcal{A}}_B$. Then Ψ and Ω have a CBPP.*

Proof. Define the map $\Gamma : \Omega(\overline{\mathcal{A}}_B) \to \mathcal{A}_{\mathcal{B}}$ by

$$\Gamma(S) = \{\theta \in \mathcal{A}_{\mathcal{B}} : \Delta(\theta, S) = \rho(\mathcal{A}, \mathcal{B})\}, \tag{13}$$

for all $S \in \overline{\mathcal{A}}_B$. The map Γ is well defined, for if $\Gamma(S) = \theta_1$ and $\Gamma(S) = \theta_2$, then $\Delta(\theta_1, S) = \rho(\mathcal{A}, \mathcal{B})$ and $\Delta(\theta_2, S) = \rho(\mathcal{A}, \mathcal{B})$. By weak Δ-property, we have $\rho(\theta_1, \theta_2) \leq \mathcal{H}(S, S) = 0$, i.e., $\theta_1 = \theta_2$.

From (13), we have $\Delta(\Gamma(\Omega\theta), \Omega\theta) = \rho(\mathcal{A}, \mathcal{B})$ and $\Delta(\Gamma(\Omega\xi), \Omega\xi) = \rho(\mathcal{A}, \mathcal{B})$ for any $\theta, \xi \in \overline{\mathcal{A}}_B$.

Again, using the weak Δ-property, we have

$$\begin{aligned}
\rho(\Gamma(\Omega\theta), \Gamma(\Omega\xi)) &\leq \mathcal{H}(\Omega\theta, \Omega\xi) \\
&\leq \frac{\lambda}{2}[\Delta(\theta, \Omega\theta) + \Delta(\xi, \Omega\xi) - 2\rho(\mathcal{A}, \mathcal{B})] \\
&\leq \frac{\lambda}{2}[\rho(\theta, \Gamma(\Omega\theta)) + \Delta(\Gamma(\Omega\theta), \Omega\theta) + \rho(\xi, \Gamma(\Omega\xi)) + \Delta(\Gamma(\Omega\xi), \Omega\xi) - 2\rho(\mathcal{A}, \mathcal{B})] \\
&= \frac{\lambda}{2}[\rho(\theta, \Gamma(\Omega\theta)) + \rho(\xi, \Gamma(\Omega\xi)) - 2\rho(\mathcal{A}, \mathcal{B})],
\end{aligned}$$

for any $\theta, \xi \in \overline{\mathcal{A}}_B$ and $\lambda \in [0, 1)$.

It means that the composition map $\Gamma o \Omega : \overline{\mathcal{A}}_B \to \overline{\mathcal{A}}_B$ is a Kannan map from $\overline{\mathcal{A}}_B$ to itself, which is a complete metric space.

Thus, $\Gamma o \Omega$ has a unique fixed point θ_1, i.e., $\Gamma o \Omega(\theta_1) = \theta_1 \in \mathcal{A}_{\mathcal{B}}$, which implies that $\Delta(\theta_1, \Omega(\theta_1)) = \rho(\mathcal{A}, \mathcal{B})$.

Similarly, we can define $\Pi : \Psi(\overline{\mathcal{A}}_B) \to \mathcal{A}_\mathcal{B}$ and obtain a unique fixed point θ_2 of $\Pi o \Psi$ and consequently $\Delta(\theta_2, \Psi(\theta_2)) = \rho(\mathcal{A}, \mathcal{B})$.

Using the weak Δ-property, we have that

$$\begin{aligned}\rho(\theta_1, \theta_2) &\leq \mathcal{H}(\Omega\theta_1, \Psi\theta_2)\\ &\leq \frac{\lambda}{2}[\Delta(\theta_1, \Omega\theta_1) + \Delta(\theta_2, \Psi\theta_2) - 2\rho(\mathcal{A}, \mathcal{B})]\\ &= 0,\end{aligned}$$

which implies that $\theta_1 = \theta_2 = \theta$ (say).

Therefore, $\Delta(\theta, \Omega(\theta)) = \Delta(\theta, \Psi(\theta)) = \rho(\mathcal{A}, \mathcal{B})$. Thus θ is a CBPP of Ω and Ψ. □

4. Conclusions

The concepts of MVBCP, MVKCP and weak Δ-property have been introduced in this paper. Using weak P-property, a CBPP result has been proved for a MVBCP and using the weak Δ-property, a similar result has been established for a MVKCP. The current study is interesting because the proof of our main theorem in Section 2 provides us with a scheme on how to find a CBPP for two multivalued maps. An application of the same has also been discussed in Example 1.

Author Contributions: Author P.D. contributed in Conceptualization, Investigation, Methodology and Writing the original draft; Author H.M.S. contributed in Investigation, Validation, Writing and Editing. The authors thank all the reviewers for their detailed comments resulting in improvement of the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The first author (P. Debnath) acknowledges the UGC-BSR Start-Up Grant vide letter No. F.30-452/2018(BSR) dated 12 Feb 2019 (Ministry of HRD, Govt. of India).

Conflicts of Interest: The authors declare no conflict of interest.

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