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Article

Product States of Infinite Tensor Product of JC-algebras

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Abstract: The objective of our study is to generalize the results on product states of the tensor product of two JC-algebras to infinite tensor product JC-algebras. Also, we characterize the tracial product state of the tensor product of two JC-algebras, and the tracial product state of infinite tensor products of JC-algebras.

Keywords: C*-algebras; JC-algebras; Jordan algebras; universal enveloping algebras; states; tensor products of operator algebras

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1. Introduction

A (complex) *C*-algebra* is a Banach *-algebra $\mathcal{A}$ over $\mathbb{C}$, and satisfies $\|xx^*\| = \|x\|^2$, for all $x \in \mathcal{A}$. If H is a complex Hilbert space then $B(H)$, the bounded linear operators on H, is a C*-algebra in the usual operator norm $\|x\| = \sup\limits_{\|\xi\|\leq 1} \|x\xi\|$, and involution x^* defined by $< x\xi, \eta >=< \xi, x^*\eta >$, for all $\xi, \eta \in H$. If $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, a mapping φ from $\mathcal{A}$ into $\mathcal{B}$ is called *a C*-algebra homomorphism (or, simply a *-homomorphism)* if it is a homomorphism (that is, it is linear, multiplicative, and carries the unit of $\mathcal{A}$ onto that of $\mathcal{B}$, if the C*-algebras have units) with the additional property that $\varphi(x^*) = \varphi(x)^*$ for each x in $\mathcal{A}$. If, in addition, φ is one-to-one (i.e., injective), it is described as a *-isomorphism. It is known that *-homomorphisms between C*-algebras are continuous ([1], Theorem 4.1.8), ([2], Proposition 1.5.2). A representation of a *C*-algebra* $\mathcal{A}$ on a Hilbert space H is a *-homomorphism π from $\mathcal{A}$ into $B(H)$. If, in addition, π is injective, it is called a *faithful representation*. An algebra $\mathcal{A}$ with the product $(x, y) \longmapsto x \circ y$ is called *a Jordan algebra* if the product satisfies $x \circ y = y \circ x$, and $(x \circ y) \circ x^2 = x \circ (y \circ x^2)$ for all $x, y \in \mathcal{A}$. It is clear that if $\mathcal{A}$ is any associative algebra, then $x \circ y = \frac{xy+yx}{2}$, $x, y \in \mathcal{A}$, defines a bilinear, commutative product on $\mathcal{A}$ that satisfies the Jordan product identities. A Jordan Banach algebra is a real Jordan algebra A equipped with a complete norm that satisfies $\|a \circ b\| \leq \|a\|\|b\|$, $a, b \in A$. The self-adjoint part $B(H)_{sa} = \{x \in B(H) : x = x^*\}$ of $B(H)$ is a Jordan Banach algebra with the Jordan product $x \circ y = \frac{xy+yx}{2}$, $x, y \in B(H)_{sa}$. A JB algebra is a Jordan Banach algebra A in which the norm satisfies the two identities: $\|a^2\| = \|a\|^2$ and $\|a^2\| \leq \|a^2 + b^2\|$, for all $a, b \in A$. If A has a unit element $\mathbf{1}$, then it is clear that $\|\mathbf{1}\| = 1$. A closed Jordan subalgebra of the self-adjoint part $B(H)_{sa}$ of all bounded linear operators $B(H)$ on a complex Hilbert space H is called *a JC-algebra*. Sometimes a JB algebra is called a JC-algebra if it is isometrically isomorphic to a JC-algebra defined in this way, and hence, any JC-algebra is a JB algebra.

A positive linear functional ρ on a C*-algebra $\mathcal{A}$ with norm 1 is called *a state*. Accordingly, the set $\mathcal{S}(\mathcal{A})$ of all states of $\mathcal{A}$ is contained in the surface of the unit ball of the dual space $\mathcal{A}^*$ of $\mathcal{A}$. It is known that $\mathcal{S}(\mathcal{A})$ is convex and weak*-closed (i.e., $\sigma(\mathcal{A}, \mathcal{A}^*)$-closed)

([1], p. 257), and hence weak*-compact. By the Krein–Milman Theorem, $\mathcal{S}(\mathcal{A})$ has an exteme point (a point $x_\circ$ of a convex set X in a locally convex space $\mathcal{Y}$ is an exteme point of X, if whenever $x_\circ$ is expressed as a convex combination $x_\circ = \alpha x_1 + (1-\alpha)x_2$, with $0 < \alpha < 1$ and $x_1, x_2 \in X$, then $x_1 = x_2 = x_\circ$). The extreme points of $\mathcal{S}(\mathcal{A})$ are called *pure states*, and $\mathcal{S}(\mathcal{A})$ is the weak*-closure $\overline{co(\mathcal{P}(\mathcal{A}))}^{w}$ of the convex hull $co(\mathcal{P}(\mathcal{A}))$ of the set $\mathcal{P}(\mathcal{A})$ of its extreme points. A state ρ on a C*-algebra $\mathcal{A}$ is said to be *tracial* if $\rho(xx^*) = \rho(x^*x)$ for all $x \in \mathcal{A}$, equivalently, if $\rho(xy) = \rho(yx)$ for all $x, y \in \mathcal{A}$ ([2], Definition 5.3.18). A state ρ on a JC-algebra A is *tracial* if $\rho((a \circ b) \circ c) = \rho(a \circ (b \circ c))$ for all $a, b, c \in A$ ([3], Defintion 5.19).

The structure of the state space of a C*-algebra can significantly influence the structure of the algebra itself, and provides valuable insights into the algebra's properties and behavior. For example, the characteristic of an element in a C*-algebra being the zero element, self-adjoint, positive or normal is determind by the values of the states of the C*-algebra of this element ([1], Theorem 4.3.4). There exists a longstanding known practice of employing the set of states of a C*-algebra as a dual object to illuminate the algebraic structure of the algebra. The exploration of state spaces within operator algebras, along with their geometric properties, holds significant interest due to their role in defining representations of the algebra. The intriguing and captivating aspect lies in how the algebraic structure of the system is intricately encoded within the geometric characteristics of its state space, and consequently, characterizing the state space of operator algebras among all convex sets is equivalent to characterizing the algebras (or their self-adjoint parts) among all ordered linear spaces. The great role of states lies in a fundamental result in the theory of C*-algebras and operator theory, called Gelfand–Naimark–Segal construction (GNS), which provides a powerful tool for representing abstract C*-algebras concretely as algebras of bounded operators on Hilbert spaces, connecting the C*-algebras to the familiar setting of operators on Hilbert spaces. The (GNS) construction asserts that if $\mathcal{A}$ is an involutive Banach algebra with unit (or, with bounded approximate identity, if it is not unital), then to each positive linear functional ρ on $\mathcal{A}$, there is a complex Hilbert space H_ρ, a unit vector ξ_ρ in H_ρ, and a *-representation π_ρ of $\mathcal{A}$ on H_ρ such that $\rho(x) =< \pi_\rho(x)\xi_\rho, \xi_\rho >$, for all $x \in \mathcal{A}$. The representation $\{\pi_\rho, H_\rho\}$ is usually denoted by $\{\pi_\rho, H_\rho, \xi_\rho\}$, and is called *the cyclic representation* of $\mathcal{A}$ induced by ρ. With H a Hilbert space, and ξ a fixed element in H, the equation $\omega_\xi(x) =< x\xi, \xi >, x \in B(H)$, defines a linear functional on $B(H)$, which is clearly bounded by Cauchy–Schwarz inequality ([1], Proposition 2.1.1), and $\omega_\xi(x) \geq 0$ whenever $x \geq 0$, that is, ω_ξ is a positive linear functional on $B(H)$ with $\omega_\xi(I) = \|\xi\|^2$, where I is the identity of $B(H)$. If $\|\xi\| = 1$, then ω_ξ is a state on $B(H)$ called *a vector state*. Having the cyclic representation $\{\pi_\rho, H_\rho, \xi_\rho\}$ induced by a state ρ of a C*-algebra $\mathcal{A}$, the representation $\Phi : \mathcal{A} \to B(H_\Phi)$, where $\Phi = \sum^{\oplus}_{\rho \in S(A)} \pi_\rho$ on the Hilbert space $H_\Phi = \sum^{\oplus}_{\rho \in S(A)} H_\rho$ is a faithful represntation of $\mathcal{A}$, and each state of the C*-algebra $\Phi(\mathcal{A})$ is a vector state ω_ξ, for some unit vector ξ in H_Φ. Hence, each state of $\mathcal{A}$ has the form $\omega_\xi \circ \Phi$. Identifying $\mathcal{A}$ with $\Phi(\mathcal{A})$, one can assume that any C*-algebra acts on some Hilbert space. Thus, states on given C*-algebras can be reconstructed as vector states using the cyclic representation spaces they induce. Undoubtedly, connecting C*-algebras to the familiar setting of operators on Hilbert spaces enables concrete calculations and analyses of abstract C*-algebras. This correspondence is crucial, and used in quantum theory, where states describe the physical properties of quantum systems, and understanding the formulation of quantum dynamics, and the analysis of physical observables in terms of operators is essential.

Product states of tensor product of two JC-algebras were studied by Jamjoom in [4], and further advancements in the theory of tensor products involving infinite families of JC-algebras were established in [5] analogous to well-known results in the context of C*-algebras. Our study aims to generalize the results delineated in [4] to encompass the domain of infinite tensor product of JC-algebras as established in [5]. Additionally, we characterize the tracial product state of the tensor product of two JC-algebras, as well as the tracial product state of infinite tensor products of JC-algebras. Before presenting our results in Section 3, we provide essential background information in Section 2 to assist readers in understanding the scope and purpose of the study.

2. Preliminaries

Tensor product of vector spaces or algebras is a fundamental concept in mathematics with broad importance and applications across various fields. It is a versatile mathematical tool with applications across a wide range of areas, including algebra, geometry, topology, physics, and engineering. It generalizes the concept of multilinear maps, and allows for the construction of a new vector space or algebra that captures multilinear relationships between vectors. In representation theory, particularly in the study of group representations and Lie algebras, tensor products provide a natural framework for combining and decomposing representations, leading to deep insights into the symmetries and structures of algebraic objects.

Let X and Y be vector spaces over a field $\mathbb{K}$ (in practice, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). The tensor product of X and Y denoted by $X \underset{\mathbb{K}}{\otimes} Y$, or just $X \otimes Y$, is characterized by the following universal propery: if Z is a $\mathbb{K}$-vector space and $\theta : X \times Y \to Z$ is a $\mathbb{K}$-bilinear map, then there exists a $\mathbb{K}$-linear map $\overline{\theta} : X \otimes Y \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} X \times Y \arrow[r, "\theta"] \arrow[d, "\gamma"] & Z \\ X \otimes Y \arrow[ur, "\overline{\theta}"'] & \end{tikzcd}$$

where $\gamma((x,y)) = x \otimes y$, $x \in X$, $y \in Y$ ([6], Section 4.7), ([7], Section 4.5). A typical tensor in $X \otimes Y$ has the form $\sum\limits_{i=1}^{n} x_i \otimes y_i$, where $x_i \in X$, $y_i \in Y$. Tensor products satisfy the associative and commutative laws, that is, if X, Y and Z are vector spaces over a field $\mathbb{K}$, then $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$, and $X \otimes Y \cong Y \otimes X$ ([6], Proposition 4.7.2), ([7], Theorems 4.5.8 and 4.5.9).

Let H and K be (complex) Hilbert spaces. Then their algebraic tensor product $H \otimes K$, as a vector space, is a pre-Hilbert space, where the inner product on $H \otimes K$ is defined by

$$< \sum_{i=1}^{n} h_i \otimes k_i, \sum_{j=1}^{m} h_j^{'} \otimes k_j^{'} > = \sum_{i=1}^{n}\sum_{j=1}^{m} < h_i, h_j^{'} >< k_i, k_j^{'} >,$$

$h_i, h_j^{'} \in H$, $k_i, k_j^{'} \in K$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. The completion of $H \otimes K$ (denoted also by $H \otimes K$) is called *the Hilbert tensor product* of H and K. Given C*-algebras $\mathcal{A}$ and $\mathcal{B}$, their algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ is a complex involutary algebra in the usual way; $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2$, and $(x \otimes y)^* = x^* \otimes y^*$ for all $x, x_1, x_2 \in \mathcal{A}$ and $y, y_1, y_2 \in \mathcal{B}$. By a representation of $\mathcal{A} \otimes \mathcal{B}$, we mean a *-homomorphism $\pi : \mathcal{A} \otimes \mathcal{B} \to B(H)$, where H is a complex Hilbert space. A pair of representations $\pi_1 : \mathcal{A} \to B(H)$ and $\pi_2 : \mathcal{B} \to B(K)$, where H and K are complex Hilbert spaces, induces a natural representation $\pi_1 \otimes \pi_2 : \mathcal{A} \otimes \mathcal{B} \to B(H) \otimes B(K) \subset B(H \otimes K)$ via

$$((\pi_1 \otimes \pi_2)(x \otimes y))(h \otimes k) = (\pi_1(x))(h) \otimes (\pi_2(y))(k),$$

for all $x \in \mathcal{A}$, $y \in \mathcal{B}$, $h \in H$, $k \in K$. Given C*-algebras $\mathcal{A}$ and $\mathcal{B}$, a *C*-norm* on $\mathcal{A} \otimes \mathcal{B}$ is a norm λ satisfying $\lambda(x^* x) = (x)^2$ for all $x \in \mathcal{A} \otimes \mathcal{B}$. The completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to λ is a C*-algebra, and will be denoted by $\mathcal{A} \underset{\lambda}{\otimes} \mathcal{B}$. It is known that every C*-norm λ on $\mathcal{A} \otimes \mathcal{B}$ is *a cross norm*, that is, $\lambda(x \otimes y) = \|x\|\|y\|$ for all x in $\mathcal{A}$ and y in $\mathcal{B}$ ([8], Corollary 11.3.10). The norm on $\mathcal{A} \otimes \mathcal{B}$ is defined by

$$\|x\|_{\min} = Sup\{\|(\pi_1 \otimes \pi_2)(x)\| : \pi_1, \pi_2 \text{ representations of } \mathcal{A}, \mathcal{B}\},$$

for all $x \in \mathcal{A} \otimes \mathcal{B}$, is *the smallest (minimum) C*-norm* on $\mathcal{A} \otimes \mathcal{B}$, and the norm on $\mathcal{A} \otimes \mathcal{B}$ is defined by

$$\|x\|_{\max} = Sup\{\|\pi(x)\| : \pi \text{ a representations of } \mathcal{A} \otimes \mathcal{B}\},$$

for all $x \in \mathcal{A} \otimes \mathcal{B}$, is *the largest (maximum) C*-norm* on $\mathcal{A} \otimes \mathcal{B}$ (see [2], Definitions 4.4.5 and 4.4.8). It is convenient sometimes to write *min* and *max* instead of $\|.\|_{\min}$ and $\|.\|_{\max}$. These norms are indeed C*-norms, and satisfy $min \leq \lambda \leq max$ for every C*-norm λ on $\mathcal{A} \otimes \mathcal{B}$ ([2], p. 216), ([8], Theorems 11.3.1 and 11.3.4). The fundamental property of the min C*-norm lies deeper, namely, given C*-algebras $\mathcal{A}_i$ and $\mathcal{B}_i$, $i = 1,2$, and a *-homomorphism $\pi_i : \mathcal{A}_i \to \mathcal{B}_i$, then the natural map $\pi_1 \otimes \pi_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{B}_1 \otimes \mathcal{B}_2$ defined by $(\pi_1 \otimes \pi_2)(x_1 \otimes x_2) = \pi_1(x_1) \otimes \pi_2(x_2)$, $x_i \in \mathcal{A}_i, i = 1,2$, extends to a (C*-algebra) homomorphism $\pi : \mathcal{A}_1 \underset{\min}{\otimes} \mathcal{A}_2 \to \mathcal{B}_1 \underset{\min}{\otimes} \mathcal{B}_2$. Further, if π_i is injective, then π is injective. Hence, if $\mathcal{A}_i, \subseteq B(H_i)$, then $\mathcal{A}_1 \underset{\min}{\otimes} \mathcal{A}_2 \subseteq B(H_1) \underset{\min}{\otimes} B(H_2) \subseteq B(H_1 \otimes H_2)$ ([2], Theorem 4.4.9 (iii) and Proposition 4.4.22). If $\mathcal{A}_i$ $(i = 1,2,\ldots,n)$ is a C*-algebra, then by taking a faithful representation $\{\pi_i, H_i\}$ of $\mathcal{A}_i$ for each $i = 1,2,\ldots,n$, and identifying $\mathcal{A}_i$ with $\pi_i(\mathcal{A}_i)$ in $B(H_i)$, the minimum C*-tensor product is seen to be associative, in the sense that there is a *-isomorphim from $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$ onto $(\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_k) \underset{\min}{\otimes} (\mathcal{A}_{k+1} \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n)$ taking $(x_1 \otimes \ldots \otimes x_n)$ to $(x_1 \otimes ... \otimes x_k) \otimes (x_{k+1} \otimes \ldots \otimes x_n)$. As a result of this expressive property, given states $\rho_1, \ldots, \rho_n$ of $\mathcal{A}_1, \ldots, \mathcal{A}_n$, respectively, a unique state ρ on $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$ is described in terms of the given states.

Theorem 1 ([8], Proposition 11.3.8)**.** *Let $\mathcal{A}_i$ be a C*-algebra, and ρ_i is a state of $\mathcal{A}_i$ $(i = 1,2,\ldots,n)$. Then there is a unique state ρ of $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$ such that $\rho(x_1 \otimes ... \otimes x_n) = \rho_1(x_1)\ldots\rho_n(x_n)$, $x_i \in \mathcal{A}_i, i = 1,2,\ldots,n$.*

This state ρ is called a product state of $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$, and is denoted by $\rho = \rho_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \rho_n$. If $\mathcal{A}_i$ is acting on a Hilbert space H_i, and ω_{ξ_i} is the vector state on $\mathcal{A}_i$, arising from a unit vector $\xi_i \in H_i$, then $\omega_{\xi_1 \otimes \ldots \otimes \xi_n} = \omega_{\xi_1} \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \omega_{\xi_n}$ is the vector state on $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$ arising from the unit vector $\xi_1 \otimes \ldots \otimes \xi_n \in H_1 \otimes \ldots \otimes H_n$, that is, for each $x_1 \otimes \ldots \otimes x_n \in \mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$

$$\begin{aligned}
\omega_{\xi_1 \otimes \ldots \otimes \xi_n}(x_1 \otimes \ldots \otimes x_n) &= \ < (x_1 \otimes \ldots \otimes x_n)(\xi_1 \otimes \ldots \otimes \xi_n), \xi_1 \otimes \ldots \otimes \xi_n > \\
&= \ < (x_1(\xi_1) \otimes \ldots \otimes x_n(\xi_n)), \xi_1 \otimes \ldots \otimes \xi_n > \\
&= \ < x_1(\xi_1), \xi_1 > \ldots < x_n(\xi_n), \xi_n > \\
&= \ \omega_{\xi_1}(x_1)\ldots\omega_{\xi_n}(x_n).
\end{aligned}$$

The importance of product states $\rho_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \rho_n$ of $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$, derived from states $\rho_1, \ldots, \rho_n$ on $\mathcal{A}_1, \ldots, \mathcal{A}_n$, respectively, is highlighted in the following theorem. This theorem establishes that the norm of an element $x \in \mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$ can be expressed in terms of product states of $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$. Using the associative property of the minimum C-tensor product $\mathcal{A}_1 \underset{\min}{\otimes} \ldots \underset{\min}{\otimes} \mathcal{A}_n$ of the C*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$, we can take $n = 2$.*

Theorem 2 (see ([2], Theorem 4.4.9 (ii)), ([8], p. 847))**.** *Let $\mathcal{A}_i$ be a C*-algebra, and ρ_i is a state of $\mathcal{A}_i$ $(i = 1,2)$. Then the minimum norm of each $x \in \mathcal{A}_1 \underset{\min}{\otimes} \mathcal{A}_2$ is given by*

$$\|x\|_{\min} = Sup\left\{ \frac{\rho_1 \underset{\min}{\otimes} \rho_2(y^* x^* x y)}{\rho_1 \underset{\min}{\otimes} \rho_2(y^* y)} : y \in \mathcal{A}_1 \underset{\min}{\otimes} \mathcal{A}_2, \rho_i \in S(\mathcal{A}_i), i = 1,2 \right\},$$

Let A and B be JC-algebras, and let $\varphi : A \to B$ be a linear map. Then φ is called a Jordan homomorphism if it preserves the Jordan product, that is, $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$ for all $a, b \in A$.

It is called faithful, if it is injective. It is known that any Jordan homomorphism between JC-algebras is continuous; further, if it is injective, then it is an isometry ([9], 3.4.2 and 3.4.3).

Let A be a JC-algebra, a C-algebra $\mathfrak{A}$ is called the universal enveloping C^*-algebra of A if there is a faithful Jordan homomorphism $\psi : A \to \mathfrak{A}_{sa}$ such that $\psi(A)$ generates $\mathfrak{A}$ as a C^*-algebra, and if $\mathcal{B}$ is a C^*-algebra and $\pi : A \to \mathcal{B}_{sa}$ is a Jordan homomorphism, then there is a *-homomorphism $\overset{\wedge}{\pi} : \mathfrak{A} \to \mathcal{B}$ such that $\pi = \overset{\wedge}{\pi}\psi$ ([3], Proposition 4.36), ([9], Theorem 7.1.8). Such C*-algebra has a unique *-antiautomorphism Φ of period two leaving all points of $\psi(A)$ fixed ([3], Proposition 4.40). The universal enveloping C^*-algebra of A will be denoted by $C^*(A)$, and A will be identified with $\psi(A)$, so that A is assumed to generate $C^*(A)$ as a C*-algebra.*

The reader is referred to [1–3,8,10] for the relevant material of C-algebras, and to [4,5,11,12] for the properties of tensor products of JC-algebras and their universal enveloping C*-algebras.*

Definition 1. *Let A and B be JC-algebras canonically embedded in the self-adjoint parts $C^*(A)_{sa}$, $C^*(B)_{sa}$ of their respective universal enveloping C*-algebras $C^*(A)$, $C^*(B)$, so that, $A \otimes B \subset C^*(A)_{sa} \otimes C^*(B)_{sa}$. Let $J(A \otimes B)$ be the Jordan algebra generated by $A \otimes B$ in $C^*(A)_{sa} \otimes C^*(B)_{sa} = (C^*(A) \otimes C^*(B))_{sa}$ (see [2], Lemma 4.4.4 (i)). If λ is any C*-norm on $C^*(A) \otimes C^*(B)$, the completion $JC(A \underset{\lambda}{\otimes} B)$ of $J(A \otimes B)$ in $C^*(A) \underset{\lambda}{\otimes} C^*(B)$ is called the JC-tensor product of A and B with respect to λ.*

Note that given JC-algebras A and B, and a C-norm λ on $C^*(A) \otimes C^*(B)$, it is not always true that $C^*(JC(A \underset{\lambda}{\otimes} B)) = C^*(A) \underset{\lambda}{\otimes} C^*(B)$ ([12], Theorem 3.4). The necessary and sufficient conditions for this equality to hold are described in ([12], Proposition 2.2).*

Theorem 3 ([12], Proposition 2.2 and Corollary 2.3)**.** *Let A and B be JC-algebras, and let $\lambda = \min$ or $\lambda = \max$ C*-cross-norm on $C^*(A) \otimes C^*(B)$. Then $C^*(JC(A \underset{\lambda}{\otimes} B)) = C^*(A) \underset{\lambda}{\otimes} C^*(B)$.*

To pass from finite to infinite tensor products of JC-algebras, we need the concept of a direct limit of a directed system of JC-algebras.

Definition 2. *(i). A directed system of JC-algebras $\{A_i, \varphi_{ji}\}_{i,j\in I}$ is a family $\{A_i : i \in I\}$ of JC-algebras in which the index set I is directed by a binary relation $\leq$, together with a family $\{\varphi_{ji} : i, j \in I\}$ of Jordan homomorphisms between the JC-algebras, with the property that whenever $i, j \in I$, $i \leq j$, there is a Jordan homomorphism φ_{ji} from A_i into A_j, and if $i, j, k \in I$ such that $i \leq j \leq k$, then $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$.*

(ii). The direct limit of a directed system $\{A_i, \varphi_{ji}\}_{i,j\in I}$ of JC- algebras, denoted by $\varinjlim A_i$, is a JC-algebra A with a family $\{\alpha_i : i \in I\}$ of Jordan homomorphisms that satisfies the following properties:

1. *$\underset{i\in I}{\cup} \alpha_i(A_i)$ is everywhere dense in A.*
2. *For each $i \in I$, there is a Jordan homomorphism $\alpha_i : A_i \to A$ such that $\alpha_i = \alpha_j \circ \varphi_{ji}$, whenever $i \leq j$.*
3. *If B is a JC-algebra, and $\{\psi_i\}_{i\in I}$ is a family of Jordan homomorphisms satisfying (1) and (2) above, then there is a unique Jordan homorphism $\phi : A \to B$ such that $\psi_i = \phi \circ \alpha_i$.*

Theorem 4 ([5], Theorem 2.2 and Corollary 2.3)**.** *Direct limits exist in the category of JC-algebras and Jordan homomorphisms for every directed system $\{A_i, \varphi_{ji}\}_{i,j\in I}$ of JC- algebras, and $C^*(\varinjlim A_i) = \varinjlim C^*(A_i)$.*

Recall that if $A_i, i = 1, 2, 3$, is a JC-algebra canonically embedded in its universal enveloping C-algebra $C^*(A_i)$, then by the associativity of the tensor product, we have $A_1 \otimes A_2 \otimes A_3 \hookrightarrow C^*(A_1) \otimes C^*(A_2) \otimes C^*(A_3)$. Since*

$$
\begin{aligned}
C^*(JC(JC(A_1 \underset{\min}{\otimes} A_2) \underset{\min}{\otimes} A_3) &= C^*(JC(A_1 \underset{\min}{\otimes} A_2)) \underset{\min}{\otimes} C^*(A_3) \\
&= C^*(A_1) \underset{\min}{\otimes} C^*(A_2) \underset{\min}{\otimes} C^*(A_3),
\end{aligned}
$$

we have $JC(JC(A_1 \underset{\min}{\otimes} A_2) \underset{\min}{\otimes} A_3)$ *as the JC-alebra* $JC(A_1 \underset{\min}{\otimes} A_2 \underset{\min}{\otimes} A_3)$ *generated by* $A_1 \otimes A_2 \otimes A_3$ *in* $C^*(A_1) \underset{\min}{\otimes} C^*(A_2) \underset{\min}{\otimes} C^*(A_3)$. *Hence, if F is a finite set, and* $\{A_i : i \in F\}$ *is a family of JC-algebras, then* $JC(\underset{\min}{\otimes} A_i)_{i \in F}$ *is the JC-algebra generated by* $(\otimes A_i)_{i \in F}$ *in* $(\underset{\min}{\otimes} C^*(A_i))_{i \in F}$.

Definition 3 ([5])**.** *Let* $\{A_i : i \in I\}$ *be a family of JC-algebras (not necessarily unitals), and let* $\mathfrak{F}$ *be the family of all finite subsets F of I. Define* $\leq$ *on* $\mathfrak{F}$ *by* $F \leq G$, *if* $F \subseteq G$, $F, G \in \mathfrak{F}$. *For each* $F \in \mathfrak{F}$, *let* $\mathcal{A}_F = JC(\underset{\min}{\otimes} A_i)_{i \in F}$ *be the JC-tensor product of* $\{A_i\}_{i \in F}$, *and note that:*

1. *Whenever* $F \subseteq G$, $F, G \in \mathfrak{F}$, *there is a natural isomorphism* σ_{GF} *from* $JC(\mathcal{A}_F \underset{\min}{\otimes} \mathcal{A}_{G \setminus F})$ *onto* $\mathcal{A}_G$, *by Theorem 1.4 and the associativity of the tensor product.*
2. *The map* $\alpha_{GF} : \mathcal{A}_F \to \mathcal{A}_G$ *defined by*

$$\alpha_{GF}(x) = \text{strong limit } \sigma_{GF}(x \otimes \nu_\beta),$$

is a Jordan homomorphism, by Theorem 3 and ([12], Proposition 1.2), where (ν_β) *is an approximate identity of* $\mathcal{A}_{G \setminus F}$.
3. *If* $F, G, H \in \mathfrak{F}$, *and* $F \subseteq G \subseteq H$, *then* $\alpha_{HF} = \alpha_{HG} \circ \alpha_{GF}$.

Hence, the family $\{\mathcal{A}_F : F \in \mathfrak{F}\}$ *with the Jordan homomorphisms* $\{\alpha_{GF} : F, G \in \mathfrak{F}\}$ *constitutes a directed system of JC-algebras. The JC-direct limit,* $\lim\limits_{\rightarrow} \mathcal{A}_F$, *of the system exists (cf. Theorem 4), and is a JC-algebra, called the tensor product of the infinite family of JC-algebras, and is denoted by* $JC(\underset{\min}{\otimes} A_i)_{i \in I}$ *([5], Definition 2.6).*

The universal enveloping C*-algebra of the tensor product of an infinite family of JC-algebras is characterized in the following:

Theorem 5 ([5], Theorem 2.7)**.** *Let* $\{A_i : i \in I\}$ *be a family of JC-algebras. Then*

$$C^*(JC(\underset{\min}{\otimes} A_i)_{i \in I}) = (\underset{\min}{\otimes} C^*(A_i))_{i \in I}$$

3. The Main Results

Within this section, our aim is to extend the scope of Theorems 4 and 5 as outlined in [5] (cf. Theorem 6) to encompass the realm of an infinite family of JC-algebras. Subsequently, we delve into providing the Jordan counterpart to Propositions 11.4.6, 11.3.2, and 11.4.7 elucidated in [8]. This endeavor seeks to offer a comprehensive understanding of the applicability and implications of these propositions within the context of JC-algebras, thereby enriching the discourse and expanding the theoretical framework within this domain.

Theorem 6 ([4], Theorems 4 and 6)**.** *Let A and B be JC-algebras, and let* ν, σ *be states of A and B, respectively. Then there is a state* ρ *of* $JC(A \underset{\min}{\otimes} B)$ *such that* $\rho(x \otimes y) = \nu(x)\sigma(y)$, *for all* $x \in A$ *and* $y \in B$.

It should be noted that Theorem 6 can be applied to any finite family $\{A_k : k = 1, ..., n\}$ of JC-algebras. The state occuring in this theorem, denoted by $\nu \underset{\min}{\otimes} \sigma$, is called *a product state of* $JC(A \underset{\min}{\otimes} B)$.

The characterization of the product state in the tensor product of an infinite family of C*-algebras is presented below.

Theorem 7 ([8], Proposition 11.4.6)**.** *Let* $\{\mathfrak{A}_i : i \in I\}$ *be a family of C*-algebras,* ρ_i *a state of* $\mathfrak{A}_i$. *Then there is a state* ρ *of the tensor product* $(\underset{\min}{\otimes} \mathfrak{A}_i)_{i \in I}$ *of the family* $\{\mathfrak{A}_i : i \in I\}$ *such that*

$$\rho(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \rho_{\alpha_1}(x_1)\rho_{\alpha_2}(x_2) \ldots \rho_{\alpha_n}(x_n),$$

whenever $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are distict elements of* I, *and* $x_i \in \mathfrak{A}_i$, $i = 1, 2, \ldots, n$.

The state ρ occurring in Theorem 7, denoted by $(\underset{\min}{\otimes} \rho_i)_{i \in I}$, is described as a product state of $(\underset{\min}{\otimes} \mathfrak{A}_i)_{i \in I}$.

The Jordan analogue of Theorem 7 for JC-algebras is given in the following result:

Theorem 8. *Let* $\{A_i : i \in I\}$ *be a family of JC-algebras, and let* ρ_i *be a state of* A_i, *for each* $i \in I$. *Then there is a state* ρ *of* $JC(\underset{\min}{\otimes} A_i)_{i \in I}$ *such that*

$$\rho(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \rho_{\alpha_1}(a_1)\rho_{\alpha_2}(a_2) \ldots \rho_{\alpha_n}(a_n),$$

whenever $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are distinct elements of* I, *and* $a_i \in A_i$, $i = 1, 2, \ldots, n$.

Proof. Let $\mathfrak{F}$ be the family of all finite subsets F of I, and consider the directed system of JC-algebras $\{\mathcal{A}_F : \alpha_{GF}\}_{F,G \in \mathfrak{F}}$, where $\mathcal{A}_F = JC(\underset{\min}{\otimes} A_i)_{i \in F}$ is the JC-tensor product of $\{A_i\}_{i \in F}$, and $\alpha_{GF} : \mathcal{A}_F \to \mathcal{A}_G$, $F \subseteq G$, is the Jordan homomorphism defined by $\alpha_{GF}(x) =$ strong limit $\sigma_{GF}(x \otimes v_\beta)$, $x \in \mathcal{A}_F$ and (v_β) is an approximate identity of $\mathcal{A}_{G \setminus F}$. By ([9], Theorem 7.1.8) and ([1], Theorem 4.3.13(i)), ρ_i extends to a state $\hat{\rho}_i$ of $C^*(A_i)$. Hence, by Theorem 7, there is a state $\hat{\rho}$ of the tensor product $(\underset{\min}{\otimes} C^*(A_i))_{i \in I}$ of the family $\{C^*(A_i) : i \in I\}$ such that for each $F \in \mathfrak{F}$, $F = \{\alpha_i : i = 1, \ldots, n\}$, and $x_i \in C^*(A_i)$, $i = 1, 2, \ldots, n$, we have

$$\hat{\rho}(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \hat{\rho}_{\alpha_1}(x_1)\hat{\rho}_{\alpha_2}(x_2) \ldots \hat{\rho}_{\alpha_n}(x_n).$$

Since $JC(\underset{\min}{\otimes} A_i)_{i \in I} \subseteq C^*(JC(\underset{\min}{\otimes} A_i)_{i \in I}) = (\underset{\min}{\otimes} C^*(A_i))_{i \in I}$ (cf. Theorem 5), the restriction $\rho = \hat{\rho}\mid_{JC(\underset{\min}{\otimes} A_i)_{i \in I}}$ is a state of $JC(\underset{\min}{\otimes} A_i)_{i \in I}$ satisfying

$$\begin{aligned} \hat{\rho}(a_1 \otimes a_2 \otimes \ldots \otimes a_n) &= \hat{\rho}_{\alpha_1}(a_1)\hat{\rho}_{\alpha_2}(a_2) \ldots \hat{\rho}_{\alpha_n}(a_n) \\ &= \rho_{\alpha_1}(a_1)\rho_{\alpha_2}(a_2) \ldots \rho_{\alpha_n}(a_n), \end{aligned}$$

for each $F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in $\mathfrak{F}$, and $a_i \in A_i$, $i = 1, 2, \ldots, n$. That is, $\rho\mid_{\mathcal{A}_F}$ is the product state $\rho_F = (\underset{\min}{\otimes} \rho_{\alpha_i})_{i \in F}$ on the JC-subalgebra $\mathcal{A}_F = JC(\underset{\min}{\otimes} A_i)_{i \in F}$ of $JC(\underset{\min}{\otimes} A_i)_{i \in I}$. Note that if $G = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \in \mathfrak{F}$, $G \supseteq F$, and if $a_i \in A_i$, $i = 1, 2, \ldots, m$, then we have

$$\begin{aligned} \rho_G(a_1 \otimes a_2 \otimes \ldots \otimes a_m) &= \rho_{\alpha_1}(a_1)\rho_{\alpha_2}(a_2) \ldots \rho_{\alpha_n}(a_n) \ldots \rho_{\alpha_m}(a_m) \\ &= \rho_F(a_1 \otimes a_2 \otimes \ldots \otimes a_n)\rho_{\alpha_{n+1}}(a_{n+1}) \ldots \rho_{\alpha_m}(a_m) \\ &= \rho_F(a_1 \otimes a_2 \otimes \ldots \otimes a_n)\rho_{G \setminus F}(a_{n+1} \otimes \ldots \otimes a_m) \\ &= (\rho_F \underset{\min}{\otimes} \rho_{G \setminus F})(a_1 \otimes a_2 \otimes \ldots \otimes a_n \otimes a_m). \end{aligned}$$

That is, $\rho_F = \rho_G\mid_{\mathcal{A}_F}$. Consequently, the map $\rho_\circ : \cup \mathcal{A}_F \to \mathbb{R}$, defined by

$$\begin{aligned} \rho_\circ(a_1 \otimes a_2 \otimes \ldots \otimes a_n) &= \rho_F(a_1 \otimes a_2 \otimes \ldots \otimes a_n) \\ &= (\underset{\min}{\otimes} \rho_{\alpha_i})_{i \in F}(a_1 \otimes a_2 \otimes \ldots \otimes a_n) \\ &= \rho_{\alpha_1}(a_1)\rho_{\alpha_2}(a_2) \ldots \rho_{\alpha_n}(a_n), \end{aligned}$$

$a_i \in A_i$, $i = 1, 2, \ldots, n$, is a linear bounded functional. Since $\|\rho_F\| = 1$, for each $F \in \mathfrak{F}$, we can easily see that $\|\rho_\circ\| = 1$. So, $\rho_\circ$ extends uniquely, by contiuity, to a state σ on $JC(\underset{\min}{\otimes} A_i)_{i \in I}$, since $\cup \mathcal{A}_F$ is everywhere dense in $JC(\underset{\min}{\otimes} A_i)_{i \in I}$, and hence, we have $\sigma = \rho$.

The state ρ occurring in Theorem 8, denoted by $(\underset{\min}{\otimes}\rho_i)_{i\in I}$, is called *a product state of* $JC(\underset{\min}{\otimes}A_i)_{i\in I}$. From the above argument, we note that given a product state on $JC(\underset{\min}{\otimes}A_i)_{i\in I}$, the component states ρ_{α_i} are uniquely determined, since $\rho_{\alpha_i} = \rho\,|_{\mathcal{A}_{\{\alpha_i\}}=A_i}$. □

Theorem 9 ([8], Propositions 11.3.2 and 11.4.7)**.** *(i) Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*- algebras, and let ν and σ be states of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then the product state $\nu \underset{\min}{\otimes} \sigma$ of $\mathfrak{A} \underset{\min}{\otimes} \mathfrak{B}$ is tracial if and only if ν and σ are tracial.*

(ii) Let $\{\mathfrak{A}_i : i \in I\}$ be a family of C-algebras, ρ_i a state of $\mathfrak{A}_i$. Then the product state $(\underset{\min}{\otimes}\rho_i)_{i\in I}$ of $(\underset{\min}{\otimes}\mathfrak{A}_i)_{i\in I}$ is pure if and only if each ρ_i is pure.*

The Jordan analogue of Theorem 9(i) for JC-algebras is given in the following result:

Theorem 10. *Let ρ and ν be states on JC-algebras A and B, respectively. Then the product state $\rho \underset{\min}{\otimes} \nu$ of $JC(A \underset{\min}{\otimes} B)$ is tracial if and only if ρ and ν are tracial.*

Proof. Suppose that ρ and ν are tracial states. By ([4], Theorems 4 and 6), $\sigma = \rho \underset{\min}{\otimes} \nu$ is a state of $JC(A \underset{\min}{\otimes} B)$, where $\sigma(a \otimes b) = \rho(a)\nu(b)$, for all $a \in A$, and $b \in B$. Now, for $i = 1, 2, 3$, let $x_i \in A \otimes B$ be a simple tensor, say $x_i = a_i \otimes b_i$, $a_i \in A$, $b_i \in B$. Since ρ and ν are tracial states, we have $\rho((a_1 \circ a_2) \circ a_3) = \rho(a_1 \circ (a_2 \circ a_3))$, and $\nu((b_1 \circ b_2) \circ b_3) = \nu(b_1 \circ (b_2 \circ b_3))$. Hence, by definition of the multiplication on $A \otimes B$, we have

$$\begin{aligned}
\sigma((x_1 \circ x_2) \circ x_3) &= \sigma((a_1 \otimes b_1 \circ a_2 \otimes b_2) \circ a_3 \otimes b_3) \\
&= \sigma((a_1 \circ a_2 \otimes b_1 \circ b_2) \circ a_3 \otimes b_3) \\
&= \sigma((a_1 \circ a_2) \circ a_3 \otimes (b_1 \circ b_2) \circ b_3) \\
&= (\rho \underset{\min}{\otimes} \nu)((a_1 \circ a_2) \circ a_3 \otimes (b_1 \circ b_2) \circ b_3)) \\
&= \rho((a_1 \circ a_2) \circ a_3)\nu((b_1 \circ b_2) \circ b_3)) \\
&= \rho(a_1 \circ (a_2 \circ a_3))\nu(b_1 \circ (b_2 \circ b_3)) \\
&= (\rho \underset{\min}{\otimes} \nu)(a_1 \circ (a_2 \circ a_3) \otimes b_1 \circ (b_2 \circ b_3)) \\
&= \sigma(x_1 \circ (x_2 \circ x_3)).
\end{aligned}$$

Now, let $x, y, z \in A \otimes B$, such that $x = \sum_{i=1}^{n} a_i \otimes b_i$, $y = \sum_{j=1}^{m} a'_j \otimes b'_j$, and $z = \sum_{k=1}^{r} a''_k \otimes b''_k$, where $a_i, a'_j, a''_k \in A$ and $b_i, b'_j, b''_k \in B$. Then it is easy to see that

$$(x \circ y) \circ z = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(a_i \circ a'_j) \circ a''_k \otimes (b_i \circ b'_j) \circ b''_k\,,$$

and

$$x \circ (y \circ z) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} a_i \circ (a'_j \circ a''_k) \otimes b_i \circ (b'_j \circ b''_k).$$

The linearity of ρ and ν implies that

$$\begin{aligned}
\sigma((x \circ y) \circ z) &= \sigma(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(a_i \circ a'_j) \circ a''_k \otimes (b_i \circ b'_j) \circ b''_k \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}\sigma((a_i \circ a'_j) \circ a''_k \otimes (b_i \circ b'_j) \circ b''_k) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}(\rho \underset{\min}{\otimes} \nu)((a_i \circ a'_j) \circ a''_k \otimes (b_i \circ b'_j) \circ b''_k) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}\rho((a_i \circ a'_j) \circ a''_k)\nu((b_i \circ b'_j) \circ b''_k) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}\rho(a_i \circ (a'_j \circ a''_k))\nu(b_i \circ (b'_j \circ b''_k)) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}\sigma(a_i \circ (a'_j \circ a''_k) \otimes b_i \circ (b'_j \circ b''_k)) \\
&= \sigma(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}a_i \circ (a'_j \circ a''_k) \otimes b_i \circ (b'_j \circ b''_k) \\
&= \sigma(x \circ (y \circ z))
\end{aligned}$$

By the contiuity of $\sigma = \rho \underset{\min}{\otimes} \nu$, we have $\sigma((x \circ y) \circ z) = \sigma(x \circ (y \circ z))$ for all $x, y, z \in JC(A \underset{\min}{\otimes} B)$, and hence the product state $\rho \underset{\min}{\otimes} \nu$ is tracial.

Conversely, suppose that the product state $\rho \underset{\min}{\otimes} \nu$ is tracial, and let (u_λ) and (w_β) be increasing approximate identities of A and B, respectively ([9], Proposition 3.5.4). Then $(u_\lambda \otimes w_\beta)$ is an increasing approximate identity of $JC(A \underset{\min}{\otimes} B)$, and $(a \circ u_\lambda \otimes b) \to a \otimes b$, and $(a \otimes b \circ w_\beta) \to a \otimes b$ in norm, for all $a \in A, b \in B$ ([12], Lemma 1.1). Let $a_i \in A, b_i \in B, i = 1, 2, 3$. Since ν is a state of B, $\lim \nu(w_\beta) = \|\nu\| = 1$ (see ([9], Lemma 3.6.3)), and since $\rho \underset{\min}{\otimes} \nu$ is a product state, we have

$$\begin{aligned}
\rho((a_1 \circ a_2) \circ a_3) &= \rho((a_1 \circ a_2) \circ a_3).1 \\
&= \rho((a_1 \circ a_2) \circ a_3). \lim \nu(w_\beta) \\
&= \lim(\rho \underset{\min}{\otimes} \nu)((a_1 \circ a_2) \circ a_3) \otimes w_\beta) \\
&= \lim(\rho \underset{\min}{\otimes} \nu)((a_1 \otimes w_\beta \circ a_2 \otimes w_\beta) \circ a_3 \otimes w_\beta) \\
&= \lim(\rho \underset{\min}{\otimes} \nu)(a_1 \otimes w_\beta \circ (a_2 \otimes w_\beta \circ a_3 \otimes w_\beta)) \\
&= \rho(a_1 \circ (a_2 \circ a_3). \lim \nu(w_\beta) \\
&= \rho(a_1 \circ (a_2 \circ a_3)\ .
\end{aligned}$$

Hence, ρ is a tracial state on A. A similar argument shows that ν is a tracial state on B. Note that the above steps are straightforward if A and B are unital JC-algebras. □

Theorem 11. *Let $\{A_i : i \in I\}$ be a family of JC-algebras, and let ρ_i be a state of A_i, for each $i \in I$. Then the product state $\rho = (\underset{\min}{\otimes} \rho_i)_{i \in I}$ of $JC(\underset{\min}{\otimes} A_i)_{i \in I}$ is tracial if and only if ρ_i is tracial for each $i \in I$.*

Proof. Consider the directed system of JC-algebras $\{\mathcal{A}_F : \alpha_{GF}\}_{F,G \in \mathfrak{F}}$, where $\mathfrak{F}$ is the family of all finite subsets F of I, $\mathcal{A}_F = JC(\underset{\min}{\otimes} A_i)_{i \in F}$ is the JC-tensor product of $\{A_i\}_{i \in F}$, and α_{GF} is a Jordan homomorphism of $\mathcal{A}_F$ into $\mathcal{A}_G$, whenever $F \subseteq G$, $F, G \in \mathfrak{F}$ (cf. begining of the proof of Theorem 8). Suppose that ρ_i is tracial for each $i \in I$. Then, by Theorem 10, the product state $\rho_F = (\underset{\min}{\otimes} \rho_i)_{i\in F}$ is a tracial state on $\mathcal{A}_F$, for each $F \in \mathfrak{F}$. Note that $\rho_F = \rho\mid_{\mathcal{A}_F}$, where $\rho = (\underset{\min}{\otimes} \rho_i)_{i\in I}$ is the product state of $JC(\underset{\min}{\otimes} A_i)_{i\in I}$ occurring in Theorem 8. If $x, y, z \in \underset{F\in\mathfrak{F}}{\cup} \mathcal{A}_F$, then $x \in \mathcal{A}_F, y \in \mathcal{A}_G, z \in \mathcal{A}_H$, for some $F, G, H \in \mathfrak{F}$, which implies that $x, y, z \in F \cup G \cup H \in \mathfrak{F}$. Since the product state $\rho_{F\cup G\cup H} = (\underset{\min}{\otimes} \rho_i)_{i\in F\cup G\cup H}$ is tracial, by Theorem 10, we have

$$\rho_{F\cup G\cup H}((x \circ y) \circ z) = \rho_{F\cup G\cup H}(x \circ (y \circ z)).$$

This implies that ρ is tricial on $\underset{F\in\mathfrak{F}}{\cup} \mathcal{A}_F$, and hence, by contiuity, ρ is tracial on the norm closure $\overline{\underset{F\in\mathfrak{F}}{\cup} \mathcal{A}_F}^n = JC(\underset{\min}{\otimes} A_i)_{i\in I}$ of $\underset{F\in\mathfrak{F}}{\cup} \mathcal{A}_F$.

The converse is immidiate, since $\rho_i = \rho_{\{i\}} = \rho\mid_{\mathcal{A}_{\{i\}}=A_i}$, for each $i \in I$.

The Jordan analogue of Theorem 9(ii) is given in the the following theorem, but first recall that if A is JB-algebra, and $r : \mathcal{S}(C^*(A)) \to S(A)$ is the restriction map of the state space $\mathcal{S}(C^*(A))$ of $C^*(A)$ onto the state space $S(A)$ of A, then by the Krein–Milman theorem, $P(A) \subseteq r(\mathcal{P}(C^*(A)))$, where $P(A), \mathcal{P}(C^*(A))$ are the set of pure states of $A, C^*(A)$, respectively. The inverse image $r^{-1}(r(\overset{\wedge}{\rho}))$ of any $\overset{\wedge}{\rho} \in \mathcal{P}(C^*(A))$ equals the line segment $[\overset{\wedge}{\rho}, \Phi^*(\overset{\wedge}{\rho})]$, which degenerates to a single point if $\Phi^*(\overset{\wedge}{\rho}) = \overset{\wedge}{\rho}$, where $\Phi^* : C^*(A)^* \to C^*(A)^*$ is the adjoint map of the *-antiautomorphism Φ of $C^*(A)$ ([10], Proposition 5.5). Consequently, if ρ is a pure state of JC-algebra A, then there is a pure state $\overset{\wedge}{\rho}$ of $C^*(A)$ such that $r(\overset{\wedge}{\rho}) = \rho$ (see also ([8], Theorem 11.3.13), ([13], Proposition 5.3.3)). $\square$

Theorem 12. *Let $\{A_i : i \in I\}$ be a family of JC-algebras, and let ρ_i be a state of A_i, for each $i \in I$. Then the product state $\rho = (\underset{\min}{\otimes} \rho_i)_{i\in I}$ of $JC(\underset{\min}{\otimes} A_i)_{i\in I}$ is pure if and only if ρ_i is pure for each $i \in I$.*

Proof. We start (as in the begining of the proof of Theorem 8) by considering the directed system of JC-algebras $\{\mathcal{A}_F : \alpha_{GF}\}_{F,G\in\mathfrak{F}}$, where $\mathfrak{F}$ is the family of all finite subsets F of I, and $\mathcal{A}_F = JC(\underset{\min}{\otimes} A_i)_{i\in F}$. Suppose that the product state $\rho = (\underset{\min}{\otimes} \rho_i)_{i\in I}$ of $JC(\underset{\min}{\otimes} A_i)_{i\in I}$ is pure. If $\rho_{i_\circ}$ is not pure for some $i_\circ \in I$, then $\rho_{i_\circ} = t\rho_1^{(i_\circ)} + (1-t)\rho_2^{(i_\circ)}$, for some states $\rho_1^{(i_\circ)}, \rho_2^{(i_\circ)}$ of $A_{i_\circ}$, $0 < t < 1$, which implies that ρ is not pure, since in this case $\rho = (t\rho_1^{(i_\circ)} \underset{\min}{\otimes} \rho_i + (1-t)\rho_2^{(i_\circ)} \underset{\min}{\otimes} \rho_i)_{i\in I, i\neq i_\circ}$. Hence, ρ_i is pure for each $i \in I$.

Conversely, suppose that ρ_i is a pure state of A_i, for each $i \in I$, and let $\overset{\wedge}{\rho}_i$ be a pure state of $C^*(A_i)$ such that $r(\overset{\wedge}{\rho}_i) = \overset{\wedge}{\rho}_i\mid_{A_i} = \rho_i$. Then $\overset{\wedge}{\rho} = (\underset{\min}{\otimes} \overset{\wedge}{\rho}_i)_{i\in I}$ is a pure state of $(\underset{\min}{\otimes} C^*(A_i))_{i\in I}$, by Theorem 9(i). Since $C^*(JC(\underset{\min}{\otimes} A_i)_{i\in I}) = (\underset{\min}{\otimes} C^*(A_i))_{i\in I}$ (cf. Theorem 5), we have $\rho = r(\overset{\wedge}{\rho})$ as a pure state of $JC(\underset{\min}{\otimes} A_i)_{i\in I}$. Note that $\overset{\wedge}{\rho}_F = (\underset{\min}{\otimes} \overset{\wedge}{\rho}_{\alpha_i})_{i\in F} = \overset{\wedge}{\rho}\mid_F$ is a pure state of $(\underset{\min}{\otimes} C^*(A_i))_{i\in F}$, for each $F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \mathfrak{F}$, by ([8], Proposition 11.3.2), where

$$\overset{\wedge}{\rho}_F(x_{\alpha_1} \otimes x_{\alpha_2} \otimes \ldots \otimes x_{\alpha_n}) = \overset{\wedge}{\rho}_{\alpha_1}(x_{\alpha_1})\overset{\wedge}{\rho}_{\alpha_2}(x_{\alpha_2}) \ldots \overset{\wedge}{\rho}_{\alpha_n}(x_{\alpha_n}),$$

and $x_{\alpha_i} \in C^*(A_{\alpha_i})$, $i = 1, 2, \ldots, n$. Since $C^*(JC(\underset{\min}{\otimes} A_{\alpha_i})_{i\in F}) = (\underset{\min}{\otimes} C^*(A_{\alpha_i}))_{i\in F}$, we have $\rho_F = r(\overset{\wedge}{\rho}_F)$ as a pure state of $JC(\underset{\min}{\otimes} A_{\alpha_i})_{i\in F}$, which satisfies

$$\begin{aligned} \rho_F(a_{\alpha_1} \otimes a_{\alpha_2} \otimes \ldots \otimes a_{\alpha_n}) &= \overset{\wedge}{\rho}_{\alpha_1}(a_{\alpha_1})\overset{\wedge}{\rho}_{\alpha_2}(a_{\alpha_2}) \ldots \overset{\wedge}{\rho}_{\alpha_n}(a_{\alpha_n}) \\ &= \rho_{\alpha_1}(a_{\alpha_1})\rho_{\alpha_2}(a_{\alpha_2}) \ldots \rho_{\alpha_n}(a_{\alpha_n}). \end{aligned}$$

That is, for each $F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \mathfrak{F}$, the pure state ρ_F is the product state $(\underset{\min}{\otimes} \rho_{\alpha_i})_{i \in F}$ of the pure states $\rho_{\alpha_1}, \rho_{\alpha_2}, \ldots, \rho_{\alpha_n}$ of $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$. It follows that

$$\rho = r(\overset{\wedge}{\rho}) = r((\underset{\min}{\otimes} \overset{\wedge}{\rho}_i)_{i \in I}) = (\underset{\min}{\otimes} r(\overset{\wedge}{\rho}_i))_{i \in I} = (\underset{\min}{\otimes} \rho_i)_{i \in I},$$

and the proof is complete. □

4. Conclusions

It is evident that product states in the infinite tensor product of JC-algebras are deeply related to representation theory and operator algebras, as each JC-algebra resides within the self-adjoint part of a C*-algebra. Consequently, studying and understanding the structure of product states in the infinite tensor product of JC-algebras can provide insights into the algebraic properties of operators acting on infinite tensor product spaces. Currently, in practice, the investigation of product states within the infinite tensor product of C*-algebras is recognized as significant across diverse domains of mathematics and theoretical physics. In addition to representation theory and operator algebras, it serves as a valuable tool for establishing a mathematical framework to comprehend the entanglement structure, thermodynamic behavior, and algebraic properties of quantum systems with infinitely many degrees of freedom. Since each JC-algebra generates a C*-algebra and JC-algebras exhibit a distinct and robust relationship with the C*-algebras they generate in many aspects, we anticipate that this study will have significant applications in mathematics and theoretical physics.

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